This is a preprint of: "Periodic solutions of Lienard differential equations via averaging theory of order two", Jaume Llibre, Douglas D. Novaes, Marco Antonio Teixeira, *An. Acad. Brasil. Ci.*, vol. 87(4), 1905–1913, 2015.
DOI: [10.1590/0001-3765201520140129]

PERIODIC SOLUTIONS OF LIENARD DIFFERENTIAL EQUATIONS VIA AVERAGING THEORY OF ORDER 2

JAUME LLIBRE[1], DOUGLAS D. NOVAES[2] AND MARCO A. TEIXEIRA[2]

Abstract. For $\varepsilon \neq 0$ sufficiently small we provide sufficient conditions for the existence of periodic solutions for the Lienard differential equations of the form

$$x'' + f(x)x' + n^2x + g(x) = \varepsilon^2 p_1(t) + \varepsilon^3 p_2(t),$$

where n is a positive integer, $f : \mathbb{R} \to \mathbb{R}$ is a C^3 function, $g : \mathbb{R} \to \mathbb{R}$ is a C^4 function, and $p_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ are continuous 2π–periodic function. The main tool used in this paper is the averaging theory of second order. We also provide one application of the main result obtained.

1. Introduction and statement of the main results

In a recent paper Ma and Wang [5] have studied the existence of periodic solutions for the class of Lienard differential equations of the form

$$x'' + f(x)x' + V'(x) + g(x) = p(t), \tag{1}$$

where $f, V, g, p : \mathbb{R} \to \mathbb{R}$ are continuous and p is 2π–periodic. They have also assumed that g and V' are locally Lipschitz; and the function V is a $2\pi/n$–isochronous potential, i.e. all nontrivial solutions of $x'' + V'(x) = 0$ are $2\pi/n$–periodic, where n is a positive integer. The authors have provided sufficient bounded conditions related with the functions involved in equation (1) to ensure the existence of periodic solutions for this equation. We shall study a particular subclass of equations (1) which such a bounded conditions are not necessary.

In this paper we consider the subclass

$$x'' + f(x)x' + n^2x + g(x) = \varepsilon^2 p_1(t) + \varepsilon^3 p_2(t), \tag{2}$$

of Lienard differential equations (1) where n is a positive integer, ε is a small parameter, $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^3$ function in a neighborhood of $x = 0$, $g : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^4$ function in a neighborhood of $x = 0$, and

2010 *Mathematics Subject Classification.* 37G15, 37C80, 37C30.
Key words and phrases. periodic solution, Lienard differential equation, averaging theory.

$p_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ are continuous 2π–periodic functions. Note that here we are taking $V(x) = n^2x^2/2$ which is a $2\pi/n$–isochronous potential, already considered in [5].

The objective of this paper is to give sufficient conditions on the functions f, g and p_i to assure the existence of periodic solutions for the equation (2). Here the functions f and g do not need to satisfy the bounded conditions of [5].

In general to obtain analytically periodic solutions of a differential system is a very difficult problem, many times impossible. Here using the averaging theory this difficult problem for the differential equations (2) is reduced to find the zeros of a nonlinear system of two functions with two unknowns. We must mention that the averaging theory for finding periodic solutions in general does not provide all the periodic solutions of the system. More precisely, the main tool used will be the averaging theory of second order for computing periodic orbits, see [3, 1, 2, 4]. This theory provides a quantitative relation between the solutions of some non–autonomous periodic differential system and the solutions of the averaged differential system, which is autonomous. In this way a finite dimensional function f is computed, the simple zeros of this function correspond with the periodic orbits of the non–autonomous periodic differential system for values of a parameter $\varepsilon \neq 0$ sufficiently small. Here a simple zero a of a function f means that the Jacobian of f at a is not zero. For a general introduction to the averaging theory see for instance the book of Sanders, Verhulst and Murdock [7].

