Non-integrability of measure preserving maps via Lie symmetries[*]

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Abstract

We consider the problem of characterizing, for certain natural number m, the local $\mathcal{C}^m$-non-integrability near elliptic fixed points of smooth planar measure preserving maps. Our criterion relates this non-integrability with the existence of some Lie Symmetries associated to the maps, together with the study of the finiteness of its periodic points. One of the steps in the proof uses the regularity of the period function on the whole period annulus for non-degenerate centers, question that we believe that is interesting by itself. The obtained criterion can be applied to prove the local non-integrability of the Cohen map and of several rational maps coming from second order difference equations.

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1 Introduction and main results

In the last years the development of criteria to determine the integrable nature of discrete dynamical systems has been the focus of an intensive research activity (see [14] and references therein), however there are very few non-integrability results for discrete dynamical systems, see for instance [8, 5, 10, 12, 25, 27] and their references. The main result of this paper, Theorem 1 below, provides a criterion to establish the local non-integrability of real planar measure preserving maps in terms of non existence of local first integrals of class $\mathcal{C}^m$, for certain $m \in \mathbb{N}$, near an *elliptic* fixed point (that is, a fixed point such that the eigenvalues of the associated linear part lie in the unit circle, but excluding the values ± 1).

We will say that a planar map is $\mathcal{C}^m$-*locally integrable at an elliptic fixed point* p if it does exist a neighborhood $\mathcal{U}$ of p and a locally non-constant real valued function $V \in \mathcal{C}^m(\mathcal{U})$, with $m \geq 2$, (called first integral) such that $V(F(x)) = V(x)$, all the level curves $\{V = h\} \cap \mathcal{U}$ are closed curves surrounding p and, moreover, p is an isolated non-degenerate critical point of V in $\mathcal{U}$.

Prior to state the main result, we recall that a map F defined on $\mathcal{U}$, an open set of $\mathbb{R}^2$, preserves an absolutely continuous measure with respect the Lebesgue's one with non-vanishing density ν, if $m(F^{-1}(B)) = m(B)$ for any measurable set B, where $m(B) = \int_B \nu(x,y)\,dxdy$, and $\nu|_{\mathcal{U}} \neq 0$. For the sake of simplicity, in this paper sometimes we will refer these maps simply as *measure preserving maps*.

When the eigenvalues $\lambda, \bar{\lambda} = 1/\lambda$ of the linear part of a $\mathcal{C}^1$-planar map F at an elliptic fixed point $p \in \mathbb{R}^2$ are not roots of unity of order ℓ for $0 < \ell \leq k$ we will say that p is *not k-resonant*. Recall that for not k-resonant elliptic fixed points a $\mathcal{C}^k$-map, F, is locally conjugated to its *Birkhoff normal form* plus some remainder terms, see [1]:

$$F_B(z) = \lambda z \left(1 + \sum_{j=1}^{[(k-1)/2]} B_j (z\bar{z})^j \right) + o(|z|^k), \tag{1}$$

where $z = x + iy$, and $[\cdot]$ denotes the integer part. It is well-known that near a locally integrable elliptic point the first non-vanishing *Birkhoff constant* B_n, if exists, must be purely imaginary. We recall a proof of this fact in Lemma 12.

The main result of this paper is the following theorem:

Theorem 1. *Let F be a $\mathcal{C}^{2n+2}$-planar map defined on an open set $\mathcal{U} \subseteq \mathbb{R}^2$ with an elliptic fixed point p, not $(2n+1)$-resonant, and such that its first non-vanishing Birkhoff constant is $B_n = i\, b_n$, for some $0 < n \in \mathbb{N}$ and $b_n \in \mathbb{R} \setminus \{0\}$. Moreover, assume that F is a measure preserving map with a non-vanishing density $\nu \in \mathcal{C}^{2n+3}$. If, for an unbounded sequence of natural numbers $\{N_k\}_k$, F has finitely many N_k-periodic points in $\mathcal{U}$ then it is not $\mathcal{C}^{2n+4}$-locally integrable at p.*

Our proof uses some of the ideas presented by G. Lowther in [20] for explaining the non-integrability of the Cohen map. As we will see, our result has also several applications for proving non-smooth integrability of several rational difference equations.

One of the main ingredients in our proof of Theorem 1 is that any integrable measure preserving map has an associated vector field X, called a *Lie Symmetry*, such that F can be expressed in terms of the flow of X, see Section 3 for further details. As we will see, to proceed with our approach, from this Lie symmetry we need to construct another one, say Y, having an isochronous center. Our construction of this vector field Y is based on the study of the regularity of the so called *period function* in a neighborhood of a non-degenerate center. Let us recall its definition.

Let p be a non-degenerate center of a smooth vector field X, that is such that $DX(p)$ has eigenvalues $\pm i\omega$ with $0 \neq \omega \in \mathbb{R}$. Let $\mathcal{V}$ be the largest neighborhood of p such that $\mathcal{V} \setminus \{p\}$ is foliated by periodic orbits. This set is usually called *period annulus*. Then for all $(x, y) \in \mathcal{V} \setminus \{p\}$, the function $T(x, y)$ giving the (minimal) period of the closed orbit passing through (x, y) can be extended continuously to p as $T(0, 0) = 2\pi/\omega$. As usual, we will call this function T, defined on the whole set $\mathcal{V}$, the *period function* of X on $\mathcal{V}$.

The regularity of the period function on $\mathcal{V}$ for non-degenerate centers of $\mathcal{C}^\infty$ or analytic planar vector field is known. It coincides, in the whole set $\mathcal{V}$, with the regularity of the corresponding vector field, see [29]. In next result we show that this is no more true for $\mathcal{C}^k$-vector fields, $k \in \mathbb{N}$.

Theorem 2. *Let X be a $\mathcal{C}^k$-vector field with $1 \leq k \in \mathbb{N} \cup \{\infty, \omega\}$ with a non-degenerate center p, and let $\mathcal{V}$ be its period annulus. Then the period function T is of class $\mathcal{C}^k$ on $\mathcal{V} \setminus \{p\}$ and, at p, it is of class $\mathcal{C}^{k-1}$, where for the sake of notation $\infty - 1 = \infty$ and $\omega - 1 = \omega$. Moreover, in general, the regularity of T at p can not be improved.*

Notice that since the period function T of a non-degenerate center on its period annulus is clearly a first integral for the corresponding vector field, a direct consequence of the above result is:

Corollary 3. *Let p be a non-degenerate center of a $\mathcal{C}^k$-vector field, $1 \leq k \in \mathbb{N} \cup \{\infty, \omega\}$, and let $\mathcal{V}$ be its period annulus. Then the vector field has a $\mathcal{C}^{k-1}$-first integral on $\mathcal{V}$.*

In fact, it is already known that if p is a center, no necessarily non-degenerate, of a $\mathcal{C}^k$-vector field ($k \in \mathbb{N} \cup \{\infty\}$), then there exists a $\mathcal{C}^k$-first integral in a small enough neighborhood of p, see [22]. Nevertheless, although the corollary gives a much weaker result, our approach is different to the one of [22].

The second ingredient is a method for checking when the discrete dynamical system generated by a map $F : \mathbb{R}^M \to \mathbb{R}^M$ has finitely many K-periodic points. Or, equivalently,

when the system

$$\mathbf{x}_1 - F(\mathbf{x}_0) = \mathbf{0}, \ \mathbf{x}_2 - F(\mathbf{x}_1) = \mathbf{0}, \dots, \mathbf{x}_{K-1} - F(\mathbf{x}_{K-2}) = \mathbf{0}, \ \mathbf{x}_0 - F(\mathbf{x}_{K-1}) = \mathbf{0},$$

has finitely many real solutions. Notice that the above system can be written in a compact for as $\widehat{G}(\mathbf{y}) = \mathbf{0}$, where $\mathbf{y} = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{K-1}) \in \mathbb{R}^N$, for some map $\widehat{G} : \mathbb{R}^N \to \mathbb{R}^N$, where $N = K\,M$. We prove the following result, that can be applied in case that all solutions of the system $\widehat{G}(\mathbf{y}) = \mathbf{0}$ are also solutions of a new system, $G(\mathbf{y}) = \mathbf{0}$, for some polynomial map $G : \mathbb{R}^N \to \mathbb{R}^N$.

Theorem 4. *Let $G : \mathbb{C}^N \to \mathbb{C}^N$ be a polynomial map of degree d. Let G_d denote the homogenous map corresponding to the degree d terms of G. If $\mathbf{y} = \mathbf{0}$ is the unique solution in $\mathbb{C}^N$ of the homogeneous system $G_d(\mathbf{y}) = \mathbf{0}$, then the polynomial system $G(\mathbf{y}) = \mathbf{0}$ has finitely many solutions.*

Although we have not found the above result in the literature, most probably it is a folklore result. In any case, we sketch a proof in Section 2.2.

