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Lie Symmetries of Birational Maps Preserving Genus 0 Fibrations.*

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Abstract

We prove that any planar birational integrable map, which preserves a fibration given by genus 0 curves has a Lie symmetry and some associated invariant measures. The obtained results allow to study in a systematic way the global dynamics of these maps. Using this approach, the dynamics of several maps is described. In particular we are able to give, for particular examples, the explicit expression of the rotation number function, and the set of periods of the considered maps.

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1 Introduction

A planar *rational* map $F : \mathcal{U} \to \mathcal{U}$, where $\mathcal{U} \subseteq \mathbb{K}^2$ is an open set and where $\mathbb{K}$ can either be $\mathbb{R}$ or $\mathbb{C}$, is called *birational* if it has a rational inverse F^{-1} defined in $\mathcal{U}$. Such a map is *integrable* if there exists a non-constant function $V : \mathcal{U} \to \mathbb{K}$ such that $V(F(x, y)) = V(x, y)$,

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which is called a *first integral* of F. If a map F possesses a first integral V, then the level sets of V are invariant under F. We say that a map F preserves a fibration of curves $\{C_h\}$ if each curve C_h is invariant under the iterates of F.

In this paper, we consider integrable birational maps that have *rational first integrals*

$$V(x,y) = \frac{V_1(x,y)}{V_2(x,y)},$$

so that they preserve the fibration given by the algebraic curves

$$V_1(x,y) - h\,V_2(x,y) = 0, \text{ for } h \in \mathrm{Im}(V).$$

The dynamics of planar birational maps on with rational first integrals can be classified in terms of the *genus* of the curves in the fibration associated to the first integral. In summary, it is known that any birational map F preserving a fibration of *nonsingular* curves of generic genus greater or equal than 2 is globally periodic. This is a consequence of Hurwitz automorphisms theorem and Montgomery periodic homeomorphisms theorem. If the genus of the preserved fibration is generically 1, then either F or F^2 are conjugate to a linear action. The reader is referred to [7, 8, 11] and references therein for more details.

We present a systematic way to study the case where $\{V = h\}_{h\in\mathrm{Im}(V)}$ is a *genus* 0 *fibration*, that is, when each curve $\{V = h\}$ have genus 0. As it is shown in the next section, by using rational parameterizations of the curves of the invariant fibration one obtains that on each curve any birational map is conjugate to a Möbius transformation. Using this fact, we will prove that any of these maps possesses a Lie symmetry with an associated invariant measure. The proofs are constructive, so we are able to give the explicit expression of the symmetry and the density of the measure for particular examples. These results permits us to give a global analysis of the dynamics of the maps under consideration. In particular, this approach allows to obtain the explicit expression of the rotation number function associated to the maps defined on an open set foliated by closed invariant curves. The explicitness of the rotation number function is a special feature of these kind of maps, on the contrary of what happens when the maps preserve genus 1 fibrations, and it facilitates the full characterization of the set of periods of the maps.

The paper is structured in two sections. In Section 2 we present the main results of the paper: the ones that ensures the existence of Lie symmetries and invariant measures, Theorem 2 and Corollary 5 respectively, and the one that characterizes the existence of conjugations of birational maps with conjugated associated Möbius maps, Proposition 6. In Section 3, we apply our approach to study the global dynamics of some particular maps and recurrences.

2 Main results

2.1 Dynamics via associated Möbius transformations

From now on we assume that F is a birational map with a rational first integral $V = V_1/V_2$, where both F and V are defined in an open set of the domain of definition of the dynamical system, the *good set* $\mathcal{G}(F) \subset \mathbb{K}^2$. We also assume that there exists a set $\mathcal{H} \subset \mathrm{Im}(V) \subset \mathbb{K}$ such that the set $\mathcal{U} := \{(x, y) \in \mathbb{K}^2 \colon V(x, y) \in \mathcal{H}\} \cap \mathcal{G}(F)$ is a non empty open set of $\mathbb{K}^2$, and each curve $C_h := \{V_1(x, y) - hV_2(x, y) = 0\}$ is irreducible in $\mathbb{C}$ and has genus 0.

A characteristics of genus 0 curves is that they are rationally parameterizable. Recall that a *rational* curve C in $\mathbb{C}^2$, is an algebraic one such that there exists rational functions $P_1(t), P_2(t) \in \mathbb{C}(t)$ such that for almost all of the values $t \in \mathbb{K}$ we have $(P_1(t), P_2(t)) \in C$; and reciprocally, for almost all point $(x, y) \in C$, there exists $t \in \mathbb{K}$ such that $(x, y) = (P_1(t), P_2(t))$. The map $P(t) = (P_1(t), P_2(t))$ is called a *rational parametrization* of C.

The relationship between rational curves and genus 0 ones is given by the Cayley-Riemann Theorem which states that an algebraic curve in $\mathbb{C}^2$ is rational if and only if has genus 0, see [14, Thm. 4.63] for instance.

Of course a rational curve has not a unique rational parametrization, however one can always obtain a *proper* one, that is, a parametrization P such that P^{-1} is also rational. In other words, one can always find a birational parametrization of a genus 0 curve C, which is unique modulus Möbius transformations, [14, Lemma 4.13].

Observe that given a birational map F preserving an algebraic curve C defined in $\mathbb{K}^2$, by using homogeneous coordinates, one can always extend them to a polynomial map $\widetilde{F}$ and an algebraic curve $\widetilde{C}$ in $\mathbb{K}P^2$. Our main tool, is the following known result:

Proposition 1. *Let F be a birational map defined on an open set $\mathcal{U} \subset \mathbb{K}^2$, preserving the fibration given by algebraic curves of genus 0, $\{\widetilde{C}_h\}_{h\in\mathcal{H}}$, where $\mathcal{H}$ is an open set of $\mathbb{K}$. Then for each $h \in \mathcal{H}$, $\widetilde{F}_{|\widetilde{C}_h}$ is conjugate to a Möbius transformation in $\widehat{\mathbb{K}} = \mathbb{K} \cup \{\infty\}$.*

Proof. Set $\mathbb{K} = \mathbb{C}$, since each curve $\widetilde{C}_h$ has a genus 0, by the Cayley-Riemmann Theorem there is a rational parametrization of $\widetilde{C}_h$, $P_h : \widehat{\mathbb{C}} \to \widetilde{C}_h \subset \mathbb{C}P^2$. This parametrization can be chosen to be proper, so that P^{-1} is also rational. In consequence, the map $\widetilde{M_h} = P_h^{-1} \circ \widetilde{F}_h \circ P_h$, defined in $\widehat{\mathbb{C}}$, is a one-dimensional complex birational map, hence a Möbius transformation.

Consider now the case $\mathbb{K} = \mathbb{R}$. Since every rational real curve can be properly parameterized over the reals [14, Thm. 7.6], then any real birational map preserving a fibration of real algebraic curves of genus 0 can be represented by real Möbius transformations. ∎

The dynamics of Möbius transformations is well understood, see Proposition 7 for instace. As a consequence of the above result, a priori, for planar birational integrable maps preserving a fibration given by genus 0 curves, there can only appear curves with periodic orbits of arbitrary large period, curves filled with dense solutions, as well as curves with one or two attractive and/or repulsive points, which could be located at the infinity line of the projective space $\mathbb{K}P^2$, as in the case the case studied in Proposition 8.

2.2 Existence of Lie symmetries and invariant measures

Some objects that can be associated to integrable birational maps preserving a genus 0 fibration are *Lie symmetries* and *invariant measures*. Formally, a Lie symmetry of a map F defined in an open set $\mathcal{U}$ is a vector field X, also defined in $\mathcal{U}$, such that for any $p \in \mathcal{U}$ it is satisfied the compatibility equation

$$X(F(p)) = DF(p)\,X(p), \tag{1}$$

see [4, 9]. From a dynamical viewpoint, if F is an integrable map with first integral V, and therefore preserving the associated fibration $\{C_h\}$ (see Section 2.1), a Lie symmetry of F is a vector field X with the same first integral V, such that the map F can be seen as the flow φ of this vector field at certain time $\tau(h)$, which only depends on each invariant curve C_h, that is: $F(p) = \varphi(\tau(h), p)$ for all $p \in C_h \cap \mathcal{G}(F)$.

We prove that for birational maps preserving genus 0 fibrations these vector fields exists, and one of them can be explicitly constructed by using a family of parameterizations of the invariant curves.