In order to present our results we need some preliminary definitions and notations. We define the constants

$$\alpha = \int_0^{2\pi} p_1(t)\sin(nt)dt \quad \text{and} \quad \beta = \int_0^{2\pi} p_1(t)\cos(nt)dt.$$

and the two functions

$$\begin{aligned} f_{21}(u,v) = \; & -\frac{n^2u^2+v^2}{24n^6}\Big(3n^2(a_1b_1-a_2n^2)u+\big(10b_1^2+n^2(a_1^2-9b_2)\big)v\Big) \\ & -\frac{1}{2\pi n^3}\int_0^{2\pi}\sin(nt)\left(\left(\left(\int_0^t p_1(s)\cos(ns)\,ds\right)\right.\right. \\ & \cdot\left(-b_1v+(n^2a_1u+b_1v)\cos(2nt)+n(a_1v-b_1u)\sin(2nt)\right) \\ & +n\left(np_2(t)+\left(\int_0^t \frac{p_1(s)\sin(ns)}{n}\,ds\right)\right. \\ & \left.\left.\left.\cdot\big(nb_1u+n(b_1u-a_1v)\cos(2nt)+(n^2a_1u+b_1v)\sin(2nt)\big)\right)\right)dt, \end{aligned}$$

and

$$\begin{aligned} f_{22}(u,v) = \;& \frac{n^2u^2+v^2}{24n^4}\Big(\big(10b_1^2+n^2(a_1^2-9b_2)\big)u+(3n^2a_2-3a_1b_1)v\Big) \\ &+\frac{1}{2\pi n^2}\int_0^{2\pi}\left(n^2p_2(t)\cos(nt)+\cos(nt)\left(\int_0^t \frac{p_1(s)\sin(ns)}{n}\,ds\right)\right. \\ &\cdot n\big(nb_1u+n(b_1u-a_1v)\cos(2nt)+(n^2a_1u+b_1v)\sin(2nt)\big) \\ &+\cos(nt)\left(\int_0^t p_1(s)\cos(ns)\,ds\right) \\ &\left.\cdot\big(-b_1v+(n^2a_1u+b_1v)\cos(2nt)+n(a_1v-b_1u)\sin(2nt)\big)\right)dt, \end{aligned}$$

where

$$a_1=f'(0),\quad a_2=\frac{1}{2}f''(0),\quad b_1=\frac{1}{2}g''(0),\quad \text{and}\quad b_2=\frac{1}{6}g'''(0).$$

Our main result is the following.

Theorem 1. *Assume that the functions f and g of the Lienard differential equation (2) satisfy that f is a $\mathcal{C}^3$ function in a neighborhood of $x=0$, g is a $\mathcal{C}^4$ function in a neighborhood of $x=0$, and $f(0)=g(0)=g'(0)=0$. Suppose also that the constants $\alpha=\beta=0$. Then for $\varepsilon\neq 0$ sufficiently small and for every simple zero (u^*,v^*) of the system*

$$f_{21}(u,v)=0,\qquad f_{22}(u,v)=0,$$

there exists a periodic solution $x(t,\varepsilon)$ of the differential equation (2) such that $x(0,\varepsilon)\approx\varepsilon u^*+\mathcal{O}(\varepsilon^2)$ and $x'(0,\varepsilon)\approx\varepsilon v^*+\mathcal{O}(\varepsilon^2)$.

Theorem 1 is proved in section 2.

In the next corollary we apply Theorem 1 to a given Lienard differential equation (2) and we show that such an equation has two periodic solutions.

Corollary 2. *Assume that*

$$f(x)=a_1x+a_2x^2,\quad g(x)=b_1x^2+b_2x^3,\quad p_1(t)=1,\ \text{ and }\ p_2(t)=\sin(nt),$$

where

$$a_1 = \frac{-7n^5+2n^3+n}{-6n^4-4n^2+2},$$
$$a_2 = \frac{n(7n^7-6n^6-2n^5+2n^4-n^3+6n^2-2)}{(n^2+1)^2(3n^2-1)^2},$$
$$b_1 = \frac{2n^5}{3n^4+2n^2-1}, \quad \textit{and}$$
$$b_2 = \frac{n^2(49n^8+132n^6-144n^5-10n^4-96n^3+4n^2+48n+1)}{36(n^2+1)^2(3n^2-1)^2}.$$