Notice also, that by Bezout's Theorem, we also know that when the hypotheses of the theorem are satisfied the maximum number of solutions of $G(\mathbf{y}) = \mathbf{0}$ is d^N. Finally observe that applying Theorem 4 when G is linear, that is $d = 1$ and $G(\mathbf{y}) = A\,\mathbf{y} + \mathbf{b}$, for some $N \times N$ matrix A, then $G_d(\mathbf{y}) = A\,\mathbf{y}$ and the condition that $G_d(\mathbf{y}) = \mathbf{0}$ if and only if $\mathbf{y} = \mathbf{0}$ reduces to $\det(A) \neq 0$. Therefore the above result can be thought as a natural extension of the well-known result: system $A\,\mathbf{y} + \mathbf{b} = \mathbf{0}$ has finitely many solutions (in fact, 0 or 1) if $\det A \neq 0$.

As a first application of Theorems 1 and 4 we recover the result of G. Lowther about the non-integrability of the Cohen map

$$F(x, y) = \left(y, -x + \sqrt{y^2 + 1} \right), \tag{2}$$

exposed in [20].

Theorem 5. *The Cohen map is not $\mathcal{C}^6$-locally integrable at its fixed point $\left(\sqrt{3}/3, \sqrt{3}/3 \right)$.*

According to M. Rychlik and M. Torgesson [25], the question about the integrability of this map was first conjectured by H. Cohen and comunicated by C. de Verdière to J. Moser in 1993. Rychlik and Torgesson showed that it has not a first integral given by algebraic functions. Nowadays this map is considered unlikely to be integrable since numerical explorations show that it has hyperbolic periodic points and chains of islands of period 14 and 23, see [20, 28].

Our second application covers a wide class of rational difference equations. Consider

$$F(x, y) = \left(y, \frac{f(y)}{x} \right), \tag{3}$$

where $f = P/Q$ is a rational map. For the sake of notation, define $\deg(f) = \deg(P) - \deg(Q)$. Its fixed points are $p = (\bar{x}, \bar{x})$, where $\bar{x}$ are the non-zero solutions of the equation $f(\bar{x}) = \bar{x}^2$ and p is an elliptic point if and only if $|f'(\bar{x})/\bar{x}| < 2$. Moreover (3) preserves the measure with density $\nu(x, y) = 1/(xy)$, that does not vanish on a neighborhood of the fixed points.

Theorem 6. *Consider the map (3), where f is a rational function with $\deg(f) > 2$. If p is an elliptic fixed point, not $(2n+1)$-resonant, and such that its first non-vanishing Birkhoff constant is $B_n = i\,b_n$, for some $0 < n \in \mathbb{N}$ and $b_n \in \mathbb{R} \setminus \{0\}$, then F is not $\mathcal{C}^{2n+4}$-locally integrable at p.*

It is interesting to notice that when $\deg(f) \leq 2$ there are integrable cases at least for $\deg(f) \in \{-1, 0, 1, 2\}$. For $k \in \{-1, 0, 1\}$ it suffices to consider the periodic maps F with $f(y) = y^k$, because all rational periodic maps are rationally integrable, see [6]. Other integrable, non-periodic cases are the well-known Lyness map, that corresponds to $f(y) = (a + y)$, see [13], or for $\deg(f) = 2$, the map studied by G. Bastien and M. Rogalski in [2], given by $f(y) = (a - y + y^2)$, which possesses the first integral $V(x, y) = (x^2 + y^2 - x - y + a)/(xy)$.

In the case with $\deg(f) = 2$, we study with more detail the family of maps

$$F(x, y) = \left(y, \frac{A + By + Cy^2}{x}\right), \quad C \neq 0, \tag{4}$$

that extends the one given in [2]. In next result we prove that in this family integrability and non-integrability coexist.

Proposition 7. *A map (4) having an elliptic fixed point p, not 5-resonant, is $\mathcal{C}^6$-locally integrable at p if and only if $C = 1$. Moreover, when $C = 1$ the map has the rational first integral $V(x, y) = (x^2 + y^2 + B(x + y) + A)/(xy)$.*

The third application deals with the area preserving maps (density $\nu = 1$),

$$F(x, y) = (y, -x + f(y)), \tag{5}$$

with f also a rational map. In this case we prove:

Theorem 8. *Consider the map (5), where f is a rational function with $\deg(f) > 1$. If p is an elliptic fixed point, not $(2n+1)$-resonant, and such that its first non-vanishing Birkhoff constant is $B_n = i\,b_n$, for some $0 < n \in \mathbb{N}$ and $b_n \in \mathbb{R} \setminus \{0\}$, then F is not $\mathcal{C}^{2n+4}$-locally integrable at p.*

Also for this family, when $\deg(f) \leq 1$, there are integrable cases at least for $\deg(f) \in \{0, -1\}$. For $\deg(f) = 0$, $f(y) \equiv k$ and the map is an involution and therefore rationally

integrable, see again [6]. For $\deg(f) = -1$ we can consider the well-known integrable McMillan-Gumowski-Mira map (for short MGM map) with $f(y) = ay/(1 + y^2)$ and first integral $V(x, y) = x^2y^2 + x^2 + y^2 - axy$, see [16, 21]. We remark that this map possesses elliptic not resonant points for many values of a, see Section 2.3.

Finally, as a consequence of Theorems 6 and 8, we prove that two celebrated integrable maps, the MGM and the Lyness ones given above, are "isolated" in a suitable set of rational maps.

Corollary 9. *Let g be a rational function. Consider the maps:*

(i) $F_\varepsilon(x, y) = \left(y, -x + \frac{ay}{1+y^2} + \varepsilon\, g(y) \right)$, *with $\deg(g) > 1$ and $a \in (-2, \infty) \setminus \{-1, 0, 2\}$.*

(ii) $G_\varepsilon(x, y) = \left(y, \frac{a+y+\varepsilon\, g(y)}{x} \right)$, *with $\deg(g) > 2$ and $a \in (-1/4, \infty) \setminus \{0, 1\}$.*

Then, for $|\varepsilon|$ small enough, F_ε or G_ε are $\mathcal{C}^6$-locally integrable at its corresponding elliptic fixed points if and only if $\varepsilon = 0$.

Similarly, we obtain:

Corollary 10. *Let g and h be rational functions with either $\deg(g) > 1$ or $\deg(h) > 2$, and $2\deg(g) \neq \deg(h)$. Then, for $|\varepsilon|$ small enough, the map*

$$H_\varepsilon(x, y) = \left(y, -x + \sqrt{y^2 + 1 + \varepsilon h(y)} + \varepsilon\, g(y) \right),$$

is not $\mathcal{C}^6$-locally integrable at its elliptic fixed point.

We remark that a similar result to the one of item (i) of Corollary 9 was obtained in [12] when $a > 2$, proving that the map F_ε, when $0 \neq |\varepsilon|$ is small enough, is not holomorphically integrable near the homoclinic loops passing through the origin. When $a < 2$ these homoclinic loops no more exist. Alternatively, our approach proves the smooth non-integrability near the elliptic points, that exist for all $a \in (-2, \infty) \setminus \{0, 2\}$. The value $a = -1$ is excluded in our study because for it the map F_0 has a 2-resonance at the origin.

The paper is structured as follows. Section 2 contains several preliminary results. More concretely, in Section 2.1 we include our results about the regularity of the period function and in particular the proof of Theorem 2. In Section 2.2 we prove Theorem 4, which recall that gives a tool for proving the existence of finitely many N-periodic points and in Section 2.3 we introduce the Birkhoff constants and compute them in some simple examples that will appear afterwards. Section 4 is devoted to prove the non-integrability of the Cohen map, that is Theorem 5. Finally, in Section 5, we apply our results to several rational maps motivated from well-known difference equations, proving Theorems 6 and 8 and their consequences.