Theorem 2. *Any birational map F with a rational first integral V, preserving the genus 0 fibration given by $\{V = h\}_{h\in\mathcal{H}}$, has a Lie symmetry. Furthermore, if $\{P_h(t)\}_{h\in\mathcal{H}}$ is a family of proper parameterizations of $\{V = h\}_{h\in\mathcal{H}}$, then there is a Lie symmetry of F in $\mathcal{U}$ given by*

$$X(x, y) = DP_h\left(P_h^{-1}(x, y)\right) \cdot Y_h\left(P_h^{-1}(x, y)\right)\big|_{h=V(x,y)} \tag{2}$$

where

$$Y_h(t) = \left(-b(h) + (d(h) - a(h))t + c(h)t^2\right) \frac{\partial}{\partial t}, \tag{3}$$

and the functions a, b, c and d are defined by the coefficients of the map

$$M_h(t) = P_h^{-1} \circ F \circ P_h(t) = \frac{a(h)t + b(h)}{c(h)t + d(h)}.$$

In the next section, by using the formula given in Equation (2), we will obtain the explicit Lie symmetries of some particular examples of birational maps, see for instance Proposition 10, Corollary 12, or Section 3.4.1. Prior to prove Theorem 2, we present the following preliminary result:

Lemma 3. *A Möbius map $M : \mathbb{K} \to \mathbb{K}$ given by*

$$M(t) = \frac{at+b}{ct+d},$$

admits the Lie symmetry given by the vector field $Y(t) = \left(-b + (d-a)t + ct^2\right) \dfrac{\partial}{\partial t}$.

The proof of the above Lemma is straightforward. We obtained the vector field Y constructively by searching a polynomial one, but to prove the result it is enough to check that it satisfies the compatibility equation $Y(M(t)) = M'(t)Y(t)$.

Proof of Theorem 2. From Proposition 1 we have that for each curve C_h the map $F_{|C_h}$ is conjugate to the Möbius transformation

$$M_h(t) = P_h^{-1} \circ F_{|C_h} \circ P_h(t) = \frac{a(h)t + b(h)}{c(h)t + d(h)}. \tag{4}$$

By Lemma 3, the map M_h has the Lie symmetry Y_h given by (3). Now, by taking the differential of each parametrization $P_h(t)$ we get that for each $(x,y) \in C_h \cap \mathcal{G}(F)$:

$$X(x,y) = DP_h(t)Y_h(t)\Big|_{\substack{t = P_h^{-1}(x,y) \\ h = V(x,y)}} = DP_h(P_h^{-1}(x,y))Y_h(P_h^{-1}(x,y))\Big|_{h = V(x,y)}.$$

Observe that, by construction, X is a vector field tangent to each curve C_h, so V is a first integral of X.

Now we check that X satisfies the compatibility equation (1). Indeed, set $h \in \mathcal{H}$ and take $(x,y) \in C_h$, then by Equation (4) we have $P_h^{-1}(F(x,y)) = M_h(P_h^{-1}(x,y))$. On the other hand, since Y_h is a Lie symmetry of M_h we have $Y(M_h(P_h^{-1}(x,y))) = M_h'(P_h^{-1}(x,y))\, Y(P_h^{-1}(x,y))$, hence

$$\begin{aligned} X(F(x,y)) &= DP_h(P_h^{-1}(F(x,y)))\, Y(P_h^{-1}(F(x,y))) \\ &= DP_h(M_h(P_h^{-1}(x,y)))\, Y(M_h(P_h^{-1}(x,y))) \\ &= DP_h(M_h(P_h^{-1}(x,y)))\, M_h'(P_h^{-1}(x,y))\, Y(P_h^{-1}(x,y))). \end{aligned}$$

Notice that, again by Equation (4), we have

$$DF(x,y) = DP_h(M_h(P_h^{-1}(x,y)))\, M_h'(P_h^{-1}(x,y))\, DP_h^{-1}(x,y),$$

hence

$$DF(x,y)\, DP_h(P_h^{-1}(x,y)) = DP_h(M_h(P_h^{-1}(x,y)))\, M_h'(P_h^{-1}(x,y)),$$

and therefore

$$X(F(x,y)) = DF(x,y)\, DP_h(P_h^{-1}(x,y))\, Y(P_h^{-1}(x,y)) = DF(x,y)\, X(x,y).$$

∎

Notice that Theorem 2 is an step towards a positive answer of the following question: *Given a birational integrable map with rational first integrals, does always exists a rational Lie symmetry?* [5, Open problem, p.252].

Lie symmetries are interesting objects in the theory of discrete integrability, [9]. In particular, their existence implies that the one-dimensional dynamics of real maps F on an invariant curve is essentially linear, [4]. Hence, as a consequence of the above result and Theorem 1 in [4], we obtain the following result:

Corollary 4. *Let F be a real birational map with a rational first integral V, preserving the associated genus 0 fibration given by $\{C_h\}_{h\in\mathcal{H}}$, and let X be a Lie symmetry of F with first integral V. Let γ_h be a connected component of $C_h \cap \mathcal{G}(F)$, then:*

(a) If γ_h of C_h is homeomorphic to $\mathbb{S}^1$, then $F_{|\gamma_h}$ is conjugate to a rotation with rotation number given by $\tau(h)/T(h)$, where $T(h)$ is the period of C_h as a periodic orbit of (2), and $\tau(h)$ is defined by the equation $F(x,y) = \varphi(\tau(h),(x,y))$, where φ denotes the flow of X, and $(x,y) \in C_h$.

(b) If γ_h is homeomorphic to $\mathbb{R}$, then $F_{|\gamma_h}$ is conjugate to a translation.

(c) If γ_h is a point, then it is a fixed point of F.

Recall that a map F defined on an open set of $\mathbb{R}^2$, preserves a measure absolutely continuous with respect the Lebesgue's one with non-vanishing density ν, if $m(F^{-1}(B)) = m(B)$ for any Lebesgue measurable set B, where $m(B) = \int_B \nu(x,y)\,dxdy$. The existence of a Lie symmetry preserving the integral V guarantees the existence of certain preserved measures as a consequence of the results in [4]:

Corollary 5. *Let F be a real birational map F, with a rational first integral V, preserving the genus 0 fibration given by $\{V = h\}_{h\in\mathcal{H}}$. Then there are some open disjoint sets $\mathcal{U}^+$ and $\mathcal{U}^- \subset \mathcal{U}$ (possibly some of them, but not both, empty) such that:*

(a) If F preserves orientation on $\mathcal{U}$, then on each set $\mathcal{U}^+$ and $\mathcal{U}^-$, it preserves an invariant measure absolutely continuous with respect the Lebesgue one.

(b) On each set $\mathcal{U}^+$ and $\mathcal{U}^-$, the map F^2 preserves an invariant measure absolutely continuous with respect the Lebesgue one.

Proof. Observe that F has the Lie symmetry given by Equation (2). By construction this symmetry preserves the curves $\{V = h\}$ for each $h \in \mathcal{H}$, hence it must be a multiple of the hamiltonian vector field associated to V, that is

$$X(x,y) = \mu(x,y)\left(-V_y(x,y)\frac{\partial}{\partial x} + V_x(x,y)\frac{\partial}{\partial y}\right). \qquad (5)$$

In this case, the compatibility equation (1), is equivalent to

$$\mu(F(x,y)) = \det(DF(x,y))\,\mu(x,y), \tag{6}$$

see [4, Theorem 12 (ii)]. Set $\mathcal{U}^{\pm} := \{(x,y) \in \mathcal{U}\colon\ \pm\mu(x,y) > 0\}$. Observe that some of them, but not both, can be empty.

Suppose now that F preserves orientation in U, then Equation (6) implies that both sets $\mathcal{U}^+$ and $\mathcal{U}^-$ are invariant by F. Taking

$$\nu_{\pm}(x,y) := \pm\frac{1}{\mu(x,y)},$$

we get some invariant measures on each set $\mathcal{U}^{\pm}$ given by $m_{\pm}(\mathcal{B}) = \int_{\mathcal{B}} \nu_{\pm}(x,y)dxdy$ for any Lebesgue measurable set $\mathcal{B} \subset \mathcal{U}^{\pm}$. Indeed, take $\mathcal{B} \in \mathcal{U}^+$ a Lebesgue measurable set, then by using the change of variables formula and Equation (6) we get:

$$\begin{aligned} m_+(F^{-1}(\mathcal{B})) &= \int_{F^{-1}(\mathcal{B})} \nu_+(x,y)dxdy = \int_{\mathcal{B}} \nu_+(F(x,y))\ \det(DF(x,y))dxdy = \\ &= \int_{\mathcal{B}} \frac{1}{\mu(F(x,y))}\ \det(DF(x,y))dxdy = \int_{\mathcal{B}} \frac{1}{\mu(x,y)}\ dxdy = m_+(\mathcal{B}) \end{aligned}$$

In an analogous way we can prove that m_- is an invariant measure on $\mathcal{U}^-$, thus proving (a), and that $m(\mathcal{B}) = \int_{\mathcal{B}} 1/\mu(x,y)dxdy$ is an invariant measure of F^2 in $\mathcal{U}^- \cup \mathcal{U}^+$, thus proving (b). ∎

2.3 Detection of conjugations via conjugations of Möbius maps

Another consequence of Proposition 1 is that it is possible to easily verify if two birational integrable maps preserving a genus 0 fibration are conjugate, by checking if their associated Möbius transformations are conjugate. This is summarized in the result below which, additionally, allows to construct the explicit conjugations. In Section 3.4.2 we use this result to detect the conjugations between the maps associated to the six recurrences presented by F. Palladino in [12].

