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Local positivity in terms of Newton–Okounkov bodies

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Abstract

In recent years, the study of Newton–Okounkov bodies on normal varieties has become a central subject in the asymptotic theory of linear series, after its introduction by Lazarsfeld–Mustaţă and Kaveh–Khovanskii. One reason for this is that they encode all numerical equivalence information of divisor classes (by work of Jow). At the same time, they can be seen as local positivity invariants, and Küronya-Lozovanu have studied them in depth from this point of view.

We determine what information is encoded by the set of all Newton–Okounkov bodies of a big divisor with respect to flags centered at a fixed point of a surface, by showing that it determines and is determined by the numerical equivalence class of the divisor up to negative components in the Zariski decomposition that do not go through the fixed point.

1 Introduction

Newton–Okounkov bodies Inspired by the work of A. Okounkov [12], R. Lazarsfeld and M. Mustaţă [11] and independently K. Kaveh and A. Khovanskii [5] introduced Newton–Okounkov bodies as a tool in the asymptotic theory of linear series on normal varieties, a tool which proved to be very powerful and in recent developments of the theory has gained a central role. An excellent introduction to the subject —not exhaustive due to the rapid development of the theory— can be found in the review [1] by S. Boucksom.

Newton–Okounkov bodies are defined as follows. Let X be a normal projective variety of dimension n. A flag of irreducible subvarieties

$$Y_\bullet = \{X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{p\}\}$$

is called full and admissible if Y_i has codimension i in X and is smooth at the point p. p is called the center of the flag. For every non-zero rational function $\varphi \in K(X)$, write $\varphi_0 = \varphi$, and for $i = 1, \ldots, n$

$$\nu_i(\varphi) \stackrel{\text{def}}{=} \operatorname{ord}_{Y_i}(\varphi_{i-1}), \qquad \varphi_i \stackrel{\text{def}}{=} \frac{\varphi_{i-1}}{g_{Y_i}^{\nu_i(\varphi)}}, \tag{$\star$}$$

where g_i is a local equation of Y_i in Y_{i-1} around p (this makes sense because the flag is admissible). The sequence $\nu_{Y_\bullet} = (\nu_1, \nu_2, \ldots, \nu_n)$ determines a rank n discrete valuation $K(X)^* \longrightarrow \mathbb{Z}^n_{\text{lex}}$ with center at p [16].

Definition 1. If X is a normal projective variety, D a big Cartier divisor on it, and $Y_\bullet$ an admissible flag, the Newton–Okounkov body of D with respect to $Y_\bullet$ is

$$\Delta_{Y_\bullet}(D) \stackrel{\text{def}}{=} \overline{\left\{\frac{\nu_{Y_\bullet}(\varphi)}{k} \,\middle|\, \varphi \in H^0(X, \mathcal{O}_X(kD)),\ k \in \mathbb{N}\right\}} \subset \mathbb{R}^n,$$

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where $\overline{\{\cdot\}}$ denotes the closure with respect to the usual topology of $\mathbb{R}^2$. Although not obvious from this definition, $\Delta_{Y_\bullet}(D)$ is convex and compact, with nonempty interior, i.e., a body (see [5], [11], [1]). A. Küronya, V. Lozovanu and C. Maclean [10] have shown that it is a polygon if X is a surface, and that in higher dimensions it can be non-polyhedral, even if X is a Mori dream space. The definition can be carried over to the more general setting of graded linear series, and also to $\mathbb{Q}$ or $\mathbb{R}$-divisors; in the absence of some bigness condition, $\Delta_{Y_\bullet}(D)$ may fail to have top dimension.

By [11, Proposition 4.1], $\Delta_{Y_\bullet}(D)$ only depends on the numerical equivalence class of D. S. Y. Jow proved in [6] that the set of all Newton–Okounkov bodies works as a complete set of numerical invariants of D, in the sense that, if D' is another big Cartier divisor with $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$ for all flags $Y_\bullet$, then D and D' are numerically equivalent.

Flags on proper models of $K(X)$ It is most natural to define Newton–Okounkov bodies with respect to any valuation v with value group equal to $\mathbb{Z}^{\dim X}_{\text{lex}}$, and not only those coming from flags on X (see [5], [1]). Thus we consider admissible flags on arbitrary birational models of X, noting that even to express the results for flags lying on X (theorem 2 below) we need to consider clusters of infinitely near points.