Then, for $\varepsilon \neq 0$ sufficiently small the Lienard differential equation (2) *has two periodic solutions $x_i(t,\varepsilon)$ for $i=1,2$ such that*

$$x_1(0,\varepsilon) \approx \varepsilon + \mathcal{O}(\varepsilon^2) \quad \textit{and} \quad x_1'(0,\varepsilon) \approx \varepsilon + \mathcal{O}(\varepsilon^2),$$

$$x_2(0,\varepsilon) \approx 2\varepsilon + \mathcal{O}(\varepsilon^2) \quad \textit{and} \quad x_2'(0,\varepsilon) \approx \mathcal{O}(\varepsilon^2).$$

Corollary 2 is proved in section 2.

In section 3 we summarize the averaging theory of second order for studying periodic solutions that we shall need for proving Theorem 1.

2. Proof of the results

In this section we shall prove Theorem 1 and Corollary 2.

Proof of Theorem 1. First we shall write the Lienard differential equations (3) in the normal form for applying the averaging theory, see Theorem 3 of the appendix.

We change the variable x by a new variable z doing the rescaling $x = \varepsilon z$. Then equation (3) becomes

$$z'' + f(\varepsilon z)z' + n^2 z + \frac{g(\varepsilon z)}{\varepsilon} = \varepsilon p_1(t) + \varepsilon^2 p_2(t). \tag{3}$$

Since f is a $\mathcal{C}^3$ function in a neighborhood of $x=0$, g is a $\mathcal{C}^4$ function in a neighborhood of $x=0$, and $f(0)=g(0)=g'(0)=0$, we can write

$$f(\varepsilon z) = \varepsilon f'(0)z + \varepsilon^2 \frac{1}{2} f''(0)z^2 + \mathcal{O}(\varepsilon^3) = \varepsilon a_1 z + \varepsilon^2 a_2 z^2 + \mathcal{O}(\varepsilon^3),$$
$$g(\varepsilon z) = \varepsilon^2 \frac{1}{2} g''(0)z^2 + \varepsilon^3 \frac{1}{6} g'''(0)z^3 + \mathcal{O}(\varepsilon^4) = \varepsilon^2 b_1 z^2 + \varepsilon^3 b_2 z^3 + \mathcal{O}(\varepsilon^4).$$

Thus

$$f(\varepsilon z)z' + \frac{g(\varepsilon z)}{\varepsilon} = \varepsilon(a_1 z z' + b_1 z^2) + \varepsilon^2(a_2 z^2 z' + b_2 z^3) + \mathcal{O}(\varepsilon^3).$$

We introduce a new variable $w = z'$. Then the differential equation (3) can be written as the differential system

$$\begin{aligned} &z' = w, \\ &w' = -n^2 z + \varepsilon(p_1(t) - a_1 z w - b_1 z^2) + \varepsilon^2(p_2(t) - a_2 z^2 w - b_2 z^3) \\ &\qquad +\mathcal{O}(\varepsilon^3). \end{aligned} \tag{4}$$

Now we change the variables (z, w) by the new variables (u, v) defined through the equality

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \cos(nt) & \dfrac{\sin(nt)}{n} \\ -n\sin(nt) & \cos(nt) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5}$$

We do this changes in order that the differential system in the new variables (u, v) starts with terms of order $\mathcal{O}(\varepsilon)$ and we can apply the averaging theory described in the appendix. Thus the differential system (4) in the new variables becomes

$$\begin{aligned} u' = {}& \varepsilon \frac{1}{n^3} G_1 \sin(nt) + \varepsilon^2 \frac{1}{n^4} G_2 \sin(nt) + \mathcal{O}(\varepsilon^3) \\ = {}& \varepsilon F_{11}(t, u, v) + \varepsilon^2 F_{21}(t, u, v) + \mathcal{O}(\varepsilon^3), \\ v' = {}& \varepsilon \frac{1}{n^3} G_1 \cos(nt) + \varepsilon^2 \frac{1}{n^4} G_2 \cos(nt) + \mathcal{O}(\varepsilon^3) \\ = {}& \varepsilon F_{12}(t, u, v) + \varepsilon^2 F_{22}(t, u, v) + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{6}$$