Prior to state the result we introduce the main assumptions and notation. In the following we assume that F and G are integrable birational maps in $\mathbb{K}$ with first integrals V and W respectively, and that there exists some nonempty open sets $\mathcal{H} \subseteq \mathrm{Im}(V)$ and $\mathcal{K} \subseteq \mathrm{Im}(W)$ such that the sets $\mathcal{U} = \{(x,y)\colon\ V(x,y) = h \in \mathcal{H}\}$ and $\mathcal{V} = \{(x,y)\colon\ W(x,y) = k \in \mathcal{K}\}$ are nonempty, and each curve $C_h = \{V = h\}$ for $h \in \mathcal{H}$ and $D_k = \{W = k\}$ for $k \in \mathcal{K}$ are irreducible in $\mathbb{C}$ and rational.

Let $\{P_h\}_{h\in\mathcal{H}}$ and $\{Q_k\}_{k\in\mathcal{K}}$ be families of proper parameterizations of the family of curves $\{C_h\}_{k\in\mathcal{K}}$ and $\{D_k\}_{k\in\mathcal{K}}$ respectively, and let $M_h = P_h^{-1} \circ F \circ P_h$ and $N_k = Q_k^{-1} \circ G \circ Q_k$ denote the Möbius transformations associated to $F_{|C_h}$ and $G_{|D_k}$, respectively. Let $\mathcal{G}(M_h)$ denote the good set of M_h in $\mathbb{K}$.

Proposition 6. *Under the previous assumptions, F is conjugate with G via a conjugation $F = \Psi^{-1} \circ G \circ \Psi$, where Ψ is a correspondence between the curves C_h in $\mathcal{U}$ and the curves D_k in $\mathcal{V}$, if and only if there exists a correspondence f between $\mathcal{H}$ and $\mathcal{K}$, such that for all $h \in \mathcal{H}$ there exists an invertible map m_h defined in $\mathcal{G}(M_h)$ such that $M_h = m_h^{-1} \circ N_k \circ m_h$ for $k = f(h)$. Furthermore, the conjugation is given by:*

$$\Psi(x, y) = Q_{f(h)} \circ m_h \circ P_h^{-1}(x, y)\big|_{h=V(x,y)} \tag{7}$$

and

$$\Psi^{-1}(u, v) = P_{f^{-1}(k)} \circ m_{f^{-1}(k)}^{-1} \circ Q_k^{-1}(u, v)\Big|_{k=W(u,v)} \tag{8}$$

Proof. Suppose that there exists a bijection $k = f(h)$ between $\mathcal{H}$ and $\mathcal{K}$, such that for all $h \in \mathcal{H}$ there exists an invertible map m_h defined in $\mathcal{G}(M_h)$ such that $M_h = m_h^{-1} \circ N_k \circ m_h$ for $k = f(h)$. Consider the map Ψ given by Equation (7), it is easy to check that, by construction, it maps any curve C_h to the curve $D_{f(h)}$. Hence $W(\Psi(x, y)) = f(h)$ for each $(x, y) \in C_h$.

Set $\Phi(u, v) = P_{f^{-1}(k)} \circ m_{f^{-1}(k)}^{-1} \circ Q_k^{-1}(u, v)\Big|_{k=W(u,v)}$. Now we prove that $\Phi \circ \Psi = \mathrm{Id}$. Indeed, take $(x, y) \in C_h$, set $k := f(h) = W(\Psi(x, y))$, then

$$\Phi \circ \Psi(x, y) = P_{f^{-1}(k)} \circ m_{f^{-1}(k)}^{-1} \circ Q_k^{-1} \circ Q_{f(h)} \circ m_h \circ P_h^{-1}(x, y) =$$

$$= P_h \circ m_h^{-1} \circ Q_k^{-1} \circ Q_k \circ m_h \circ P_h^{-1}(x, y) = (x, y),$$

and analogously $\Psi \circ \Phi = \mathrm{Id}$, hence $\Psi^{-1} = \Phi$.

Now we prove that Ψ is a conjugation. Indeed,

$$\Psi^{-1} \circ G \circ \Psi(x, y) = P_h \circ m_h^{-1} \circ Q_k^{-1} \circ G \circ Q_k \circ m_h \circ P_h^{-1}(x, y) =$$

$$= P_h \circ m_h^{-1} \circ N_k \circ m_h \circ P_h^{-1}(x, y) = P_h \circ M_h \circ P_h^{-1}(x, y) = F(x, y).$$

Conversely, suppose that F is conjugate with G in $\mathcal{U}$, via a conjugation $F = \Psi^{-1} \circ G \circ \Psi$ such that for all $h \in \mathcal{H}$, there exists $k \in \mathcal{K}$ such that $\Psi(C_h) = D_k$ and reciprocally, for all $k \in \mathcal{K}$, there exists $h \in \mathcal{H}$ such that $\Psi^{-1}(D_k) = C_h$. This fact allows to introduce a correspondence f between $\mathcal{H}$ and $\mathcal{K}$ via $k = W \circ \Psi_{|C_h}$ and $h = V \circ \Psi_{|D_k}^{-1}$.

Take $(x, y) \in C_h$, set $k := f(h) = W(\Psi(x, y))$, then

$$F(x, y) = P_h \circ M_h \circ P_h^{-1}(x, y) = \Psi^{-1} \circ G \circ \Psi,$$

and therefore for all $t = P_h^{-1}(x, y) \in \mathcal{G}(M_h)$:

$$M_h(t) = P_h^{-1} \circ \Psi^{-1} \circ G \circ \Psi \circ P_h(t) = P_h^{-1} \circ \Psi^{-1} \circ Q_k \circ N_k \circ Q_k^{-1} \circ \Psi \circ P_h(t).$$

So there exists a conjugation $m_h = Q_{f(h)}^{-1} \circ \Psi \circ P_h(t)$ between M_h and N_k for $k = f(h)$. ∎

Observe that if the dependence on the parameters h and k in the terms of the expression (7) and (8) is rational, and f is a birational function, then the conjugation Ψ is a birational map, and therefore global. This is the case of the conjugations constructed in the proof of Proposition 15. However, notice that, this is not the general situation since not every proper parametrization of a curve of the form $\{V = h\}$ has a rational dependence on the parameter h. This is the case of the ones obtained when we use the parametrization algorithms implemented in some computer algebra software, [10].

3 Applications

In this section we show how the results in Section 2 can be used to analyze the global dynamics of birational maps preserving genus 0 fibrations in a unified way. The considered examples include a one-parameter family of maps previously studied by G. Bastien and M. Rogalski in [1], a map introduced by S. Saito and N. Saitoh in [13], and the maps associated to some difference equations considered by F. Palladino in [12]. The method can also be applied to study some other maps appearing in the literature, for instance the ones in [2] and [15].

As we will need to recall it in every example, and although it is well-known, for convenience of the reader we summarize the dynamics of Möbius maps in the following well-know result, see [3, Section 2.2] for instance.

Proposition 7. *Consider the map* $M(t) = (at+b)/(ct+d)$*, where* $a, b, c, d \in \mathbb{K}$*, with* $c \neq 0$*, defined for* $t \in \widehat{\mathbb{K}}$*. Set* $\Delta = (d-a)^2 + 4bc$*, and* $\xi = (a+d+\sqrt{\Delta})/(a+d-\sqrt{\Delta})$*.*

- *(a) If* $\Delta \neq 0$*, then there are two fixed points* t_0 *and* t_1 *in* $\widehat{\mathbb{K}}$*, furthermore*
  - (a_1) *When* $|\xi| \neq 1$*, one of the fixed points, say* t_j*, is an attractor of* M *in* $\widehat{\mathbb{K}} \setminus \{t_{j+1\,(\mathrm{mod}\,2)}\}$*.*
  - (a_2) *When* $|\xi| = 1$*, then* M *is conjugated to a rotation in* $\widehat{\mathbb{K}}$ *with rotation number* $\theta := \arg(\xi)\,(\mathrm{mod}\,2\pi)$*. In particular,* M *is periodic with minimal period* p *if and only if* ξ *is a primitive root of the unity. Furthermore,* $W(t) = |(t-t_0)/(t-t_1)|$ *is a first integral of* M*.*
- *(b) If* $\Delta = 0$*, then there is a unique fixed point* t_0 *which is a global attractor in* $\widehat{\mathbb{K}}$ *of* M*.*