Definition 2. We call admissible flag for X any admissible flag $Y_\bullet$ on $\tilde{X}$ where $\pi : \tilde{X} \to X$ is a proper birational morphism. Whenever we need to specify the map we will use the notation

$$Y_\bullet = \left\{ X \xleftarrow{\pi} \tilde{X} = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{p\} \right\}$$

but mostly we omit an explicit mention of the model $\tilde{X}$ on which p and Y_i lie. The point $\pi(p) = O \in X$ will be called the center of the flag on X; if π contracts the whole flag, i.e., $\pi(Y_1) = \pi(p) = O$ then we say that $Y_\bullet$ is an infinitesimal flag, and if $\operatorname{codim} \pi(Y_i) = i$ then it is a proper flag. If $\tilde{X} = X$, $\pi = \mathrm{id}_X$, we say that the flag is smooth at O. The corresponding Newton–Okounkov bodies will be also called infinitesimal, proper or smooth accordingly.

Already Lazarsfeld–Mustaţă [11] considered Newton–Okounkov bodies of D defined by flags on varieties birational to X —more precisely, flags contained in the exceptional divisor of a blowup of X, with the goal of making a canonical choice of "generic infinitesimal" flag and getting rid of the arbitrariness of the choice of a flag—. A. Küronya and V. Lozovanu [9] have pushed forward the study of infinitesimal flags, with the philosophical viewpoint that the "local positivity" (see [8, Chapter 5] of D at a smooth point O should be governed by the set of Newton–Okounkov bodies $\Delta_{Y_\bullet}(D)$ where the flag $Y_\bullet$ is centered at O. This raises the question of what information on L is contained in the set of Newton–Okounkov bodies $\Delta_{Y_\bullet}(D)$ with fixed center, analogously to Jow's result for the set of all Newton–Okounkov bodies. In the case when X is a surface, we provide a complete answer which supports the "local positivity" viewpoint, and we prove that Newton–Okounkov bodies given by infinitesimal flags suffice to determine all Newton–Okounkov bodies given by flags centered at O.

Clusters of infinitely near points Fix X a projective surface, and $O \in X$ a smooth point. A point infinitely near to O is a smooth point $p \in \tilde{X}$, where $\pi : \tilde{X} \to X$ is a proper birational morphism, such that $\pi(p) = O$.

A finite or infinite set K of points equal or infinitely near to O, such that for each $p \in K$, K contains all points to which p is infinitely near, is called a cluster of points infinitely near to O. We now review a few facts on clusters that we need, referring to E. Casas-Alvero's book [2] for details and proofs. The simplest example of a cluster is the sequence of images of a point $p \in \tilde{X}$ infinitely near to O: π_p can be factored as a sequence of k point blowups $\pi = \mathrm{bl}_O \circ \mathrm{bl}_{p_1} \circ \cdots \circ \mathrm{bl}_{p_{k-1}}$,

$$X = X_0 \xleftarrow{\mathrm{bl}_O} X_1 \xleftarrow{\mathrm{bl}_{p_1}} \ldots \xleftarrow{\mathrm{bl}_{p_{k-1}}} X_k = \tilde{X} = X_p$$

and then

$$K(p) = \{O, bl_{p_1} \circ \cdots \circ bl_{p_{k-1}}(p), \ldots, bl_{p_{k-1}}(p), p\}$$

is a cluster. A priori, infinitely near points belong to different surfaces, but we consider the points $p \in X_p \xrightarrow{\pi_p} X$ and $p' \in X_{p'} \xrightarrow{\pi_{p'}} X$ to be the same point when there is a birational map defined in a neighborhood of p, $X_p \supset U_p \to X_{p'}$, which commutes with $\pi_p, \pi_{p'}$, maps p to p' and is an isomorphism in a (possibly smaller) neighborhood of p. Then we can safely assume that the sequence of points blown up to get the surface where p lies is formed exactly by the points in $K(p)$ except p itself: $K(p) = \{O, p_1, \ldots, p_{k-1}, p\}$. In this sense, every infinitely near point p has a well defined predecessor, namely the last blown up point p_{k-1}.

Points infinitely near to O are classified as satellite if $p \in \mathrm{Sing}(\pi^{-1}(O))$ and free otherwise. We shall call a cluster K free if every $p \in K$ is free. A relevant fact when dealing with smooth flags is that there is a smooth curve through O whose birational transform in $\tilde{X}$ contains the infinitely near point $p \in \tilde{X}$ if and only if the cluster $K(p)$ is free. It is customary to say that a curve goes through an infinitely near point p (or has multiplicity m there) if its birational transform does so; we will follow this convention without further notice.

