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NEWTON–OKOUNKOV BODIES SPROUTING ON THE VALUATIVE TREE

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Abstract. Given a smooth projective algebraic surface X, a point $O \in X$ and a big divisor D on X, we consider the set of all Newton–Okounkov bodies of D with respect to valuations of the field of rational functions of X centred at O, or, equivalently, with respect to a flag (E, p) which is infinitely near O, in the sense that there is a sequence of blowups $X^0 \to X$, mapping the smooth, irreducible rational curve $E \subset X^0$ to O. The main objective of this paper is to start a systematic study of the variation of these infinitesimal Newton–Okounkov bodies as (E, p) varies, focusing on the case $X = \mathbb{P}^2$.

Contents

1. Introduction

The concept of Newton–Okounkov bodies originates in Okounkov's work [25]. Relying on earlier work of Newton and Khovanskii, Okounkov associates convex bodies to ample line bundles on homogeneous spaces from a representation-theoretic point of view. In the generality we know them today, Newton–Okounkov bodies have been introduced by Lazarsfeld–Mustaţă [24] and Kaveh–Khovanskii [18].

Given an irreducible normal projective variety X of dimension r defined over an algebraically closed field K of characteristic 0, a big divisor D and a maximal rank valuation v on the function field $K(X)$ (or, equivalently, an admissible/good flag of subvarieties on some proper birational model of X (see §2.1)), a convex body $\Delta_v(D)$ is attached to these data which encodes in its convex geometric structure the asymptotic vanishing behaviour of the linear systems $|dD|$ for d 0 with respect to v.

Newton–Okounkov bodies contain a lot of information: from a conceptual point of view, they serve as a set of 'universal numerical invariants' according to a result of Jow [17]. From a more practical angle, they reveal information about the structure of the Mori Cone of X or of its blowups, about positivity properties of divisors (ampleness, nefness, and the like, see for instance Theorem 2.22, Remark 2.23 and [20]), and invariants like the volume or Seshadri constants (see [20, 21]).

Not surprisingly, the determination of Newton–Okounkov bodies is extremely complicated in dimensions three and above. They can be non-polyhedral even if D is ample and X is a Mori dream space (see [22]). We point out that the shape of $\Delta_v(D)$ depends on the choice of v to a large extent: an adequate choice of a valuation can guarantee a more regular Newton–Okounkov body [1]. The case of surfaces, though not easy at all, is reasonably more tractable: the Newton–Okounkov bodies are polygons with rational slopes, and they can be computed in terms of Zariski decompositions (see §2.3).

In this paper we are mainly interested in infinitesimal Newton–Okounkov bodies, which arise from valuations determined by flags (E, p), with $p \in E$, which are infinitely near a point of the surface X, i.e. there is a birational morphism $X^0 \to X$ mapping the smooth, irreducible rational curve $E \subset X^0$ to O. These Newton–Okounkov bodies have already been studied in [20, 21], and their consideration is implicit in [11]. Here we intend to connect the discussion in [11] to infinitesimal Newton–Okounkov bodies.

One of the main underlying ideas of [11] is to study the invariant þ (see §3.4 for the definition), which is roughly speaking an asymptotic multiplicity for quasi-monomial valuations. As such, it can be interpreted as a function on the topological space QM, the valuative tree of quasi-monomial valuations. Spaces of valuations were introduced by Zariski, and the topology we are interested in was originally considered in the celebrated work of Berkovich [4], see also [12]. The tree QM is rooted, and the root corresponds to the multiplicity valuation centred at O, with infinite maximal arcs homeomorphic to $[1, \infty)$ starting from the root, and arcs sprouting from vertices corresponding to integer points (see [12] and Remark 3.6). The function þ is continuous along the arcs of QM. Interestingly enough, infinitesimal Newton–Okounkov bodies can be interpreted as 2-dimensional counterparts of þ.

Here we will focus on the case $X = P^2$; the same questions on other surfaces (general surfaces of degree d in P^3 for instance) are likely to be equally interesting, but we do not treat them in this work in the hope that we will come back to them in the future. A basic property of $\hat\mu$, pointed out in [11], is that

$$\hat\mu(s) \geq \sqrt{s}$$

assuming that $s \in [1, +\infty)$ is an appropriately chosen parameter on an arc of $\mathbf{QM}$. Furthermore, equality holds unless there is a good geometric reason for the contrary, in the form of a curve C_s on X_s (for X_s the appropriate minimal blow-up of X where the related flag shows up) such that the corresponding valuation takes a value higher than $\deg(C_s) \cdot \sqrt{s}$. Such a curve is called supraminimal (see §5.3). Supraminimal curves are geometrically very particular, and give information on the Mori cone of X_s; for instance, it is conjectured in [11] (see also Conjecture 3.13 below) that along sufficiently general arcs of $\mathbf{QM}$, all supraminimal curves are (-1)-curves. If so,

$$\hat\mu(s) = \sqrt{s} \quad \text{for every } s \geq 8 + 1/36,$$

which among others implies Nagata's celebrated conjecture claiming that the inverse of the t-point Seshadri constant of $\mathbb{P}^2$ equals $\sqrt{t}$ for $t \geq 9$.

In this paper we associate a Newton–Okounkov body to each point of the valuative tree and investigate how they change along the arcs of $\mathbf{QM}$. We start by taking a quasimonomial valuation $v(C, s) \in \mathbf{QM}$, where C is a curve defining an arc in $\mathbf{QM}$ and $s \in [1, \infty)$ defines a point on the arc, and associating with it a rank 2 valuation. We take $v = (v_1, v_2)$, where $v_1 = v(C, s)$ and v_2 is the right (or left) derivative of $v(C, s)$ with respect to s. The process is described in §3.3 (precisely in Proposition 3.10). We obtain two valuations: $v_+(C, s)$ and $v_-(C, s)$, by taking respectively the right or the left derivative. Once we fix D to be a line in the plane the two valuations lead to the Newton–Okounkov bodies Δ_{C,s_+} and Δ_{C,s_-}. We focus on describing the properties of Δ_{C,s_+}, however the cases are conceptually very similar and many results are obtained simultaneously for both.

The study of the variation of Newton–Okounkov bodies is a natural extension of $\hat\mu$. In particular the projection of Δ_{C,s_+} to the first axis is $[0, \hat\mu(C, s)]$. Moreover, the convex geometric behavior of Δ_{C,s_+} is essentially simple (it is a certain triangle) whenever $\hat\mu$ has the expected value (see §5.1). Otherwise Δ_{C,s_+} exhibits more complicated features. The interesting phenomenon which we study is that, while $\hat\mu$ is continuous on $\mathbf{QM}$, the corresponding Newton–Okounkov bodies are not. We explain the concept of continuity and discontinuity (i.e. mutation) of Newton–Okounkov bodies in §5.3 (in particular Definition 5.16).

