

BIFURCATION OF THE SEPARATRIX SKELETON IN SOME 1-PARAMETER FAMILIES OF PLANAR VECTOR FIELDS.

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ABSTRACT. This article deals with the bifurcation of polycycles and limit cycles within the 1-parameter families of planar vector fields X_m^k , defined by $\dot{x} = y^3 - x^{2k+1}$, $\dot{y} = -x + my^{4k+1}$, where m is a real parameter and $k \geq 1$ integer. The bifurcation diagram for the separatrix skeleton of X_m^k in function of m is determined and the one for the global phase portraits of $(X_m^1)_{m \in \mathbb{R}}$ is completed. Furthermore for arbitrary $k \geq 1$ some bifurcation and finiteness problems of periodic orbits are solved. Among others, the number of periodic orbits of X_m^k is found to be uniformly bounded independent of $m \in \mathbb{R}$ and the Hilbert number for $(X_m^k)_{m \in \mathbb{R}}$, that thus is finite, is found to be at least one.

1. INTRODUCTION

This article concerns periodic orbits and separatrix cycles for the 1-parameter families $(X_m^k)_{m \in \mathbb{R}}$, where X_m^k are planar polynomial vector fields of degree $4k+1$, given by

$$(1) \quad \dot{x} = y^3 - x^{2k+1}, \quad \dot{y} = -x + my^{4k+1}$$

depending on the parameter $m \in \mathbb{R}$, for arbitrary but fixed $k \geq 1$. Here both the nilpotent center-focus problem as well as the existential part of Hilbert’s sixteenth problem for $(X_m^k)_{m \in \mathbb{R}}$ are approached.

The study of the particular family (1) is motivated by the questions raised in [11, 12, 13]. The authors in these papers presumed that the change of stability of the focus of (1) announces the birth of a connection between the two saddles. In this paper this presumption is confirmed qualitatively. Besides system (1) is a simple mathematical model whose study is not trivial and it gives the opportunity to illustrate a whole arsenal of methods classically used in the field. Next theorem summarizes the results from [11, 12, 13].

Theorem 1 ([13]). *Let X_m^1 be defined by (1). For $m \leq 0$ the origin is a global attractor for X_m^1 . For $m > 0$ the global phase portrait of X_m^1 is topologically equivalent to one of the four drawn in Figure 3; in particular,*

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- (1) *there are three singularities: a nilpotent focus at $(0,0)$, which is stable for $0 < m < 3/5$ and unstable for $m \geq 3/5$, and two hyperbolic saddle points at $\mathbf{p}_\pm \equiv \mathbf{p}_\pm(m) = (\pm m^{-1/4}, \pm m^{-1/4})$.*
- (2) *For $m < 547/1000$ or $m \geq 3/5$ no limit cycles nor polycycles do exist.*
- (3) *For $547/1000 \leq m < 3/5$ at most one limit cycle and polycycle exist and both cannot coexist. The limit cycle, if it exists, is hyperbolic and unstable. There exist $n \in \mathbb{N}$, $547/1000 < m_C^1 < \dots < m_C^n < 3/5$ such that for $m = m_C^j$, $1 \leq j \leq n$, a heteroclinic 2-saddle cycle is formed.*

From numerical simulations the authors of [13] presumed that there is exactly one parameter value m_C for which X_m^1 presents a 2-saddle cycle. However the authors emphasize that a rigorous proof for its unicity is missing.

This article provides with an analytic confirmation of the unicity (see Theorem 5) and the bifurcation diagram of global phase portraits of X_m^1 , $m > 0$ can thus be completed. Furthermore, here the case $k \geq 2$ is considered.

For $k \geq 2$ the bifurcation diagram of global phase portraits for $(X_m^k)_{m \in \mathbb{R}}$ is completed up to configurations of limit cycles of X_m^k . The analyses involves the control of separatrix and limit cycles, which are of global nature and therefore difficult to trace.

Recently, in [14], a technique is developed to localize separatrix bifurcations, which is applied in [15] to give fine estimates for the Bogdanov-Takens separatrix cycle. This technique does not apply for the family $(X_m^k)_m$. However the family transforms into a semi-complete family of indefinitely rotated vector fields $X_m^{k,R}$. Then the existence of the 2-saddle cycle is obtained from the behavior of the limit vector fields, both being strip flows with an algebraic curve of singularities. This argument differs from the one applied in [13] for the case $k = 1$, where one relies on Poincaré-Bendixson Theorem and limit cycle results. Next the uniqueness is proven exploiting the principles of the rotated property owned by $X_m^{k,R}$. Of course the monotonic movement is not necessary conserved by the separatrices of X_m^k . Nevertheless this has no influence on the bifurcation of the separatrix skeleton of X_m^k , $m > 0$.

In this article, for all $k \geq 1$, the relative movement of the separatrices at the hyperbolic saddles of X_m^k is controlled with increasing $m > 0$ and the bifurcation diagram for the separatrix skeleton of X_m^k with varying m thus is obtained (see Theorem 3). Furthermore, the absence of limit cycles is proven for m sufficiently small and m sufficiently large, that permits to apply the Roussarie compactification-localization method in the treatment of Hilbert's 16th problem for $(X_m^k)_{m \in \mathbb{R}}$ (see Theorem 4 and [27]).

Recall that Hilbert 16th Problem asks for the maximal number \mathcal{H}_n of limit cycles of a planar polynomial vector field $\dot{x} = P_n(x, y)$, $\dot{y} = Q_n(x, y)$, only depending on the degree of the polynomials P_n, Q_n (see e.g. [27, 17, 29, 28, 4, 5]). The so-called existential part deals with the finiteness of the

Hilbert number \mathcal{H}_n and is still to be answered beyond the field of linear vector fields. Dulac's problem, which concerns the finiteness of the number of limit cycles for *individual* analytic vector fields, is solved independently by Ilyashenko and Ecalle [17]. There are several lower bounds known for \mathcal{H}_n ; best lower bounds until now grow at order $(n+1)^2 \ln(n+1)$, see [6, 18, 19], e.g., $\mathcal{H}_n \geq 4(n+1)^2(1.442695 \ln(n+1) - 1/6) + n - 2/3$ [18]. In this article we provide with an example of 1-parameter family $(X_m^k)_{m \in \mathbb{R}}$ of polynomial vector fields of degree $4k+1$ for which the Hilbert number is finite. There are only a few concrete families known for which the Hilbert number is finite. When restricting the family of planar vector fields of degree n to *bounded* classical Liénard equations of degree n , i.e. $\mathcal{L}_{n,K} \leftrightarrow \dot{x} = y, \dot{y} = -x + \sum_{i=0}^{n-1} a_i x^i y$, where $|a_i| \leq K, \forall 0 \leq i \leq n-1$, for some arbitrary $K > 0$, then the number of limit cycles of $\mathcal{L}_{n,K}$ is bounded uniformly (only depending on n and K , see [28] for n odd and [4] for n even). Putting a bound K on the family of Liénard equations corresponds to staying at a distance from slow-fast systems, where more limit cycles can be created (see [9]).

For the Center-Focus Problem we refer to [26, 20, 22, 2, 1] and recall that a singularity is said to be a (topological) center if it has a punctured neighborhood full of concentric (non-isolated) periodic orbits. It aims at deciding whether a singularity is a focus or a center. Classically this problem deals with singularities being a center for the linearization of a polynomial or an analytic vector field (i.e. having purely imaginary eigenvalues), and is referred to as the center problem of Poincaré. The analytic linear type center is proved to be a topological center if an analytic first integral exists; see [26, 20, 22]. In [1, 13] an algebraic algorithm is provided for solving the analytic nilpotent center-focus problem that is encountered in $(X_m^k)_{m \in \mathbb{R}}$. The problem thus is algebraically solvable, however computations become too complicated at the bifurcation value when the focus changes stability for general values of the degree parameter $k \geq 2$. To overcome these difficulties here we additionally rely on the separatrix skeleton.

2. STATEMENT OF THE RESULTS AND ORGANIZATION OF THE ARTICLE

Throughout the paper we assume that $k \geq 1$ is an arbitrary fixed integer. To precisely state the results in this article we recall the definition of *separatrix* and *separatrix skeleton* used in present article (see [8]).

Definition 2. *Let X be a continuous planar vector field having only isolated singularities. An orbit Γ of X is called separatrix if it is homeomorphic to \mathbb{R} and if for each neighborhood \mathcal{N} of Γ there exists $q \in \mathcal{N}$ such that $\alpha(q) \neq \alpha(\Gamma)$ or $\omega(q) \neq \omega(\Gamma)$. The closure of the union of separatrices is called the separatrix skeleton of X . The union of the separatrix skeleton, limit cycles and topological sinks and sources of X is called the extended separatrix skeleton of X . Maximal connected components in the complement of the extended separatrix skeleton are called canonical regions of X . The*

union of the extended separatrix skeleton together with one orbit from each of the canonical regions is called the completed separatrix skeleton.

Recall that a canonical region is found to be parallel, i.e. given either by a strip, an annular or spiral flow (see [8]). Furthermore, a *limit cycle* is a periodic orbit γ that is isolated, meaning that there does exist a neighborhood of γ in the Hausdorff sense with no other periodic orbits. According to Definition 2 a limit cycle is not a separatrix and it is not included in the separatrix skeleton. It is included in the extended separatrix skeleton while a non-isolated periodic orbit is not. If γ is a non-isolated periodic orbit, then it belongs to an open annulus full of concentric non-isolated periodic orbits, that we call a *period annulus*. A (maximal) period annulus is an example of canonical region and γ is a possible characteristic orbit for it. A non-isolated periodic orbit is only included in the completed separatrix skeleton. Furthermore, topological sinks and sources are considered as degenerate limit cycles and therefore not included in the separatrix skeleton.

The first result describes the separatrix skeleton for varying $m \in \mathbb{R}$ and for arbitrary fixed $k \geq 1$.

Theorem 3. *Let $(X_m^k)_{m \in \mathbb{R}}$ be the family of vector fields in (1). For $m \leq 0$ the origin is the only singularity for X_m^k and it is a global attractor. For $m > 0$ there are three finite singularities, a nilpotent focus $\mathbf{p}_0 = (0, 0)$ and two hyperbolic saddles \mathbf{p}_\pm , given by*

$$(2) \quad \mathbf{p}_\pm = (\pm m^{-\frac{3}{2(4k-1)(k+1)}}, \pm m^{-\frac{2k+1}{2(4k-1)(k+1)}}).$$

With increasing $m > 0$, the phase portrait of X_m^k undergoes a separatrix bifurcation passing through a unique parameter value $m_C^k > 0$, giving rise to three separatrix skeletons. In particular, for $m = m_C^k$ the phase portrait of X_m^k exhibits a 2-saddle cycle that gets broken for $m \neq m_C^k$ and according to the sign of $m - m_C^k$, the separatrices are rearranged as in Figure 1.

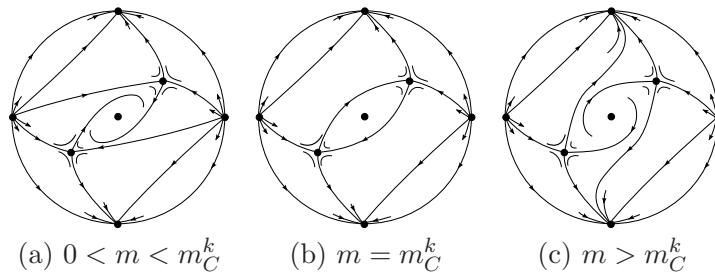


FIGURE 1. Bifurcation of separatrix skeleton for $(X_m^k)_{m>0}$ (see Theorem 3).

Next result adds information on periodic orbits and limit cycles of X_m^k .