A weighted cluster is a pair $\mathcal{K} = (K, m)$ where K is a cluster and m is a map $m : K \longrightarrow \mathbb{Z}$. A typical example is, given a proper birational morphism $\tilde{X} \xrightarrow{\pi} X$ (factored as above) and a curve C through O, the set of all points infinitely near to O in $\bigcup_{i=0}^{k} X_i$ that belong to C, weighted with $m(p) = \mathrm{mult}_p(\tilde{C})$.

Let $C \subset X$ be a curve through O which has no smooth branch through O. There exists a minimal model $\pi : \tilde{X} \longrightarrow X$ such that, denoting $\tilde{C}$ the strict transform of C, all of the (finitely many) points of $\tilde{C}$ infinitely near to O (i.e., $\pi^{-1}(O) \cap \tilde{C}$) are satellite. For any factorization of such a π as a sequence of point blowups, the centers of the blowups form a free cluster. This cluster, weighted with the multiplicities of C at its points, will be called the cluster of initial free points of C and denoted $\mathcal{F}_C$. Remark that an equality $\mathcal{F}_C = \mathcal{F}_{C'}$ means that the minimal model such that the strict transform of C has no free point infinitely near to O is also the minimal model such that the strict transform of C' has no free point infinitely near to O, and moreover the multiplicities of the strict transforms of C and C' at each blown up point coincide.

Local numerical equivalence on surfaces Let still X be a normal projective surface. Every pseudoeffective $\mathbb{Q}$-divisor D admits a unique Zariski decomposition $D = P + N$, where P, N are $\mathbb{Q}$-divisors with P nef, N effective, the components N_i of N have negative definite intersection matrix, and $P \cdot N_i = 0$. Zariski showed in [15] that a unique such decomposition exists for any effective divisor D on a smooth surface —in what can be considered a foundational work of the asymptotic theory of linear systems. The generalization to pseudoeffective $\mathbb{Q}$-divisors is due to Fujita [4]. The result then carries over to normal surfaces using the intersection theory developed by Sakai in [14], see [13, Theorem 2.2]. One should bear in mind that in this case P and N are in general Weil divisors only, even if D is Cartier.

Definition 3. Fix $O \in X$, and let D be a divisor on X, with Zariski decomposition $D = P + N$. We decompose the negative part as

$$N = N_O + N_O^c$$

where the support of N_O are exactly the divisors in N which go through O. We say that

$$D = P + N_O + N_O^c$$

is the refined Zariski decomposition at O.

Definition 4. Given two divisors D, D' on X with refined Zariski decompositions at O

$$D = P + N_O + N_O^c, \qquad D' = P' + N'_O + N'^c_O$$

we say that D and D' are numerically equivalent near O if

$$P \equiv P' \text{ and } N_O = N'_O. \qquad (\dagger)$$

The main results of this paper show that the information contained in the set of all Newton–Okounkov bodies of a big Cartier divisor D with center at a smooth point O of a surface is exactly the numerical equivalence class near O of D in the sense above.

Theorem 1. Let D, D' be big Cartier divisors on a normal projective surface X, and let $O \in X$ a smooth point. The following are equivalent:

1. D and D' are numerically equivalent near O, i.e., their Zariski decompositions satisfy ($\dagger$).
2. For all admissible flags with center O, $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$.
3. For all infinitesimal admissible flags with center O, $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$.
4. For all proper admissible flags with center O, $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$.

It is obvious that (2) is equivalent to [(3) and (4)]. The skeleton of our proof is as follows:

$$(1) \Rightarrow (2), \qquad (3) \Rightarrow (4) \Rightarrow (1).$$

Each implication follows from one or two of the lemmas in section 2; some of the lemmas are actually stronger than is required and may be interesting for themselves.

Remark that it is not enough to know the Newton–Okounkov bodies of D with respect to all flags lying on X with center at O (smooth flags) in order to recover the numerical equivalence class near O. The information contained in this smaller collection of Newton–Okounkov bodies is determined in the next theorem, after which it will be easy to give examples. Assume D is a divisor with refined Zariski decomposition

$$D = P + N_O + N_O^c,$$

and decompose further $N_O = N_O^{sing} + N_O^{sm}$ where N_O^{sm} is formed by all components with at least one smooth branch through O. Then the result can be expressed in terms of the clusters of initial free points $\mathcal{F}_{N_O^{sing}}$ and $\mathcal{F}_{N'^{sing}_O}$:

Theorem 2. Fix $O \in X$ a smooth point. Let D, D' be big Cartier divisors on X, with refined Zariski decompositions

$$D = P + N_O^{sing} + N_O^{sm} + N_O^c, \quad D' = P' + N'^{sing}_O + N'^{sm}_O + N'^c_O$$

The following are equivalent:

1. $P \equiv P'$, $N_O^{sm} = N'^{sm}_O$ and $\mathcal{F}_{N_O^{sing}} = \mathcal{F}_{N'^{sing}_O}$.
2. For almost all infinitesimal admissible flags $\{\tilde{X} \supset E \supset \{p\}\}$ with center O such that the cluster $K(p)$ is free, $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$.
3. For all smooth admissible flags with center O, $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$.

The easiest example in which the set of all smooth Newton–Okounkov bodies with center at O does not determine the numerical equivalence class near O is given by two big Cartier divisors D, D' whose negative parts N, N' are distinct irreducible curves with ordinary cusps at the same point O and with the same tangent direction (it is not difficult to construct such divisors on suitable blowups of $\mathbb{P}^2$). In that case $F_{N_O^{sing}} = F_{N_O'^{sing}}$ consists of two points: O and the point infinitely near to it in the direction tangent to the cusps, with multiplicities 2 and 1 respectively. Therefore all Newton–Okounkov bodies with respect to smooth flags centered at O coincide, but $N_O = N = N' = N'_O$.

The proof, contained in the lemmas of section 2 follows the same structure as for theorem 1. The main ingredient in both cases is the computation of Newton–Okounkov bodies in terms of Zariski decompositions which can be found as Theorem 6.4 in [11]. Although Lazarsfeld and Mustaţa proved this fact for smooth surfaces, the result applies on a normal surface X as long as the flag is centered at a smooth point O of X. Indeed, using a resolution of singularities $\pi : \tilde{X} \to X$ which is an isomorphism in a neighbourhood of O, one may apply [11, Theorem 6.4] to the pullback divisor $\pi^* D$ with respect to the pulled back flag, because Zariski decompositions agree via pull-backs (see [13, 2.3]) and intersection numbers agree by the projection formula.

Higher dimension Our results depend on the existence of a Zariski decomposition. Decompositions with similar properties exist on some higher dimensional varieties as well (for instance, on toric varieties) and in that case one can expect the behaviour of Newton–Okounkov bodies to be related to the decomposition similarly to what happens for surfaces.

Given a Cartier $\mathbb{R}$-divisor D on X, a Zariski decomposition of D in the sense of Cutkosky-Kawamata-Moriwaki (or simply a CKM-Zariski decomposition) is an equality

$$\pi^* D = P + N$$

on a smooth birational modification $\pi : \tilde{X} \to X$ such that

1. P is nef,
2. N is effective,
3. all sections of multiples of D are carried by P, i.e., the natural maps

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lfloor kP \rfloor)) \longrightarrow H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lfloor k\pi^* D \rfloor)) = H^0(X, \mathcal{O}_X(\lfloor kD \rfloor))$$

   are bijective for all $k \geq 0$.

See Y. Prokhorov's survey [13] for more on CKM-Zariski decompositions and other generalizations. Such decompositions don't always exist [3] and when they do, P and N may be irrational even if D is an integral divisor. But if they do exist, for instance if X is a toric variety [7], Newton–Okounkov bodies centered at a given point O will be governed by the Zariski decomposition:

Proposition 5. Let D, D' be big Cartier divisors on a variety X, admitting a CKM-Zariski decomposition and let $O \in X$ a point. If D and D' are numerically equivalent near O, i.e., their CKM-Zariski decompositions satisfy (†). Then for all admissible flags with center O, $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$.

It should be expected that a converse statement similar to what holds for surfaces be valid in higher dimension. In fact, the proof of lemma 11 below can be easily adapted to the higher dimensional setting, so N_O is indeed determined by the Newton–Okounkov bodies centered at O. The methods of this note are however not sufficient to show that the positive part is also determined by the Newton–Okounkov bodies centered at O.

We work over an algebraically closed field.