Taking into account the relation between Newton–Okounkov bodies and variation of Zariski decompositions (see Theorem 2.17 and [3, Theorem 1]), some discontinuity phenomenon is not unexpected, related to non-differentiability of Zariski decompositions in the big cone (Remark 2.20). We would also like to point out a plausible alternative explanation, provided by higher rank nonarchimedean analytifications, which we however do not use at all in this work. Just as $\mathbf{QM}$ parameterizes quasimonomial valuations (of rank 1), topological spaces that parameterize valuations of arbitrary rank have been introduced in the literature, starting with the Zariski Riemann space [30, VI, §17] (whose topology is however unrelated to the Berkovich topology of $\mathbf{QM}$ and so not suitable for our purposes) and most recently and notably the Huber analytification [16] and the Hahn analytification [13] of $\mathbb{P}^2$ (which admit continuous maps to the Berkovich analytification that contains $\mathbf{QM}$). Assigning the rank 2 valuation $v_+(C, s)$ to the point $v(C, s) \in \mathbf{QM}$

determines a map from the tree of quasimonomial valuations to the higher rank analytification, but this map turns out to be nowhere continuous [8]. From this point of view, discontinuities are unsurprising; what is remarkable is the piecewise continuity described in §5.3 and §5.4.

The main object of our interest is the study of mutations when the valuations move away from the root of $\mathcal{QM}$ along a fairly general route, and the results we have been able to obtain are collected in §5. Our results are partial in the sense that there are intervals in which we have been unable to obtain the appropriate information about mutations occurring there. Our manuscript is far from conclusive, it is simply devoted to lay the ground for future research on the subject.

The paper is organized as follows. In §2 we collect some basic definitions and results about valuations and Newton–Okounkov bodies, which we recall here to make the paper as self contained as possible. In §3 we focus on the two dimensional case, and specifically on quasi-monomial valuations, their interpretation in terms of the classical Newton–Puiseux algorithm, and the related clusters of centres. In this section (precisely in Remark 3.6) we briefly recall the structure of the valuation tree $\mathcal{QM}$. In §5 we provide our computations about the infinitesimal Newton–Okounkov bodies.

In what follows we will mainly work over the field of complex numbers.

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2. Preliminaries

Newton–Okounkov bodies in the projective geometric setting have been treated in [24], hence this is the source we will primarily follow.

Let X be an irreducible normal projective variety of dimension r defined over an algebraically closed field K of characteristic 0 (we will usually have the case $K = \mathbb{C}$ in mind), and let D be a big Cartier divisor (or line bundle; we may abuse terminology and identify the two concepts) on X.

Although one first introduces Newton–Okounkov bodies for Cartier divisors, the notion is numerical, even better, it extends to big classes in $N^1(X)_{\mathbb{R}}$ (see [24, Proposition 4.1]). Newton–Okounkov bodies are defined with respect to a rank r valuation of the field of rational functions $K(X)$ of X. We refer to [30, Chapter VI and Appendix 5] and [9, Chapter 8] for the general theory of valuations.

2.1. Basics on Valuations.

Definition 2.1. A valuation on $K(X)$ is a map $v : K(X)^* \to G$ where G is an ordered abelian group satisfying the following properties:

(1) $v(fg) = v(f) + v(g)$, $\forall f, g \in K(X)^*$,
(2) $v(f + g) \geq \min(v(f), v(g))$, $\forall f, g \in K(X)^*$,
(3) v is surjective,
(4) $v(a) = 0$, $\forall a \in K^*$.

G is called the value group of the valuation. Two valuations v, v' with value groups G, G' respectively are said to be equivalent if there is an isomorphism $\iota : G \to G'$ of ordered groups such that $v' = \iota \circ v$.

The subring

$$R_v = \{f \in K(X) \mid v(f) > 0\}$$

is a valuation ring, i.e., for all $f \in K(X)$, if $f \notin R_v$ then $f^{-1} \in R_v$; its unique maximal ideal is $m_v = \{f \in K(X) \mid v(f) > 0\}$ and the field $K_v = R_v/m_v$ is called the residue field of v. Two valuations v, v' are equivalent if and only if $R_v = R_{v'}$ [30, VI, §8].

Definition 2.2. The rank of a valuation v is the minimal non-negative integer r such that the value group is isomorphic to an ordered subgroup of $\mathbb{R}^r_{lex}$ (i.e. $\mathbb{R}^r$ with the lexicographic order). One can then write

$$v(f) = (v_1(f), v_2(f), \dots, v_r(f))$$

with $v_i : K(X)^* \to \mathbb{R}$ for every integer i with $1 \leq i \leq r$.

For every integer i with $1 \leq i \leq r$, the i-th truncation of v is the rank i valuation

$$v|_i(f) = (v_1(f), v_2(f), \dots, v_i(f)).$$

The trivial valuation, defined as $v(f) = 0$ for all $f = 0$, has rank zero; it can be considered as the 0-th truncation of all valuations v.

Remark 2.3. The rank of every valuation on $K(X)$ is bounded by $r = \dim(X)$, and every valuation of maximal rank is discrete, i.e., it has a value group isomorphic to $\mathbb{Z}^r_{lex} \subset \mathbb{R}^r_{lex}$ [30, VI, §10 and §14]. Whenever v is a valuation of maximal rank, one may assume that the value group of v equals $\mathbb{Z}^r_{lex}$ up to equivalence under the action of some order-preserving (i.e., lower-triangular) element of $GL(r, \mathbb{R})$.

Remark 2.4. The rational rank of the valuation v is the dimension of the $\mathbb{Q}$-vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$, where G is the value group of v; it is well known that the rational rank is bounded below by the rank of v, and above by the dimension of X (see [30, VI, §10, p. 50]). A valuation v can be of rank 1 and rational rank $r > 1$. The standard example is in [30, VI, §14, Example 1, p. 100] (see also Remark 3.3 below).

By [30, VI, §10, Theorem 15], the rank of a valuation v equals the Krull dimension of its valuation ring R_v. More precisely, the ideals in R_v are totally ordered by inclusion, and if the rank is r, then the prime ideals of R_v are

$$0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_r = m_v, \quad \text{where} \quad \mathfrak{p}_i = \{f \in R_v \mid v|_i(f) > 0\}.$$

The valuation rings of the truncations satisfy reverse inclusions

$$K(X) = R_{v|_0} \supsetneq R_{v|_1} \supsetneq \dots \supsetneq R_{v|_r} = R_v$$

as they are the localizations $R_{v|_i} = (R_v)_{\mathfrak{p}_i}$.