Theorem 4. Let X_m^k and m_C^k be defined in (1) and Theorem 3 respectively. Then there exist $0 < m_0^k < m_\infty^k < \infty$ such that periodic orbits can only exist for X_m^k with $m_0^k < m < m_\infty^k$ and they are isolated. Furthermore,

- (1) The number of periodic orbits of $(X_m^k)_{m \in \mathbb{R}}$ is uniformly bounded, and the Hilbert number $\mathcal{H}((X_m^k)_{m \in \mathbb{R}})$ is at least one: $1 \leq \mathcal{H}((X_m^k)_{m \in \mathbb{R}}) < \infty$.
- (2) The phase portrait of X_m^k in the Poincaré disc is presented in Figure 2 for $m < m_0^k$ and $m > m_\infty^k$.

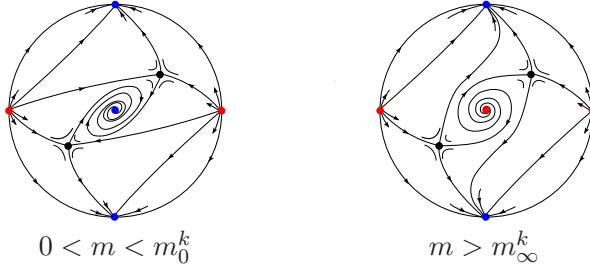


FIGURE 2. Global phase portraits of X_m^k for small and large m (see Theorem 4).

Furthermore we obtain the following estimates for $m_0^k, m_\infty^k, k \geq 1$: $m_\infty^1 = \frac{3}{5}$, $\frac{547}{1000} < m_0^1 = m_C^1 < \frac{3}{5}$, $m_\infty^k \geq \max(m_C^k, \frac{(2k+1)!!}{(4k+1)!!!!})$, $\left(\frac{k \sqrt[4]{2(k-1)(2k+1)}}{2(k-1)(4k+1)}\right)^{k+1} \leq m_0^k \leq \min(m_C^k, \frac{(2k+1)!!}{(4k+1)!!!!})$, $\forall k \geq 2$. Moreover we find the complete bifurcation diagram of global phase portraits of X_m^k for $k = 1$.

Theorem 5. There exists a unique $547/1000 < m_C^1 < 3/5$ such that the global phase portraits of X_m^1 are presented in Figure 3 in function of increasing $m > 0$. In particular,

- (1) At $m = 3/5$ a Hopf-like bifurcation takes place: the origin is attracting for all $m < 3/5$ and repelling for all $m \geq 3/5$.
- (2) For all $m < m_C^1$ or $m \geq 3/5$ there are no limit cycles nor polycycles.
- (3) For $m = m_C^1$ there exists a repelling hyperbolic 2-saddle cycle Γ .
- (4) For all $m_C^1 < m < 3/5$ there exists a repelling hyperbolic limit cycle, that shrinks from Γ to the origin when m increases from m_C^1 to $3/5$.

Recall that the global phase portraits are obtained by extending the phase portraits to infinity which is represented by the equator on the Poincaré disc \mathbb{D}^1 (see e.g. [8]). The phase portraits sketched in Figures 1, 2, 3, 4, 8 and 9 are on \mathbb{D}^1 . Two vector fields X and Y are said to be topologically equivalent on \mathbb{D}^1 if there exists a homeomorphism $h : \mathbb{D}^1 \rightarrow \mathbb{D}^1$ sending orbits of X to orbits of Y preserving the orientation. In case that the homeomorphism h also is a linear isomorphism, then we say that X and Y are

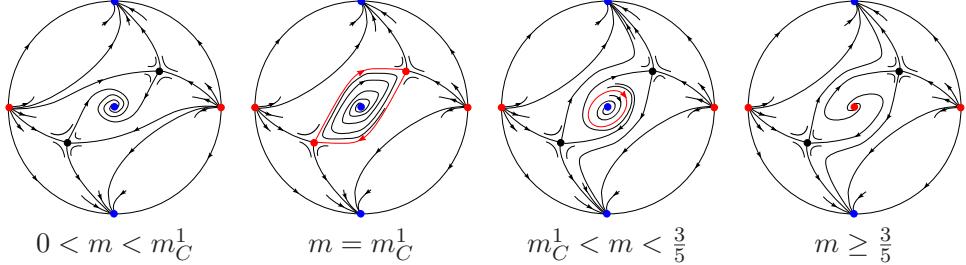


FIGURE 3. Bifurcation of global phase portraits for $(X_m^1)_{m>0}$ (see Theorem 5).

linearly equivalent. Notice that the phase portraits (a) and (d) in Figure 3 are not topologically equivalent since the orientation is not preserved. In the topological classification of a continuous planar vector field with only finitely many singularities, we rely on the Theorem of Markus, Neumann and Peixoto, saying that its topological equivalence class is determined by the completed separatrix skeleton (see [8]).

The article is organized as follows. In Section 3 singularities of X_m^k are localized and their topological type is analyzed for all $m \in \mathbb{R}$. From this analysis it already turns out that the case $m \leq 0$ is completely understood by Lyapunov Stability Theorem (Proposition 6) and that separatrices only show up in the case that $m > 0$. In Section 4 the vector field X_m^k is transformed into $X_m^{k,R}$ by linear equivalence, thus generating an analytic semi-complete family of indefinitely rotated vector fields $X_m^{k,R}$. Definition and properties of such vector fields are quickly recalled by which the movement of the separatrices then is controlled. In further sections the statements of Theorems 3, 4 and 5 are proven for $X_m^{k,R}$. Using the equivalence between both families these statements can be transferred to X_m^k .

Next, in Section 5, since we are interested in global phase portraits of $X_m^{k,R}$, the behavior of $X_m^{k,R}$ near infinity is analyzed by means of Poincaré compactification. In Section 6 the absence of limit cycles in the phase portrait of $X_m^{k,R}$ is shown for sufficiently small values of m (see Theorem 16) and a unique global phase portrait is obtained up to topological equivalence, thus proving the statement for $0 < m < m_0^k$ in Theorem 4. In Section 7 the relative positions of the separatrices for $X_m^{k,R}$ is determined for sufficiently large m . Using the results from Sections 5, 6 and 7 the existence of a unique $m_C^k > 0$ for $k \geq 1$ is proven in Section 8 with the properties described in Theorem 3 (see Theorem 19).

Then Theorem 5 is obtained as a corollary of Theorems 1 and 3, the Poincaré-Bendixson Theorem and the Planar Termination Principle of Perko-Wintner. In Section 9 the statement for $m > m_\infty^k$ in Theorem 4 is found using a quasi-homogenous desingularization followed by the Roussarie localization-compactification method to deduce the absence of limit cycles for sufficiently

large m (see Corollary 24). In Section 10 the finiteness result in Theorem 4 is deduced by similar techniques (see Theorems 26 and 27) and finally the center-focus problem for the nilpotent singularity of the family $(X_m^k)_{m \in \mathbb{R}}$ is solved in an analytic-geometric way (see Theorem 25).

3. FINITE SINGULARITIES

Clearly, the flow of X_m^k is invariant with respect to the transformation

$$(3) \quad (t, x, y) \mapsto (t, -x, -y),$$

and throughout the article we rely on it to reduce computations.

Proposition 6. *For $m \leq 0$, the vector field X_m^k has exactly one singularity, $\mathbf{p}_0 = (0, 0)$, and it is a (nilpotent) global attractor. In particular X_m^k does not have limit cycles nor polycycles for $m \leq 0$.*

Proof. Consider $V(x, y) = 2x^2 + y^4 \geq 0$ for $(x, y) \in \mathbb{R}^2$. Then, for $m \leq 0$, $\dot{V} = \langle X_m^k(x, y), \nabla V(x, y) \rangle = -4(x^{2k+2} - my^{4k+4}) \leq 0, \forall (x, y) \in \mathbb{R}^2$. Hence for $m < 0$ the statement directly follows from the Lyapunov Stability Theorem. For $m = 0$ we notice that $V(x, y) \rightarrow \infty$ for $|(x, y)| \rightarrow \infty$ and that the maximal invariant subset of $\{\dot{V} = 0\}$ is $\{(0, 0)\}$; then the Lyapunov Stability Theorem also applies. \square

By Proposition 6 the phase portrait for $m \leq 0$ is completely understood. From now on we only consider the case $m > 0$. Then the Lyapunov method cannot be applied to determine the stability type of the singularity at the origin. Therefore, in [13], a generalization of Lyapunov quantities for nilpotent singularities are computed. To describe the corresponding result, we recall some generalizations of the factorial function $n!$. Given $n \in \mathbb{N} \setminus \{0\}$ the quantities $n!!$ and $n!!!!$ are defined by the following recurrence equations,

$$n!! = n \times (n - 2)!! \text{ and } n!!!! = n \times (n - 4)!!!!,$$

with $j!! = j$ for $j = 1, 2$ and $j!!!! = j$ for $1 \leq j \leq 4$.

Lemma 7. *For $m > 0$ the vector field X_m^k has three singularities, $\mathbf{p}_0 = (0, 0)$ and \mathbf{p}_\pm , defined in (2). The singularities \mathbf{p}_\pm are hyperbolic saddles and the nilpotent singularity \mathbf{p}_0 is an attracting (resp. repelling) focus for $m < m_S^k$ (resp. $m > m_S^k$) where*

$$(4) \quad m_S^k \equiv \frac{(2k+1)!!}{(4k+1)!!!!} \text{ for } k \geq 1.$$

If $k = 1$ the origin also is a repelling focus for $m = m_S^1 = 3/5$.

Proof. The stability of \mathbf{p}_0 is established in [13]. The hyperbolicity and topological type of \mathbf{p}_\pm follow from straightforward calculation of the determinant of the linearization of $X_m^{k,R}$ at \mathbf{p}_\pm , which is $-2(k+1)(4k-1)m^{-\frac{2k+1}{(4k-1)(k+1)}}$, and thus clearly negative for $k \geq 1$. \square

From the results in [13] one cannot decide whether the origin is a center or focus if $m = m_S^k, k > 1$. Then the calculation of the second Lyapunov quantity is particularly delicate because m_S^k becomes exponentially small if k grows large: $\lim_{k \rightarrow \infty} m_S^k = 0$, where m_S^k is defined in (4). Moreover,

$$m_S^{k+1} \leq m_S^k \leq m_S^1 = 3/5 \text{ and } 2^{-k} \leq m_S^k \leq \frac{3}{5} \cdot \left(\frac{3}{4}\right)^{k-1}.$$

Nevertheless in Section 9 we show, relying on the uniqueness of the 2-saddle cycle, that if $m = m_S^k, k > 1$ the origin cannot be a center.

4. SEMI-COMPLETE FAMILY OF INDEFINITELY ROTATED VECTOR FIELDS

From (2) it is seen that, fixing $k \geq 1$, the distance of the saddles \mathbf{p}_\pm of X_m^k to the origin decreases from ∞ to 0 as m increases from 0 to ∞ . Now by a parameter dependent rescaling the singularities \mathbf{p}_\pm can be fixed at $(\pm 1, \pm 1)$ while the singularity \mathbf{p}_0 remains at the origin. In this way the family is reduced to a semi-complete family of indefinitely rotated vector fields, whose definition we recall from [24].

Definition 8. Let $E \subset \mathbb{R}^2$ be connected, $I \subset \mathbb{R}$ an interval and $f = (f_1, f_2) : E \times I \rightarrow \mathbb{R}^2$, $G : E \rightarrow \mathbb{R}$ analytic functions such that $G^{-1}(0)$ does not contain any cycle of the vector fields $X_\lambda \leftrightarrow \dot{x} = f(x, \lambda)$. Then, $(X_\lambda)_{\lambda \in I}$ is said to be

- (1) a semi-complete family of positively (resp. negatively) rotated vector fields (mod $G = 0$) on E if $(f_1 \frac{\partial f_2}{\partial \lambda} - f_2 \frac{\partial f_1}{\partial \lambda})(x, \lambda) > 0$ (resp. < 0) at all $(x, \lambda) \in E \times I$ for which $f(x, \lambda)G(x) \neq 0$ and the singularities of X_λ do not move with $\lambda \in I$.
- (2) a semi-complete family of indefinitely rotated vector fields (mod $G = 0$), if $(X_\lambda)_{\lambda \in I}$ is a semi-complete family of positively or negatively rotated vector fields on any connected component of $E \setminus G^{-1}(0)$.