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2 Proofs

Local numerical equivalence implies equal Newton–Okounkov bodies Let us first prove (1) ⇒ (2) in theorems 1 and 2, and at the same time proposition 5. So assume that D and D' are big Cartier divisors on a variety X, numerically equivalent near O, i.e., with refined CKM-Zariski decompositions satisfying (†):

$$\pi^* D = P + N_O + N_O^c, \qquad \pi^* D' = P' + N_O + N'^c_O$$

with $P \equiv P'$. By Lazarsfeld–Mustaţă [11, Theorem 4.1], all Newton-Okounkov bodies of D' and

$$D'' = D' + (P - P') = P + N_O + N'^c_O$$

coincide, because D'' and D' are numerically equivalent. Thus for the proof of proposition 5 and (1) ⇒ (2) in theorem 1 it is not restrictive to assume that $P = P'$. Then there is a sequence of divisors

$$D = D_0, D_1, \dots, D_k = D'$$

whose CKM-Zariski decompositions $\pi^* D_i = P + N_i$ have the same positive part and N_i differs from N_{i+1} in a multiple of a base divisor E_i with $O \notin \pi(E_i)$. Thus the desired implication follows from the following:

Lemma 6. Let D, D' be two big Cartier divisors with respective refined Zariski decompositions

$$\pi^* D = P + N, \quad \pi^* D' = P + (N + \lambda E)$$

with $\lambda \in \mathbb{R}$, and O any point $O \notin \pi(E)$. Then for all admissible flags with center O, $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$.

Proof. An equation of E is invertible in a neighborhood of O, and therefore also in a neighborhood of every point p infinitely near to O. So for every flag $Y_\bullet$ centered at O, $\nu_{Y_\bullet}(E) = 0$. Since global sections of $\lfloor kD \rfloor$ and $\lfloor kD' \rfloor$ differ exactly in $\lfloor k\lambda E \rfloor$, their values under $\nu_{Y_\bullet}$ agree, and therefore the Newton–Okounkov bodies are the same. □

Now the following lemma is enough to finish the proof of (1) ⇒ (2) in theorem 2:

Lemma 7. Let D, D' be two big Cartier divisors on a normal surface X with respective refined Zariski decompositions

$$D = P + (N + \lambda C), \quad D' = P + (N + \lambda' C')$$

with $\lambda, \lambda' \in \mathbb{R}$, and C, C' irreducible curves with no smooth branch through O, and satisfying $\lambda \mathcal{F}_C = \lambda' \mathcal{F}_{C'}$ Then for all infinitesimal admissible flags $\{X \xleftarrow{\pi} \tilde{X} \supset E \supset \{p\}\}$ with center at a given smooth point O such that the cluster $K(p)$ is free and $p \notin \mathcal{F}_C$, $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$.

Note that the cluster $\mathcal{F}_C = \mathcal{F}_{C'}$ is finite, and its weights are the multiplicities of C (equivalently, C') at each $p \in \mathcal{F}_C$. The equality $\lambda \mathcal{F}_C = \lambda' \mathcal{F}_{C'}$ means that both clusters consist of the same points, and their respective weights m, m' satisfy the proportionality $\lambda m(p) = \lambda' m'(p)$ for all $p \in \mathcal{F}_C$.

Proof. Let $\{X \xleftarrow{\pi} \tilde{X} \supset E \supset \{p\}\}$ be an infinitesimal admissible flag with center O such that the cluster $K(p) = \{O = p_0, p_1, \dots, p_{k-1}, p_k = p\}$ is free and $p \notin \mathcal{F}_C$. Let E' be the birational transform of E in the blowup X_p:

$$X = X_0 \xleftarrow{bl_O} X_1 \xleftarrow{bl_{p_1}} \dots \xleftarrow{bl_{p_{k-1}}} X_k = X_p .$$

Since $p \in E'$, E' is an irreducible curve (is not contracted in X_p), and since E' contracts to the smooth point O, it must be one of the exceptional components; in fact it must be the last, $E^0 = E_{p_k - 1}$, which is the only one containing p. Thus it is not restrictive to assume that $\tilde{X} = X_p$, $\pi = bl_O \circ bl_{p_1} \circ \cdots \circ bl_{p_{k-1}}$, and $E = E_{p_{k-1}}$.

The order of vanishing of $\pi^* C$ along E is

$$\mathrm{ord}_E(\pi^* C) = \sum_{i=0}^{k-1} \mathrm{mult}_{p_i} \tilde{C} = \sum_{i=0}^{k-1} m(p_i)$$

(where $m(p_i) = 0$ if $p_i \notin \mathcal{F}_C$) because the p_i are free; similarly, $\mathrm{ord}_E(\pi^* C') = \sum_{i=0}^{k-1} m'(p_i)$. Moreover, C and C' do not pass through p. So $\lambda\nu_{Y_\bullet}(C) = \lambda'\nu_{Y_\bullet}(C')$. Therefore we conclude as in the previous lemma: since global sections of $\lfloor kD \rfloor$ and $\lfloor kD' \rfloor$ differ exactly in $\lfloor k\lambda(C - C') \rfloor$, their values under $\nu_{Y_\bullet}$ agree, and therefore the Newton–Okounkov bodies are the same.