By the valuative criterion of properness [15, II, 4.7], since X is projective, there is a (unique) morphism

$$\sigma_{X,v} : \operatorname{Spec}(R_v) \to X$$

which, composed with $\operatorname{Spec}(K(X)) \to \operatorname{Spec}(R_v)$, identifies $\operatorname{Spec}(K(X))$ as the generic point of X. The image in X of the closed point of $\operatorname{Spec}(R_v)$ (or the irreducible subvariety which is its closure) is called the centre of v in X, and we denote it by $\operatorname{centre}_X(v)$. When the variety X is understood, we shall write $\operatorname{centre}(v) = \operatorname{centre}_X(v)$. A valuation v of rank r determines a flag

(1) $$X = \operatorname{centre}(v|_0) \supsetneq \operatorname{centre}(v|_1) \supseteq \dots \supseteq \operatorname{centre}(v|_r) = \operatorname{centre}(v),$$

and $\operatorname{centre}(v|_i) = \overline{\sigma_{X,v}(\mathfrak{p}_i)}$. Note that some of the inclusions may be equalities.

For a valuation of rank $r > 1$, the centre of the first truncation $v|_1$ is called the home of v, following [6].

Example 2.5. (Divisorial valuations) If centre(v) is a divisor V, then v is equivalent to the valuation that assigns to each rational function its order of vanishing along V. Moreover, the residue field K_v is the function field of V (see [30, VI, §14]).

Remark 2.6. Let L be a line bundle on X. On an affine neighborhood U of the centre of v (considered as a schematic point; equivalently, an affine neighborhood of the generic point of centre(v)) L is trivial, and so any section $s \in H^0(X, L) - \{0\}$ restricted to U can be seen as a non-zero element $f \in K(X)$, and one can set $v(s) = v(f)$. A different choice of U would give an element in $K(X)$ differing by a factor of value 0, so $v(s)$ is well defined. By setting $v(D) = v(s)$ whenever $D = (s)$, $s \in H^0(X, \mathcal{O}_X(D))$, valuations can be considered to assume values on divisors; effective divisors take nonnegative values.

Example 2.7. (Valuation associated to an admissible flag) A full flag $Y_\bullet$ of irreducible subvarieties

$$X = Y_0 \supset Y_1 \supset \ldots \supset Y_{r-1} \supset Y_r \tag{2}$$

is called admissible, if $\operatorname{codim}_X(Y_i) = i$ for all $0 \leqslant i \leqslant \dim(X) = r$, and Y_i is normal and smooth at the point Y_r, for all $0 \leqslant i \leqslant r - 1$. The flag is called good if Y_i is smooth for all $i = 0, \ldots, r$.

Let $\varphi \in K(X)$ be a non-zero rational function, and set

$$v_1(\varphi) \stackrel{\text{def}}{=} \operatorname{ord}_{Y_1}(\varphi) \quad \text{and} \quad \varphi_1 \stackrel{\text{def}}{=} \frac{\varphi}{g_1^{v_1(\varphi)}}\Big|_{Y_1}$$

where $g_1 = 0$ is a local equation of Y_1 in Y_0 in an open Zariski subset around the point Y_r. Continuing this way via

$$v_i(\varphi) \stackrel{\text{def}}{=} \operatorname{ord}_{Y_i}(\varphi_{i-1}) \ , \ \varphi_i \stackrel{\text{def}}{=} \frac{\varphi_{i-1}}{g_i^{v_i(\varphi_{i-1})}}\Big|_{Y_i} \qquad \text{for all} \quad i = 2, \ldots, r,$$

where $g_i = 0$ is a local equation of Y_i on Y_{i-1} around Y_r, we arrive at a function

$$\varphi \mapsto v_{Y_\bullet}(\varphi) \stackrel{\text{def}}{=} (v_1(\varphi), \ldots, v_r(\varphi)) .$$

One verifies that $v_{Y_\bullet}$ is a valuation of maximal rank, and that the flag (1) given by the centres of the truncations of $v_{Y_\bullet}$ coincides with the flag $Y_\bullet$ in (2).

Proposition 2.8. Let v be a valuation of maximal rank $r = \dim(X)$ whose flag of centres $Y_\bullet$ in (1) is admissible. Then v is equivalent to the flag valuation $v_{Y_\bullet}$.

Proof. By induction on r. For $r = 0$ there is nothing to prove, so assume $r > 1$. Remark 2.5 tells us that $v|_1 = v_{Y_\bullet}|_1$ (up to equivalence), and that their common residue field is $K(Y_1)$, with $Y_1 = \operatorname{centre}_X(v|_1)$. The valuation v (resp. $v_{Y_\bullet}$) induces a valuation $\bar{v}$ (resp. $\bar{v}_{Y_\bullet}$) on $K(Y_1) = R_{v|_1}/m_{v|_1}$ as follows. For any

$$0 = \bar{f} \in R_{v|_1}/m_{v|_1},$$

there is an $f \in K(X)$ sitting in $R_{v|_1}$ whose class modulo $m_{v|_1}$ is $\bar{f}$. Then one sets $\bar{v}(\bar{f}) = v(f)$ (similarly for $\bar{v}_{Y_\bullet}$) and verifies that this is well defined. The value group of $\bar{v}$ is the subgroup of the value group of v determined by $v_1 = 0$ (the "maximal isolated subgroup" in the language of [30, VI, §10]) and so it has rank

$r-1$ (maximal for valuations of $K(Y_1)$); it is easy to see that its flag of centres is $Y_\bullet$, with $Y_i = Y_{i+1}$ for $i = 0, \dots, r-1$. But $\bar{v}_{Y_\bullet}$ has maximal rank $r-1$ and flag of centres is $Y_\bullet$ as well, so by induction $\bar{v}$ and $\bar{v}_{Y_\bullet}$ are equivalent.

Finally, the valuation ring of v (resp. of $v_{Y_\bullet}$) consists of those f in $K(X)$ with $v|_1(f) = v_{Y_\bullet}|_1(f) > 0$ and of those f in $K(X)$ with $v|_1(f) = v_{Y_\bullet}|_1(f) = 0$ and $v(f) > 0$ (resp. $v_{Y_\bullet}(f) > 0$). Now $v|_1(f) = v_{Y_\bullet}|_1(f) = 0$ means that $f \in R_{v|_1} \setminus m_{v|_1}$, so for f satisfying this equality, $\bar{f} \in R_{v|_1}/m_{v|_1}$ is well defined, and $v(f) > 0$ (resp. $v_{Y_\bullet}(f) > 0$) is equivalent to $\bar{v}(\bar{f}) > 0$ (resp. $\bar{v}_{Y_\bullet}(\bar{f}) > 0$). Since $\bar{v}$ and $\bar{v}_Y$ are equivalent, then the valuation rings of v and $v_{Y_\bullet}$ are the same, as claimed.

Valuations of maximal rank are very well known (see [30, VI, §14], [29, Examples 5 and 6]) and Theorem 2.9 below is presumably obvious for experts working in the area of resolution of singularities. We include a proof as we lack a precise reference for it. For the case of surfaces, see §4 below.

Theorem 2.9. Let X be a normal projective variety, and v a valuation of the field $K(X)$ of maximal rank $r = \dim(X)$. There exist a proper birational morphism $\pi : \tilde{X} \to X$ and a good flag

$$Y_\bullet : \tilde{X} = Y_0 \supset Y_1 \supset \dots \supset Y_r$$

such that v is equivalent to the valuation associated to $Y_\bullet$.