2 Preliminary results

2.1 Regularity of the period function. Proof of Theorem 2

Observe that from the implicit function theorem, and the regularity of the flow of X, the function $T(x, y)$ is of class $\mathcal{C}^k$ in $\mathcal{V} \setminus \{p\}$. So, to prove Theorem 2 it only remains to study the regularity of T at p. In the cases $k \in \{\infty, \omega\}$ this is already done in [29]. Let us consider $k \in \mathbb{N}$. It is well-known that, since p is a non-degenerate center, the vector field in a neighborhood of p is $\mathcal{C}^k$-conjugated to its Poincaré normal form , see [15]. Its corresponding differential equation is

$$\dot{z} = iz \left(\omega + \sum_{j=1}^{s} a_{2j} (z\bar{z})^j \right) + o(|z|^k).$$

where $z = x + iy \in \mathbb{C}$, $\omega, a_{2j} \in \mathbb{R}$, and $s = [(k-1)/2]$. Taking polar coordinates we get:

$$\begin{cases} \dot{r} = o_\theta(r; k), \\ \dot{\theta} = \omega + \sum_{j=1}^{s} a_{2j} r^{2j} + o_\theta(r; k-1), \end{cases}$$

where $o_\theta(r; m)$ stands for a $\mathcal{C}^m$ function of the form $f(r, \theta) \, r^m$ such that $\lim_{r \to 0} f(r, \theta) = 0$, uniformly in θ. Notice that all the derivatives of $o_\theta(r; m)$ of order less or equal than m at the origin are zero. Hence

$$\frac{dr}{d\theta} = \frac{o_\theta(r; k)}{\omega + \sum\limits_{j=1}^{s} a_{2j} r^{2j} + o_\theta(r; k-1)} = o_\theta(r; k).$$

The solution of the above equation with initial condition $r(\alpha) = \rho$, is $r(\theta; \rho, \alpha) = \rho + o_{\theta,\alpha}(\rho; k)$, where here $o_{\theta,\alpha}(r; m)$ stands for a $\mathcal{C}^m$ function of the form $r^m \, g(r, \theta, \alpha)$ and $\lim_{r \to 0} g(r, \theta, \alpha) = 0$, uniformly in both variables θ and α. Moreover, the period function expressed in the polar coordinates (ρ, α), is

$$\begin{aligned} \widetilde{T}(\rho, \alpha) &= \int_\alpha^{\alpha + 2\pi} \frac{d\theta}{\omega + \sum\limits_{j=1}^{s} a_{2j} r^{2j}(\theta; \rho, \alpha) + o_{\theta,\alpha}(r(\theta; \rho, \alpha); k-1)} = \\ &= \int_\alpha^{\alpha + 2\pi} \frac{d\theta}{\omega + \sum\limits_{j=1}^{s} a_{2j} \rho^{2j} + o_{\theta,\alpha}(\rho; k-1)} \\ &= \int_\alpha^{\alpha + 2\pi} \frac{1}{\omega} + \sum_{j=1}^{s} \tau_{2j} \rho^{2j} + o_{\theta,\alpha}(\rho; k-1) \, d\theta \\ &= \frac{2\pi}{\omega} + \sum_{j=1}^{s} T_{2j} \rho^{2j} + \int_\alpha^{\alpha+2\pi} o_{\theta,\alpha}(\rho; k-1) \, d\theta, \end{aligned}$$

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where T_{2j} are some real constants. Notice also that

$$\int_\alpha^{\alpha+2\pi} o_{\theta,\alpha}(\rho; k-1)\, d\theta = o_\alpha(\rho; k-1),$$

because if $o_{\theta,\alpha}(\rho; k-1) = \rho^{k-1} g(\rho,\theta,\alpha)$, then the function $g(\rho,\theta,\alpha)$ tends to zero, when ρ goes to zero, uniformly in θ and α, and therefore,

$$\widetilde{T}(\rho,\alpha) = \frac{2\pi}{\omega} + \sum_{j=1}^s T_{2j}\rho^{2j} + \rho^{k-1} h(\rho,\alpha),$$

for some h such that $\lim_{\rho\to 0} h(\rho,\alpha) = 0$, uniformly in α. Then, the period function at the point $z = x + iy$ is $T(x,y) = \widetilde{T}(\sqrt{x^2+y^2}, \arg(x+iy))$, so

$$T(x,y) = \frac{2\pi}{\omega} + \sum_{j=1}^s T_{2j}(x^2+y^2)^j + (x^2+y^2)^{\frac{k-1}{2}} H(x,y),$$

with $H(x,y) = h(\sqrt{x^2+y^2}, \arg(x+iy))$, satisfying $\lim_{(x,y)\to(0,0)} H(x,y) = 0$. This implies that $(x^2+y^2)^{\frac{k-1}{2}} H(x,y)$ is of class $\mathcal{C}^{k-1}$ at the origin, with all the derivatives zero, as we wanted to show.

Finally we give some examples which prove that the regularity given at p can not be improved. For $a \in \mathbb{R}$ consider the vector field, with associated differential system,

$$\begin{cases} \dot{x} = -y\left(1 + (x^2+y^2)^a\right), \\ \dot{y} = x\left(1 + (x^2+y^2)^a\right), \end{cases}$$

Its period function is

$$T(x,y) = \frac{2\pi}{1 + (x^2+y^2)^a}.$$

Taking $a = k/2$ when k is odd, or $a = k/(k+1)$ when k is even, we obtain $\mathcal{C}^k$-vector fields such that its corresponding period function is of class $\mathcal{C}^{k-1}$, and not of class $\mathcal{C}^k$ at the origin.

2.2   Existence of finitely many periodic points. Proof of Theorem 4

We start by proving Theorem 4. Then, in Proposition 11, we adapt it for maps coming from difference equations.

Proof of Theorem 4. Set $\mathbf{x} = (x_0, \ldots, x_{N-1})$. We want to prove that when $\mathbf{0}$ is the unique solution of $G_d(\mathbf{x}) = \mathbf{0}$ then $G(\mathbf{x}) = \mathbf{0}$ has finitely many solutions. Observe that the system $G(\mathbf{x}) = \mathbf{0}$ defines an algebraic set $\mathcal{X} \subset \mathbb{C}P^N$ given by $\tilde{G}(\mathbf{z}) = \mathbf{0}$, where $\tilde{G} = (\tilde{G}_0, \ldots, \tilde{G}_{N-1}, x_N^d)$, $\tilde{G}_i(\mathbf{z})$ is the corresponding homogenization of G_i, and $\mathbf{z} = [x_0 : \ldots : x_{N-1} : x_N] \in \mathbb{C}P^N$ where $\{x_N = 0\}$ is the hyperplane at infinity.

We will use that given an algebraic set $\mathcal{X}$ such that $\dim(\mathcal{X}) = r$, then $\dim(\mathcal{X} \cap \{x_N = 0\}) \geq r - 1$. This fact is a consequence of the well-known Projective Dimension Theorem on dimension theory of algebraic varieties, see [18, Th. 7.2], also [24, Cor. 3.14].

Assume that $G(\mathbf{x}) = \mathbf{0}$ has infinitely many solutions. Then $\dim(\mathcal{X}) \geq 1$. By the above result, $\dim(\mathcal{X} \cap \{x_N = 0\}) \geq 0$. This inequality implies that $\mathcal{X}$ intersects the hyperplane of infinity. This fact is precisely equivalent to say that $G_d(\mathbf{x}) = \mathbf{0}$ has some non-zero solution, as we wanted to prove. ∎

Another proof of Theorem 4 holds by using the following consequence of Chevalley's Theorem (see [18, Ex. 3.19]): the first coordinate projection of $\mathcal{X}$ is either finite or dense in $\mathbb{C}$. Then, under our hypotheses, this projection is dense in $\mathbb{C}$, implying that $\mathcal{X}$ reaches infinity at some points that produce non-zero solutions of $G_d(\mathbf{x}) = \mathbf{0}$. Clearly, Theorem 4 does not hold for real algebraic varieties, as the circle in $\mathbb{R}^2$, $x_1^2 + x_2^2 = 1$, shows.