□

Equality of infinitesimal bodies implies equality of proper bodies Now we prove (3) ⇒ (4) in theorem 1 and (2) ⇒ (3) in theorem 2, namely we need to show that if $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$ for all infinitesimal admissible flags $Y_\bullet$ with center O (resp. for almost all infinitesimal admissible flags $\{\tilde{X} \supset E \supset \{p\}\}$ with center O such that the cluster $K(p)$ is free) then the same equality holds for all proper admissible flags $Y_\bullet$ with center O, (resp. for all smooth admissible flags $Y_\bullet$ with center O).

Given a curve C through O, an infinite cluster $K = \{p_0 = O, p_1, \ldots, p_k, \ldots\}$ will be called a branch cluster for C if each p_i is infinitely near to p_{i-1} and all of them belong to C. Note that in a branch cluster, at most finitely many points are satellite, and C has a smooth branch at O if and only if it admits a branch cluster which is free.

Associated to each branch cluster there is a sequence of flags

$$Y_\bullet^{(k)} = \{X_k \supset E_{p_{k-1}} \supset \{p_k\}\}, \qquad (\ddagger)$$

and a corresponding sequence of valuations $\nu^{(k)} = \nu_{Y_\bullet^{(k)}}$.

Lemma 8. Let $(\nu^{(k)})_{k\in\mathbb{N}}$ be the valuations associated to a branch cluster for the irreducible curve C through O. Let k_0 be such that the birational transform of C at p_{k_0} is smooth, and let $Y_\bullet = \{\tilde{X} \supset \tilde{C} \supset \{p_k\}\}$ be the corresponding proper admissible flag. Then for every $\varphi \in K(X)$ and every $k \geq 0$ there is an equality

$$\nu^{(k)}(\varphi) = \begin{pmatrix} k - k_0 & 1 \\ 1 & 0 \end{pmatrix} \nu_{Y_\bullet}(\varphi).$$

Proof. Assume without loss of generality that φ is a regular function on a neighbourhood of p_{k_0}. Recall the definition of $\nu_{Y_\bullet}(\varphi)$: $\nu_1(\varphi) = \mathrm{ord}_C(\varphi)$, $\varphi_1 = \varphi/g^{\nu_1(\varphi)}$, where g is a local equation of $\tilde{C}$ at p_{k_0}, and $\nu_2(\varphi) = \mathrm{ord}_{p_{k_0}}(\varphi_1|_C)$. φ_1 is the local equation of some effective divisor D which does not contain C, and hence, by Noether's formula for intersection multiplicities [2, Theorem 3.3.1], there is k_1 such that D does not go through any point $p_k, k \geq k_1$ and $\nu_2(\varphi) = \sum_{i=k_0}^{k_1} \mathrm{mult}_{p_i} D = \mathrm{ord}_{E_{p_{k-1}}} \varphi_1$ for all $k \geq k_1$. On the other hand, it is immediate that $\mathrm{ord}_{E_{p_{k-1}}} g = k - k_0$ for all $k \geq \max\{k_0, 1\}$, so $\nu_1^{(k)}(\varphi) = \mathrm{ord}_{E_{p_{k-1}}}(\varphi) = (k - k_0)\nu_1(\varphi) + \nu_2(\varphi)$ for all $k \geq \max\{k_1, 1\}$. Similarly, $\nu_1^{(k)}(\varphi_1) = 0$ for $k \geq k_1$ and $\nu_1^{(k)}(g) = 1$, and the claim follows. □

Corollary 9. Let D, D′ be arbitrary Cartier divisors on a surface X, and $O \in X$ a smooth point. Given a proper admissible flag $Y_\bullet = \{X \xleftarrow{\pi} \tilde{X} \supset C \supset \{p\}\}$ centered at O, and a branch cluster $K = \{p_0, \ldots, p_k, \ldots\}$ for $\pi(C)$, denote $Y_\bullet^{(k)}$ the sequence of flags (‡). If the set of indices k with $\Delta_{Y_\bullet^{(k)}}(D) = \Delta_{Y_\bullet^{(k)}}(D')$ is infinite, then $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$.