Proof. Denote by $\zeta \in X$ the generic point, and set $K = K(\zeta) = K(X)$. Let

$$0 = p_0 \subset p_1 \subset \dots \subset p_r$$

be the maximal chain of prime ideals in R_v, and choose $f_1, \dots, f_r \in R_v \subset K$ so that each $f_i \in p_i \setminus p_{i-1}$. Fix projective coordinates $[x_0 : \dots : x_r]$ in $P^r_K \subset P^r_K \times X$, and let $\xi = [1 : f_1 : \dots : f_r] \in P^r_K$. Let X_0 be the Zariski closure of ξ in $P^r_K \times X$. Since its generic point is ξ (which is a closed K-point in P^r_K), it has residue field equal to K, and the induced projective morphism $X_0 \to X$ is birational.

For $i = 1, \dots, r$, the restriction of the rational function x_i/x_0 to X_0 is f_i, which has positive v-value. Therefore the centre of v in X_0 lies in $[1 : 0 : \dots : 0] \times X$, and its local ring $\mathcal{O}_{\mathrm{centre}_{X_0}(v)}$ contains $f_1, \dots, f_r$. Hence

$$f_i \in p_i \cap \mathcal{O}_{\mathrm{centre}_{X_0}(v)} \setminus p_{i-1} \cap \mathcal{O}_{\mathrm{centre}_{X_0}(v)} ,$$

so that $p_i \cap \mathcal{O}_{\mathrm{centre}_{X_0}(v)} \neq p_{i-1} \cap \mathcal{O}_{\mathrm{centre}_{X_0}(v)}$. Since $p_i \cap \mathcal{O}_{\mathrm{centre}_{X_0}(v)} = \sigma_{X_0,v}(p_i)$ as schematic points in X_0, it follows that the centres of the truncations of v are all distinct. Since there are as many truncations as the dimension of X, the flag (1) in X_1 is a full flag, i.e., $\dim(\mathrm{centre}_{X_0}(v|_i)) = r - i$ for $i = 0, \dots, r$. Every birational model of X dominating X_0 will again have this property.

The flag of centres of the truncations in X_0 is usually not good (or even admissible), as X_0 is not necessarily smooth (not even normal) at $\mathrm{centre}_{X_0}(v)$. Using Hironaka's resolution of singularities we know that there is a birational morphism $X_1 \to X_0$, obtained as a composition of blowups along smooth centres, with X_1 a smooth projective variety. On X_1 we have a full flag like (1) whose codimension 1 term, $\mathrm{centre}_{X_1}(v|_1)$, may be singular. But again there is a composition of blowups along smooth centres (contained in $\mathrm{centre}_{X_1}(v|_1)$) that desingularizes it; we apply these blowups to X_1, to obtain $X_2 \to X_1$. Since the blowup of a smooth variety along a smooth centre is again smooth, X_2 stays smooth, and the divisorial part of the full flag (1) in X_2 is now also smooth. By resolving sequencially the singularities of $\text{centre}_{X_2}(v|_2)$, . . . , $\text{centre}_{X_{r-1}}(v|_{r-1})$ we arrive at a model $\tilde{X} = X_r$ where the flag

$$Y_\bullet : \tilde{X} = \text{centre}_{\tilde{X}}(v|_0) \supsetneq \text{centre}_{\tilde{X}}(v|_1) \supsetneq \ldots \supsetneq \text{centre}_{\tilde{X}}(v|_r) = \text{centre}_{\tilde{X}}(v),$$

is good. Now by Proposition 2.8, the valuation v is equivalent to $v_{Y_\bullet}$ as claimed.

Remark 2.10. We work here in characteristic 0, but a suitable (weaker) version of Theorem 2.9 still holds in any characteristic. The same proof works, by replacing Hironaka's resolution with a sequence of blowups along nonsingular centres given by Urabe's resolution of maximal rank valuations [28]. The members of the resulting flag are not necessarily smooth, but they are non-singular at the centre.

In the situation of Theorem 2.9, we call $Y_\bullet$ the good flag associated to v in the model $\tilde{X}$. The choice of a flag is not unique, but for two models, the induced rational map between them maps the associated flags into one another.

2.2. Newton–Okounkov bodies.

Definition 2.11. Let X be an irreducible normal projective variety, D a big divisor on X, and v a valuation of $K(X)$ of maximal rank $r = \dim(X)$. Define the Newton–Okounkov body of D with respect to v as follows

$$\Delta_v(D) \overset{\text{def}}{=} \overline{\text{convex hull}\left(\bigcup_{k\in\mathbb{Z}_{>0}} \left\{ \frac{v(f)}{k} \;\middle|\; f \in H^0(X, \mathcal{O}_X(kD)) - \{0\} \right\} \right)}. \tag{3}$$

The points in $\Delta_v(D) \cap \mathbb{Q}^r$ of the form $\frac{v(f)}{k}$ with $f \in H^0(X, \mathcal{O}_X(kD)) - \{0\}$ for some integer $k > 0$ are called valuative points.

Remark 2.12. The properties of valuations yield that if A, B are two distinct valuative points, then any rational point on the segment joining A and B is again a valuative point. This implies that valuative points are dense in $\Delta_v(D)$ (see [19, Corollary 2.10], for the surface case; the proof is analogous in general). Therefore in (3) it suffices to take the closure in the Euclidean topology of $\mathbb{R}^r$.

Alternatively, one defines the Newton–Okounkov body of D with respect to v as

$$\Delta_v(D) \overset{\text{def}}{=} \overline{v\left(\{D' \mid D' \equiv D \text{ effective } \mathbb{Q}\text{-divisor}\}\right)},$$

where $\equiv$ is the $\mathbb{Q}$-linear equivalence relation. By [24, Proposition 4.1], one can replace $\mathbb{Q}$-linear equivalence by numerical equivalence. Hence, one can define $\Delta_v(\zeta)$ for any numerical class ζ in the big cone $\text{Big}(X) \subset N^1(X)_\mathbb{R}$ of X.

Our definition differs from the one in [24] in that we use valuations of maximal rank instead of those defined by admissible flags on X. But, an admissible flag on X gives rise to a valuation of maximal rank on $K(X)$ by Example 2.7 (see also [18]). Conversely, by Theorem 2.9, any valuation of maximal rank arises from an admissible flag on a suitable proper birational model of X; thus maximal rank valuations are the birational version of admissible flags. In conclusion, all known results for Newton–Okounkov bodies defined in terms of flag valuations carry over to Newton–Okounkov bodies in terms of valuations of maximal rank, modulo passing to some different birational model.

In [5, 18] one considers Newton–Okounkov bodies defined by valuations of maximal rational rank, an even more general situation which we will not consider here.