3.1 The Bastien and Rogalski map revisited

We consider the planar birational map defined in $\mathbb{R}^{2,+}$, given by

$$F_a(x,y) = \left(y, \frac{a - y + y^2}{x}\right) \tag{9}$$

with $a > 1/4$. The periodic structure of this map was characterized by G. Bastien and M. Rogalski as a part of the study of the difference equation $u_{n+2} = (a - u_{n+1} + u_{n+1}^2)/u_n$, see [1]. The map F_a possess the rational first integral

$$V_a(x,y) = \frac{x^2+y^2-x-y+a}{xy},$$

and it preserves the fibration of $\mathbb{R}^{2,+}$ given by the algebraic curves of genus 0 (conics):

$$C_h = \{x^2+y^2-x-y+a-hxy=0\}, \tag{10}$$

where $h \in (2-1/a, \infty)$. The map has a unique fixed point in $\mathbb{R}^{2,+}$, given by (a,a) with energy level $h_c := V(a,a) = 2-1/a$. This point is an elliptic one since $a > 1/4$. Let $\widetilde{C}_h = \{[x:y:z],\, x^2+y^2-hxy-xz-yz+az^2=0\}$ and $\widetilde{F}_a([x:y:z]) = [xy : az^2-yz+y^2 : xz]$, denote the extensions of C_h and F_a to $\mathbb{R}P^2$. We prove:

Proposition 8. *Set $a > 1/4$ and $h_c = 2-1/a$, then the following statement hold:*

(a) For $h > 2$, C_h is a hyperbola, $F_{a|C_h}$ is conjugated to a translation, and there are two fixed points of $\widetilde{F}_{a|\widetilde{C}_h}$ at infinity given by $[2 : h \pm \sqrt{h^2-4} : 0]$, an attractor and a repeller.

(b) For $h = 2$, C_h is a parabola, $F_{a|C_h}$ is conjugated to a translation, and the point at the infinity $[1:1:0]$ is a global attractor of $\widetilde{F}_{a|\widetilde{C}_h}$.

(c) For $h_c < h < 2$, C_h is an ellipse, $F_{a|C_h}$ is conjugated to a rotation with rotation number

$$\theta(h) = \arg\left(\frac{h - i\sqrt{4-h^2}}{2}\right) (\operatorname{mod} 2\pi). \tag{11}$$

Using the rotation number function (11) we reobtain the results of Bastien and Rogalski in [1], concerning the set of periods of each particular map F_a, and of the family of maps.

Proposition 9 ([1])**.** *Set $a > 1/4$, $h_c = 2-1/a$, and $\theta_a = \frac{1}{2\pi}\arg\left(\frac{2a-1-i\sqrt{4a-1}}{2a}\right)$. Then:*

(a) For any fixed $a > 1/4$ and any natural number $p \geq \mathrm{E}\left(1/(1-\theta_a)\right)+1$ there exists $h_p \in (h_c, 2)$ such that C_{h_p} is filled of p-periodic orbits.

(b) For all $p \in \mathbb{N}$, $p \geqslant 3$ there exists $a > 1/4$ and $h_p \in (h_c, 2)$ such that C_{h_p} is filled of p-periodic orbits.

Prior to prove the above results we find a real proper parametrization the curves (10). In our case, we will use the method of *parametrization by lines* [14, Section 4.6], to obtain the proper parametrization. First, for each curve in (10), we consider the point

$$(x_0, y_0) = \left(a, \frac{ah+1+\delta}{2}\right) \in C_h,$$

where $\delta = \sqrt{(ah+1)^2 - 4a^2} \in \mathbb{R}$. Taking the new variables $x = u + x_0, y = v + y_0$ we bring this point to the origin, so that, each curve in the new variables is defined by $f_1(u,v) + f_2(u,v) = 0$ where f_k stands for the homogeneous part of degree k. In our case:

$$\begin{aligned} f_1(u,v) &= \left(-ah^2/2 - (1+\delta)\,h/2 - 1 + 2a\right) u + \delta v, \\ f_2(u,v) &= u^2 + v^2 - huv. \end{aligned}$$

We compute the intersection points of these curves with the the lines $v = t\,u$,

$$\begin{cases} v = tu, \\ f_2(u,v) + f_1(u,v) = 0, \end{cases}$$

obtaining an affine parametrization $(u(t), v(t))$, so that the parametrization of the corresponding curve C_h is the *real* one given by $P_h(t) = (P_{1,h}(t), P_{2,h}(t)) = (u(t) + x_0, v(t) + y_0)$ where

$$\begin{aligned} P_{1,h}(t) &= \frac{2\delta t - ah^2 - (1+\delta)h - 2 + 4a}{2(-t^2 + ht - 1)} + a, \\ P_{2,h}(t) &= \frac{(-ah + \delta - 1)\,t^2 + (4\,a - 2)\,t - ah - \delta - 1}{2(-t^2 + ht - 1)}. \end{aligned}$$

It is straightforward to check (see Theorem 18 in the Appendix) that for each $h > h_c$, the above parameterizations of C_h is a proper one. Furthermore, by using the method described in Theorem 19 and Equation (20), one gets that its inverse is given by

$$P_h{}^{-1}(x,y) = \frac{-2\,\delta\,x + \left(ah^2 + (\delta+1)h - 4\,a + 2\right) y - ah + 2\,a + \delta - 1}{(ah^2 + (1-\delta)h - 4\,a + 2)\,x + 2\,\delta\,y + ah - 2\,a - \delta + 1}.$$

The parametrization $P_h(t)$ is defined in $\widehat{\mathbb{R}}$ if $h_c < h < 2$, and in $\widehat{\mathbb{R}} \setminus \{t_0, t_1\}$ where

$$t_j := \frac{h + (-1)^j \sqrt{h^2 - 4}}{2}, \tag{12}$$

if $h \geq 2$. Observe that this values of the parameters do not correspond to affine points of C_h. Indeed, each curve extends to $\mathbb{R}P^2$ as $\widetilde{C}_h = \{[x : y : t],\; x^2 + y^2 - hxy - xt - yt + at^2 = 0\}$, which is properly parameterized for $t \in \widehat{\mathbb{R}}$ by

$$\begin{aligned} \widetilde{P}_h(t) = \; & \left[-2at^2 + (2ah + 2\delta)\,t - ah^2 - h\delta + 2a - h - 2 \, : \right. \\ & \left. (-ah + \delta - 1)\,t^2 + (4a - 2)\,t - ah - \delta - 1 : 2(-t^2 + ht - 1)\right]. \end{aligned}$$

The values of the parameters t_j correspond with the infinite points

$$Q_{j,h} = [2 : h + (-1)^j \sqrt{h^2 - 4} : 0]$$

which are fixed points of the extension of F to $\mathbb{R}P^2$ given by $\widetilde{F}_a([x : y : z]) = [xy : az^2 - yz + y^2 : xz]$. Of course, if $h = 2$, then $t_1 = t_0 = 1$ and $Q_{1,h} = Q_{0,h} = [1 : 1 : 0]$.

As an immediate consequence of the above arguments, Theorem 2 and Corollary 5, we have the following result:

Proposition 10. *Each map (9) possesses the Lie symmetry*

$$X(x,y) = \frac{x^2-y^2+a-x}{y}\frac{\partial}{\partial x} + \frac{x^2-y^2-a+y}{x}\frac{\partial}{\partial y}, \tag{13}$$

and preserves a measure absolutely continuous with respect the Lebesgue one in $\mathbb{R}^{2,+}$ *given by*

$$m(\mathcal{B}) = \int_{\mathcal{B}} \frac{1}{x\,y}\,dx\,dy$$

for any Lebesgue measurable set $\mathcal{B}$ *of* $\mathbb{R}^{2,+}$.

Proof. From the above arguments, we have that $\{P_h(t)\}_{h\in\mathcal{H}}$, where $\mathcal{H} := \{h > h_c\}$, is a family of proper affine parameterizations of $\{C_h\}_{h\in\mathcal{H}}$. Now a computation shows that

$$M_h(t) = P_h^{-1}\circ F_a \circ P_h(t) = \frac{(h+1)t-1}{t+1}, \tag{14}$$

so $F_{|C_h}$ is conjugate to $M_{h|\widehat{\mathbb{R}}}$ if $h_c < h < 2$, and $M_{h|\widehat{\mathbb{R}}\setminus\{t_0,t_1\}}$ otherwise. By Lemma 3, each map M_h has the Lie symmetry $Y_h(t) = \left(1-ht+t^2\right)\dfrac{\partial}{\partial t}$. Observe that, by setting $P_h(t) = (P_{1,h}(t), P_{2,h}(t))$, then Equation (2) in Theorem 2 gives the vector field $X = X_1\partial/\partial x + X_2\partial/\partial y$ where :

$$X_1(x,y) = P'_{1,h}(P_h^{-1}(x,y))Y_h(P_h^{-1}(x,y))\Big|_{h=V(x,y)} = (x^2-y^2+a-x)/y$$
$$X_2(x,y) = P'_{2,h}(P_h^{-1}(x,y))Y_h(P_h^{-1}(x,y))\Big|_{h=V(x,y)} = (x^2-y^2-a+y)/x$$

Hence, we get that F_a has the Lie symmetry (13), also defined in $\mathbb{R}^{2,+}$.