The proof of the corollary is straightforward and is left to the reader.

Now the desired implications in theorem 1 and 2 follow, because every curve C through O (resp. smooth at O) admits a branch cluster (resp. a free branch cluster), and statement (3) in theorem 1 (resp. (2) in theorem 2) imply the infiniteness needed in corollary 9.

Equality of proper bodies implies local numerical equivalence Finally we prove (4) ⇒ (1) in theorem 1 and (3) ⇒ (1) in theorem 2. We deal separately with the positive and negative parts, because for the positive part it is enough to consider smooth proper flags:

Lemma 10. Let D and D^0 be big Cartier divisors with Zariski decompositions

$$D = P + N, \quad D^0 = P^0 + N^0.$$

Assume that, for all curves $C \subset X$ smooth at O, the bodies $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D^0)$ agree for the flag O}}. Then $P \equiv P^0$.

Proof. Choose ample divisor classes $L_1, \ldots, L_\rho$ whose Q-span is all of the rational Néron-Severi space $N^1(X)_{\mathbb{Q}}$. Replacing each L_i by a suitable multiple, we can assume that it is the class of an irreducible curve C_i smooth at O, whose tangent direction there is different from every tangent direction to a component of the augmented base locus which may pass through O. (This is well known, and can be proved as follows: by Serre vanishing there exist k such that $H^1(X, \mathbf{I}_O^2(L_i^{\otimes k})) = 0$ where $\mathbf{I}_O$ denotes the ideal sheaf of the point O in X. Then the exact sequence in cohomology determined by

$$0 \longrightarrow \mathbf{I}_O^2 \otimes L_i^{\otimes k} \longrightarrow \mathcal{O}_X(L_i^{\otimes k}) \longrightarrow (\mathcal{O}_X/\mathbf{I}_O^2)(L_i^{\otimes k}) \longrightarrow 0$$

shows that $H^0(X, \mathcal{O}_X(L_i^{\otimes k}))$ surjects onto $H^0(X, \mathcal{O}_X/\mathbf{I}_O^2)$ and in particular it is possible to find a section in $H^0(X, \mathcal{O}_X(L_i^{\otimes k}))$ which vanishes at O to order exactly 1 and has preassigned image=tangent direction).

So for each $i = 1, \ldots, \rho$, we can compute Newton–Okounkov bodies of D and D^0 with respect to the flag $Y_\bullet^{(i)} = \{X \supset C_i \supset \{O\}\}$, and by [11, Theorem 6.4], the height of the intersection of $\Delta_{Y_\bullet^{(i)}}(D)$ with the second coordinate axis equals $P \cdot L_i$; since by hypothesis the bodies $\Delta_{Y_\bullet^{(i)}}(D) = \Delta_{Y_\bullet^{(i)}}(D^0)$ coincide, it follows that $P \cdot L_i = P^0 \cdot L_i$ for all i. Therefore $P \equiv P^0$. □

Lemma 11. Let D and D^0 be big Cartier divisors with refined Zariski decompositions

$$D = P + N_O + N_O^c, \quad D^0 = P^0 + N_O^0 + N_O^{0c},$$

and assume that for all proper admissible flags with center O, $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D^0)$. Then $N_O = N_O^0$.

Proof. Let C be a component of N_O, and let $\pi : \tilde{X} \longrightarrow X$ be a proper birational morphism such that the strict transform $\tilde{C}$ of C is nonsingular at $\pi^{-1}(O)$; let p be a point in $\tilde{C} \cap \pi^{-1}(O)$, and $Y_\bullet = p \in \tilde{C} \subset \tilde{X}$, which is a proper admissible flag with center at O. The first coordinate of the leftmost point in $\Delta_{Y_\bullet}(D)$ is the coefficient of C in N_O by the proof of [11, Theorem 6.4], so this is also the coefficient of C in N_O^0. Doing this for each component, we obtain $N_O = N_O^0$ as claimed. □

This finishes the proof of (4) ⇒ (1) in theorem 1; to conclude with (3) ⇒ (1) in theorem 2 we need another lemma:

Lemma 12. Fix $O \in X$ a smooth point. Let D, D^0 be big Cartier divisors on X, with refined Zariski decompositions

$$D = P + N_O^{sing} + N_O^{sm} + N_O^c, \quad D^0 = P^0 + N_O^{0sing} + N_O^{0sm} + N_O^{0c},$$

and assume that for all smooth admissible flags with center O, $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D^0)$. Then $N_O^{sm} = N_O^{0sm}$ and $F_{N_O^{sing}} = F_{N_O^{0sing}}$.