2.3. **Some properties of Newton–Okounkov bodies.** A very important feature of Newton–Okounkov bodies is that they give rise to a 'categorification' of various asymptotic invariants associated to line bundles (see for instance [19, Theorem C] for the corresponding statement for moving Seshadri constants). Recall that the volume of a Cartier divisor D on an irreducible normal projective variety X of dimension r is defined as

$$\mathrm{vol}(D) \overset{\mathrm{def}}{=} \limsup_{m\to\infty} \frac{\dim H^0(X, \mathcal{O}_X(mD))}{m^r/r!} .$$

Theorem 2.13 (Lazarsfeld–Mustaţa, [24, Theorem 2.3]). *Let X be an irreducible normal projective variety of dimension r, let D be a big divisor on X, and let v be a valuation of the field $K(X)$ with value group $\mathbb{Z}^r_{\mathrm{lex}}$. Then*

$$\mathrm{vol}(\Delta_v(D)) = \frac{1}{r!}\mathrm{vol}(D),$$

where the volume on the left-hand side denotes the Lebesgue measure in $\mathbb{R}^r$.

Remark 2.14. Although the proof of Theorem 2.3 from [24] takes the admissible flags viewpoint, the statement remains valid for Newton–Okounkov bodies defined in terms of valuations of maximal rational rank (with value group equal to $\mathbb{Z}^r$) by the remark above (see also [5, Corollaire 3.9]).

Since the main focus of our work is on the surface case, we will concentrate on surface-specific properties of Newton–Okounkov bodies.

Theorem 2.15 (Küronya–Lozovanu–MacLean, [22]). *If $\dim(X) = 2$, then every Newton–Okounkov body is a polygon.*

If $\dim X = 2$, then an admissible flag is given by a pair (C, x), where C is a curve, and $x \in C$ a smooth point. If D is a big divisor on X, the corresponding Newton–Okounkov body will be denoted by $\Delta_{(C,x)}(D)$.

Remark 2.16. In fact one can say somewhat more about the convex geometry of Newton–Okounkov polygons, see [22, Proposition 2.2]. First, all the slopes of its edges are rational. Second, if one defines

$$\mu_C(D) \overset{\mathrm{def}}{=} \sup\{t > 0 \mid D - tC \text{ is big}\} ,$$

then all the vertices of $\Delta_{(C,x)}(D)$ are rational with possibly two exceptions, i.e. the points of this convex set lying on the line $\{\mu_C(D)\} \times \mathbb{R}$.

Lazarsfeld and Mustaţă observe in [24, Theorem 6.4] that variation of Zariski decomposition [3, Theorem 1] provides a recipe for computing Newton–Okounkov bodies in the surface case. Let $D = P + N$ be the Zariski decomposition of D (for the definition and basic properties see [2, 14]), where the notation, here and later, is the standard one: P is the nef part $\mathrm{Nef}(D)$ and N the negative part $\mathrm{Neg}(D)$ of the decomposition. Denote by $\nu = \nu(D, C)$ the coefficient of C in N and $\mu = \mu_C(D)$ whenever there is no danger of confusion. Let also $\mathrm{Null}(D)$ be the divisor (containing $\mathrm{Neg}(D)$) given by the union of all irreducible curves E on X such that $\mathrm{Nef}(D) \cdot E = 0$. Note that by Nakamaye's theorem [23, 10.3], $\mathrm{Null}(D)$ coincides with the augmented base locus of D, $\mathbf{B}_+(D) = \mathbf{B}(D - A)$ where A is any ample divisor and is a sufficiently small positive real number.

For any $t \in [\nu, \mu]$, set $D_t = D - tC$ and let $D_t = P_t + N_t$ be the Zariski decomposition of D_t. Consider the functions $\alpha, \beta : [\nu, \mu] \to \mathbb{R}^+$ defined as follows

$$\alpha(t) \stackrel{\text{def}}{=} \operatorname{ord}_x(N_{t|C}), \quad \beta(t) \stackrel{\text{def}}{=} \alpha(t) + P_t \cdot C \ .$$

Theorem 2.17 (Lazarsfeld–Mustaţă, [24, Theorem 6.4]). *If C is not a component of $\operatorname{Null}(D)$, then*

$$\Delta_{(C,x)}(D) = \left\{ (t, u) \in \mathbb{R}^2 \,|\, \nu \leqslant t \leqslant \mu,\ \alpha(t) \leqslant u \leqslant \beta(t) \right\} .$$

Remark 2.18. Note that all the results concerning Newton–Okounkov bodies use Zariski decomposition in Fujita's sense, i.e. for pseudo-effective $\mathbb{R}$-divisors.

As an immediate consequence we have:

Corollary 2.19. *In the above setting the lengths of the vertical slices of $\Delta_{(C,x)}(D)$ are independent of the (smooth) point $x \in C$.*

Remark 2.20. (See [22, proof of Proposition 2.2]) In the above setting, the function $t \to N_t$ is nondecreasing on $[\nu, \mu]$, i.e. $N_{t_2} - N_{t_1}$ is effective whenever $\nu \leqslant t_1 \leqslant t_2 \leqslant \mu$. This implies that a vertex (t, u) of $\Delta_{(C,x)}(D)$ may only occur for those $t \in [\nu, \mu]$ where the ray $D - tC$ crosses into a different Zariski chamber, in particular, where a new curve appears in N_t.

Given three real numbers $a > 0, b > 0, c > 0$, we will denote by $\Delta_{a,b,c}$ the triangle with vertices $(0, 0)$, $(a, 0)$ and (b, c). We set $\Delta_{a,c} := \Delta_{a,0,c}$ and $\Delta_{a,a} := \Delta_a$. Note that the triangle $\Delta_{a,b,c}$ degenerates into a segment if $c = 0$.

Example 2.21. In the above setting suppose that D is an ample divisor. Then, by Theorem 2.17, the Newton–Okounkov body $\Delta_{(C,x)}(D)$ contains the triangle $\Delta_{\mu_C(D), D\cdot C}$, and by Theorem 2.13 one has

$$\mu_C(D) \leqslant \frac{D^2}{D \cdot C} \ .$$

Equality holds if and only if $\Delta_{(C,x)}(D) = \Delta_{\mu_C(D), D\cdot C}$. In particular, if $X = \mathbb{P}^2$, C is a curve of degree d, and D a line, then $\Delta_{(C,x)}(D) = \Delta_{\frac{1}{d}, d}$.

Theorem 2.22 (Küronya–Lozovanu, [19, Theorem 2.4, Remark 2.5]). *Let X be a smooth projective surface, D be a big divisor on X and $x \in X$ a point. Then:*

(i) *$x \notin \operatorname{Neg}(D)$ if and only if for any admissible flag (C, x) one has $(0, 0) \in \Delta_{(C,x)}(D)$;*
(ii) *$x \notin \operatorname{Null}(D)$ if and only if for any admissible flag (C, x) there is a positive number λ such that $\Delta_\lambda \subseteq \Delta_{(C,x)}(D)$.*

Remark 2.23. The divisor D is nef (resp. ample) if and only if $\operatorname{Neg}(D) = \emptyset$ (resp. $\operatorname{Null}(D) = \emptyset$), so that Theorem 2.22 provides nefness and ampleness criteria for D detected from Newton–Okounkov bodies.