Given a difference equation such that its periodic solutions satisfy certain *algebraic recurrence* $g(x_n, \ldots, x_{n+k}) = 0$. The existence of N-periodic orbits can be characterized by a system of N algebraic equations

$$G(\mathbf{x}) = \mathbf{0}, \quad \text{where} \quad G = (g_0, g_1, \ldots, g_{N-1}), \tag{6}$$

$\mathbf{x} = (x_0, \ldots, x_{N-1})$ and $g_i(\mathbf{x}) := g(x_{i \bmod N}, \ldots, x_{i+k \bmod N})$. If $d = \deg(g)$ then $\deg(G) = d$ and $G_d = (g_{0,d}, g_{1,d}, \ldots, g_{N-1,d})$, where $g_{i,d}$ is the homogenous part of degree d of g_i,

$$g_{i,d}(\mathbf{x}) := g_d(x_{i \bmod N}, x_{i+1 \bmod N}, \ldots, x_{i+k \bmod N}). \tag{7}$$

Therefore, Theorem 4 for difference equations reads as follows:

Proposition 11. *Consider a difference equation of order k such that its associated sequence satisfies an algebraic recurrence $g(x_n, \ldots, x_{n+k}) = 0$ of degree d. Then it has finitely many N-periodic solutions if $x = \mathbf{0}$ is the unique solution in $\mathbb{C}^N$ of the system*

$$g_{0,d}(\mathbf{x}) = 0, \; g_{1,d}(\mathbf{x}) = 0, \; \ldots, \; g_{N-1,d}(\mathbf{x}) = 0. \tag{8}$$

2.3 Birkhoff normal forms.

The computation of the Birkhoff normal form is a well-known issue and we address the reader to [1] for further references. In particular the expression of the first Birkhoff constant of a map with a not 3-resonant fixed point at the origin,

$$F(x, y) = \left(\lambda x + \sum_{i+j \geq 2} f_{i,j} x^i y^j, \frac{1}{\lambda} y + \sum_{i+j \geq 2} g_{i,j} x^i y^j \right), \tag{9}$$

where $\lambda \in \mathbb{C}$, $|\lambda| = 1$, is

$$B_1 = \frac{\mathcal{P}_1(F)}{\lambda^2 (\lambda - 1) (\lambda^2 + \lambda + 1)}, \tag{10}$$

where

$$\begin{aligned}
\mathcal{P}_1(F) = & \ (f_{11}g_{11} + f_{21}) \lambda^4 - f_{11} (2f_{20} - g_{11}) \lambda^3 + (2f_{02}g_{20} - f_{11}f_{20} + f_{11}g_{11}) \lambda^2 \\
& - (f_{11}f_{20} + f_{21}) \lambda + f_{11}f_{20},
\end{aligned}$$

see for instance [9, Sect. 4]. The general expression of B_2 is

$$B_2 = \frac{\mathcal{P}_3(F)}{\lambda^4 (\lambda - 1)^3 (\lambda^2 + \lambda + 1)^3 (\lambda^2 + 1) (\lambda + 1)},$$

where $\mathcal{P}_3(F)$ is a huge polynomial expression that can be found in [9, App. A].

A well-known result is the following:

Lemma 12. *For $1 < n \in \mathbb{N}$, consider a $\mathcal{C}^{2n+2}$-map F with an elliptic fixed point $p \in \mathcal{U}$, not $(2n + 1)$-resonant. Let B_n be its first non-vanishing Birkhoff constant. If $\mathrm{Re}(B_n) < 0$ (resp. $\mathrm{Re}(B_n) > 0$), then the point p is a local attractor (resp. repeller) point. In particular the map is not $\mathcal{C}^2$-locally integrable at p.*

Proof. It suffices to prove that $\mathrm{Re}(B_k) \neq 0$ implies that the function $V(z) = z\bar{z} = |z|^2$ is a strict Lyapunov function at the origin for the normal form map F_B of F, given in (1). For instance, when $\mathrm{Re}(B_n) < 0$,

$$V(F_B(z)) = |z|^2 \left| 1 + 2\mathrm{Re}(B_n)(z\bar{z})^n + o(|z|^{2n}) \right| < V(z),$$

for z in a small enough neighborhood of p, as we wanted to prove. Clearly these maps can not be locally integrable at p. ∎

As an example of computation of B_1, consider the elliptic fixed points of the following class of area preserving integrable MGM maps ([12, 16]),

$$F(x, y) = \left(y, -x + \frac{ay}{1 + y^2} \right), \quad a \in \mathbb{R}. \tag{11}$$

Lemma 13. *The elliptic fixed points of (11) are the origin, when $|a| < 2$, and $(\pm z(a), \pm z(a))$, when $a > 2$, where $z(a) = \sqrt{(a - 2)/2}$. Moreover,*

(i) When $|a| < 2$ and $a \notin \{-1\}$ the first Birkhoff constant of the origin is

$$B_1 = \frac{3a}{\sqrt{4 - a^2}} i.$$

(ii) When $a > 2$ the first Birkhoff constant of the points $(\pm z(a), \pm z(a))$ is

$$B_1 = -\frac{4\sqrt{2}(a+4)}{a^2\sqrt{a-2}}\, i.$$

Proof. The characterization of the elliptic fixed points of (11) is straightforward.

(i) The eigenvalues of the Jacobian matrix $DF(0,0)$ satisfy $\lambda^2 - a\lambda + 1 = 0$, and their are a couple of conjugate pure imaginary values if and only if $|a| < 2$. If, in addition, $a \neq -1$ then the origin is not 3-resonant. We introduce the rational parametrization $a = \lambda + 1/\lambda$, with $\lambda = \mathrm{e}^{i\theta}, \theta \in \mathbb{R} \setminus \{0\}$, that covers all the values $|a| < 2$ because $a = 2\cos\theta$. Then, the linear transformation $\Phi(x,y) = (x/\lambda + \lambda y, x + y)$ gives a conjugation between F and a map of the form (9). Using the expression (10), after some computations we get

$$B_1 = -\frac{3(\lambda^2+1)}{\lambda^2-1} = \frac{3\cos\theta}{\sin\theta}\, i = \frac{3a}{\sqrt{4-a^2}}\, i.$$

(ii) To study the Birkhoof constants at the points $(\pm z(a), \pm z(a))$ it is convenient in this case to introduce the new parametrization $a = 2\mu^2 + 2, \mu > 0$. Then the fixed points are $(\pm\mu, \pm\mu)$ and after translating each of them to the origin we can proceed as in item (i). For both points we get

$$B_1 = -\frac{2(\mu^2+3)}{\mu(\mu^2+1)^2}\, i = -\frac{4\sqrt{2}(a+4)}{a^2\sqrt{a-2}}\, i,$$

as we wanted to prove. ∎

The Birkhoff constants given in the above lemma are also obtained in [19] to study the stability of the elliptic fixed points.

Next two results are stated without detailing the proofs. The first one follows after simple computations. The second one is as consequence of the computations in [9, Sec. 6.1], because the so called *periodicity conditions*, introduced and given in that paper, essentially coincide with the Birkhoff constants.

Lemma 14. *The first Birkhoff constant of the Cohen map (2) at the elliptic fixed point $(\sqrt{3}/3, \sqrt{3}/3)$ is $B_1 = i\, 135/256$.*

Lemma 15. *The first Birkhoff constant of the Lyness map $F(x,y) = (y, (a+y)/x)$, with $a \in (-1/4, \infty) \setminus \{0, 1\}$, at its elliptic fixed points is $B_1 = i\, b_1 \neq 0$, for some $b_1 \in \mathbb{R}$.*

In fact, it is well known that the map has elliptic fixed points only when $a > -1/4$. Moreover, when $a = 0$ (resp. $a = 1$), it is 6-periodic (resp. 5-periodic) and so linearizable. Hence, when $a \in \{0, 1\}$, $B_n = 0$ for all $n \in \mathbb{N}$.

3 Proof of Theorem 1

One key step in the proof of Theorem 1 is that, under its hypotheses, the map F should have a *Lie Symmetry*. A vector field X is said to be a Lie symmetry of a map F if it satisfies the condition

$$X(F(x)) = DF(x) \cdot X(x), \tag{12}$$

where DF is the jacobian matrix of F, [7, 17]. The vector field X is related with the dynamics of the map since F maps any orbit of the differential system determined by the vector field, to another orbit of this system, see [7]. In the integrable case, where the dynamics are in fact one dimensional, the existence of a Lie symmetry fully characterizes the dynamics. In [7, Thm. 1] we prove:

Theorem 16. *Let X be a $\mathcal{C}^1$-Lie Symmetry of a $\mathcal{C}^1$-diffeomorphism $F : \mathcal{U} \to \mathcal{U}$. Let γ be an orbit of X invariant under F. Then, $F|_\gamma$ is the τ-time map of the flow of X, that is $F(x) = \varphi(\tau, x)$, $x \in \gamma$. Moreover,*

(a) *If $\gamma \cong \{p\}$ then p is a fixed point of F.*

(b) *If $\gamma \cong \mathbb{S}^1$, then $F|_\gamma$ is conjugated to a rotation, with rotation number $\theta = \tau/T$, where T is the period of γ.*

(c) *If $\gamma \cong \mathbb{R}$, then $F|_\gamma$ is conjugated to a translation on the line.*

It can be seen that if F has a first integral $V \in \mathcal{C}^{m+1}(\mathcal{U})$ on $\mathcal{U} \subset \mathbb{R}^2$ and it preserves a measure absolutely continuous with respect the Lebesgue measure with non-vanishing density $\nu \in \mathcal{C}^m$ in $\mathcal{U}$, it holds that

$$X(x,y) = \mu(x,y) \left(-V_y(x,y)\frac{\partial}{\partial x} + V_x(x,y)\frac{\partial}{\partial y} \right), \quad \text{where} \quad \mu(x,y) = \frac{1}{\nu(x,y)}, \tag{13}$$

is a Lie symmetry of F in $\mathcal{U}$, see again [7]. Observe that in case that F is an area preserving map then $\mu \equiv 1$ and the symmetry (13) is simply the hamiltonian vector field associated to V.