where

$$\begin{aligned} G_1 &= -n^2 p_1(t) + A\big(B_1 \cos(nt) + C_1 \sin(nt)\big), \\ G_2 &= -n^3 p_2(t) + A^2\big(B_2 \cos(nt) + C_2 \sin(nt)\big), \\ A &= nu \cos(nt) + v \sin(nt), \\ B_i &= n(b_i u + a_i v), \\ C_i &= -n^2 a_i u + b_i v, \end{aligned}$$

In short, the differential system (6) is in the normal form for applying the averaging theory described in Theorem 3 of the appendix. Using the notation of the appendix we have that $T = 2\pi$.

Let $F_i(t,u,v) = (F_{i1}(t,u,v), F_{i2}(t,u,v))$ for $i = 1,2$. Now we compute the function $f_1(u,v)$ defined in the appendix and we get, from the assumptions, that

$$\begin{aligned} f_1(u,v) = & \ \frac{1}{2\pi}\left(\int_0^{2\pi} F_{11}(t,u,v)dt, \int_0^{2\pi} F_{12}(t,u,v)dt\right) \\ = & \ \frac{1}{2\pi}\left(-\frac{\alpha}{n}, \beta\right) = (0,0). \end{aligned}$$

Since the function $f_1(u,v) = (0,0)$ we shall apply the apply the averaging theory of second order. So we first compute

$$\begin{aligned} \int_0^t F_{11}(s,u,v)ds = & -\int_0^t \frac{p_1(s)\sin(ns)}{n}\,ds \\ & -\frac{3}{12n^4}\Big((b_1u^2n^2 - 2a_1uvn^2 + 3b_1v^2)\cos(nt) \\ & +(b_1u^2n^2 + 2a_1uvn^2 - b_1v^2)\cos(3nt) \\ & -4\big((-a_1u^2n^3 + a_1v^2n + 2b_1uvn)\sin^3(nt) \\ & +b_1n^2u^2 + 2b_1v^2 - a_1n^2uv\big)\Big), \\ \int_0^t F_{21}(s,u,v)ds = & \int_0^t p_1(s)\cos(ns)\,ds \\ & +\frac{1}{12n^3}\Big(-n(a_1n^2u^2 - 2b_1vu - a_1v^2) \\ & \cdot(3\cos(nt) + \cos(3nt) - 4) \\ & -3\big(2a_1uvn^2 + b_1(3n^2u^2 + v^2)\big)\sin(nt) \\ & +\big(b_1(v^2 - n^2u^2) - 2a_1n^2uv\big)\sin(3nt)\Big), \\ D_{(u,v)}(F_1(t,u,v)) = & \begin{pmatrix} \dfrac{\partial F_{11}(t,u,v)}{\partial u} & \dfrac{\partial F_{11}(t,u,v)}{\partial v} \\ \dfrac{\partial F_{21}(t,u,v)}{\partial u} & \dfrac{\partial F_{21}(t,u,v)}{\partial v} \end{pmatrix} \end{aligned}$$

Now we are ready to compute the function

$$f_2(u,v) = (f_{21}(u,v), f_{22}(u,v)), \tag{7}$$

defined in the appendix, and we get the functions $f_{21}(u,v)$ and $f_{21}(u,v)$ which appear in the statement of the theorem.