Note that Theorem 2.22 has a version in higher dimension (see [20]). The same papers [19, 20] explain how to read the moving Seshardi constant of D at a point $x \notin \operatorname{Neg}(D)$ from Newton–Okounkov bodies.

3. Valuations in dimension 2

3.1. Quasimonomial valuations. We will mainly treat the case $X = \mathbb{P}^2$ and D a line, leaving to the reader to make the obvious adaptations for other surfaces.

Let O denote the origin $(0,0) \in \mathbb{A}^2 = \operatorname{Spec}(K[x,y]) \subset \mathbb{P}^2 = \operatorname{Proj}(K[X,Y,Z])$ with $x = X/Z$, $y = Y/Z$, and let $K = K(\mathbb{P}^2) = K(x,y)$ be the field of rational functions in two variables. We will focus on Newton–Okounkov bodies of D with respect to rank 2 valuations $v = (v_1, v_2)$ with centre at O, with the additional condition that either the home of v is a smooth curve through O, or it is equal to O (in which case we call the corresponding body an infinitesimal Newton–Okounkov body) and v_1 is a quasimonomial valuation.

Fix a smooth germ of curve C through O; we can assume without loss of generality that C is tangent to the line $y = 0$; hence C can be locally parameterized by $x \mapsto (x, \xi(x)) \in \mathbb{A}^2$, where $\xi(x) \in K[[x]]$ with $\xi(0) = \xi'(0) = 0$.

Definition 3.1. Given a real number $s > 1$ and any $f \in K^*$, set

$$v_1(C, s; f) := \operatorname{ord}_x(f(x, \xi(x) + \theta x^s)) , \tag{4}$$

where θ is transcendental over $\mathbb{C}$. Equivalently, expand f as a Laurent series

$$f(x,y) = \sum a_{ij} x^i (y - \xi(x))^j . \tag{5}$$

One has

$$v_1(C, s; f) = \min\{i + sj \,|\, a_{ij} = 0\} . \tag{6}$$

Then $f \mapsto v_1(C, s; f)$ is a rank 1 valuation which we denote by $v_1(C, s)$. Such valuations are called *monomial* if C is the line $y = 0$ (i.e., $\xi = 0$), and *quasimonomial* in general. The point O is the *centre* of the valuation.

We call (5) the C-expansion of f. Slightly abusing language, s will be called the *characteristic exponent* of $v_1(\xi, s)$ (even if it is an integer).

Example 3.2. The valuation $v_O := v_1(C, 1)$ is the O-adic valuation or multiplicity valuation : if f is a non–zero polynomial, then $v_O(f)$ is the multiplicity $\operatorname{mult}_O(f)$ of $f = 0$ at O.

Remark 3.3. The value group of $v_1(C, s)$ is:

- $\mathbb{Z}\frac{1}{q} \subset \mathbb{Q}$ if s is a rational number $s = \frac{p}{q}$ with $\gcd(p,q) = 1$;
- $\mathbb{Z} + \mathbb{Z}s \subset \mathbb{R}$ if s is an irrational number.

So the rank of $v_1(C, s)$ is 1, but in the latter case the valuation has rational rank 2. We will be mostly concerned with the rational case. Note that $v_1(C, s)$ is discrete if and only if s is rational.

Remark 3.4. The valuation $v_1(C, s)$ depends only on the $\lfloor s \rfloor$-th jet of C, so for fixed s the series ξ can be assumed to be a polynomial; however, later on we shall let s vary for a fixed C, so we better keep $\xi(x)$ a series.

Example 3.5. If $f = 0$ is the equation of C (supposed to be algebraic, which, for fixed s is no restriction by Remark 3.4), then by plugging $y = \xi(x)$ in (5) we have $\sum_{i=1}^{\infty} a_{i0} x^i \equiv 0$, hence $a_{i0} = 0$, for all $i > 0$. Then $f(x,y) = (y - \xi(x)) \cdot g(x,y)$ where $g(0,0) = 0$. This implies that $v_1(C, s; f) = s$, which can be also deduced from (4) by expanding $f(x, \xi(x) + \theta x^s)$ in Taylor series with initial point $(x, \xi(x))$.

Remark 3.6. (See [12]) The set $\mathcal{QM}$ of all quasi-monomial valuations with centre at O has a natural topology, namely the coarsest topology such that for all $f \in K^*$, $v \mapsto v(f)$ is a continuous map $\mathcal{QM} \to \mathbb{R}$. This is called the weak topology. For a fixed C, the map $s \mapsto v_1(C, s)$ is continuous in $[1, +\infty)$.

There is however a finer topology of interest on the valuative tree $\mathcal{QM}$: the finest topology such that $s \mapsto v_1(C, s)$ is continuous in $[1, +\infty)$ for all C. This latter is called the strong topology. With the strong topology, $\mathcal{QM}$ is a profinite $\mathbb{R}$-tree, rooted at the O-adic valuation (see [12] for details). To avoid confusion with branches of curves, we will call the branches in $\mathcal{QM}$ arcs. Maximal arcs of the valuative tree are homeomorphic to the interval $[1, \infty)$, parameterized by $s \mapsto v_1(C, s)$ where C is a smooth branch of curve at O.

The arcs of $\mathcal{QM}$ share the segments given by coincident jets, and separate at integer values of s; these correspond to divisorial valuations on an appropriate birational model.

Though we will not use this fact, note that $\mathcal{QM}$ is a sub-tree of a larger $\mathbb{R}$-tree $\mathcal{V}$ with the same root, called the valuation tree, which consists of all real valuations of K with centre O. Ramification on $\mathcal{V}$ occurs at all rational points of the arcs, rather than only at integer points, because of valuations corresponding to singular branches. The tree $\mathcal{QM}$ is obtained from $\mathcal{V}$ by removing the arcs corresponding to singular branches and all ends (see [12, Chapter 4] for details).

3.2. Quasimonomial valuations and the Newton–Puiseux algorithm.

We recall briefly the Newton–Puiseux algorithm (see [9, Chapter 1] for a full discussion).

Given $f(x, y) \in K(x, y) - \{0\}$ (we may in fact assume that f belongs to $K[[x, y]]$), and a curve C as in §3.1, we want to investigate the behavior of the function

$$v_1(C; f) : s \in [1, +\infty) \mapsto v_1(C, s; f) \in \mathbb{R}$$
.

Returning to (5), consider the convex hull $\overline{NP}(C, f)$ in $\mathbb{R}^2$ (with (t, u) coordinates) of all points $(i, j) + v \in \mathbb{R}^2$ such that $a_{ij} = 0$, and $v \in_+ \mathbb{R}^2$. The boundary of $NP(C, f)$ consists of two half-lines parallel to the t and u axes, respectively, along with a polygon $NP(C, f)$, named the Newton polygon of f with respect to C
.