Proof. The equality $N_O^{sm} = N_O^{\prime sm}$ follows with the same proof of the previous lemma. Let us now show that $F_{N_O^{sing}} = F_{N_O^{\prime sing}}$. Without loss of generality we may assume that $N_O^{sm} = N_O^{\prime sm} = 0$.

For each point $p \in F_{N_O^{sing}}$, let C_p be a smooth curve going through p and missing all points in $F_{N_O^{sing}} \cup F_{N_O^{\prime sing}}$ infinitely near to p, and let $Y_\bullet = \{X \supset C \supset \{O\}\}$. By [11, Theorem 6.4], the least α such that $(0, \alpha) \in \Delta_{Y_\bullet}$ is $\mathrm{ord}_O (N_O^{sing}|_{C_p}$. So the hypothesis tells us that

$$\mathrm{ord}_O (N_O^{sing}|_{C_p} = \mathrm{ord}_O (N_O^{\prime sing}|_{C_p} \quad (\S)$$

If $p = O$, then the two sides of the preceding inequality equal the weights of O in $F_{N_O^{sing}}$ and $F_{N_O^{\prime sing}}$ respectively.

If $p = O$, let q be the point preceding p and C_q the corresponding curve. The weights of p in $F_{N_O^{sing}}$ and $F_{N_O^{\prime sing}}$ now equal the differences

$$m(p) = \mathrm{ord}_O (N_O^{sing}|_{C_p}) - \mathrm{ord}_O (N_O^{sing}|_{C_q}),$$
$$m^{\prime}(p) = \mathrm{ord}_O (N_O^{\prime sing}|_{C_p}) - \mathrm{ord}_O (N_O^{\prime sing}|_{C_q})$$

respectively so they also coincide.

We have proved that every point of the cluster $F_{N_O^{sing}}$ appears in $F_{N_O^{\prime sing}}$ with the same weight; by symmetry, we conclude that both weighted clusters are in fact equal. □

References

[1] Boucksom, S. Corps D'Okounkov, Séminaire Bourbaki.

[2] Casas-Alvero, E., Singularities of plane curves, London Math. Soc. Lecture Note Ser., vol. 276, Cambridge University Press, 2000.

[3] Cutkosky, S. D., Zariski decomposition of divisors on algebraic varieties, Duke Math. J. 53 (1986), no. 1, 149–156

[4] Fujita, T. On Zariski problem, Proc. Japan Acad. Ser. A Math. Sci. 55 (1979), no. 3, 106–110.

[5] Kaveh, K., and Khovanskii, A.: Newton–Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. Annals of Mathematics 176, 2012, 925–978.

[6] Jow, S-Y. Okounkov bodies and restricted volumes along very general curves, Adv. Math. 223 (2010), no. 4, 1356–1371.

[7] Kawamata, Y., The Zariski decomposition of log-canonical divisors, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 425–433, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[8] Lazarsfeld, R., Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series.], 48. Springer-Verlag, Berlin, 2004. xviii+387 pp. ISBN: 3-540-22533-1 .

[9] Küronya, A., and Lozovanu, V.: Local positivity of linear series on surfaces preprint arXiv:1411.6205.

[10] Küronya, A. Lozovanu, V. and Maclean, C.: Convex bodies appearing as Okounkov bodies of divisors Adv. Math. 229 (2012), 2622-2639.

[11] Lazarsfeld, R., and Mustaţă: Convex bodies associated to linear series. Ann. Sci. Éc. Norm. Supér. (4) 42 (5) 2009, 783-835,

[12] Okounkov, A. Brunn-Minkowski inequality for multiplicities, Invent. Math. 125 (1996), no. 3, 405–411.

[13] Prokhorov, Yu. G., On the Zariski decomposition problem, Tr. Mat. Inst. Steklova 240 (2003), no. Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 43–72 (Russian, with Russian summary); English transl., Proc. Steklov Inst. Math. 1 (240) (2003), 37–65.

[14] Sakai, F., Weil divisors on normal surfaces, Duke Math. J. 51 (1984), no. 4, 877–887.

[15] Zariski, O. The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, Ann. of Math. 76 (1962) 560–615.

[16] Zariski, O., Samuel, P., Commutative algebra. Vol. II. Reprint of the 1960 edition. Graduate Texts in Mathematics, Vol. 29. Springer-Verlag, New York-Heidelberg, 1975. x+414 pp.

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