We will use the following corollary of the above results.

Corollary 17. *Let F be a $\mathcal{C}^2$-measure preserving map with an invariant measure with non-vanishing density $\nu \in \mathcal{C}^1(\mathcal{U})$ and with a first integral $V \in \mathcal{C}^2(\mathcal{U})$. If V has a connected component γ of an invariant level set $\{V(x) = h\}$, which is invariant by F and diffeomorphic to $\mathbb{S}^1$, and $F|_\gamma$ has rotation number $\theta = \frac{\tau}{T} = p/q \in \mathbb{Q}$, with $\gcd(p,q) = 1$, then F has a continuum of q-periodic points in $\gamma \subset \mathcal{U}$.*

Notice that if p is a fixed point of F, then from (12), $X(p) = DF(p) X(p)$. If the matrix $DF(p)$ has no the eigenvalue $\lambda = 1$ then $X(p)$ must be zero, that is, p is a singular point of the vector field.

Lemma 18. *Let F be a $\mathcal{C}^m$-planar map that preserves an absolutely continuous measure with respect the Lebesgue's one with non-vanishing density $\nu \in \mathcal{C}^m$, with $m \geq 2$, which is locally integrable at an elliptic fixed point p with a $\mathcal{C}^m$-local first integral V. Then its associated Lie symmetry (13) has a non-degenerate center at p.*

Proof. Without loss of generality, we assume that $p = 0$, $V(0) = 0$ and $V(x, y) > 0$ in a neighborhood of the origin. Since 0 is a non-degenerate critical point of V we get $V(x, y) = ax^2 + bxy + cy^2 + o(|x, y|^3)$, with $= 4ac - b^2 > 0$.

Moreover $\mu_0 := \mu(0) \neq 0$. Hence, from (13) we have that

$$X(x, y) = -\mu_0 (bx + 2cy) + o(|x, y|^2) \frac{\partial}{\partial x} + \mu_0 (2ax + by) + o(|x, y|^2) \frac{\partial}{\partial y}$$

and therefore $\mathrm{Spec}(DX(0)) = \{\pm i\mu_0 \sqrt{4ac - b^2} \neq 0\}$. Since $\{V = h\}$ are closed curves for $h > 0$ small enough, we get that X has a non-degenerate center at 0. ∎

Lemma 19. *Let F be a $\mathcal{C}^m$-planar map that preserves an absolutely continuous measure with respect the Lebesgue's one with non-vanishing density $\nu \in \mathcal{C}^m$, with $m \geq 2$, which is locally integrable at an elliptic fixed point p with a $\mathcal{C}^m$-local first integral V. Let $\theta(h)$ and $T(h)$ denote, respectively, the rotation number of F and the period function of the Lie symmetry (13) evaluated on $\{V(x, y) = h\}$. Set $h_p = V(p)$. Then*

$$DF(p) = e^{\tau_p \, DX(p)}, \tag{14}$$

where $\tau_p = \theta_p T_p$, with $T_p = \lim\limits_{h \to h_p} T(h)$, and $\theta_p = \lim\limits_{h \to h_p} \theta(h)$ is the rotation number of the linear map $L(q) = DF(p) \, q$.

Proof. Since the vector field (13) is a Lie symmetry of F, we know by Theorem 16 that

$$F(q) = \varphi(\tau(h), q) \text{ for all } q \in \mathcal{U}, \text{ where } h = V(q).$$

By differentiation we arrive to

$$\begin{aligned} DF(q) &= \frac{d\varphi}{dt}(\tau(V(q)), q)\tau'(V(q))\vec{\nabla}V(q) + D_q\varphi(\tau(V(q)), q) \\ &= X(F(q))\tau'(V(q))\vec{\nabla}V(q) + D_q\varphi(\tau(V(q)), q). \end{aligned}$$

Taking the limit as $q \to p$ and using that $X(p) = 0$ we get

$$DF(p) = \lim\limits_{q \to p} D_q\varphi(\tau(V(q)), q). \tag{15}$$

Using the variational equations we know that $M(t) := D_q\varphi(t, q)$, is the solution of

$$\dot{M}(t) = DX(\varphi(t,q)) \, M(t), \quad M(0) = \, Id.$$

Considering q as a parameter in the above equation and using the theorem of continuity respect parameters, when $q \to p$, the solution of the above equation tends to the solution of

$$\dot{M}(t) = DX(p) \, M(t), \quad M(0) = \, Id,$$

that is, $\lim_{q \to p} D_q\varphi(t, q)$ is the fundamental matrix of the above linear system (which has constant coefficients) that at $t = 0$ is the identity. As usual, we denote this matrix by $e^{tDX(p)}$.

Now recall that $\theta(V(q)) = \tau(V(q))/T(V(q))$ and that, from Lemma 18, p is a non-degenerate center, and hence there exists $\lim_{q \to p} T(V(q)) = T_p \neq 0$. Moreover, from [3, Prop. 8], $\lim_{h \to h_p} \theta(h) = \theta_p$, where θ_p is the rotation number of the linear map $L(q) = DF(p) \, q$. Hence, from (15) we have $DF(p) = e^{\tau_p DX(p)}$, with $\tau_p = \lim_{q \to p} \tau(V(q)) = \lim_{q \to p} \theta(V(q)) \, T(V(q)) = \theta_p T_p$, as we wanted to prove. ∎

Lemma 20. *Let X be a Lie Symmetry of class $\mathcal{C}^m(\mathcal{U})$ of a map F defined in an open set $\mathcal{U} \subset \mathbb{R}^2$, having a non-degenerate center at $p \in \mathcal{U}$ and let T be its corresponding period function. Then*

$$Y(x,y) = T(x,y) \, X(x,y)$$

is a $\mathcal{C}^{m-1}(\mathcal{U})$ Lie Symmetry of F, having an isochronous center at p.

Proof. From Theorem 2, we can ensure that $Y \in \mathcal{C}^{m-1}(\mathcal{U})$. A trivial computation shows that it has a non-degenerate *isochronous* center at p. Now, the chain of equalities

$$Y(F(q)) = T(F(q)) \, X(F(q)) = T(q) \, X(F(q)) = T(q) \, DF(q) \, X(q) = DF(q) \, Y(q),$$

show that Y satisfies equation (12), so it is another Lie Symmetry of F. ∎

Proof of Theorem 1. Assume that the map has not continua of periodic points for a sequence of unbounded positive integer numbers $\{N_k\}_k$. Assume also that F is locally integrable at p with a $\mathcal{C}^{2n+4}$ first integral V. By definition, the level curves $\{V = h\} \subset \mathcal{U}$ contain closed curves surrounding p. Since F has the associated Lie Symmetry $X \in \mathcal{C}^{2n+3}(\mathcal{U})$ given by (13), and the energy level curves $\{V = h\}$ are also integral curves of X, the local integrability condition also implies that the curves $\{V = h\}$ surrounding p have no singular points of X.

From Theorem 16 (b), the map $F|_{\{V=h\} \cap \mathcal{U}}$ is conjugate to a rotation with associated rotation number $\theta(h) = \tau(h)/T(h)$, where $T(h)$ is the period of each curve $\{V = h\}$ as an

orbit of X, and $\tau(h)$ is defined by the equation $F(q) = \varphi(\tau(h), q)$, where φ is the flow of X. This last assertion ensures, in particular, that $\theta(h)$ is a continuous function.

Let $h_0 = V(p)$ be the energy of the fixed elliptic point. It is not restrictive to assume that $V(q) \geq h_0$ on a neighborhood of p. Let us see that the proof follows by using the next claim:

Claim: $\theta(h)$ is a nonconstant continuous function on a neighborhood of h_0.

Assuming the above claim, there is a non-degenerate rotation interval $I = \mathrm{Image}(\theta(h))$ for $h \gtrsim h_0$, and therefore there exists $M_1 \in \mathbb{N}$ such that for all $N \geq M_1$ there exists $M \in \mathbb{N}$ coprime with N such that $h_N = M/N \in I$, with $\{V = h_N\} \cap \mathcal{U} \neq \emptyset$ and such that $\theta(h_N) = M/N$. By Corollary 17, the set $\{V = h_N\} \cap \mathcal{U}$ is full of N-periodic points of F, in contradiction with our hypotheses. So F is not $\mathcal{C}^{2n+4}$-locally integrable at p.