Observe that using the notation in Section 2.2, this vector is exactly the multiple of the hamiltonian vector field (5) associated to V_a, with $\mu(x,y) = xy$. So by Corollary 5 it defines a measure absolutely continuous with respect the Lebesgue one in $\mathbb{R}^{2,+}$ given by $m(\mathcal{B}) = \int_{\mathcal{B}}(1/\mu)\,dx\,dy$, for any Lebesgue measurable set $\mathcal{B}$ of $\mathbb{R}^{2,+}$. ∎

Proof of Proposition 8. It is straightforward to check that for any fixed $a > 1/4$, the curves C_h are ellipses for $h_c < h < 2$, a parabola when $h = 2$, and a branch of a hyperbola when $2 < h < \infty$, see [1]. By using Corollary 4 and the existence of the Lie symmetry (13), we get that the affine dynamics of each map $F_{a|C_h}$ is conjugate to a translation when $h \geq 2$, and conjugate to a rotation when $h_c < h < 2$.

As shown above, each map $\widetilde{F}_{a|C_h}$ is conjugate to the Möbius one M_h given in (14). Observe that for each $h \geq 2$, the map $M_h(t)$ has the fixed points t_j given in (12) which correspond with the infinite points $Q_{j,h}$, which are fixed points of $\widetilde{F}_a$.

Following the notation of Proposition 7, for the map M_h one gets $\Delta = h^2 - 4$ and $\xi = (h+2-\sqrt{h^2-4})/(h+2+\sqrt{h^2-4})$. Now,

- If $h > 2$, then $\Delta > 0$ and $\xi < 1$. In this case t_0 is an attractor of M_h in $\widehat{\mathbb{R}} \setminus t_1$ and t_1 is a repeller, so $Q_{0,h}$ is an attractor of $\widetilde{F}_{a|\widetilde{C}_h}$ in $\widetilde{C}_h \setminus Q_{1,h}$, and $Q_{1,h}$ a repeller.

- If $h = 2$, then $\Delta = 0$, so the the point $t_0 = h/2 = 1$ is a global attractor of M_2 in $\widehat{\mathbb{R}}$, thus $Q_{0,2} = [1 : 1 : 0]$ is a global attractor of $\widetilde{F}_{a|\widetilde{C}_2}$ in $\widetilde{C}_2$.

- If $h_c < h < 2$, then $\Delta < 0$, hence M_h is conjugated to a rotation in $\widehat{\mathbb{R}}$, with rotation number

$$\theta(h) = \arg\left(\frac{h+2-i\sqrt{4-h^2}}{h+2+i\sqrt{4-h^2}}\right) \pmod{2\pi} = \arg\left(\frac{h-i\sqrt{4-h^2}}{2}\right) \pmod{2\pi},$$

and therefore $F_{a|C_h}$ is a conjugate to a rotation with the same rotation number (recall that, in this case, the curves C_h are affine ellipses with no points at the infinity line).

∎

Proof of Proposition 9. (a) In the proof of Proposition 8 we have seen that if $h_c < h < 2$, then each map $F_{a|C_h}$ is conjugate to a rotation with rotation number given by Equation (11). It is straightforward to check that for a fixed $a > 1/4$ this function $\theta(h)$ grows monotonically from θ_a to 1 for $h \in (h_c, 2)$, where

$$\theta_a = \lim_{h\to h_c^+} \theta(h) = \frac{1}{2\pi}\arg\left(\frac{h_c - i\sqrt{4-h_c^2}}{2}\right) = \frac{1}{2\pi}\arg\left(\frac{2a-1-i\sqrt{4a-1}}{2a}\right).$$

Therefore, for all $\theta \in (\theta_a, 1)$ there exists $h \in (h_c, 2)$ such that $\theta(h) = \theta$, so to characterize the set of periods of any map F_a in $\mathbb{R}^{2,+}$, we need to know which are the irreducible fractions $q/p \in (\theta_a, 1)$.

Observe that if an irreducible fraction $q/p \in (\theta_a, 1)$, then $1 \le q \le p-1$ and therefore $(p-1)/p \in (\theta_a, 1)$, because

$$\theta_a < \frac{q}{p} \le \frac{p-1}{p} < 1.$$

Hence we only need to characterize which are integer numbers p such that $\theta_a < (p-1)/p$. For such a number, one easily obtains $p > 1/(1-\theta_a)$, hence

$$p \ge E\left(\frac{1}{1-\theta_a}\right) + 1.$$

(b) A computation shows that when $1/4 < a \le 1/2$, h_c varies monotonically from -2, to 0, and therefore $2\pi\theta_a$ is an angle in the third quadrant that varies from π to $3\pi/4$. If $a > 1/2$ then $h_c > 0$ and $2\pi\theta_a$ is an angle of the fourth quadrant. Hence

$$I := \bigcup_{a>1/4} (\theta_a, 1) = \left(\frac{1}{2}, 1\right).$$

Observe that $1/p \in I$ for all $p \ge 3$, and therefore the result follows. ∎

3.2 The Saito and Saitoh map

In [13], S. Saito and N. Saitoh considered the map

$$F(x,y) = \left(xy, \frac{y(1+x)}{1+xy}\right), \tag{15}$$

defined in the open set $\mathcal{G}(F) = \mathbb{C}^2 \setminus \{\bigcup_{n\geq 0} F^{-n}(x,-1/x)\}$, and they found a continuum of 3-periodic orbits. Here we describe the global dynamics of F, and we notice that there are continua of periodic orbits of minimal period p for all $p \geq 2$, which have the simple expression $y(1+x) = h_p$, being h_p any primitive p-root of the unity.

As already noticed in [13], this map has the first integral $V(x,y) = y(1+x)$, hence the affine sets $\gamma_h := \{y(1+x) = h,\, h \in \mathbb{C}\} \cap \mathcal{G}(F)$ are invariant. The extension to $\mathbb{C}P^2$ of all these sets contains the affine points of γ_h plus the infinity points $[1:0:0]$ and $[0:1:0]$. But these infinity points belong to the so called *indeterminacy locus* of the extension of F to $\mathbb{C}P^2$, $\widetilde{F}([x:y:z]) = [xy(xy+z^2) : yz^2(x+z) : (xy+z^2)z^2]$. That is: $\widetilde{F}([1:0:0]) = \widetilde{F}([0:1:0]) = [0:0:0]$, see [6, 7] for further details. This is the reason why we will describe the dynamics of F only in the affine space $\mathbb{C}^2$. The global dynamics is given by next result.

Proposition 11. *Consider the map (15), then:*

(a) Any solution with initial condition on γ_0 reaches the point $(0,0)$, which is a global attractor of $F_{|\gamma_0}$ in finite time.

(b) For $h \neq 1$, on any invariant set γ_h there are two fixed points of F, given by $Q_0 = (0,h)$ and $Q_1 = (h-1,1)$. Furthermore,

(i) If $|h| < 1$, then the point Q_0 is the attractor of $F_{|\gamma_h}$ in $\gamma_h \setminus Q_1$, and Q_1 is a repeller. Conversely if $|h| > 1$, then the point Q_0 is the repeller and Q_1 an attractor of $F_{|\gamma_h}$ in $\gamma_h \setminus Q_0$.

(ii) If $|h| = 1$, then either any initial condition (x_0,y_0) in γ_h gives rise to a periodic orbit with minimal period $p \geq 2$, when h is a primitive p-root of the unity, or it gives rise to an orbit which densely fills the set $\gamma_h \cap \{W(x,y) = W(x_0,y_0)\}$, where $W(x,y) = |x/(x-h+1)|$, otherwise.

(c) For $h = 1$, the point $p = (0,h)$ is the unique fixed point of F in γ_1, which is a global attractor of $F_{|\gamma_1}$.

Proof. The set γ_0 is given by the lines $x = -1$ and $y = 0$. Observe that $(0,0) \in \gamma_0$ is a fixed point of F. A simple computation shows that $F^2(-1,y) = (0,0)$ and $F(x,0) = (0,0)$ for all $x, y \in \mathbb{C}$, which proves statement (a).