For $m \geq 0$ we define $X_m^{k,R}, Y_\eta^{k,R}$ and $G_k : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(5) \quad X_m^{k,R} \leftrightarrow \bar{x}' = \bar{y}^3 - \bar{x}^{2k+1}, \bar{y}' = m^{\frac{1}{k+1}}(-\bar{x} + \bar{y}^{4k+1}),$$

$$(6) \quad Y_\eta^{k,R} \leftrightarrow \bar{x}' = \eta^{1/(k+1)}(\bar{y}^3 - \bar{x}^{2k+1}), \bar{y}' = -\bar{x} + \bar{y}^{4k+1} \text{ and}$$

$$(7) \quad G_k(\bar{x}, \bar{y}) = (\bar{y}^3 - \bar{x}^{2k+1})(\bar{y}^{4k+1} - \bar{x}).$$

By Definition 8 it is seen that $(X_m^{k,R})_{m \geq 0}$ is a semi-complete family of indefinitely rotated vector fields (mod $G_k = 0$), that is positively rotated in $G_k^{-1}[0, \infty)$ and negatively rotated in $G_k^{-1}(-\infty, 0]$; for $(Y_\eta^{k,R})_{\eta \geq 0}$ holds the same property, reversing positively by negatively and viceversa. Furthermore for $m > 0$ the vector fields X_m^k , $X_m^{k,R}$ and $Y_{1/m}^{k,R}$ are topologically equivalent; in particular the phase portraits of $X_m^{k,R}$ and $Y_{1/m}^{k,R}$ are identical. In fact, after coordinate transformation $x = m^{-3/[2(4k-1)(k+1)]}\bar{x}$, $y = m^{-(2k+1)/[2(4k-1)(k+1)]}\bar{y}$ and time rescaling $t = m^{6k/[2(4k-1)(k+1)]}\bar{t}$, the vector field X_m^k is reduced to $X_m^{k,R}$ and the saddles \mathbf{p}_\pm are fixed at $(\pm 1, \pm 1)$. Therefore the analysis of limit cycle and separatrix bifurcations for X_m^k is replaced

by the one for $X_m^{k,R}$. To analyze the flow of $X_m^{k,R}$ for small $m > 0$ (resp. large $m > 0$), one relies on the limiting vector field for $m \rightarrow 0$ (resp. $m \rightarrow \infty$). The behavior of $X_m^{k,R}$ for large m can be understood from $Y_\eta^{k,R}$, that is obtained from $X_m^{k,R}$ by introducing the new parameter variable $\eta = 1/m$, rescaling time $t = \eta^{1/(k+1)}\bar{t}$, and taking the limit for $\eta \rightarrow 0$. Notice that for $m \geq 0, \eta \geq 0$ the flow of $X_m^{k,R}$ and $Y_\eta^{k,R}$ remains invariant with respect to (3).

For both $m \rightarrow 0$ as well as $m \rightarrow \infty$ the limiting vector fields are the layer equations of slow-fast systems for small m and small $\eta = 1/m$ respectively. Their phase portrait in the Poincaré disc away from the singular locus are so-called strip flows, i.e. topologically equivalent to $\bar{x}' = 1, \bar{y}' = 0$, as drawn in Figures 4(a) and (c) respectively. The bifurcation problem of limit cycles of $X_m^{k,R}$ for $m \downarrow 0$ (resp. $m \rightarrow \infty$) thus corresponds to cyclicity problems in slow-fast systems.

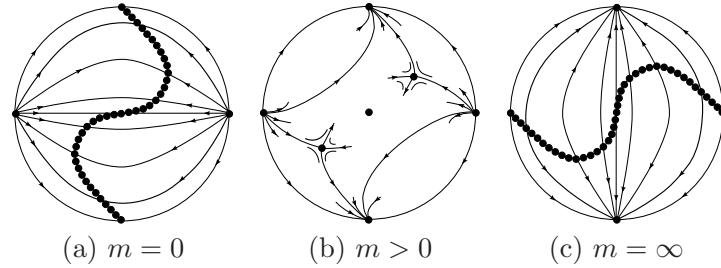


FIGURE 4. Global phase portraits of $X_m^{k,R}$. (a) Horizontal (resp. (c) vertical) strip flow with curve of singularities; (b) Qualitative behavior near infinity (see Proposition 13).

Using coordinates (\bar{x}, \bar{y}) not only the saddles \mathbf{p}_\pm are fixed also possible limit cycles and polycycles are captured in a fixed compact region independent of (m, k) , as is stated in next proposition.

Proposition 9. *Let $X_m^{k,R}, m > 0$ be given in (5). Any polycycle or limit cycle of $X_m^{k,R}$ is contained in the cube $C \equiv [-1, -1] \times [-1, 1]$. Moreover a polycycle necessarily is a 2-saddle cycle.*

Proof. Clearly the saddles are situated at two corner points of C and the direction of the flow along the sides of the cube C is as in Figure 5(b). It is to say, along the sides $\bar{x} = \pm 1$ the flow of $X_m^{k,R}$ points inward to C , while along the sides $\bar{y} = \pm 1$ the flow points outward to C . From the Poincaré-Hopf formula it is known that the sum of the indices surrounded by a periodic orbit or polycycle Γ is 1. Therefore Γ surrounds only the singularity at the origin and hence Γ remains in the cube. The remaining statement follows from the invariance of the flow with respect to (3). \square

Next we have a hyperbolicity criterion for the 2-saddle cycle, if it exists.

Lemma 10. *Let $X_m^{k,R}$, $m > 0$ be as defined in (5) and let μ_S^k be defined by*

$$(8) \quad \mu_S^k = \left(\frac{2k+1}{4k+1} \right)^{k+1} \text{ for all } k \geq 1.$$

If Γ is a polycycle of $X_m^{k,R}$, then Γ is a hyperbolic 2-saddle cycle that is attracting for $m < \mu_S^k$ and repelling for $m > \mu_S^k$.

Proof. Recall that the ratio of hyperbolicity of Γ equals $\alpha = \alpha_- \alpha_+$, where α_{\pm} are the ratios of hyperbolicity of the saddles \mathbf{p}_{\pm} . Here $\alpha_- = \alpha_+$, therefore

$$(9) \quad \alpha = \alpha(m, k) = \left(\frac{-(2k+1) + (4k+1)m^{1/(k+1)} + B(m, k)}{(2k+1) - (4k+1)m^{1/(k+1)} + B(m, k)} \right)^2, \text{ where}$$

$B(m, k) = \sqrt{(2k+1)^2 + m^{1/(k+1)}[m^{1/(k+1)}(4k+1)^2 + 2(4k+5)(2k-1)]}$. It is seen that $\alpha < 1$ if and only if $m < \mu_S^k$ and $\alpha > 1$ if and only if $m > \mu_S^k$. \square

To describe the motion of the separatrices at the hyperbolic saddles \mathbf{p}_{\pm} of $X_m^{k,R}$ with increasing $m > 0$ we recall two principles from [25] about non-intersection of separatrices and splitting of hyperbolic saddle connections for semi-complete families of rotated vector fields.

Theorem 11. *[see [25]] Assume that $(X_{\lambda})_{\lambda \in I}$ is an analytic semi-complete family of positively rotated vector fields.*

- (1) *If $S(\lambda)$ is a separatrix at a hyperbolic saddle of $(X_{\lambda})_{\lambda \in I}$, then it follows that $S(\lambda_1) \cap S(\lambda_2) = \emptyset$ for $\lambda_1 \neq \lambda_2$. Furthermore the tangent line to $S(\lambda)$ rotates monotonically in the positive sense as λ increases.*
- (2) *Assume that $S^{\pm}(\lambda)$ are separatrices at the hyperbolic saddles \mathbf{p}_{\pm} of (X_{λ}) , $\lambda \in I$, and that there is a saddle connection at $\lambda = \lambda_0$, i.e. $S^+(\lambda_0) = S^-(\lambda_0)$. Then, as λ varies from λ_0 , the saddle connection splits and if Σ is a smooth curve transverse to $S^+(\lambda_0)$, the separatrices $S^+(\lambda)$ and $S^-(\lambda)$ move in opposite directions along Σ as λ increases.*

Let $m > 0$ arbitrary but fixed. If the separatrices at the hyperbolic saddle \mathbf{p}_+ are denoted by $\Gamma_+^i = \Gamma_+^i(m)$, $i = 1, 2, 3, 4$, then we use the invariance of the flow of $X_m^{k,R}$ with respect to (3) to denote the corresponding separatrices as Γ_-^i , $i = 1, 2, 3, 4$, at \mathbf{p}_- as illustrated in Figure 5(a) and (c). In particular, if the stable and unstable manifold at \mathbf{p}_{\pm} respectively are denoted by $\mathcal{W}_{\pm}^s = \mathcal{W}_{\pm}^s(m)$ and $\mathcal{W}_{\pm}^u = \mathcal{W}_{\pm}^u(m)$, then

$$(10) \quad \begin{aligned} \alpha(\Gamma_{\pm}^2) = \alpha(\Gamma_{\pm}^4) &= \{\mathbf{p}_{\pm}\}, \omega(\Gamma_{\pm}^1) = \omega(\Gamma_{\pm}^3) = \{\mathbf{p}_{\pm}\}, \text{ or} \\ \mathcal{W}_{\pm}^u &= \Gamma_{\pm}^2 \cup \Gamma_{\pm}^4 \cup \{\mathbf{p}_{\pm}\} \text{ and } \mathcal{W}_{\pm}^s = \Gamma_{\pm}^1 \cup \Gamma_{\pm}^3 \cup \{\mathbf{p}_{\pm}\}. \end{aligned}$$

Clearly the separatrices move when varying m ; this movement is described more precisely in Proposition 12. Let the algebraic sets A_{\pm} be the connected components of $G_k^{-1}(0, \infty)$, indicated in Figure 5, such that

$$(11) \quad G_k^{-1}(0, \infty) = A_+ \cup A_-, \{0\} \times (0, \infty) \subset A_+ \text{ and } \{0\} \times (-\infty, 0) \subset A_-.$$

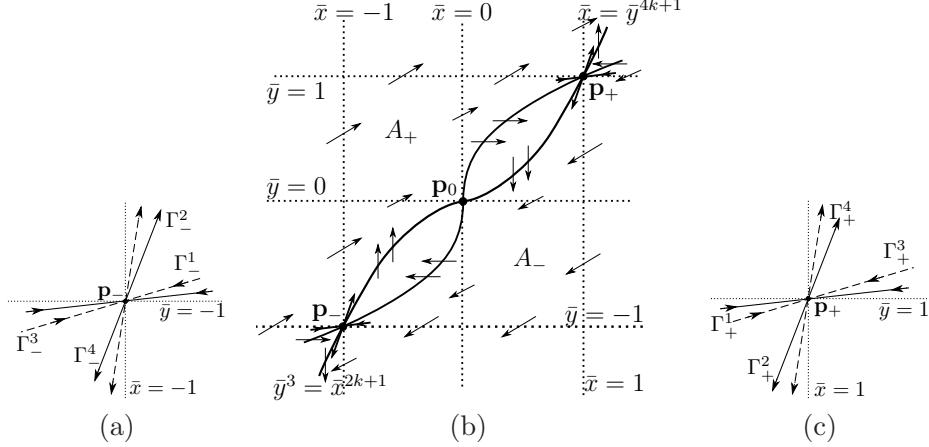


FIGURE 5. Proposition 12. (a,c) Movement of tangent vectors to the separatrices $\Gamma_{\pm}^i, 1 \leq i \leq 4$ of $X_m^{k,R}$ at \mathbf{p}_{\pm} for increasing m in dashed line. (b) Direction of the flow and relative position of the separatrices of $X_m^{k,R}$ with respect to the graphs $\bar{x} = \bar{y}^{4k+1}$ and $\bar{y}^3 = \bar{x}^{2k+1}$.