Now, from Theorem 3, we obtain that for every $\varepsilon \neq 0$ sufficiently small and for every simple zero (u^*,v^*) of the system (7), i.e. satisfying that

$$\det \left(\begin{array}{cc} \dfrac{\partial f_{21}(u,v)}{\partial u} & \dfrac{\partial f_{21}(u,v)}{\partial v} \\ \dfrac{\partial f_{22}(u,v)}{\partial u} & \dfrac{\partial f_{22}(u,v)}{\partial v} \end{array} \right)\Bigg|_{(u,v)=(u^*,v^*)} \neq 0,$$

there exists a periodic solution $(u(t,\varepsilon),v(t,\varepsilon))$ of the differential system (6) such that

$$(u(0,\varepsilon),v(0,\varepsilon)) \to (u^*,v^*) \quad \text{when } \varepsilon \to 0.$$

Going back through the change of variables (5) the periodic solution $(u(t,\varepsilon),v(t,\varepsilon))$ of the differential system (6) becomes the periodic solution

$$z(t,\varepsilon) = \cos(nt)u(t,\varepsilon) + \frac{1}{n}\sin(nt)v(t,\varepsilon),$$

$$w(t,\varepsilon) = -n\sin(nt)u(t,\varepsilon) + \cos(nt)v(t,\varepsilon),$$

of the differential system (4) such that

$$(z(0,\varepsilon),w(0,\varepsilon)) \to (u^*,v^*) \quad \text{when } \varepsilon \to 0.$$

Finally, since $x = \varepsilon z$ the periodic solution $(z(t,\varepsilon),w(t,\varepsilon))$ of the differential system (4) provides the periodic solution

$$x(t,\varepsilon) = \varepsilon\left(\cos(nt)u(t,\varepsilon) + \frac{1}{n}\sin(nt)v(t,\varepsilon)\right),$$

of the Lienard differential equation (2) such that

$$x(0,\varepsilon) \approx \varepsilon u^* + \mathcal{O}(\varepsilon^2).$$

Moreover, since $x' = \varepsilon w$ the theorem follows. $\square$

Proof of Corollary 2. We shall apply the results of Theorem 1 to the Lienard differential equation of the statement of Corollary 2. So we compute the functions $f_{21}(u,v)$ and $f_{22}(u,v)$ defined just before the statement of Theorem 1, and we obtain

$$\frac{-\left(u^3-7u+6\right)n^4+\left(u^3-2vu^2-\left(v^2+2\right)u+8v-4\right)n^2-2v^3+u\left(v^2-1\right)+2}{4n\left(3n^4+2n^2-1\right)},$$

$$\frac{\left(7v+u\left(2u^2-vu-8\right)\right)n^4+v\left(u^2+2vu-v^2-2\right)n^2+v^3-v}{4n\left(3n^4+2n^2-1\right)},$$

respectively.

Doing the resultant of the functions f_{21} and f_{22} with respect to the variable v we obtain a cubic polynomial in the variable u which has the following three roots

$$u_1 = 1, \quad u_2 = 2, \quad u_3 = \frac{-3n^6 + 31n^4 - 25n^2 + 5}{(n^2+1)^3}.$$

In a similar way doing the resultant of the functions f_{21} and f_{22} with respect to the variable u we obtain another cubic polynomial in the variable v which has the following three roots

$$v_1 = 1, \quad v_2 = 0, \quad v_3 = \frac{-15n^6 + 35n^4 - 13n^2 + 1}{(n^2+1)^3}.$$

From the properties of the resultants it follows that all the solutions (u^*, v^*) of the system $f_{21}(u,v) = 0$, $f_{22}(u,v) = 0$ are of the form (u_i, v_j) being u_i and v_j some of the above roots. Trying the nine possible solutions, we obtain only two solutions for the system $f_{21}(u,v) = 0$, $f_{22}(u,v) = 0$, namely

$$(u_1^*, v_1^*) = (1,1) \quad \text{and} \quad (u_2^*, v_2^*) = (2,0).$$

Then applying Theorem 1 the corollary follows. □

For more information about the resultants see for instance [6].

3. Appendix: The averaging theory of second order

In this section we recall the averaging theory of second order to find periodic orbits.