Now we prove the claim by contradiction. Assume that the rotation number is a constant function $\theta(h) \equiv \theta$. Then each map $F|_{\{V=h\} \cap \mathcal{U}}$ is conjugate to a rotation with the same rotation number θ. We will prove that F is globally $\mathcal{C}^{2n+2}$-conjugate on $\mathcal{U}$ to the linear map $L(q) = DF(p)\,q$.

From Lemma 18, X has a non-degenerate center at p. In consequence, by Theorem 2, the period function $T(x, y) \in \mathcal{C}^{2n+2}(\mathcal{U})$ and $T(0, 0) > 0$.

Now we consider the vector field

$$Y(x, y) = T(x, y)\, X(x, y),$$

which, by Lemma 20, is again a Lie Symmetry of F of class $\mathcal{C}^{2n+2}(\mathcal{U})$, having an isochronous center at p with period function identically 1.

By using the isochronicity of Y and the fact that the rotation number is constant, one gets $F(q) = \widetilde{\varphi}(\widetilde{\tau}, q)$ for all $q \in \mathcal{U}$, where $\widetilde{\tau}$ is a constant, and $\widetilde{\varphi}$ is the flow of Y.

Now we can prove that the map given by

$$\Phi(q) = \int_0^1 \mathrm{e}^{-DY(p)\,s}\, \widetilde{\varphi}(s, q)\, ds, \tag{16}$$

is the desired conjugation between F and the linear map L. We remark that this linearization is the one given in the proof of the classical Bochner Theorem ([4] and [23, Chap. V, Thm. 1]). Also notice that $\Phi \in \mathcal{C}^{2n+2}(\mathcal{U})$ because of the regularity of $\widetilde{\varphi}$. Indeed, using that $DF(p) = \mathrm{e}^{DY(p)\widetilde{\tau}}$, see (14) in Lemma 19, we get

$$
\begin{aligned}
\Phi \circ F(q) \;&= \int_0^1 \mathrm{e}^{-DY(p)\,s}\, \widetilde{\varphi}(s, \widetilde{\varphi}(\widetilde{\tau}, q))\, ds = \int_0^1 \mathrm{e}^{-DY(p)\,s}\, \widetilde{\varphi}(s + \widetilde{\tau}, q)\, ds \\
&= \int_{\widetilde{\tau}}^{\widetilde{\tau}+1} \mathrm{e}^{-DY(p)(u-\widetilde{\tau})}\, \widetilde{\varphi}(u, q)\, du = \mathrm{e}^{DY(p)\widetilde{\tau}} \int_{\widetilde{\tau}}^{\widetilde{\tau}+1} \mathrm{e}^{-DY(p)\,u}\, \widetilde{\varphi}(u, q)\, du \\
&\overset{(*)}{=} \mathrm{e}^{DY(p)\widetilde{\tau}} \int_0^1 \mathrm{e}^{-DY(p)\,u}\, \widetilde{\varphi}(u, q)\, du \\
&= \mathrm{e}^{DY(p)\widetilde{\tau}}\, \Phi(q) = DF(p)\, \Phi(q) = L \circ \Phi(q),
\end{aligned}
\tag{17}
$$

where in the equality marked with (∗) we have used that both functions $e^{-DY(p)u}$ and $\widetilde{\varphi}(u, q)$ are 1-periodic with respect the variable u.

Hence, on one hand we have proved that F is $\mathcal{C}^{2n+2}$-conjugate to the linear map L, and on the the other hand, F is $\mathcal{C}^{2n+2}$-conjugate its Birkhoff normal form

$$F_B(z) = e^{\theta i} z \left(1 + i\, b_n\, |z|^{2n} + o(|z|^{2n}) \right),$$

where $b_n \neq 0$ and $\lambda = e^{\theta i}$, a contradiction because the vanishing of the first non-zero Birkhoff constant B_n is an invariant by $\mathcal{C}^{2n+2}$-conjugations. ∎

4   Non-integrability of the Cohen map

The proof of Theorem 5 follows as a consequence of Theorem 1, by using Lemma 14 and the next Proposition, which states that there are not continua of periodic orbits of the Cohen map for all arbitrary large period. This result is already given in [20]. We give a proof based on Theorem 4.

Proposition 21. *There are finitely many N-periodic points for the Cohen map when $N \neq \dot{3}$.*

Proof. The Cohen map has the associated second order difference equation $x_{n+2} = -x_n + \sqrt{1 + x_{n+1}^2}$. An straightforward computation shows that any N-periodic orbit of the Cohen map also satisfies the multivalued difference equation

$$g(x_n, x_{n+1}, x_{n+2}) = (x_n + x_{n+2})^2 - x_{n+1}^2 - 1 = 0, \tag{18}$$

or equivalently, the system (6):

$$\begin{cases} (x_0 + x_2)^2 - x_1^2 - 1 = 0, \\ (x_1 + x_3)^2 - x_2^2 - 1 = 0, \\ \quad\vdots \\ (x_{N-2} + x_0)^2 - x_{N-1}^2 - 1 = 0, \\ (x_{N-1} + x_1)^2 - x_0^2 - 1 = 0. \end{cases}$$

Using Proposition 11, there will be a finite number of N periodic orbits of the multivalued equation (18) if $\mathbf{0}$ is the unique solution of all the associated linear systems (8):

$$\begin{cases} x_0 + x_2 = \pm x_1, \\ x_1 + x_3 = \pm x_2, \\ \quad\vdots \\ x_{N-2} + x_0 = \pm x_{N-1}, \\ x_{N-1} + x_1 = \pm x_0, \end{cases}$$

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or equivalently by setting $\mathbf{x} = (x_0, \ldots, x_{N-1})$, if $\mathbf{0}$ is the unique solution of the linear systems $A_N(\varepsilon_0, \ldots, \varepsilon_{N-1})\mathbf{x} = \mathbf{0}$, where $A_N(\varepsilon_0, \ldots, \varepsilon_{N-1})$ are the $N \times N$ matrices

$$
A_N(\varepsilon_0, \ldots, \varepsilon_{N-1}) =
\begin{pmatrix}
1 & \varepsilon_0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & \varepsilon_1 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \varepsilon_2 & 1 & \cdots & 0 \\
 & & & & & & \\
0 & 0 & 0 & \cdots & 1 & \varepsilon_{N-3} & 1 \\
1 & 0 & 0 & \cdots & 0 & 1 & \varepsilon_{N-2} \\
\varepsilon_{N-1} & 1 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix},
$$

with $\varepsilon_j \in \{-1, 1\}$, for each $j = 0, \ldots, N - 1$. Proposition 21 holds if we prove that for $N \neq \dot{3}$ and every choice of $\varepsilon_j \in \{-1, 1\}$, with $j = 0, \ldots, N - 1$, $\det(A_N(\varepsilon_0, \ldots, \varepsilon_{N-1})) \neq 0$,

To prove this fact, observe first that

$$\det(A_N(\varepsilon_0, \ldots, \varepsilon_{N-1})) \equiv \det(A_N) \bmod 2, \tag{19}$$

where $A_N := A_N(1, \ldots, 1)$. This is a consequence of the following simple observation: If $M = (m_{i,j})$ and $M' = (m'_{i,j})$ are two square matrices such that $m'_{i,j} \equiv m_{i,j} \bmod 2$, then $\det(M) \equiv \det(M') \bmod 2$.

Therefore, by (19), the proposition will follow once we show that

$$a_n = \det(A_N) = \begin{cases} 3 & \text{if } N \neq \dot{3}, \\ 0 & \text{if } N = \dot{3}. \end{cases} \tag{20}$$

To prove (20) we introduce $t_n = \det(T_n[1, 1])$ where $T_n[1, 1]$ is the $n \times n$ Toepliz matrix

$$
T_n[1, 1] =
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & 0 & \cdots & 0 \\
 & & & & & & \\
0 & & \cdots & 0 & 1 & 1 & 1 \\
0 & & \cdots & 0 & 0 & 1 & 1
\end{pmatrix}.
$$

We will use the next claims, which are inspired in the results in [11]:

Claim 1: The sequence a_n satisfies $a_n = (-1)^{n-1} t_{n-1} + 2(-1)^n t_{n-2} + 2$.

Claim 2: The sequence t_n satisfies $t_n = t_{n-1} - t_{n-2}$, with $t_1 = 1$ and $t_2 = 0$, and therefore it is the 6-periodic sequence $\{1, 0, -1, -1, 0, 1\}$.