To prove statements (b) and (c) we fix $h \neq 0$. Observe that the set γ_h is given by the intersection of the curve $C_h = \{y(1+x) = h\}$ with $\mathcal{G}$. Any curve C_h admits the trivial proper affine parametrization

$$P_h(t) = \left(t, \frac{h}{t+1}\right) \text{ for } t \neq -1.$$

Observe that parameter $t = -1$ correspond with the excluded infinity point $[0 : 1 : 0]$. Observe that $P_h^{-1}(x, y) = x$. A computation gives.

$$M_h(t) = P_h^{-1} \circ F \circ P_h(t) = \frac{ht}{t+1}.$$

It is easy to check that $\gamma_h = C_h \setminus \{\bigcup_{n\geq 0} F^{-n}(-1/(h+1), h+1)\}$ for $h \neq -1$; that $\gamma_{-1} = C_{-1}$; and that for any $h \in \mathbb{C}$, the map $F_{|\gamma_h}$ is conjugate to the $M_{h|\mathcal{G}(M_h)}$, where $\mathcal{G}(M_h) = \mathbb{C}\setminus\{\cup_{n\geq 0} M_h^{-n}(-1)\}$ is the good set of M_h in $\mathbb{C}$. In particular $\mathcal{G}(M_{-1}) = \mathbb{C}\setminus\{-1\}$.

Each map M_h has two fixed points $t_0 = 0$ and $t_1 = h-1$ corresponding to the parameters of the points $Q_0, Q_1 \in \mathbb{C}^2$ respectively. Now, the result follows directly from Proposition 7. In particular, for $|h| = 1$ with $h \neq 1$, and since $\Delta = (1-h)^2$ and $\xi = 1/h$, one easily gets that the map M_h is conjugate to a rotation in $\mathbb{C}$ with rotation number

$$\theta(h) = \arg\left(\frac{1}{h}\right) \pmod{2\pi} = \arg\left(\bar{h}\right) \pmod{2\pi}.$$

Hence the map $F_{|\gamma_h}$ has continua of periodic orbits of all minimal periods $p \geq 2$, located at the hyperbolae $y(1+x) = h_p$, being h_p a primitive p-root of the unity. Furthermore, in this case

$$W(x, y) = \left|\frac{x}{x-h+1}\right|,$$

is a first integral of $F_{|\gamma_h}$. Otherwise, if h is not a p-root of the unity, then for each initial condition on γ_h, the associated orbit of $F_{|\gamma_h}$ fills densely the set $\gamma_h \cap \{W = W(x_0, y_0)\}$. ∎

Notice that, as a consequence of the above result, the only periodic orbits that can be observed in $\mathbb{R}^2$ are the 2-periodic orbits that are located in the real hyperbola $y(1+x) = -1$.

Finally, as a direct consequence of Theorem 2, we have:

Corollary 12. *The map (15) possesses the Lie symmetry*

$$X(x, y) = -x\,(1+x)\,(y-1)\,\frac{\partial}{\partial x} + xy\,(y-1)\,\frac{\partial}{\partial y}.$$

Proof. By Lemma 3, the Möbius map $M_h(t) = ht/(t+1)$ has the Lie symmetry $Y_h(t) = \left((1-h)t + t^2\right)\frac{\partial}{\partial t}$. Now the Lie symmetry of F is obtained from Equation (2), by using the parametrization $P_h(t) = (t, h/(t+1))$ and taking $h = y(1+x)$. ∎

Notice that the above vector field is a multiple of the hamiltonian one (5) with $\mu(x, y) = x(y-1)$.

3.3 A difference equation with the Saito and Saitoh invariant

We consider the *real* difference equation

$$u_{n+2} = \frac{u_{n+1}(1+u_n)}{1+u_{n+1}}. \tag{16}$$

This equation possesses the invariant given by $I(u_n, u_{n+1}) = u_{n+1}(1+u_n)$. We found this difference equation when looking for a recurrence with the same invariant as the Saito and Saitoh map. Latter, we knew that this equation is a particular case of the six equations introduced in [12], and also considered in Section 3.4.

For each initial condition u_0, u_1 in $\mathcal{G} \subset \mathbb{R}$, the good set of the equation (16), set $I := I(u_0, u_1) = u_1(1+u_0)$ and

$$t_j = t_j(u_0, u_1) := \frac{-1+(-1)^j\sqrt{1+4\,I}}{2}, \text{ and } \xi = -\frac{1+2\,I+i\sqrt{-1-4\,I}}{2\,I}.$$

Proposition 13. *Let $\{u_n\}$ be a solution of Equation (16) with initial condition u_0, u_1 in $\mathcal{G}$, then:*

(a) If $u_1 = u_0$ and $u_0 \neq -1$, then the solution is constant.

(b) If $u_1(1+u_0) \geq -1/4$ with $u_0 \neq t_1$ and $u_1 \neq t_1$, then the solution converges to $u = t_0$.

(c) If $u_1(1+u_0) < -1/4$ then the solution is either p-periodic, if ξ is a p-root of unity, or such that the set of accumulation points of $\{u_n\}_{n\in\mathbb{N}}$ is $\mathbb{R}$, otherwise. Furthermore, the Equation (16) has p-periodic solutions for all period $p \geq 3$.

The proof of the above proposition follows directly from the study of the planar real dynamical system given by the map

$$F(x,y) = \left(y, \frac{y(1+x)}{1+y}\right), \tag{17}$$

by noticing that $(u_n, u_{n+1}) = F^n(u_0, u_1)$. This map is defined in $\mathcal{G}(F) = \mathbb{R}^2 \backslash \{\bigcup_{n\geq 0} F^{-n}(x,-1)\}$. As the case of the map (15), the map F has the first integral $V(x,y) = y(1+x)$, hence the sets $\gamma_h := \{y(1+x) = h,\, h \in \mathbb{R}\} \cap \mathcal{G}(F)$ are invariant. Observe that the fixed points of F have the form (x,x), with $x \neq -1$. For this map we have the following result:

Proposition 14. *Consider the map (17), then*

(a) Any solution with initial condition on γ_0 reaches the point $(0,0)$ in finite time.

(b) For $h \neq 0$, the dynamics in the invariant set γ_h is the following:

(i) If $h > -1/4$, then there exists two fixed points of F in γ_h, $Q_{j,h} = (x_j, x_j)$, with $j = 0, 1$, where $x_j = (-1 + (-1)^j\sqrt{1+4h})/2$. Furthermore Q_0 is a global attractor of $F_{|\gamma_h \setminus Q_1}$, and Q_1 is a repeller.

(ii) If $h = -1/4$, then there exists a unique fixed points of F in $\gamma_{-\frac{1}{4}}$, $Q = (-1/2, -1/2)$ which is a global attractor of $F_{|\gamma_{-\frac{1}{4}}}$.

(iii) If $h < -1/4$, then either any initial condition in γ_h gives rise to a periodic orbit with minimal period $p \geq 3$ or it gives rise to an orbits which densely fills this set, depending on whether $(-1 - 2h - i\sqrt{-1-4h})/(2h)$ is a primitive p-root of the unity or not.

Proof. First observe that after one or two iterations any orbit with initial condition in $\gamma_0 = \{\{x = -1\} \cup \{y = 0\}\} \cap \mathcal{G}(F)$ reaches the point $(0, 0)$. Consider $h \neq 0$, and recall that any hyperbola $C_h = \{y(1+x) = h\}$ admits the proper parametrization $P_h(t) = (t, h/(t+1))$ for $t \neq -1$. Some computations give that $\gamma_h = C_h \setminus \{\bigcup_{n\geq 0} F^{-n}(-h-1, -1)\}$; that

$$M_h(t) = P_h^{-1} \circ F \circ P_h(t) = \frac{h}{t+1},$$

and that $F_{|\gamma_h}$ is conjugate to $M_{h|\mathcal{G}(M_h)}$, where $\mathcal{G}(M_h) := \mathbb{R} \setminus \{\cup_{n\geq 0} M_h^{-n}(-1)\}$.

Using Proposition 7, and setting $\Delta = 1+4h$ and $\xi = (-1-2h-\sqrt{1+4h})/(2h)$, we have that if $h > -1/4$, the map M_h has two fixed points $t_{0,h}$ (an attractor) and $t_{1,h}$ (a repeller) given by

$$t_{j,h} = \frac{-1 + (-1)^j\sqrt{1+4h}}{2}.$$

So on each set γ_h there are two fixed points $Q_{0,h} = (t_{0,h}, t_{0,h})$ and $Q_{1,h} = (t_{1,h}, t_{1,h}) \in \mathbb{R}^2$, which are an attractor and a repeller of $F_{|\gamma_h}$, respectively. If $h = -1/4$, then $M_{-\frac{1}{4}}$ has a unique fixed point $t = -1/2$ which is a global attractor in $\mathcal{G}(M_{-\frac{1}{4}})$, hence the map $F_{|\gamma_{-\frac{1}{4}}}$ has a global attractor in the point $(-1/2, -1/2)$. If $h < -1/4$, then the map M_h defined on $\widehat{\mathbb{R}}$ is conjugate to a rotation with rotation number

$$\theta(h) = \arg\left(-\frac{1+2h}{2h} - i\frac{\sqrt{-1-4h}}{2h}\right) (\operatorname{mod} 2\pi).$$

An straightforward computation shows that

$$I = \left\{\text{Image}(\theta(h)),\, h < -\frac{1}{4}\right\} = \left(\frac{1}{2}, 1\right).$$

So there are irreducible fractions with denominator $p \in I$ if and only if $p \geq 3$ and therefore the map F possesses periodic orbits for all period $p \geq 3$, and they are located in the region $\{y(1+x) < -1/4\}$. Furthermore the sets γ_{h_p} where are located the periodic orbits with minimal period p, are those such that $\xi_{|h=h_p} = (-1-2h_p-i\sqrt{-1-4h_p})/(2h_p)$ is a primitive p-root of the unity. Observe that F has not 2-periodic orbits. If ξ is not a root of unity then any orbit in γ_h densely fills this set. ∎

Observe that the proof of Proposition 13 also follows straightforwardly from the analysis of the Möebius maps M_h by noticing that due to the particular form of the family of parameterizations P_h, we have that $u_n = M_h^n(u_0)$.