Theorem 3. *Consider the differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{8}$$

where $F_1,\ F_2 : \mathbb{R}\times D \to \mathbb{R}^n$, $R : \mathbb{R}\times D\times(-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions, T-periodic in the first variable, and* D *is an open subset of* $\mathbb{R}^n$*. Assume that the following hypothesis* (i) *and* (ii) *hold.*

(i) $F_1(t,\cdot) \in C^1(D)$ *for all* $t \in \mathbb{R}$, F_1, F_2, R *and* D_xF_1 *are locally Lipschitz with respect to* x*. We define* $f_1,\ f_2 : D \to \mathbb{R}^n$ *as*

$$f_1(z) = \frac{1}{2\pi}\int_0^T F_1(s,z)ds,$$

$$f_2(z) = \frac{1}{2\pi}\int_0^T [D_zF_1(s,z)\int_0^s F_1(t,z)dt + F_2(s,z)]ds.$$

(ii) *For* $V \subset D$ *an open and bounded set and for each* $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\backslash\{0\}$, *there exist* $a \in V$ *such that* $f_1(a)+\varepsilon f_2(a) = 0$ *and* $d_B(f_1+\varepsilon f_2) \neq 0$ *(see its definition later on).*

Then for $|\varepsilon| > 0$ *sufficiently small, there exists a* $T-$*periodic solution* $\varphi(\cdot, \varepsilon)$ *of the system such that* $\varphi(0, \varepsilon) \to a$ *when* $\varepsilon \to 0$.

As usual we have denoted by $d_B(f_1 + \varepsilon f_2)$, the Brouwer degree of the function $f_1 + \varepsilon f_2 : V \to \mathbb{R}^n$ at its fixed point a. A sufficient condition for showing that the Brouwer degree of a function f at its fixed point a is non–zero, is that the Jacobian of the function f at a (when it is defined) is non–zero.

If the function f_1 is not identically zero, then the zeros of $f_1 + \varepsilon f_2$ are mainly the zeros of f_1 for ε sufficiently small. In this case Theorem 3 provides the so-called *averaging theory of first order*.

If the function f_1 is identically zero and f_2 is not identically zero, then the zeros of $f_1 + \varepsilon f_2$ are the zeros of f_2. In this case Theorem 3 provides the so-called *averaging theory of second order*.

Acknowledgements

The first author is partially supported by a MINECO/FEDER grant MTM2008–03437, an AGAUR grant number 2009SGR-0410, an ICREA Academia and FP7–PEOPLE–2012–IRSES–316338 and 318999. The second author is partially supported by a FAPESP–BRAZIL grant 2012/10231–7. The third author is partially supported by a FAPESP–BRAZIL grant 2007/06896–5. The first and third authors are also supported by the joint project CAPES-MECD grant PHB-2009-0025-PC.

References

[1] A. Buica and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bulletin des Sciences Mathemàtiques **128** (2004),7–22.

[2] J. Giné, M. Grau and J. Llibre, *Averaging theory at any order for computing periodic orbits*, Physica D **250** (2013), 58–65.

[3] J. Llibre, *Averaging theory and limit cycles for quadratic systems*, Radovi Matematicki **11** (2002), 215–228.

[4] J. Llibre, D.D. Novaes and M.A. Teixeira, *Higher order averaging theory for finding periodic solutions via Brouwer degree*, preprint, 2013.

[5] T. Ma and Z. Wang, *Periodic solutions of Lienard equations with resonant isochronous potentials*, Discrete and Continuous Dynamical Systems, **33** (2013), 1563–1581.

[6] P. Olver, *Classical invariant theory*, London Math. Soc. Student Texts, Vol. 44, Cambbridge Univ. Press, New York, 1999.

[7] J.A. Sanders, F. Verhulst and J. Murdock, Averaging methods in nonlinear dynamical systems. Second edition. Applied Mathematical Sciences, **59**. Springer, New York, 2007.

[1] Departament de Matematiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail address: jllibre@mat.uab.cat

[2] Departamento de Matematica, Universidade Estadual de Campinas, Caixa Postal 6065, 13083–859, Campinas, SP, Brazil
E-mail address: ddnovaes@ime.unicamp.br, teixeira@ime.unicamp.br