By using them, a straightforward computation shows that $a_{n+6} = a_n$ and therefore a_n is 6-periodic. Finally, since $a_3 = 0$, $a_4 = 3$, $a_5 = 3$, $a_6 = 0$, $a_7 = 3$ and $a_8 = 3$, a_n is 3-periodic and the equality (20) holds.

Now we prove Claim 1: Let $M_{i,j}$ be the (i,j)-minor of A_n. By using the Laplace expansion of the last row of a_n we get

$$a_n = (-1)^{n+1}M_{n,1} + (-1)^{n+2}M_{n,2} + (-1)^{2n}M_{n,n} = (-1)^{n+1}M_{n,1} + (-1)^n M_{n,2} + M_{n,n}.$$

It is straightforward to check that $M_{n,1} = \det(T_{n-1}[1,1]) = t_{n-1}$. Observe that

$$M_{n,2} = \det \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 \\ & & & & & & \\ 0 & & \cdots & 0 & 1 & 1 & 1 \\ 1 & & \cdots & 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } M_{n,n} = \det \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 \\ & & & & & & \\ 0 & 0 & \cdots & & 0 & 1 & 1 \\ 1 & 0 & \cdots & & 0 & 0 & 1 \end{pmatrix}.$$

By using again the Laplace expansion on the first column of $M_{n,2}$ we get that $M_{n,2} = \det(T_{n-2}[1,1]) + (-1)^n \det(L_{n-2})$, where L_{n-2} is a lower triangular $(n-2) \times (n-2)$ matrix such that all the diagonal entries are 1. Hence $M_{n,2} = t_{n-2} + (-1)^n$. Analogously, $M_{n,n} = \det(U_{n-2}) + (-1)^n \det(T_{n-2}[1,1])$, where U_{n-2} is an upper triangular matrix such that all the diagonal entries are 1. Therefore, $M_{n,n} = 1 + (-1)^n t_{n-2}$. Hence

$$\begin{aligned} a_n &= (-1)^{n+1}t_{n-1} + (-1)^n (t_{n-2} + (-1)^n) + 1 + (-1)^n t_{n-2} \\ &= (-1)^{n-1}t_{n-1} + 2(-1)^n t_{n-2} + 2, \end{aligned}$$

and the claim is proved.

Claim 2 follows by applying once again the Laplace expansion of t_n, obtaining that it satisfies the linear difference equation $t_n = t_{n-1} - t_{n-2}$ with initial conditions $t_1 = 1$ and $t_0 = 0$. It is a simple computation to check that it is 6-periodic. ∎

Remark 22. *A different proof of Proposition 21 follows using previous results that involve the celebrated Fibonacci numbers F_N. From [11, p.78] it holds that $\mathrm{per}(A_N)$, the permanent of A_N, satisfies $\mathrm{per}(A_N) = F_N + 2F_{N-1} + 2$, where the permanent of a $n \times n$ matrix $M = (m_{i,j})$ is given by $\mathrm{per}(M) = \sum_{\sigma \in \Sigma_n} \prod_{i=1}^n m_{i,\sigma(i)}$. Since $\mathrm{per}(M) \equiv \det(M) \bmod 2$, see for instance [26], we know that $\det(A_N) \equiv F_N + 2F_{N-1} + 2 \bmod 2$. Finally, the Fibonacci numbers (mod 2) are $1,1,0,1,1,0,\ldots$, giving our desired result.*

5 Other applications

Proof of Theorem 6. We want to apply Theorem 1. Let p be an elliptic fixed point of F, not $(2n+1)$-resonant, and such that its first non-vanishing Birkhoff constant is B_n is purely

imaginary. We already know that F preserves the measure with density $\nu(x, y) = 1/(xy)$, which is analytic in a neighborhood of p. So, it only remains to check whether the hypothesis about the finiteness of periodic points of F holds.

Observe that any N-periodic point of F is a periodic orbit of the second order recurrence

$$g(x_n, x_{n+1}, x_{n+2}) = x_{n+2}x_n Q(x_{n+1}) - P(x_{n+1}) = 0, \tag{21}$$

where $f = P/Q$. Hence the N-periodic points are the solutions of

$$\begin{cases} x_{n+2}x_n Q(x_{n+1}) - P(x_{n+1}) = 0 \ , \ n = 0, 1, \dots, N-3, \\ \qquad x_0 x_{N-2} Q(x_{N-1}) - P(x_{N-1}) = 0, \\ \qquad x_1 x_{N-1} Q(x_0) - P(x_0) = 0. \end{cases}$$

Setting $P(x) = \sum_{j=0}^p a_j x^j$ and $Q(x) = \sum_{j=0}^q b_j x^j$, since $\deg(f) = p - q > 2$, the system (8) associated to the above one is

$$a_p x_i^p = 0, \ \text{for } i = 0, 1, \dots, N-1.$$

and $\mathbf{0}$ is its unique solution. Then, by Proposition 11, we have that for each N there are finitely many N-periodic points of F. Therefore all the hypotheses of Theorem 1 hold and, as a consequence, F is not $\mathcal{C}^{2n+4}$-locally integrable at p, as we wanted to prove. ∎

It is interesting to notice that when $\deg(f) = p - q < 2$ Proposition 11 never applies. Indeed, in this case, using the notation introduced in (7),

$$g_{i,d}(x_{i+2 \bmod N}, x_{i+1 \bmod N}, x_{i \bmod N}) = b_q\, x_{i \bmod N} \cdot x_{i+2 \bmod N} \cdot x_{i+1 \bmod N}^q,$$

so $\mathbf{0}$ is not an isolated zero of system (8). As we have already explained in the introduction, in this subfamily there are several integrable cases.

When $\deg(f) = p - q = 2$, then

$$g_{i,d}(x_{i+2 \bmod N}, x_{i+1 \bmod N}, x_{i \bmod N}) = \begin{aligned} & b_q\, x_{i \bmod N} \cdot x_{i+2 \bmod N} \cdot x_{i+1 \bmod N}^q \\ & -a_p\, x_{i+1 \bmod N}^p. \end{aligned}$$

and there are examples where Proposition 11 applies and other where it does not. The same happens with Theorem 1, as the proof of Proposition 7 shows.

Proof of Proposition 7. First notice that when $C = 1$ it is easy to check that the function V given in the statement is in fact a first integral of the map. So we proceed with $C \neq 1$. In this case we also want to apply Theorem 1 for proving local non-integrability. We already know that F preserves the measure with density $\nu(x, y) = 1/(xy)$. To continue our proof, we have to show that F has an elliptic fixed point with suitable non-zero Birkhoff constant

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and moreover that for an unbounded sequence of natural number $\{N_k\}_k$, F has not continua of N_k-periodic points in a neighborhood of this point. We start proving this second fact.

Each map (4) has the associated difference equation

$$g(x_n, x_{n+1}, x_{n+2}) = x_{n+2}x_n - (A + Bx_{n+1} + Cx_{n+1}^2) = 0,$$

with corresponding system (8):

$$\begin{cases} x_0\, x_2 - C\, x_1^2 = 0, \\ x_1\, x_3 - C\, x_2^2 = 0, \\ \vdots \\ x_{N-3}\, x_{N-1} - C\, x_{N-2}^2 = 0, \\ x_{N-2}\, x_0 - C\, x_{N-1}^2 = 0, \\ x_{N-1}\, x_1 - C\, x_0^2 = 0. \end{cases} \tag{22}$$

Observe that $\mathbf{0}$ is always a solution of (22). Let us prove that since $C \neq 1$, there are no other solutions. Assume that the system has some non-zero solution $(y_0, \ldots, y_{N-1}) \neq \mathbf{0}$. Straightforward computations show that then $y_i \neq 0$ for all $i = 0, \ldots, N-1$. Moreover, from system (22) one easily gets

$$C\, (y_0 \cdots y_{N-1})^2 = (y_0 \cdots y_{N-1})^2 \neq 0,$$

fact that is in contradiction with $C \neq 1$. Therefore, by Proposition 11, we have proved that when $C \neq 1$, for each $N \in \mathbb{N}$ the map has finitely many N-periodic points.

Finally, let us study the existence of elliptic fixed points and compute their corresponding Birkhoff constants when $C \neq 1$.