We will denote by $V(C, f)$ (resp. by $E(C, f)$) the set of vertices (resp. of edges) of $NP(C, f)$, ordered from left to right, i.e.,

$$V(C, f) = (v_0, \ldots, v_h), \quad E(C, f) = (l_1, \ldots, l_h),$$

where l_k is the segment joining v_{k-1} and v_k, for $k = 1, \ldots, h$, and $v_k = (i_k, j_k)$.

We will denote by $\mathcal{V}$ the germ of the curve $f = 0$. Then,

$$\mathrm{mult}_O(\mathcal{V}) = \min_k \{i_k + j_k\}, \ \mathrm{ord}_C(\mathcal{V}) = j_h, \ (\mathcal{V} - j_h C, C)_O = i_h,$$

where $(\mathcal{V} - j_h C, C)_O$ is the local intersection number of two effective cycles with distinct support $\mathcal{V} - j_h C$ and C at the origin O.

The numbers $n := w(C, f) := i_h - i_0$ and $m := h(C, f) := j_0 - j_h$ are usually called the width and the height of $NP(C, f)$. Analogously, one defines the width $n_k := w(l_k)$ and the height $m_k := h(l_k)$ for any edge l_k in the obvious way, so that l_k has slope $s_k := sl(l_k) = -\frac{m_k}{n_k}$, for $k = 1, \ldots, h$.

Let $\mathcal{B}(\mathcal{V})$ be the set of branches of $\mathcal{V}$. Then the Newton–Puiseux algorithm as presented in [9, §1.3] (with suitable modifications due to the fact that (5) is not the standard expansion of $f(x, y)$ as a power series in x and y) yields a surjective map

$$\phi_{\mathcal{V}} : \mathcal{B}(\mathcal{V}) \to E(C, f)$$

such that whenever γ is a branch of V whose Puiseux expansion with respect to C starts as

$$y - \xi(x) = ax^{\tau} + \dots, \text{ with } \tau \in \mathbb{Q} \text{ and } \tau > 1$$

(i.e., γ is not tangent to the $x = 0$ axis nor contained in C) then the edge $l = \phi_V(\gamma)$ has slope $sl(l) = -\frac{1}{\tau} > -1$. Moreover

$$\tau = \frac{(\gamma, C)_O}{\mathrm{mult}_O(\gamma)},$$

and if $\gamma \in B(V)$ is the unique branch with $sl(\phi_V(\gamma)) = -\frac{1}{\tau}$, then in fact $h(\phi_V(\gamma)) = \mathrm{mult}_O(\gamma)$ is the multiplicity of γ (at O), whereas $w(\phi_V(\gamma)) = (\gamma, C)_O$ is the local intersection multiplicity of γ with C at O.

Consider now the line ℓ_s with equation $t + su = 0$ and slope $-\frac{1}{s}$. By (6), the valuation $v_1(C, s; f)$ is computed by those vertices in $V(C, f)$ with the smallest distance to ℓ_s, i.e. for any such vertex $v = (i, j)$, one has $v_1(C, s; f) = i + sj$. Note that there will be only one such point, unless ℓ_s is parallel to one of the edges $l \in E(C, f)$ (hence s is rational), in which case there will be two: the vertices of l, whose slope $sl(l) = -\frac{1}{s}$.

From the above discussion its not hard to deduce the following statement:

Proposition 3.7. For any curve C smooth at O and $f \in K(x, y) - \{0\}$ regular at O (i.e., f is defined at O) one has:
(i) $v_1(C, \cdot; f) : \mathbb{R} \to \mathbb{R}$ is continuous in $[1, +\infty)$, piecewise linear, non-decreasing, concave and its graph consists of finitely many (one more than the number of edges in $E(C, f)$ with slope greater than -1) linear arcs with rational slopes (i.e., $v_1(C, s; f)$ is a tropical polynomial in s);
(ii) the points where the derivative of $v_1(C, \cdot; f)$ is not defined are

$$s_k = -\frac{1}{sl(l_k)}, \quad \text{for} \quad k = 1, \dots, h;$$

(iii) if the curve V with equation $f = 0$ does not contain C, then

(7) $$v_1(C, s; f) = (V, C)_O \quad \text{for} \quad s \quad 1.$$

Example 3.8. Let C be the conic $x^2 - 2y = 0$, so that $\xi(x) = x^2/2$, and let

$$f = (x^2 + y^2)^3 - 4x^2y^2.$$

The C-expansion of f is then

$$\begin{aligned} f = {} & (y - \xi(x))^6 + 3x^2(y - \xi(x))^5 + \frac{15x^4(y-\xi(x))^4}{4} + 3x^2(y - \xi(x))^4 \\ & + \frac{5x^6(y-\xi(x))^3}{2} + 6x^4(y - \xi(x))^3 + \frac{15x^8(y-\xi(x))^2}{16} + \frac{9x^6(y-\xi(x))^2}{2} \\ & + 3x^4(y - \xi(x))^2 - 4x^2(y - \xi(x))^2 + \frac{3x^{10}(y-\xi(x))}{16} + \frac{3x^8(y-\xi(x))}{2} \\ & + 3x^6(y - \xi(x)) - 4x^4(y - \xi(x)) + \frac{x^{12}}{64} + \frac{3x^{10}}{16} + \frac{3x^8}{4} \end{aligned}$$

The Newton polygon of f with respect to C is depicted in Figure 1. It has three sides and four vertices, corresponding to the "monomials" $(y-\xi(x))^6$, $x^2(y-\xi(x))^2$, $x^4(y - \xi(x))$ and x^8. The curve $V : f(x, y) = 0$ has four branches through O, all smooth; two of them are transverse to C and map to the first side of the Newton

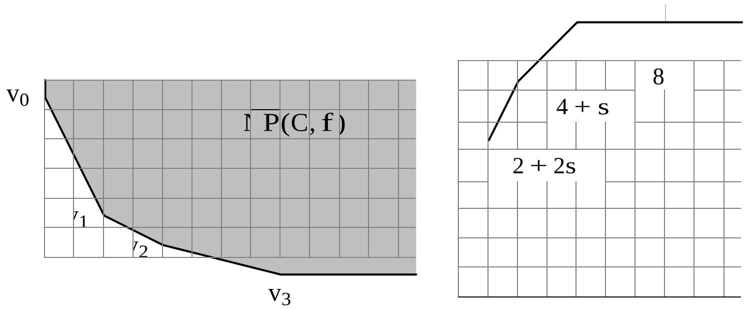

Figure 1. The Newton polygon of Example 3.8. Each dot represents a "monomial" in the C-expansion of f; only four of them create vertices of the polygon. At the right-hand side, the corresponding function $v_1(C, s; f)$ for $s > 1$. The three linear pieces correspond to the vertices $v_1 = (2, 2)$, $v_2 = (4, 1)$, $v_3 = (8, 0)$, as described in Proposition 3.7.

polygon; one of them is tangent to C with intersection multiplicity 2, and maps to the second side; the last one is tangent to C and has intersection multiplicity 4 with it, and maps to the third side.