Proposition 12. Let $X_m^{k,R}, m > 0, G_k : \mathbb{R}^2 \rightarrow \mathbb{R}, \mathcal{W}_{\pm}^s, \mathcal{W}_{\pm}^u, A_{\pm}$ be as defined in (5), (7), (10), (11) and $\Gamma_{\pm}^i, 1 \leq i \leq 4$ as sketched in Figure 5.

- (1) Denote by $\mathcal{C}(\Gamma_{\pm}^i), 1 \leq i \leq 4$ the maximal connected component of the intersection of the corresponding separatrices $\Gamma_{\pm}^1, \Gamma_{\mp}^2, \Gamma_{\mp}^3, \Gamma_{\pm}^4$ with A_{\pm} adhering at \mathbf{p}_{\pm} . Then, for all $1 \leq i \leq 4$, $\mathcal{C}(\Gamma_{\pm}^i)$ rotates monotonically in positive direction when m increases. Furthermore, $\mathcal{C}(\Gamma_{\pm}^i) = \Gamma_{\pm}^i$ for $i = 3, 4$ and the separatrices Γ_{\pm}^3 are unbounded in backward time and the separatrices Γ_{\pm}^4 are unbounded in forward time.
- (2) There is at most one parameter value $m = m_C^k$ for which there is a connection between the saddles \mathbf{p}_{\pm} of $X_m^{k,R}$. For such m_C^k , if it exists, one has $\Gamma_{\pm}^1 = \Gamma_{\mp}^2$ and a 2-saddle cycle is formed. As m varies from m_C^k , the saddle connection splits and the separatrices Γ_{\pm}^1 and Γ_{\mp}^2 move in opposite directions along both $\{(\bar{x}, 0) : -1 < \bar{x} < 0\}$ and $\{(0, \bar{y}) : 0 < \bar{y} < 1\}$, depending on the sign of $m - m_C^k$. See Figure 6.
- (3) $\Gamma_{\pm}^1 \cap \{0 \leq \bar{x} < 1\}$ tends to the graph of $\bar{x} = \bar{y}^{4k+1}$ for $m \rightarrow \infty$ and $\Gamma_{\mp}^2 \cap \{-1 < \bar{x} \leq 0\}$ tends to the graph of $\bar{y}^3 = \bar{x}^{2k+1}$ for $m \downarrow 0$.
- (4) For $m \downarrow 0$ the tangent to \mathcal{W}_{\pm}^u at \mathbf{p}_{\pm} tends to the tangent of $\bar{y}^3 = \bar{x}^{2k+1}$ at \mathbf{p}_{\pm} ; for $m \rightarrow \infty$ this tangent tends to the vertical line $\bar{x} = \pm 1$. For $m \downarrow 0$ the tangent to \mathcal{W}_{\pm}^s at \mathbf{p}_{\pm} tends to the horizontal line $\bar{y} = \pm 1$; for $m \rightarrow \infty$ this tangent tends to the tangent of $\bar{x} = \bar{y}^{4k+1}$ at $\bar{x} = \pm 1$.

Proof. By the invariance with respect to (3) it suffices to concentrate on the separatrices at \mathbf{p}_+ . Since Γ_+^4 is contained in the positively invariant set $A_+ \cap (1, \infty)^2$, it follows that Γ_+^4 is unbounded in forward time. Analogously, Γ_+^3 is unbounded in backward time. Obviously along the graphs of $\bar{y}^3 = \bar{x}^{2k+1}$ and $\bar{x} = \bar{y}^{4k+1}$ one has that $\bar{x}' = 0$ and $\bar{y}' = 0$ respectively. By analyzing the direction field corresponding to $X_m^{k,R}$ one deduces $\mathcal{C}(\Gamma_\pm^i) \neq \emptyset, 1 \leq i \leq 4$. Furthermore,

$$(12) \quad \begin{aligned} \mathcal{C}(\Gamma_+^1), \mathcal{C}(\Gamma_+^4), \mathcal{C}(\Gamma_-^2), \mathcal{C}(\Gamma_-^3) &\subset A_+ = \{(\bar{x}, \bar{y}) : \bar{y}^3 > \bar{x}^{2k+1}, \bar{y}^{4k+1} > \bar{x}\}, \\ \mathcal{C}(\Gamma_+^2), \mathcal{C}(\Gamma_+^3), \mathcal{C}(\Gamma_-^1), \mathcal{C}(\Gamma_-^4) &\subset A_- = \{(\bar{x}, \bar{y}) : \bar{y}^3 < \bar{x}^{2k+1}, \bar{y}^{4k+1} < \bar{x}\}, \\ \mathcal{C}(\Gamma_+^1), \mathcal{C}(\Gamma_+^2) &\subset (-\infty, 1)^2, \mathcal{C}(\Gamma_-^1), \mathcal{C}(\Gamma_-^2) \subset (-1, \infty)^2, \\ \mathcal{C}(\Gamma_\pm^i) &= \Gamma_\pm^i \subset (-\infty, -1)^2 \cup (1, \infty)^2, \text{ for } i = 3, 4. \end{aligned}$$

Then by Theorem 11 the first two items follow. The fourth item follows from straightforward calculations (See Figures 5(a) and (c)). So we are left with the third item. Analyzing the direction of $X_m^{k,R}$ it is seen that for all $m > 0$ the separatrix Γ_+^1 has a backward intersection with $\bar{x} = a$ at some point $(a, \gamma_+^1(a, m))$ and that the separatrix Γ_-^2 has a forward intersection with $\bar{y} = -a$ at some point $(\gamma_-^2(a, m), -a)$. Now it suffices to show that

$$(13) \quad \begin{aligned} \lim_{m \rightarrow 0} \gamma_+^1(a, m) &= 1 \text{ and } \lim_{m \rightarrow \infty} \gamma_+^1(a, m) = a^{1/(4k+1)}, \\ \lim_{m \rightarrow 0} \gamma_-^2(a, m) &= -|a|^{3/(2k+1)} \text{ and } \lim_{m \rightarrow \infty} \gamma_-^2(a, m) = -1. \end{aligned}$$

Since the stable and unstable manifolds at hyperbolic singularities of an analytic family can locally be written as graphs of analytic functions of the phase as well as parameter variable, it is found that the mappings $\gamma_-^2(a, \cdot), \gamma_+^1(a, \cdot)$ are analytic (see e.g. Appendix II in [24]). By Theorem 11 it follows that they are as well strictly decreasing with increasing $m > 0$. Since $\sqrt[4k+1]{a} < \gamma_+^1(a, m) < 1, \forall m > 0$ and since $\gamma_+^1(a, \cdot)$ is strictly increasing, the limits in (13) concerning $\gamma_+^1(a, \cdot)$ do exist and

$$(14) \quad \forall m > 0 : \lim_{m \rightarrow \infty} \gamma_+^1(a, m) \leq \gamma_+^1(a, m) \leq \lim_{m \downarrow 0} \gamma_+^1(a, m).$$

Using the notation $\bar{x}_-(m) = \gamma_-^2(0, m), \bar{y}_+(m) = \gamma_+^1(0, m)$ in Figure 6 the case $a = 0$ is illustrated. By continuous dependence of solutions of $X_m^{k,R}$ on $m = 0$, it follows directly that $\lim_{m \downarrow 0} \gamma_+^1(a, m) = 1$. As exposed before to consider the limits for $m \rightarrow \infty$ we work with $\eta = 1/m$ and $Y_\eta^{k,R}$ for $\eta \rightarrow 0$.

As seen in Figure 4(c), $Y_0^{k,R}$ defines a vertical flow with the algebraic curve $\bar{x} = \bar{y}^{4k+1}$ full of singularities. Clearly the limit of $\gamma_-^2(a, \cdot)$ for $m \rightarrow \infty$ is obtained by continuous dependence on $\eta = 0$. This reasoning cannot be used to determine the limit of $\gamma_+^1(a, \cdot)$ for $m \rightarrow \infty$, since then $\bar{x} = a$ is not a transversal section for the flow of $Y_0^{k,R}$. Assume that there exists $0 \leq a_0 < 1$ such that $\lim_{m \rightarrow \infty} \gamma_+^1(a_0, m) = b_0 > \sqrt[4k+1]{a_0}$. Then, by the Flow Box Theorem it follows that there exists $\eta_0 > 0$ such that for all $0 \leq \eta < \eta_0$ the orbit of $Y_\eta^{k,R}$ through (a_0, b_0) is unbounded in forward time.

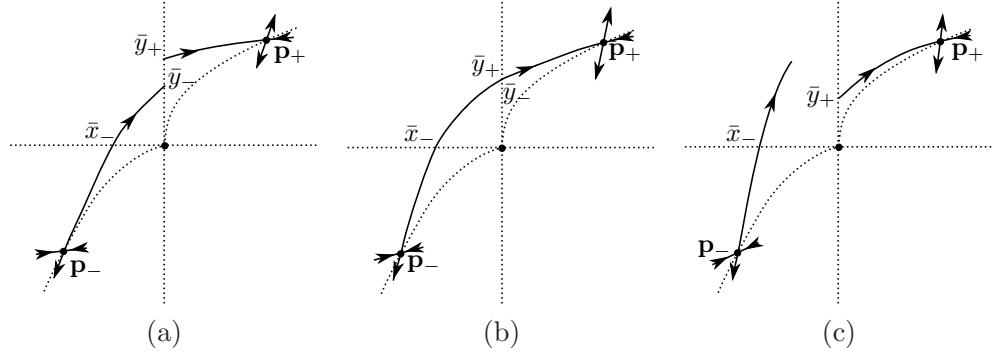


FIGURE 6. Possible relative positions of separatrices Γ_+^1 and Γ_-^2 of $X_m^{k,R}$ as m increases (see Proposition 12).

As a consequence for all $m > 1/\eta_0$ and for all $b \geq b_0$, the orbit of $X_m^{k,R}$ through (a_0, b) is unbounded in forward time. But from (14) it follows that for all $m > m_0 : \gamma_+^1(a, m) \geq b_0$ and by definition the positive orbit through $\gamma_+^1(a, m)$ of $X_m^{k,R}$ corresponds to $\Gamma_+^1 \cap \{a_0 \leq \bar{x} < 1\}$, which is bounded in forward time. Therefore the assumption that for some $0 \leq a_0 < 1$ the limit $\lim_{m \rightarrow \infty} \gamma_+^1(a_0, m)$ remains strictly above the graph of $\bar{x} = \bar{y}^{4k+1}$ is false. The limit of $\gamma_-^2(a, \cdot)$ for $m \downarrow 0$ in (13) is obtained by a similar reasoning. \square

In [7] Duff described the global behavior of any one-parameter family of limit cycles generated by a family of rotated vector fields. From this it follows that limit cycles that are completely contained in a region where the vector field is rotated in one sense (either in the positive or the negative sense), also possess the non-intersection property with increasing m . Furthermore a stable (resp. unstable) limit cycle of a positively rotated vector field contracts (resp. expands) with increasing m , and a stable (resp. unstable) limit cycle of a negatively rotated vector field expands (resp. contracts) with increasing m . Now since limit cycles of $X_m^{k,R}$ have to surround the origin, they run alternatingly through regions where the vector field rotates in the positive and negative sense. Hence the non-intersection principle for limit cycles does not apply here. Nevertheless, the *Planar Termination Principle* or also called *Wintner-Perko Termination Principle* gives explicit information on how any one-parameter family of limit cycles $(\gamma_m)_{m \in \mathcal{M}}$ of planar vector fields terminates (see [23]): ‘The family $(\gamma_m)_{m \in \mathcal{M}}$ is open or cyclic. If it is open, then either \mathcal{M} is unbounded or $(\gamma_m)_{m \in \mathcal{M}}$ terminates at a critical point or graphic of the system on the Poincaré sphere.’