Also notice that the Lie symmetry of the map (17) is given in Section 3.4.1.

3.4 The Palladino's recurrences

In [12], F. Palladino presented the analysis of the *forbidden set* of a list of six difference equations in $\mathbb{C}$ with rational invariants. These invariants allow the author to make an order reduction, so that the dynamics of the equation can be described via a family of Riccati equations. In fact all the Palladino's equations have an associated integrable birational map preserving a genus 0 fibration. Hence, the order reduction observed by Palladino in these equations are a consequence of the general fact observed in Proposition 1.

In the next subsections we will compute the associated Lie symmetries of the maps associated to the Palladino's recurrences, we study the conjugations between these maps and, as a last example, we study the dynamics of a representative of each set of non-conjugate recurrences.

3.4.1 Lie symmetries for the maps associated to the Palladino's recurrences.

In this section we consider the maps F_j, $j = 1, \ldots, 6$, associated to each Palladino recurrence. For these maps we give families of proper parametrizations associated to each fibration $C_{j,h} = \{V_{j,1} - hV_{j,2} = 0\}$ where $V_j = V_{j,1}/V_{j,2}$ is the first integral corresponding with the invariants given in [12], we give the Möbius transformations associated to the map, the parametrization, and the associated Lie symmetries as well.

1. The equation $u_{n+2} = \frac{u_{n+1}}{1+b\,(u_n-u_{n+1})}$ with $b \in \mathbb{C}\setminus\{0\}$, has the following associated objects, which characterize its dynamics:

Associated map: $F_1(x,y) = \left(y, \dfrac{y}{1+b\,(x-y)}\right)$	First integral: $V_1(x,y) = \dfrac{1+bx+by+b^2xy}{y}$	Parametrization of $C_{1,h}$: $P_{1,h}(t) = \left(t, \dfrac{-bt-1}{b^2t+b-h}\right)$
Möbius map: $M_{1,h}(t) = \dfrac{-bt-1}{b^2t+b-h}$	Lie symmetry of $M_{1,h}$: $Y_{1,h} = \left(1+(2\,b-h)\,t+b^2t^2\right)\dfrac{\partial}{\partial t}$	Lie symmetry of F_1: $X_1 = -\dfrac{(x-y)(bx+1)}{y}$ $X_2 = -b(x-y)(by+1)$

2. Objects associated with the recurrence $u_{n+2} = \frac{u_n}{1+b\,(u_{n+1}-u_n)}$ with $b \in \mathbb{C} \setminus \{0\}$:

Associated map: $F_2(x,y) = \left(y, \dfrac{x}{1+b\,(y-x)}\right)$	First integral: $V_2(x,y) = \dfrac{1+by+xy}{xy}$	Parametrization of $C_{2,h}$: $P_{2,h}(t) = \left(t, \dfrac{1}{(h-1)t-b}\right)$
Möbius map: $M_{2,h}(t) = \dfrac{1}{(h-1)t-b}$	Lie symmetry of $M_{2,h}$: $Y_{2,h} = \left(-1-bt+(h-1)t^2\right)\dfrac{\partial}{\partial t}$	Lie symmetry of F_2: $X_1 = \dfrac{x-y}{y}$ $X_2 = -\dfrac{(x-y)(by+1)}{x}$

3. Objects associated with the recurrence $u_{n+2} = \frac{b\,(u_{n+1}-u_n)+u_{n+1}^2}{u_n}$ with $b \in \mathbb{C}$:

Associated map: $F_3(x,y) = \left(y, \dfrac{-bx+by+y^2}{x}\right)$	First integral: $V_3(x,y) = \dfrac{y+b}{x}$	Parametrization of $C_{3,h}$: $P_{3,h}(t) = (t, ht-b)$
Möbius map: $M_{3,h}(t) = ht-b$	Lie symmetry of $M_{3,h}$: $Y_{3,h} = (b+(1-h)t)\dfrac{\partial}{\partial t}$	Lie symmetry of F_3: $X_1 = x-y$ $X_2 = \dfrac{(x-y)(b+y)}{x}$

4. Objects associated with the recurrence $u_{n+2} = \frac{bu_{n+1}+u_{n+1}^2}{u_n+b}$ with $b \in \mathbb{C} \setminus \{0\}$:

Associated map: $F_4(x,y) = \left(y, \dfrac{by+y^2}{x+b}\right)$	First integral: $V_4(x,y) = \dfrac{x+b}{y}$	Parametrization of $C_{4,h}$: $P_{4,h}(t) = \left(t, \dfrac{t+b}{h}\right)$
Möbius map: $M_{4,h}(t) = \dfrac{t+b}{h}$	Lie symmetry of $M_{4,h}$: $Y_{4,h} = \left(-\dfrac{b}{h}+\left(1-\dfrac{1}{h}\right)t\right)\dfrac{\partial}{\partial t}$	Lie symmetry of F_4: $X_1 = x-y$ $X_2 = \dfrac{y(x-y)}{x+b}$

5. Objects associated with the recurrence $u_{n+2} = \frac{bu_{n+1}+u_nu_{n+1}}{u_{n+1}+b}$ with $b \in \mathbb{C} \setminus \{0\}$:

Associated map: $F_5(x,y) = \left(y, \dfrac{by+xy}{y+b}\right)$	First integral: $V_5(x,y) = y(x+b)$	Parametrization of $C_{5,h}$: $P_{5,h}(t) = \left(t, \dfrac{h}{t+b}\right)$
Möbius map: $M_{5,h}(t) = \dfrac{h}{t+b}$	Lie symmetry of $M_{5,h}$: $Y_{5,h} = \left(-h+bt+t^2\right)\dfrac{\partial}{\partial t}$	Lie symmetry of F_5: $X_1 = (x-y)(b+x)$ $X_2 = -y(x-y)$

Observe that the recurrence (16) studied in Section 3.3, corresponds with this recurrence when $b = 1$.

6. Objects associated with the recurrence $u_{n+2} = \frac{bu_n - bu_{n+1} + u_n u_{n+1}}{u_{n+1}}$ with $b \in \mathbb{C} \setminus \{0\}$:

Associated map: $F_6(x,y) = \left(y, \dfrac{bx - by + xy}{y}\right)$	First integral: $V_6(x,y) = x(y+b)$	Parametrization of $C_{6,h}$: $P_{6,h}(t) = \left(t, \dfrac{-bt+h}{t}\right)$
Möbius map: $M_{6,h}(t) = \dfrac{-bt+h}{t}$	Lie symmetry of $M_{6,h}$: $Y_{6,h} = \left(-h + bt + t^2\right) \dfrac{\partial}{\partial t}$	Lie symmetry of F_6: $X_1 = x(x-y)$ $X_2 = -(x-y)(y+b)$

3.4.2 Conjugations in the set of Palladino's maps

In this Section we apply Proposition 6 to detect the conjugations between the set of Palladino's maps, obtaining:

Proposition 15. *(a) The maps F_1, F_2, F_5 and F_6 are birationally conjugate.*

(b) The maps F_3 and F_4 are birationally conjugate.

(c) Any map in the set $\{F_1, F_2, F_5, F_6\}$ is not conjugate with any map in the set $\{F_3, F_4\}$ via a conjugation which is a correspondence between the respective invariant fibrations $\{C_{j,h}\}$.