To facilitate the computations we introduce the change of variables $u = \alpha x$, $v = \alpha y$ where $A\alpha^2 + B\alpha + (C-1) = 0$. It conjugates F with the map

$$\bar{F}(x, y) = \left(y, \frac{a + (1 - a - c)y + cy^2}{x} \right), \quad c \neq 0, \tag{23}$$

where $a = \alpha^2 A$, $c = C$. Observe that if F has a simple fixed point then $B^2 - 4A(C-1) > 0$, so α is actually a real number. With this change of variables any fixed point of our initial F is brought to the fixed point $p = (1, 1)$ of this new F, which is elliptic if and only if

$$-1 < a - c < 3. \tag{24}$$

Introducing the parameter $r^2 = (3 - a + c)/(a + 1 - c)$, it is easy to see that r is a real number and that the eigenvalues associated to p are $\lambda = (r^2 - 1 \pm 2r\, i)/(r^2 + 1)$. Straightforward computations give:

- $\lambda = 1$ if and only if $a - c = -1$.

- $\lambda^2 = 1$ with $\lambda \neq 1$, or equivalently, $r = 0$ if and only if $a - c = 3$.

- $\lambda^3 = 1$ with $\lambda \neq 1$ if and only if $r^2 = \frac{1}{3}$, or equivalently, $a - c = 2$.

- $\lambda^4 = 1$ with $\lambda^2 \neq 1$ if and only if $r = 1$, or equivalently, $a - c = 1$.

- $\lambda^5 = 1$ with $\lambda \neq 1$ if and only if $r^2 = 1 \pm 2\sqrt{5}/5$, or equivalently, $a - c = (3 \mp \sqrt{5})/2$.

Therefore, 1 or 2-resonances can not appear in this new map.

After a change of variables bringing p at the origin, and computing the first Birkhoff constant, see the equation (10), we get

$$B_1 = \frac{i\, Q_1(a, c)\, (1 + r^2)^3}{16\, r\, (1 - 3\, r^2)},$$

where

$$Q_1(a, c) := a^4 - 3\, a^3 c + 3\, a^2 c^2 - ac^3 - 4\, a^3 + 5\, a^2 c - 2\, ac^2 + c^3 + 4\, a^2 + 4\, ac - 2\, c^2 - a + c.$$

When $\lambda^3 \neq 1$ and $Q_1(a, c) \neq 0$, from Theorem 1, the map is not $\mathcal{C}^6$-locally integrable at the elliptic point.

Now assume that $Q_1(a, c) = 0$ and consider the second Birkhoff constant. When $(\lambda^4 - 1)(\lambda^5 - 1) \neq 0$, tedious calculations using the formula given in [9, App. A], lead to

$$B_2 = \frac{(1 + r^2)^8\, (r + i)^2\, i\, Q_2(a, c)}{r^3\, (3r^2 - 1)^3\, (r^2 - 1)},$$

where $Q_2(a, c)$ is a real polynomial of degree 11, that we omit for the sake of simplicity. In fact, the above mentioned computations show that when $Q_1(a, c) \neq 0$, then B_2 has real part different from 0, and therefore by Proposition 12, the elliptic point is attractor or repeller and the map is not $\mathcal{C}^2$-locally integrable at this point.

If $Q_1(a, c) = 0$ and $Q_2(a, c) = 0$ we have to deal with a finite number of values of a and c. All these values, when $c \neq 0$, are:

$$(a, c) \in \left\{ (4, 1),\, (2 - 1),\, (3, 1),\, (0, 1),\, \left(\frac{5 \pm 3\sqrt{5}}{4}, \frac{\pm\sqrt{5} - 1}{4} \right) \right\}.$$

The first two cases are not under condition (24) and hence do not give rise to elliptic fixed points. The third and fourth ones are inside the integrable case $c = 1$. Finally, for the last two pairs, the corresponding eigenvalues are

$$\lambda = \frac{\sqrt{5}}{\sqrt{5} - 5} \pm \frac{\sqrt{25 - 10\sqrt{5}}}{\sqrt{5} - 5}\, i$$

which satisfy $\lambda^5 = 1$. These cases correspond to 5-resonances and they are not covered by our approach. ∎

Remark 23. *From the proof of the above theorem we get a slightly stronger result. If $\lambda^3 \neq 1$ and $B_1 \neq 0$, the map is not $\mathcal{C}^6$-locally integrable at the elliptic fixed point, still if $\lambda^k = 1$ for $k = 4$ or 5.*

Proof of Theorem 8. Its proof is similar to the one of Theorem 6 and we omit it for the sake of shortness. In this case, the condition $\deg(f) > 1$ is the one that allows to apply Proposition 11 to ensure that, for each $N \in \mathbb{N}$, the map (5) has finitely many N-periodic points, and then apply Theorem 1. ∎

The following remark shows a relation between the two families of maps (3) and (5).

Remark 24. *Consider the diffemorphism $\Psi : \mathbb{R}^2 \to (\mathbb{R}^+)^2$, $\Psi(x, y) = (\mathrm{e}^x, \mathrm{e}^y)$ with inverse $\Psi^{-1}(x, y) = (\log x, \log y)$. It holds that*

(i) If $F(x, y) = (y, f(y)/x)$ then $\Psi^{-1} \circ F \circ \Psi(x, y) = \big(y, -x + \log f\big(\mathrm{e}^y\big)\big)$.

(ii) If $F(x, y) = (y, -x + f(y))$ then $\Psi \circ F \circ \Psi^{-1}(x, y) = \Big(y, \dfrac{\mathrm{e}^{f(\log y)}}{x}\Big)$.

Therefore, Theorems 6 and 8 can also be applied to some non-rational maps of the forms (3) or (5). More concretely, the maps such that, via the changes of variables Ψ or Ψ^{-1}, can be transformed into rational maps of the other family.

Proof of Corollary 9. (i) Consider the integrable MGM map F_0. By Lemma 13, when $a \in (-2, \infty) \setminus \{-1, 0, 2\}$ it has at least one elliptic fixed point, say p_0, such that it is not 3-resonant and with non-zero purely imaginary first Birkhoff constant B_1. When $0 \neq |\varepsilon|$ is small enough, by continuity with respect to ε and since F_ε is area preserving, the map F_ε also has an elliptic fixed point, say p_ε, satisfying the same properties, that is, being not 3-resonant and with non-zero purely imaginary first Birkhoff constant $B_1(\varepsilon)$. Moreover, since

$$\deg\Big(\frac{ay}{1 + y^2} + \varepsilon\, g(y)\Big) = \deg(g(y)) > 1,$$

we are under the hypotheses of Theorem 8 for $n = 1$ and, therefore, F_ε is not $\mathcal{C}^6$-locally integrable at p_ε, as we wanted to prove.

(ii) Its proof follows exactly the same steps that the proof of item (i), changing Lemma 13 and Theorem 8, by the corresponding results Lemma 15 and Theorem 6. ∎

Proof of Corollary 10. When $\varepsilon = 0$ the result follows from Theorem 5. When $\varepsilon \neq 0$, the proof follows the same scheme of the one of Corollary 9. The main difference is the way we show that, for each $N \in \mathbb{N}$, H_ε has finitely many N-periodic orbits. Similarly that in the

case $\varepsilon = 0$, we have to study the number of solutions of

$$
\begin{cases}
(x_0 + x_2 - \varepsilon g(x_1))^2 - x_1^2 - 1 - \varepsilon h(x_1) = 0, \\
(x_1 + x_3 - \varepsilon g(x_2))^2 - x_2^2 - 1 - \varepsilon h(x_2) = 0, \\
\qquad\qquad\vdots \\
(x_{N-2} + x_0 - \varepsilon g(x_{N-1}))^2 - x_{N-1}^2 - 1 - \varepsilon h(x_{N-1}) = 0, \\
(x_{N-1} + x_1 - \varepsilon g(x_0))^2 - x_0^2 - 1 - \varepsilon h(x_0) = 0.
\end{cases}
$$

Writing $g = P/Q$ and $h = R/S$ with $P(x) = \sum_{j=0}^{p} a_j x^j$, $Q(x) = \sum_{j=0}^{q} b_j x^j$, $R(x) = \sum_{j=0}^{r} c_j x^j$, $S(x) = \sum_{j=0}^{s} d_j x^j$, the above set of rational equations can be transformed into a polynomial one. Recall that $\deg(g) = p - q$ and $\deg(h) = r - s$. Assume for instance that $\deg(g) > 1$ and $2\deg(g) > \deg(h)$; the other cases can be studied similarly. It holds that the system of equations (7) associated to the above one is

$$
d_s a_p^2 x_j^{2p+s} = 0, \text{ for } j = 0, 1, \ldots N - 1,
$$

which trivially has the only solution $\mathbf{x} = \mathbf{0}$. Hence by Proposition 11, the map H_ε has finitely many N-periodic orbits, as we wanted to prove. ∎

Remark 25. *When in Corollary 10 it holds that $\deg(h) = 2\deg(g) > 2$, by using the same approach we obtain that the same result holds when $d_s a_p^2 \neq c_r b_q^2$.*

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