5. COMPACTIFICATION OF $X_m^{k,R}$

To describe the relative movement of the separatrices $\Gamma_\pm^i, i = 1, 2$, the monotonicity property for rotated vector fields is used as seen in Section 4. However, to guarantee the crossing of separatrices, it is necessary to find two

opposite relative positions in case of sufficiently small and sufficiently large m . To that end, we rely on the Poincaré-Bendixson Theorem, that guarantees, under compactness assumptions, that the α - and ω -limit sets of the orbits are singular points, periodic orbits or separatrix cycles. To uniquely determine the asymptotic structure for small and large m one can study limit cycles in the global plane for $X_m^{k,R}$ or perturbations of the global phase portraits for the limiting vector fields $X_m^{k,R}$ for $m \rightarrow 0$ and $m \rightarrow \infty$. For these studies we compactify parameter space and phase plane by adding ∞ to both spaces. Clearly, by adding $X_\infty^{k,R} \equiv Y_0^{k,R}$, the analytic family $(X_m^{k,R})_{m \geq 0}$ is analytically extended to $(X_m^{k,R})_{0 \leq m \leq \infty}$, thus compactifying parameter space. Analogously, parameter space is compactified for $(Y_\eta^{k,R})_{\eta > 0}$.

We speak of a *compact family of planar vector fields* $(X_\lambda)_\lambda$ if the vector fields are defined on a compact metric space D , and depend on a parameter λ , that also belongs to a compact metric space P . Below we consider the compactification of $(X_m^{k,R})_{0 \leq m \leq \infty}$ by extending the vector fields $X_m^{k,R}$ analytically to the equator on the Poincaré disc \mathbb{D}^1 , with analytic dependence on $m \in [0, \infty]$. The analysis of the critical points at infinity gives the asymptotic behavior of trajectories that become unbounded. Furthermore we obtain the knowledge near infinity in a uniform way (i.e. outside a fixed compact set, which does not change when the parameter is changed). This is important when replacing the study of global phase portraits of $X_m^{k,R}$ by the study of bifurcations inside $(X_m^{k,R})_{m > 0}$. For instance, it helps in the detection of limit cycles escaping to infinity (so-called large amplitude limit cycles), to control the movement of the separatrices in the global plane for all $m > 0$ and to localize the global problem of limit cycles for large m .

Proposition 13. *The families of vector fields $(X_m^{k,R})_{m > 0}$ and $(Y_\eta^{k,R})_{\eta > 0}$ defined in (5) and (6) extend analytically to compact families $(\hat{X}_m^{k,R})_{0 \leq m \leq \infty}$ and $(\hat{Y}_\eta^{k,R})_{0 \leq \eta \leq \infty}$ respectively on the Poincaré disc.*

- (1) *The topological behavior of $\hat{X}_m^{k,R}$ near the equator on the Poincaré disc for $m > 0$ is sketched in Figure 4(b), exhibiting two non-elementary repelling nodes and two hyperbolic attracting nodes along the equator, being the α - resp. ω -limit sets of the separatrices Γ_\pm^3 resp. Γ_\pm^4 .*
- (2) *The analytic extension of the vector fields $X_0^{k,R}$ and $X_\infty^{k,R} = Y_0^{k,R}$ to the Poincaré disc is sketched in Figures 4(a) and (c), presenting two degenerate singularities -corresponding to the singular locus at infinity- and two hyperbolic nodes (repelling and attracting resp.) along the equator.*

Proof. In the charts $(\bar{x}, \bar{y}) = (v/z, 1/z)$ and $(\bar{x}, \bar{y}) = (1/z, v/z)$ and after multiplication by z^{4k} , the vector field $X_m^{k,R}$ reads as

$$(15) \quad \begin{aligned} z' &= -m^{1/(1+k)}(z + z^{4k+1}v), \\ v' &= -m^{1/(1+k)}v + z^{4k-2} + m^{1/(k+1)}z^{4k+1}v - z^{6k}v^{2k+1}. \end{aligned}$$

and respectively

$$(16) \quad \begin{aligned} z' &= z^{2k+1}(1 - z^{2k-2}v^3), \\ v' &= z^{2k}v - m^{1/(1+k)}z^{4k} + m^{1/(1+k)}v^{4k+1} - z^{4k-2}v^4. \end{aligned}$$

Clearly (15) has only one singularity along $z = 0$; it is situated at $(0, 0)$ and is a hyperbolic attracting node. Next (16) has only one singularity along $z = 0$ also situated at $(0, 0)$ and it is non-elementary. By a z -directional blow up, introducing coordinates (w, z) with $zw = v$, and multiplication by z^{-2k-1} , the blown up equations read as

$$(17) \quad \begin{aligned} z' &= 1 - z^{2k+1}w^3, \\ w' &= m^{1/(1+k)}z^{2k-2}(-1 + zw^{4k+1}), \end{aligned}$$

and do not have singularities at $z = 0$. By a w -directional blow up, introducing coordinates (v, w) by $vw = z$, and multiplying by $v^{-(2k+1)}$, (16) becomes

$$(18) \quad \begin{aligned} v' &= m^{1/(1+k)}v^{2k} + w^{2k} - v^{2k+1}w^{4k-2} - m^{1/(1+k)}v^{2k-1}w^{4k}, \\ w' &= m^{1/(1+k)}v^{2k-2}w(-v + w^{4k}). \end{aligned}$$

The origin is the unique singularity at $v = 0$ for (18) and it is non-elementary. Its type is determined by a v -directional blow up, using coordinates (v, t) where $w = vt$; after multiplication by v^{-2k-1} , (18) reads as

$$\begin{aligned} v' &= v(m^{1/(k+1)} + t^{2k} - t^{4k-2}v^{4k-1} - m^{1/(k+1)}v^{4k-1}t^{4k}), \\ t' &= t(-2m^{1/(k+1)} + m^{1/(k+1)}v^{4k-1}t^{4k} - t^{2k} + t^{4k-2}v^{4k-1} + m^{1/(k+1)}v^{4k-1}t^{4k}), \end{aligned}$$

and has a unique singularity along $v = 0$ at $(0, 0)$, being a hyperbolic saddle. Returning to the original coordinates and taking into account the time reparameterizations the topological type of $(0, 0)$ for (16) is found to be a repelling node. Then by Proposition 12 and the Poincaré-Bendixson Theorem the α - (resp. ω)-limit set of Γ_{\pm}^3 (resp. Γ_{\pm}^4) are determined as in Figure 4. \square

6. NO LIMIT CYCLES NOR POLYCYCLES FOR SMALL $m > 0$

To rule out limit cycles we rely on a generalization of the Bendixson-Dulac criterion from [13] and stability arguments (see Theorem 16). We define

$$(19) \quad \mu_0^1 = 9/25 \text{ and } \mu_0^k = \left(\frac{k \sqrt[4]{2(k-1)(2k+1)}}{2(k-1)(4k+1)} \right)^{k+1} \text{ for all } k \geq 2.$$

As an exercise on elementary analysis we can estimate the bifurcation values at which the stability changes for the polycycle and the focus \mathbf{p}_0 and prove the positivity of the Bendixson-Dulac function used in Theorem 16.

Lemma 14. *Let m_S^k , μ_S^k and μ_0^k be as defined in (4), (8) and (19) respectively. Then, $\mu_0^1 = \mu_S^1$, $(8k+2)^{-(k+1)} < \mu_0^k < \mu_S^k < 2^{-k} < m_S^k$ for all $k \geq 1$.*

Lemma 15. *Let μ_0^k be as defined in (19). Let $M : \mathbb{R}^2 \times (0, \infty) \rightarrow \mathbb{R}$ and the zero set $\mathcal{Z}(m)$ of $M(\cdot, \cdot, m) : \mathbb{R}^2 \rightarrow \mathbb{R}$ for $m > 0$ be defined as*

$$M(\bar{x}, \bar{y}, m) = m^{1/(k+1)} [4\bar{x}^{2k}\bar{y}^4 + \frac{2}{2k+1}\bar{y}^{4k+4} - m^{1/(k+1)} \frac{4(4k+1)}{2k+1} \bar{x}^2 \bar{y}^{4k}],$$

$$\mathcal{Z}(m) = \{(\bar{x}, \bar{y}) \in \mathbb{R}^2 : M(\bar{x}, \bar{y}, m) = 0\}, \text{ for } m > 0,$$

Then, for all $0 < m \leq \mu_0^k$, the function $M(\cdot, \cdot, m)$ is non-negative and $\{(0, 0)\}$ is the maximal invariant set for X_m^k that is contained in $\mathcal{Z}(m)$.

Proof. If $k = 1$, then $M(\bar{x}, \bar{y}, m) = 2\sqrt{m}\bar{y}^4[2(1 - \frac{5}{3}\sqrt{m})\bar{x}^2 + \frac{1}{3}\bar{y}^4]$, and therefore the result is trivial. For the case $k \geq 2$ we remark that for $\bar{y} \neq 0$, we can write $M(\bar{x}, \bar{y}, m) = 2m^{1/(k+1)}\bar{y}^{4k-4}P(\frac{\bar{x}^2}{\bar{y}^4}, m)$, where

$$P(t, m) = 2t^k - m^{1/(k+1)} \frac{2(4k+1)}{2k+1} t + \frac{1}{2k+1}.$$

Elementary calculations show that for $t \geq 0$ the graph of $P(\cdot, m)$ is concave up with a minimum at $t_* = t_*(m)$, defined by

$$t_*(m) = \sqrt[k-1]{\frac{(4k+1)m^{1/(k+1)}}{(2k+1)k}} \text{ and } P(t_*, m) = \frac{1}{2k+1} - \frac{2(k-1)(4k+1)}{2k+1} t_*.$$

Clearly, outside $\mathcal{X} \equiv \{(\bar{x}, 0) : \bar{x} \in \mathbb{R}\}$, it follows that $M(\bar{x}, \bar{y}, m) \geq 0$ if and only if $P(t, m) \geq 0$ for $t \geq 0$. Now $P(t, m) \geq 0, \forall t \geq 0$ is equivalent to $P(t_*, m) \geq 0$, which in turn is equivalent to $m \leq \mu_0^k$. Then, $\mathcal{Z}(m)$ is given by \mathcal{X} for $0 < m < \mu_0^k$ and by $\mathcal{X} \cup \{(\bar{x}, \bar{y}) : \bar{x}^2 = t_*(\mu_0^k)\bar{y}^4\}$ for $m = \mu_0^k$. \square

Theorem 16. *Let $X_m^{k,R}$ and μ_0^k be defined by (5) and (19). Then there exists $m_0^k \geq \mu_0^k$ such that $X_m^{k,R}$ has no limit cycles nor polycycles for $0 < m < m_0^k$.*

Proof. From Proposition 9 we know that limit cycles are situated in $[-1, 1] \times [-1, 1]$. Consider the function $V_m(\bar{x}, \bar{y}) = 2m^{1/(k+1)}\bar{x}^2 + \bar{y}^4$ and define

$$M(\bar{x}, \bar{y}, m) = \langle X_m^{k,R}(\bar{x}, \bar{y}), \nabla V_m(\bar{x}, \bar{y}) \rangle - \frac{2}{2k+1} V_m(\bar{x}, \bar{y}) \operatorname{div} X_m^{k,R}(\bar{x}, \bar{y}),$$

where ∇ and div denote the gradient and divergence respectively. It is straightforward that M has the expression given in Lemma 15, which thus satisfies the conditions of the second statement of the generalized Bendixson-Dulac Theorem given in [13]. This implies that for given $0 \leq m < \mu_0^k$ the vector field $X_m^{k,R}$ has at most one limit cycle or polycycle in \mathbb{R}^2 , and they cannot coexist. Furthermore if it has a limit cycle, it is hyperbolic and attracting since $-VM \leq 0$. Now we prove that neither limit cycles nor polycycles are possible, thus finishing the proof. Suppose that there does exist a limit cycle of $X_m^{k,R}$. Then, by Lemma 14, we know that the origin is attracting, and hence the limit cycle bounds an annular region \mathcal{A} that is negatively invariant and does not contain singularities of $X_m^{k,R}$. By the Poincaré-Bendixson Theorem it is found that there are at least two limit cycles, hence leading to a contradiction. Suppose now that $X_m^{k,R}$ does have a polycycle. By Lemmas 10 and 14 the polycycle is attracting. Then

by Poincaré-Bendixson Theorem a limit cycle co-exists with the polycycle, which leads to a contradiction. \square