Proof. (a) A computation shows that the maps $M_{1,h}$ and $M_{2,k}$ defined in the above section are conjugate via

$$m_h(t) = \frac{h}{b(b^2t + b - h)}$$

with the correspondence between the level sets given by

$$k = f(h) := -\frac{b^3 - h}{h}.$$

Hence, by using Equation (7) in Proposition 6, we obtain that $F_1 = \Psi^{-1} F_2 \Psi$, where

$$\Psi(x,y) = \Psi(x,y) = P_{2,f(h)} \circ m_h \circ P_{1,h}^{-1}(x,y)_{|h=V_1(x,y)} = \left(-\frac{by+1}{b}, -\frac{bx+1}{b(bx-by+1)}\right),$$

and

$$\Psi^{-1}(x,y) = P_{1,f^{-1}(k)} \circ m_{f^{-1}(k)} \circ P_{2,k}^{-1}(x,y)_{|k=V_2(x,y)} = \left(-\frac{b^2xy + 2\,by + 1}{b\,(by+1)}, -\frac{bx+1}{b}\right).$$

Analogously, the maps $M_{2,h}$ and $M_{5,k}$ are conjugate via the map $m_h(t) = -1/(t+b)$, with the correspondence given by $k = f(h) := h - 1$. Hence $F_2 = \Psi^{-1} F_5 \Psi$, where $\Psi(x,y) = (-(bx+1)/x, -(by+1)/y)$, and $\Psi^{-1}(x,y) = (-1/(x+b), -1/(y+b))$.

The maps $M_{6,h}$ and $M_{5,k}$ are conjugate via the map $m_h(t) = -h/t$ with the correspondence $k = f(h) = h$. Again we get $F_6 = \Psi^{-1} F_5 \Psi$ with $\Psi(x,y) = (-y-b, -x(y+b)/y)$, and $\Psi^{-1}(x,y) = (-y\,(x+b)\,/x, -x-b)$.

(b) The maps $M_{3,h}$ and $M_{4,k}$ are conjugate via the map $m_h(t) = b\,t/((h-1)t-b)$ with the correspondence $k = f(h) = h$, so $F_3 = \Psi^{-1} F_4 \Psi$ with

$$\Psi(x,y) = P_{4,f(h)} \circ m_h \circ P_{3,h}^{-1}(x,y)_{|h=V_3(x,y)} = \left(\frac{b\,x}{y-x}, \frac{-b\,xy}{(x-y)(y+b)}\right),$$

and $\Psi^{-1}(x,y) = (bxy/((x-y)(x+b)), by/(x-y))$.

(c) The statement follows from Proposition 6, by taking into account the fact that when we look for a conjugation between any of the Möbius maps M_i with $i = 1,2,5,6$, and any M_j with $j = 3,4$, we obtain that there exists such conjugations, but there is not a bijection between the level sets of V_3 and V_4, so it is not possible to construct conjugations between the maps in the set $\{F_1, F_2, F_5, F_6\}$ and the maps in the set $\{F_3, F_4\}$, via a conjugation which is a correspondence between the respective associated invariant fibrations. ∎

3.4.3 Analysis of the Palladino's recurrences number 3 and 5

From Proposition 15, the Palladino recurrences number $1, 2, 5$ and 6 on one hand, and number 3 and 4 on the other, have the same dynamics from a qualitative viewpoint. We characterize the dynamics of a representative of each set of recurrences, by studying the maps M_3 and M_5. First, we consider the difference equation

$$u_{n+2} = \frac{b\,(u_{n+1} - u_n) + u_{n+1}^2}{u_n}, \text{ with } b \in \mathbb{C}. \tag{18}$$

For each initial condition u_0, u_1 in its good set $\mathcal{G} \subset \mathbb{C}$, set $I = V_3(u_0, u_1) = (u_1 + b)/u_0$. Then:

Proposition 16. *Let $\{u_n\}$ be a solution of Equation (18) with initial condition u_0, u_1 in $\mathcal{G}$, then:*

(a) If $|I| < 1$, then the solution converges to $u = b/(I-1)$.

(b) If $|I| > 1$ or $I = 1$ and $b \neq 0$, then the solution is unbounded.

(c) If I is a p-root of the unity with $I \neq 1$, then the solution is p-periodic.

(d) If $|I| = 1$ and I is not a p-root of the unity, then $\{u_n\}$ is conjugate to a sequence generated by an irrational rotation of angle $\arg(I)$, and the set of accumulation points of $\{u_n\}$ is a the circle in $\mathbb{C}$ with center $z = b/(I-1)$ and radius $|u_0 - b/(I-1)|$.

(e) If $I = 1$ and $b = 0$, then the solution is constant .

Proof. The proof follows straightforwardly by noticing that due to the particular form of the family of parameterizations $\{P_{3,h}\}$, we have that $u_n = M_{3,h}^n(u_0)$, for $h = V_3(u_0, u_1)$. Now, if $h = 1$, then $M_{3,h}$ is the identity if $b = 0$ and each orbit of $M_{3,1}$ is unbounded if $b \neq 0$. If $h \neq 1$, then $M_{3,h}$ has a unique fixed point $t_h := b/(h-1)$. This point is a global attractor if $|h| < 1$, and a repeller (and the orbits are unbounded) if $|h| > 1$. If $|h| = 1$ with $h \neq 1$ then $M_{3,h}$ is conjugate to the rotation given by $z \to h\,z$, with $z \in \mathbb{C}$. ∎

Now, we consider the difference equation

$$u_{n+2} = \frac{bu_{n+1} + u_n u_{n+1}}{u_{n+1} + b}, \text{ with } b \in \mathbb{C} \setminus \{0\} \tag{19}$$

For each initial condition u_0, u_1 in its good set $\mathcal{G} \subset \mathbb{C}$, set

$$t_j = t_j(u_0, u_1) := \frac{-b + (-1)^j \sqrt{b^2 + 4\,u_1(u_0 + b)}}{2} \text{ and } \xi = -\frac{\left(b + \sqrt{b^2 + 4\,u_1(u_0 + b)}\right)^2}{4u_1(u_0 + b)}.$$

Proposition 17. *Let $\{u_n\}$ be a solution of Equation (19) with initial condition u_0, u_1 in $\mathcal{G}$, then:*

(a) When $u_1(u_0 + b) = -b^2/4$, the solution converges to $u = -b/2$.

(b) When $u_1(u_0 + b) \neq -b^2/4$, then:

(i) If $u_1 = u_0 = t_j$, for any $j = 1, 2$, then the solution is constant.

(ii) If $|\xi| < 1$, $u_0 \neq t_1$ and $u_1 \neq t_1$, then the solution converges to $u = t_0$; and if $|\xi| > 1$, $u_0 \neq t_0$ and $u_1 \neq t_0$, then the solution converges to $u = t_1$.

(iii) If $|\xi| = 1$, then the solution is either p-periodic if ξ is a p-root of unity, or such that is conjugate to a sequence generated by an irrational rotation of angle $\arg(\xi)$, and the set of accumulation points of $\{u_n\}$ is a set homeomorphic to $\mathbb{S}^1$ in $\widehat{\mathbb{C}}$.

Proof. The proof is a direct application of Proposition 7 since, from the particular form of the parameterizations $P_{5,h}$, we have that $u_n = M_{5,h}^n(u_0)$. ∎

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Appendix: Proper parameterizations and its inversion

Suppose that C is a rational curve. Observe that $\mathbb{K}(C)[t]$, the polynomials with coefficients in the field of rational functions in C, is a Euclidean domain, so the Euclidean algorithm can be applied to compute the greatest common divisor. Let $P(t)$ be a rational affine parametrization of C over $\mathbb{K}$ defined as

$$P(t) = \left(\frac{P_{11}(t)}{P_{12}(t)}, \frac{P_{21}(t)}{P_{22}(t)} \right),$$

where $P_{ij}(t) \in \mathbb{K}[t]$ and $\gcd(P_{1i}, P_{2i}) = 1$, that is $P(t)$ is in *reduced form.* The next results give a quick way to check whether a parametrization is proper or not:

Theorem 18. *([14, Thm. 4.21]) Let C be an affine rational curve defined over $\mathbb{K}$ with defining polynomial $f(x, y) \in \mathbb{K}[x, y]$, and let $P(t)$ be a parametrization of C. Then $P(t)$ is proper if and only if*

$$deg(P(t)) = max\{deg_x(f), deg_y(f)\},$$

where the degree of $P(t)$ is the maximum of the degrees of its rational components.

The next result allow us to compute the inverse of $P(t)$.

Theorem 19. *([14, Thm. 4.37]) Let $P(t)$ be a proper parametrization in reduced form with nonconstant components of a rational curve C. Let*

$$H_1(t, x) := x\, P_{12}(t) - P_{11}(t),$$
$$H_2(t, y) := y\, P_{22}(t) - P_{21}(t),$$

be considered as polynomials in $\mathbb{K}(C)[t]$. Set $M(x, y, t) := \gcd_{\mathbb{K}(C)[t]}(H_1, H_2)$, then, $\deg_t(M(x, y, t)) = 1$. Moreover, its single root in t, is the inverse of P.

As a consequence of the above result $M(x, y, t)$ is a linear polynomial in t, so setting $M(x, y, t) = D_1(x, y)\, t - D_0(x, y)$ the inverse of the parametrization P is given by

$$P^{-1}(t) = \frac{D_0(x, y)}{D_1(x, y)}. \tag{20}$$