Corollary 17. *There exists $m_0^k > 0$ such that for $0 < m < m_0^k$ the phase portrait of X_m^k on the Poincaré disc is drawn in Figure 2(a). The relative positions of Γ_\pm^1 and Γ_\pm^2 are uniquely determined. Moreover, Γ_\pm^1 are unbounded and Γ_\pm^2 spiral towards the origin: $\alpha(\Gamma_\pm^1) = \emptyset$ and $\omega(\Gamma_\pm^2) = \{(0, 0)\}$.*

Proof. Let $0 < m < m_0^k$ where m_0^k is defined by Theorem 16, and therefore X_m^k has no limit cycles and the origin is attracting. There are two possible relative positions for $\Gamma_\pm^i, i = 1, 2$ as sketched in Figure 7. The case drawn

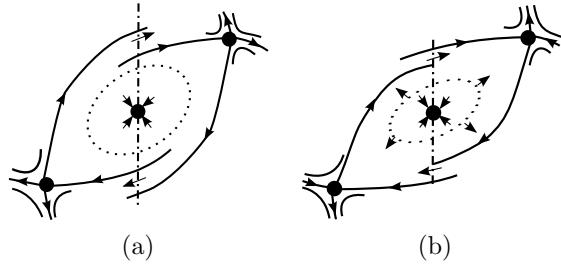


FIGURE 7. (a) Negatively and (b) positively invariant regions for $X_m^{k,R}$ with $0 < m < m_S^k$.

in Figure 7(a) is excluded by the Poincaré-Bendixson Theorem and Theorem 16. Hence the relative positions of Γ_\pm^1 and Γ_\pm^2 are as claimed. In case of Figure 7(b), if the separatrices Γ_\pm^2 do not spiral towards the origin, then one can construct a positively invariant region leading to a contradiction by the same arguments. Hence $\omega(\Gamma_\pm^2) = \{(0, 0)\}$, and then by Poincaré-Bendixson Theorem the phase portrait of X_m^k is as in Figure 2(a). \square

7. THE ω -LIMIT OF THE SEPARATRICES Γ_\pm^2 FOR LARGE $m > 0$.

In this section we deal with the case that m is large (i.e. $m \geq m_{u2}^k$ for some $m_{u2} > 0$) and aim at an analogous version of Corollary 17. It seems to be a hard task to find a convenient V defining M with the good properties to apply the generalized Bendixson-Dulac criterion as we did for small $m > 0$. For large m we use a different approach and we start by showing that for sufficiently large m the separatrix $\Gamma_+^2(m)$ is unbounded, and then the relative positions of the separatrices $\Gamma_\pm^i, i = 1, 2$ of $X_m^{k,R}$ are uniquely determined.

Proposition 18. *There exists $m_{u2}^k > 0$ such that for all $m > m_{u2}^k$ the separatrices Γ_\pm^2 are unbounded and Γ_\pm^1 are bounded, i.e. $\omega(\Gamma_\pm^2) = \emptyset$ and $\alpha(\Gamma_\pm^1) \neq \emptyset$, see Figure 8; moreover, the separatrix skeleton of $X_m^{k,R}$ is as in Figure 1(c).*

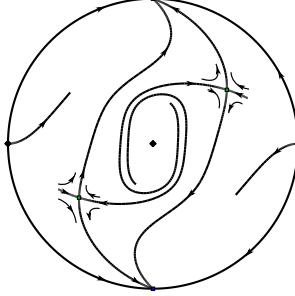


FIGURE 8. The global phase portrait of $X_m^{k,R}$ for $m > m_{u2}^k$ (Proposition 18).

Proof. For large m , instead of working with $X_m^{k,R}$ we consider the equivalent vector fields $Y_\eta^{k,R}$ defined in (6) with $\eta = 1/m > 0$ small but bounded. By invariance of the flow of $Y_\eta^{k,R}$ with respect to (3) and the Poincaré-Bendixson Theorem we only need to prove this statement concerning Γ_-^2 . By Proposition 12 the forward intersection points $(\bar{x}_-(m_\eta), 0)$ of $\Gamma_-^2(m_\eta)$ with the negative \bar{x} -axis define a decreasing sequence for $\eta \downarrow 0$:

$$-1 < \bar{x}_-(m_{\eta_2}) < \bar{x}_-(m_{\eta_1}) < 0 \text{ for } \eta_1 > \eta_2 > 0.$$

So intersection points for $m > 0$ remain at positive distance from the origin. Then $Y_0^{k,R}$ is a vertical flow in $\{(\bar{x}, \bar{y}) : -1 \leq \bar{x} \leq \bar{x}_0, -|\bar{x}_0|^{(2k+1)/3}/2 \leq \bar{y} \leq 3/2\}$. By continuous dependence on initial conditions and parameter there exists $\eta_0 > 0$ such that for all $0 < \eta < \eta_0$ the orbit of $Y_\eta^{k,R}$ through $(\bar{x}_-(m_\eta), 0)$ leaves the box $[-1, 0] \times [0, 1]$ through the boundary $\bar{y} = 1$ and hence, by Proposition 9, $\Gamma_-^2(m_\eta)$ is unbounded in forward time. \square

8. EXISTENCE AND UNICITY OF 2-SADDLE CYCLE

In this section Theorems 3 and 5 are proven up to the center-focus problem, replacing X_m^k by the equivalent vector field $X_m^{k,R}$. In determining the separatrix skeleton we do not rely on the topological behavior of the nilpotent singularity at $(0,0)$, which we know is a focus for $k = 1$ and $k \geq 2, m \neq m_S^k$ from Lemma 7. The center-focus problem for $k \geq 2, m = m_S^k$ will be treated in Theorem 25 in Section 10 ruling out the center case. For its proof we rely only on the separatrix skeleton obtained here and not on results on periodic orbits obtained in Section 9. Its proof is postponed because it makes use of classical tools such as the Poincaré map, that is introduced in Section 9.

Theorem 19. *Theorem 3 holds when X_m^k is replaced by $X_m^{k,R}$ defined in (5), up to deciding whether $(0,0)$ is a center or focus for $m = m_S^k, k \geq 2$.*

Proof. Recall that a 2-saddle cycle is formed if and only if the separatrices Γ_+^1 and Γ_-^2 coincide. By Corollary 17 and Proposition 18 two opposite relative positions of Γ_+^1 and Γ_-^2 are realized for sufficiently small and sufficiently large m . Then the existence of such $m_C^k \in (m_0^k, m_\infty^k)$ is ensured by continuous dependence of solutions on m . By the second item of Proposition 12, the uniqueness of such m_C^k follows. By adding the behavior near infinity from Proposition 13 and Figure 4 the separatrix skeleton of $X_m^{k,R}$ evolves in function of m as presented in Figure 1. \square

Proof of Theorem 5. The stability of the focus at $(0, 0)$ is due to Theorem 1. The uniqueness of m_C^1 , presenting a 2-saddle cycle, follows from Theorem 3. By Theorem 1 limit cycles or polycycles can only exist for $547/1000 < m < 3/5$; furthermore they cannot coexist. As a consequence, it is immediate that $547/1000 < m_C^1 < 3/5$ and for $m = m_C^1$ there are no limit cycles. For $m_C^1 < m < 3/5$ the Poincaré-Bendixson Theorem ensures the existence of a limit cycle (see Figure 7(a)) and then by Theorem 1 it is unique and repelling. On the other hand the existence of a limit cycle for $547/1000 < m < m_C^1$ would imply the existence of another one by the Poincaré-Bendixson Theorem (see Figure 7(b)), and hence would contradict Theorem 1. Furthermore by Lemma 10 the 2-saddle cycle is repelling since $m_C^1 > \mu_S^1 = 9/25$. Then by Proposition 13 and Figure 4 the global phase portrait of X_m^1 evolves in function of m as presented in Figure 3. We thus are left to prove the shrinking property of the limit cycles of X_m^k . The limit cycles occur for a finite range of the parameter values and are bounded by Proposition 9. Therefore, by the Planar Termination Principle, that is recalled at the end of Section 4, the one-parameter family of limit cycles $(\gamma_m)_{m_C^1 < m < 3/5}$ can only terminate at the singularity in the origin or at the 2-saddle cycle. We claim that $(\gamma_m)_{m_C^1 < m < 3/5}$ cannot be cyclic. Indeed it cannot terminate at the origin for both endpoints $m = m_C^1$ and $m = 3/5$, since small amplitude limit cycles only appear for $m \rightarrow 3/5$ (this follows from the proof of the stability result of the origin in [13]). Neither can it terminate at the 2-saddle cycle for both endpoints, since the 2-saddle cycle only exists for $m = m_C^1$. Therefore the family $(\gamma_m)_{m_C^1 < m < 3/5}$ has to be open and it has to terminate at the 2-saddle cycle for $m = m_C^1$ and at the singular point $(0, 0)$ for $m = 3/5$. \square

9. NO LIMIT CYCLES FOR LARGE $m > 0$

Here we prove the statements for large m in Theorem 4, working with vector fields $Y_\eta^{k,S}$ for small η , that are equivalent to $Y_\eta^{k,R}$ in (6).

To prove the absence of limit cycles for sufficiently large m we apply the Roussarie compactification-localization method. Usually this method is applied to solve global finiteness problems of limit cycles from local ones. Here we use this method to obtain the global absence of limit cycles uniformly from local absence results (see Proposition 20). This localization method

is described in terms of limit periodic sets, whose definition we recall from [27].

Let $(X_\lambda)_{\lambda \in P}$ be an analytic family of planar vector fields defined on $D \subset \mathbb{R}^2$, where $P \subset \mathbb{R}^p$ and let $\lambda_0 \in P$. Then we say that a compact set Γ is a limit periodic set of X_λ for $\lambda \rightarrow \lambda_0$ if and only if there exists a sequence $(\lambda_n)_{n \geq 1}$ with $\lambda_n \rightarrow \lambda_0$ for $n \rightarrow \infty$ such that for all $n \geq 1$ there exists a limit cycle γ_n of X_{λ_n} with $\gamma_n \rightarrow \Gamma$ when $n \rightarrow \infty$ (for the Hausdorff distance on the set of compact subsets of D). There exists an analogue of the Poincaré-Bendixson Theorem determining the structure of limit periodic sets, in case that the analytic family $(X_\lambda)_\lambda$ has only a finite number of singularities. In that case, a limit periodic set is either a singular point, a periodic orbit or a graphic of X_{λ_0} . A proof of this structure theorem can be found in [27] or [3].

Working with a compact analytic family of planar vector fields $(X_\lambda)_\lambda$, there exists the following equivalence between the global and local bounds for limit cycles (see [27] or [3]): the number of limit cycles of X_λ in D is bounded uniformly with respect to $\lambda \in P$ if and only if for every limit periodic set of $(X_\lambda)_\lambda$ there are only finitely many limit cycles bifurcating from Γ . By analogous compactness arguments one obtains the following localization method to rule out limit cycles globally in a uniform way with respect to the parameter. For a limit periodic set Γ of (X_λ) for $\lambda \rightarrow \lambda_0$, we say that no limit cycles bifurcate from Γ if and only if there exists a neighborhood V_Γ of Γ in the Hausdorff sense and there exists a neighborhood $W_\Gamma \subset \mathbb{R}^p$ of λ_0 such that for all $\lambda \in W_\Gamma$ the vector field X_λ has no limit cycles in V_Γ .

Proposition 20. *Let $P \subset \mathbb{R}^p$ compact and let $(X_\lambda)_{\lambda \in P}$ be a compact analytic family of planar vector fields on a compact subset D of \mathbb{R}^2 . If $\forall \lambda_0 \in P$ and for all limit periodic sets $\Gamma \subset D$ of $(X_\lambda)_{\lambda \in P}$ no limit cycles bifurcate from Γ for $\lambda \rightarrow \lambda_0$, then there exists a neighborhood W_0 of λ_0 in \mathbb{R}^p such that for all $\lambda \in W_0$ the vector field X_λ globally has no limit cycles in D .*

According to the size of the limit periodic set, three types of bifurcation phenomena of limit cycles are distinguished: large, medium and small amplitude limit cycles. Large amplitude limit cycles are limit cycles that grow arbitrarily large in some direction for $\eta \rightarrow 0$ (see [4]). It is rather quickly seen that $(Y_\eta^{k,R})$ has no large nor medium limit cycles; however to rule out small amplitude limit cycles directly for $(Y_\eta^{k,R})$ when $\eta \downarrow 0$ one has to deal with a cyclicity problem from a slow-fast system. To deal with this kind of problems one desingularizes the vector fields $Y_\eta^{k,R}$ (see A).

Here we avoid the cyclicity problem from slow-fast systems by applying Proposition 20 not to $(\hat{Y}_\eta^{k,R})_\eta$ but to a family of vector fields $(\hat{Y}_\eta^{k,S})_\eta$, for which each individual $Y_\eta^{k,S}$ is equivalent to $Y_\eta^{k,R}$, i.e. for each fixed $\eta > 0$. Of course, if then for no $0 < \eta < \eta_0$ do exist limit cycles of $Y_\eta^{k,S}$ globally, then neither do exist limit cycles for $Y_\eta^{k,R}$ with $0 < \eta < \eta_0$ globally. Notice that

as a byproduct, we then also have proven that there are no small amplitude limit cycles inside $(Y_\eta^{k,R})_\eta$ for $\eta \downarrow 0$.

For all $\eta > 0$ the rescaling $\bar{x} = \eta^{\frac{4k+1}{(8k-2)(k+1)}} \bar{x}$, $\bar{y} = \eta^{\frac{1}{(8k-2)(k+1)}} \bar{y}$ transforms $Y_\eta^{k,R}$ into the topologically equivalent vector field $Y_\eta^{k,S}$, defined by

$$(20) \quad Y_\eta^{k,S} \leftrightarrow \frac{d\bar{x}}{d\tau} = \bar{y}^3 - \eta \bar{x}^{2k+1}, \quad \frac{d\bar{y}}{d\tau} = -\bar{x} + \bar{y}^{4k+1},$$

with rescaled time $\bar{\tau} \mapsto \tau$ defined by $\frac{d\bar{\tau}}{d\tau} = \eta^{-\frac{2k-2}{(4k-1)(k+1)}}$. It is straightforward that the flow of $(Y_\eta^{k,S})_{\eta \geq 0}$ is invariant with respect to $(\tau, \bar{x}, \bar{y}) \mapsto (\tau, -\bar{x}, -\bar{y})$. Besides for fixed $\eta = 1/m > 0$ the singularities of $Y_\eta^{k,S}$ read as

$$\mathbf{p}_0 = (0, 0) \text{ and } \mathbf{p}_\pm = (\pm \eta^{\frac{-(4k+1)}{2(4k-1)(k+1)}}, \pm \eta^{\frac{-1}{2(4k-1)(k+1)}}).$$

Similarly to Propositions 6 and 13, for each $\eta_1 > 0$, the family $(Y_\eta^{k,S})_{0 < \eta \leq \eta_1}$ is analytically extended to an analytic family $(\hat{Y}_\eta^{k,S})_{0 \leq \eta \leq \eta_1}$ of vector fields defined on the Poincaré disc, whose topological behavior near infinity is presented in Figure 4(b) for $\eta > 0$. For $\eta = 0$ the extension $\hat{Y}_0^{k,S}$ has one finite singularity at $(0, 0)$, which is a nilpotent global repeller, and four infinite singularities as sketched in Figure 9.

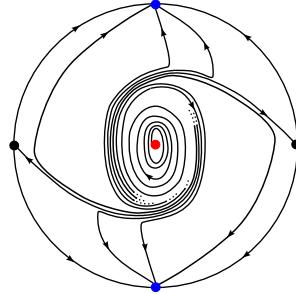


FIGURE 9. The origin of $Y_0^{k,S}$ is a global repeller. Along the equator $\hat{Y}_0^{k,S}$ has two hyperbolic attracting nodes in the y -direction and two non-elementary singularities in the x -direction.

In next proposition, using classical bifurcation techniques, it is shown that $Y_\eta^{k,S}$ has no small amplitude limit cycles for $\eta \downarrow 0$.

Proposition 21. *The family $(Y_\eta^{k,S})_{|\eta| \leq \eta_0}$, defined in (20), does not exhibit small amplitude limit cycles for $\eta \rightarrow 0$. It is to say there exist $\eta_1 > 0$ and a neighborhood V_1 of the origin in \mathbb{R}^2 such that $Y_\eta^{k,S}$ does not have limit cycles in V_1 for none of the parameter values $0 \leq \eta < \eta_1$.*

Proof. We show that for $\rho_0, \eta_0 > 0$ small enough the Poincaré first return map associated to $(Y_\eta^{k,S})_{|\eta| < \eta_0}$ is defined in $\Sigma_0 = \{(0, y_0) : |y_0| < \rho_0\}$ and

analytic on $\Sigma_0 \times (-\eta_0, \eta_0)$. Furthermore, using cartesian coordinates along the \bar{y} -axis, we show that $P : (-\rho_0, \rho_0) \times (-\eta_0, \eta_0) \rightarrow \mathbb{R}$ is given by

$$(21) \quad P(\bar{y}_0, \eta) = \bar{y}_0 + a(\eta)\bar{y}_0^{4k} + O(\bar{y}_0^{8k-1}) \text{ for } \bar{y}_0 \rightarrow 0,$$

and that there exists $0 < \eta_1 < \eta_0$ such that $a(\eta) > 0$ for all $|\eta| \leq \eta_1$. Clearly periodic orbits of $Y_\eta^{k,S}$ passing through $(0, \bar{y}_0)$ correspond to zeroes \bar{y}_0 of the associated displacement map, i.e. $\delta(\bar{y}_0, \eta) = P(\bar{y}_0, \eta) - \bar{y}_0$. From this it then follows that for some $0 < \rho_1 < \rho_0$ the vector field $Y_\eta^{k,S}$ has no limit cycles passing through $\Sigma_1 = \{(0, \bar{y}_0) : |\bar{y}_0| < \rho_1\}$ for none of the values $|\eta| < \eta_0$, which proves the proposition. To study the behavior of the flow in the neighborhood of the origin, in [13], a quasi-homogenous blow up of the nilpotent singularity at $(0, 0)$ is performed by means of generalized polar coordinates (r, θ) , that were first introduced by Lyapunov (see [21]). We follow this method to describe the Poincaré map of $Y_\eta^{k,S}$ in terms of the generalized radial coordinate r . Therefore we consider the (1,2)-trigonometric functions Sn, Cs determined by $\text{Cs}'\theta = -\text{Sn}\theta, \text{Sn}'\theta = \text{Cs}^3\theta$ with $\text{Cs}0 = 1$ and $\text{Sn}0 = 0$. It can be checked that $2\text{Sn}^2\theta + \text{Cs}^4\theta = 1$ and that Sn and Cs are periodic with period

$$(22) \quad T = \sqrt{2} \int_0^1 (1 - \theta)^{-1/2} \theta^{-3/4} d\theta = 2\sqrt{\pi^3} \Gamma(3/4)^{-2},$$

where Γ stands for the Gamma function $\Gamma(z) = \int_0^\infty \exp(-t)t^{1-z} dt$. Then we perform the parameter independent coordinate transformation

$$\bar{x} = r^2 \text{Sn}\theta, \bar{y} = r \text{Cs}\theta, r \geq 0, 0 \leq \theta \leq T.$$

The transformed differential equations are obtained by using

$$r^3 r' = \bar{y}^3 \bar{y}' + \bar{x} \bar{x}', r^3 \theta' = \bar{y} \bar{x}' - 2\bar{x} \bar{y}'$$

and after time rescaling (division by r) they read as

$$(23) \quad \begin{aligned} r' &= r^{4k} (\text{Cs}^{4k+4}\theta - \eta \text{Sn}^{2k+2}\theta), \\ \theta' &= 1 - r^{4k-1} \text{Sn}\theta \text{Cs}\theta (2\text{Cs}^{4k}\theta + \eta \text{Sn}^{2k}\theta). \end{aligned}$$

For $\eta = 0$, it follows that the radial velocity in backward time is negative, $\frac{dr}{d(-t)} \leq 0$, and the angular velocity can be written as $\frac{d\theta}{d(-t)} = -1 + O(r^{4k-1})$, $r \rightarrow 0$. Hence for r_0 sufficiently small, along the solution in backward time, the radius decreases and the angular velocity is negative and bounded away from 0. Noticing that $r^4 = 2\bar{x}^2 + \bar{y}^4$, it then follows that the angular velocity along the negative orbit of $(0, r_0)$ for $Y_\eta^{k,S}$ does not vanish. As a consequence, in backward time the negative orbit returns into the \bar{y} -axis, which is, outside the origin, transversal to the flow of $Y_\eta^{k,S}$. By continuous dependence on the initial value and the parameter, there exists $\rho_0, \eta_0 > 0$ such that for all $|\eta| < \eta_0$ and for all $|r_0| < \rho_0$, the positive orbit of $(0, r_0)$ for $Y_\eta^{k,S}$ returns into the \bar{y} -axis at some point $(0, P(r_0, \eta))$ and the angular velocity does not vanish along this orbit between $(0, r_0)$ and $(0, P(r_0, \eta))$. Therefore, for $|\eta| < \eta_0, |r_0| < \rho_0$, the solution of $Y_\eta^{k,S}$ passing through $(\bar{x}, \bar{y}) = (0, r_0)$ can

be written as a graph $r = r(r_0, \eta, \theta)$, where $r(r_0, \eta, \cdot)$ satisfies the differential equation

$$(24) \quad \frac{dr}{d\theta} = \frac{r^{4k}(\text{Cs}^{4k+4}\theta - \eta \text{Sn}^{2k+2}\theta)}{1 - r^{4k-1}\text{Sn}\theta\text{Cs}\theta(2\text{Cs}^{4k}\theta + \eta \text{Sn}^{2k}\theta)}$$

with $r(r_0, \eta, 0) = r_0$. Then, for $|\eta| < \eta_0$, the Poincaré map for $Y_\eta^{k,S}$ associated to Σ is determined by $P(r_0, \eta) = r(r_0, \eta, T)$. Next we prove the asymptotic expansion claimed in (21). For $|\eta| < \eta_0$ and $|r_0| < \rho_0$ the total solution $(\theta, \eta, r_0) \mapsto r(\theta, \eta, r_0)$ of (24) is analytic and can be written as Taylor series

$$r(\theta, \eta, r_0) = \sum_{i=0}^{\infty} u_i(\theta, \eta) r_0^i, \text{ for } r_0 \text{ near } 0,$$

for some analytic functions $u_i, i \in \mathbb{N}$. By substitution of the solution in (24) it is found that $(u_i(\theta, \eta) \equiv 0 \text{ for } i = 0 \text{ or } 1 < i < 4k)$ and $u_1(\theta, \eta) \equiv 1$. Furthermore, $u_{4k}(\theta, \eta) = \int_0^\theta \text{Cs}^{4k+4}z dz - \eta \int_0^\theta \text{Sn}^{2k+2}z dz$. By the technique of partial integration and the definition of the periodic functions Cs, Sn with period T , defined in (22), the first generalized Lyapunov quantity reads as

$$(25) \quad a(\eta) = u_{4k}(T, \eta) = \frac{(4k+1)!!!! - \eta(2k+1)!!}{(4k+3)!!!!} T,$$

and hence does not vanish for $|\eta|$ sufficiently small. Therefore the expansion in (21) is obtained ending the proof. \square

Remark 22. *From (25) the bifurcation value $m_S^k = 1/\eta_S^k$ is recovered through which the nilpotent focus at the origin changes stability (see Lemma 7). Clearly, for $\eta = \eta_S^k$, the first Lyapunov quantity vanishes and thus does not give information on the stability of $(0, 0)$.*

Proposition 23. *Let $(\hat{Y}_\eta^{k,S})_{\eta \geq 0}$ as defined in (20). Then there exists $\eta_2 > 0$ such that $(\hat{Y}_\eta^{k,S})_{0 \leq \eta \leq \eta_2}$ does not present large nor medium amplitude limit cycles. It is to say for each neighborhood V of the origin in \mathbb{R}^2 , there exists $0 < \eta_V < \eta_2$ such that for all $|\eta| < \eta_V$ the vector field $\hat{Y}_\eta^{k,R}$ has no limit cycles in the complement of V .*

Proof. Since $Y_0^{k,S}$ the origin is a global repeller for $\hat{Y}_0^{k,S}$, there are no periodic orbits nor polycycles (see Figure 9). Therefore by the Poincaré-Bendixson Theorem for limit periodic sets on the Poincaré disc there exists $\eta_0 > 0$ such that $\{(0, 0)\}$ is the only candidate limit periodic set for $(\hat{Y}_\eta^{k,S})_{|\eta| \leq \eta_0}$. \square

Corollary 24. *There exists $\eta_\infty^k > 0$ such that for all $0 < \eta < \eta_\infty^k$, the vector field $Y_\eta^{k,S}$ has no limit cycles nor polycycles in the global plane and its phase portrait is as in Figure 2 in the case $m > m_\infty^k$.*

Proof. By Proposition 21 there exists $\eta_1 > 0$ and a neighborhood V of the origin in \mathbb{R}^2 such that $Y_\eta^{k,S}$ has no limit cycles in V when $|\eta| < \eta_1$. By Proposition 23 there exists $\eta_2 > 0$ such that $Y_\eta^{k,S}$ has no limit cycles outside

V when $|\eta| < \eta_2$. Take $0 < \eta_\infty^k \leq \min(\eta_1, \eta_2, \frac{(4k+1)!!}{(2k+1)!!})$. Then $Y_\eta^{k,S}$ has no limit cycles in the global plane for $0 < \eta < \eta_\infty^k$ by Proposition 20. We claim that $Y_\eta^{k,S}$ neither presents polycycles and that the global phase portrait is topologically determined by Figure 2(b) for $0 < \eta < \eta_\infty^k$. Indeed if instead there was a polycycle Γ or if instead Γ_-^2 was bounded and Γ_+^1 unbounded, then it would follow by the choice of $\eta_\infty^k > 0$ that $(0, 0)$ and also Γ , if existing, would be repelling by Lemmas 7, 10 and 14. Then the Poincaré-Bendixson Theorem would imply that $Y_\eta^{k,S}$ would have at least one limit cycle for $0 < \eta < \eta_\infty^k$, contradicting the result obtained earlier in this proof. \square

Proof of Theorem 4 for m large. Since X_m^k and $Y_{1/m^k,S}$ are topologically equivalent for all $m > 0$, the statement for large m follows from Corollary 24. \square

To end we want to stress that the mere absence of small amplitude limit cycles inside $(Y_\eta^{k,S})_{0 \leq \eta \leq \eta_\infty}$ does not translate into the absence of small amplitude limit cycles inside $(Y_\eta^{k,R})_{0 \leq \eta \leq \eta_\infty}$. The reason lies in the fact that the variables are rescaled with η , in a way that the neighborhood $(-\bar{x}_0, \bar{x}_0) \times (-\bar{y}_0, \bar{y}_0) \times (-\eta_0, \eta_0)$ of $(\bar{x}, \bar{y}) = (0, 0)$ and $\eta = 0$ corresponds to a conic set $\{|\bar{x}| < \bar{x}_0 \eta^{\frac{(4k+1)}{2(4k-1)(k+1)}}, |\bar{y}| < \bar{y}_0 \eta^{\frac{1}{2(4k-1)(k+1)}}, |\eta| < \eta_0\}$. However the *global absence of limit cycles* for $(Y_\eta^{k,S})_{0 \leq \eta \leq \eta_\infty}$ guarantees the *absence of small amplitude limit cycles* for $(Y_\eta^{k,R})_{0 \leq \eta \leq \eta_\infty}$ for $\eta \downarrow 0$.

10. HILBERT'S 16TH PROBLEM AND CENTER-FOCUS PROBLEM FOR $(X_m^k)_{m \in \mathbb{R}}$

In next theorem the remaining center-focus problem from Theorem 3 is solved by relating it to the bifurcation of the separatrix skeleton.

Theorem 25. *The nilpotent singularity $(0, 0)$ of $X_{m_S^k}^k$, defined by (1) and (4), is a repelling or attracting focus.*

Proof. Clearly, it suffices to prove that the $(0, 0)$ is not a center for $Y_\eta^{k,S}$ where $\eta = 1/m_S^k$. The Poincaré return map $P(\cdot, \eta_S^k)$ is well-defined and analytic on a transversal section $0 \times [0, \bar{y}_1]$, for some $\bar{y}_1 > 0$; for its definition we refer to the proof of Proposition 21. Clearly, the origin is a center if and only if the Poincaré map $P(\cdot, \eta_S^k)$ is the identity: $P(\bar{y}_0, \eta_S^k) \equiv \bar{y}_0$. Using the analyticity of the Poincaré map, the origin is a center if and only if all coefficients vanish in the Taylor expansion of $P(\cdot, 1/m_S^k) - \text{Id}$. For all $k \geq 1$ the coefficient $a(\eta)$ appearing in the asymptotics in (21) vanishes if and only if $\eta = \eta_S^k$. For $k = 1$ there exists $V_2 > 0$ such that the asymptotics of the Poincaré map reads as $P(\bar{y}_0, \eta_S^k) = \bar{y}_0 + V_2 \bar{y}_0^{10} + O(\bar{y}_0^{11}), \bar{y}_0 \rightarrow 0$, and the result follows. For general $k \geq 2$ it is a challenging problem to calculate the first non-vanishing coefficient in the Taylor expansion of $P(\cdot, 1/m_S^k) - \text{Id}$. Now suppose that the origin is a center. By analyticity the period annulus extends until it reaches the saddles \mathbf{p}_\pm in its boundary. By continuous dependence on the initial values the boundary of the period annulus must

be a 2-saddle cycle. Hence by Theorem 3 it follows that $m_S^k = m_C^k$. From Lemmas 10 and 14 the 2-saddle cycle then is hyperbolic and repelling. This is in contradiction with the fact that the 2-saddle cycle is accumulated by non-isolated periodic orbits. As a consequence the origin cannot be a center. Hence the Poincaré map is not the identity and since $\dot{\theta} = 1 + O(r^{4k-1})$, $r \rightarrow 0$, it follows that the origin is an attracting or repelling focus. \square

Theorem 26. *Let X_m^k be defined in (1) for $m \in \mathbb{R}$. Periodic orbits only exist for $m > 0$. There exists $N(k) < \infty$ such that the number of limit cycles for X_m^k is uniformly bounded by $N(k)$.*

Proof. It follows from Proposition 6 that periodic orbits can only exist for $m > 0$. Of course we can replace X_m^k by $X_m^{k,R}$. We reduce the global finiteness of limit cycles for the compact analytic family $(\hat{X}_m^{k,R})_{m_0^k \leq m \leq m_\infty^k}$ to local finiteness problems for $(X_m^{k,R})_{m_0^k \leq m \leq m_\infty^k}$ following the Roussarie compactification-localization method. For a given $m_*^k > 0$ possible limit periodic sets for $(\hat{X}_m^{k,R})_{m > 0}$ for $m = m_*^k$ are $(0, 0)$, a periodic orbit or a 2-saddle cycle (in the latter case necessary $m_*^k = m_C^k$). By Theorem 25 and the principle of non-accumulation of zeroes of analytic functions, it is immediately seen that the number of limit cycles bifurcating from $(0, 0)$ or a periodic orbit is finite. From [16] and by Lemmas 10 and 14, it furthermore follows that the number of limit cycles bifurcating from a hyperbolic 2-saddle cycle Γ inside $(X_m^{k,R})_{m > 0}$ for $m \rightarrow m_C^k$ also is finite. It is to say, there exist an integer $N(k, \Gamma)$, positive constants m_1^k, m_2^k such that $m_C^k \in (m_1^k, m_2^k)$ and a neighborhood \mathcal{V} of Γ in the Hausdorff sense such that X_m^k has at most $N(k, \Gamma)$ limit cycles in \mathcal{V} for all $m \in (m_1^k, m_2^k)$. Therefore all limit periodic sets generate at most a finite number of limit cycles in the family $(\hat{X}_m^{k,R})_{m_0^k \leq m \leq m_\infty^k}$. Therefore the Roussarie compactification-localization method guarantees the existence of a uniform upper bound $N(k) < \infty$. \square

In fact for $k = 1$ previous theorem follows from Theorem 5 with optimal upper bound $N(1) = 1$. In Theorem 27 we show that $N(k) \geq 1$ for all $k \geq 2$.

Theorem 27. *Let X_m^k be defined in (1) for $m \in \mathbb{R}$. Periodic orbits of X_m^k are isolated, whenever they exist. Furthermore, there exists $m_0^k < m_e^k < m_\infty^k$ such that $X_{m_e^k}^k$ has at least one periodic orbit.*

Proof. If X_m^k has non-isolated periodic orbits for some $m > 0$, then by analyticity X_m^k has a period annulus reaching at the origin; this is impossible by Theorem 25. The existence of m_e^k follows from the proof of Theorem 25, where it is established that a Hopf-like bifurcation of limit cycles takes place for η passing through $\eta_S^k = 1/m_S^k$. \square

APPENDIX A. ALTERNATIVE PROOF FOR THEOREM 4, PART LARGE m .

Here we sketch an alternative proof for the absence of limit cycles for X_m^k when m is sufficiently large, by working directly with $Y_\eta^{k,R}$ and $\eta =$

$1/m$ sufficiently small. Using Proposition 9 and similar reasoning as for Proposition 18 it is readily seen that $(Y_\eta^{k,R})_\eta$ has no medium nor large amplitude limit cycles for $\eta \rightarrow 0$. More concretely, for each neighborhood W_0 of $(0,0)$ in \mathbb{R}^2 there exists $\eta_0 > 0$ such that for all $|\eta| < \eta_0$ the vector field $Y_\eta^{k,R}$ has no limit cycles outside W_0 . Hence, by Proposition 20, the global absence of limit cycles for $Y_\eta^{k,R}$ when η is sufficiently small follows if $(Y_\eta^{k,R})_\eta$ does not have limit cycles bifurcating from $(0,0)$ for $\eta \downarrow 0$.

Since for $\eta = 0$ the vector field has a line of singularities passing through $(0,0)$, one needs to blow up the family (i.e. x, y and η) as it is explained for instance in [10], taking for instance the blow up formulas: $\bar{x} = \rho^{4k+1}\bar{x}$, $\bar{y} = \rho\bar{y}$, $\eta = \rho^{(8k-2)(k+1)}\bar{\eta}$ with $(\bar{x}, \bar{y}, \bar{\eta})$ in the half-sphere S_+^2 and $\rho \geq 0$ small. The boundary of the blown-up space for $\eta \rightarrow 0$ is a plane with a hole replaced by the critical set S_+^2 , attached transversally along a circle γ (the boundary of the disk in Figure 9). A neighborhood of the critical set is made by the union of two charts: (a) The chart I: $\{\rho = 1\}$ where the blown up field is the rescaled one: $Y_\eta^{k,S}$, using that $\bar{\eta} = \eta$ in this chart. (b) The chart II: $\{\bar{x}^2 + \bar{y}^2 = 1\}$, which is a neighborhood of the circle γ . Lifted in the blown-up space, each Poincaré-disk $\{\eta = Cst > 0\}$ is partitioned in three parts: the exterior region, an annulus A_η in the chart II and a disk D_η in the chart I. Using that the slow dynamics has no singular point outside the origin and the fact that the slow curve is attracting, we easily see that the trajectories in the exterior part converge towards the exterior boundary of A_η . An easy study of the dynamics in the chart II shows that orbits in A_η go from the exterior boundary of A_η to the interior one. The dynamics in D_η is a global attraction towards the origin (of the critical locus), as explained in Section 9. Therefore $Y_\eta^{k,R}$ cannot give rise to small amplitude limit cycles for $\eta \downarrow 0$.

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