3.3. Quasimonomial valuations and the associated rank 2 valuations. We keep the above notation. As we saw in §3.2, we have a finite sequence

$$s_0 := 1 < s_1 < \ldots < s_h < s_{h+1} := +\infty$$

such that $v_1(C, s; f)$ is linear (hence differentiable) in each of the intervals (s_k, s_{k+1}), for $k = 0, \ldots, h$. The derivative in these intervals is constant and integral. At s_k, with $k = 0, \ldots, h+1$, there are the right and left derivatives of $v_1(C, s; f)$ (at $s_0 = 1$ (resp. at $s_{h+1} = +\infty$) there is only the right (resp. left) derivative). So we have:

Corollary 3.9. For any curve C smooth at O and $f \in K(x, y) - \{0\}$ regular at O, the function $v_1(C, \cdot; f)$ has everywhere in $(1, +\infty)$ (resp. in $[1, +\infty)$) left (resp. right) derivative. We will denote them by $\partial_- v_1(C; f)$ (resp. $\partial_+ v_1(C; f)$).

Proposition 3.10. For any curve C smooth at O, every $s \in \mathbb{Q}$, $s > 1$ and every $f \in K(x, y) - \{0\}$ set

$$v_-(C, s; f) := (v_1(C, s; f), -\partial_- v_1(C; f)(s))$$
$$v_+(C, s; f) := (v_1(C, s; f), \partial_+ v_1(C; f)(s)).$$

This defines two rank 2 valuations $v_-(C, s)$ and $v_+(C, s)$ with home at O. For $s = 1$, the valuation $v_+(C, s)$ defined as above is also a rank 2 valuation with home at O.

Proof. Let $f \in K[x, y]$ and let $(x, \xi(x))$ be a local parametrization of C. With notation as in (5), then (6) holds, thus

$$\partial_- v_1(C; f)(s) = \max\{j \mid \exists i : a_{ij} = 0 , i + sj = v_1(C, s; f)\}$$

and

$$\partial_+ v_1(C; f)(s) = \min\{j \mid \exists i : a_{ij} = 0 , i + sj = v_1(C, s; f)\}. \tag{8}$$

The fact that both $v_-(C, s), v_+(C, s) : K^* \longrightarrow \mathbb{Q}^2_{lex}$ are valuations follows from basic properties of multiplication of Laurent series and min and is left to the reader. Furthermore, if f is regular at O then $v_1(C, s; f) > 0$ if and only if $f(O) = 0$. This implies that O is the home of $v_-(C, s)$ and $v_+(C, s)$.

Obviously $v_-(C,s)$ and $v_+(C,s)$ have rank at most 2. We will show that they have rank greater than 1. Let $f_0 \in K[x,y]$ be such that $f_0 = 0$ is an equation of C (this, for fixed s, is no restriction by Remark 3.4). We have $v_\pm(C,s;f_0) = (s, \pm 1)$ (see Example 3.5). Moreover if $s = \frac{p}{q}$ for coprime positive integers p, q and $f_1 = \frac{f_0^q}{x^p}$ then $v_\pm(C,s;f_1) = (0, \pm q)$. Thus for every positive integer k we have

$$(0,0) < \pm k v_\pm(C,s;f_1) < v_\pm(C,s;x),$$

which is impossible for a rank 1 valuation.

Remark 3.11. For irrational s, the expressions v_- and v_+ (as defined in Proposition 3.10), are valuations with home at O, but they are both equivalent to v_1 (and so have real rank 1 and rational rank 2). We will not need this fact, and we leave the proof to the interested reader.

Remark 3.12. Write $s = \frac{p}{q}$ with p, q coprime positive integers. Then the value group of $v_-(C,s)$ and $v_+(C,s)$ is $(\mathbb{Z}\frac{1}{q} \times \mathbb{Z})_{lex} \subset \mathbb{Q}^2_{lex}$. In this case, we will denote by

$$\Delta_{C,s_+} \subseteq \mathbb{R}^2_+, \qquad \Delta_{C,s_-} \subseteq \mathbb{R}_+ \times \mathbb{R}_-$$

the Newton–Okounkov bodies associated to the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ with respect to the valuation $v_-(C,s)$ and $v_+(C,s)$ respectively.

Since $v_\pm(C,s)$ have maximal rank but their value groups do not equal $\mathbb{Z}^2_{lex}$, the volumes of Newton–Okounkov bodies associated to these valuations need not satisfy Theorem 2.13. However, there are order preserving elements of $GL(2,\mathbb{Q})$ relating the $v_\pm$ valuations to valuations with values in $\mathbb{Z}^2_{lex}$. In §4.3 below we compute these lower triangular matrices, which turn out to have determinant 1, and so preserve the volume. Therefore Theorem 2.13 also applies to $v_\pm(C,s)$, and

$$\operatorname{vol}\Delta_{C,s_-} = \operatorname{vol}\Delta_{C,s_+} = \frac{\operatorname{vol}(\mathcal{O}_{\mathbb{P}^2}(1))}{2} = \frac{1}{2}.$$

3.4. **The $\hat\mu$ invariant.** Let v_1 be a rank 1 valuation centred at a smooth point x of a normal irreducible projective surface X, and let D be a big Cartier divisor on X. Following [11], we set

$$\mu_D(v_1) \stackrel{\text{def}}{=} \max\{v_1(f) \mid f \in H^0(X, \mathcal{O}_X(D)) - \{0\}\}, \text{ and } \hat\mu_D(v_1) \stackrel{\text{def}}{=} \lim_{k\to\infty} \frac{\mu_{kD}(v_1)}{k}.$$

If $v = (v_1, v_2)$ is a valuation of rank 2 centred at x, then $\Delta_v(D)$ lies in the strip

$$\{(t,u) \in \mathbb{R}^2 \mid 0 \leq t \leq \mu_D(v_1)\};$$

and its projection to the t-axis lies the interval $[0, \hat\mu_D(v_1)]$, coinciding with it if and only if $x \notin \operatorname{Neg}(D)$ (see Theorem 2.22).

In order to simplify notation, we will set

$$\mu_D(C,s) = \mu_D(v_1(C,s)).$$

If $X = \mathbb{P}^2$, $x = O$ and D is a line, we drop the subscript D for $\hat\mu_D(C,s)$ and we write $\mu_d(C,s)$ instead of $\mu_{dD}(C,s)$ for any non-negative integer d.

From [11] we know that the function $\hat\mu : QM \to \mathbb{R}$ is lower semicontinuous for the weak topology and continuous for the strong topology, i.e., $\hat\mu(C,s)$ is continuous for $s \in [1, +\infty)$ (see [11, Proposition 3.9]). Moreover $\hat\mu(C,s) \geq \sqrt{s}$ [11]. If $\hat\mu(C,s) = \sqrt{s}$, then $v_1(C,s)$ is said to be minimal (the concept of minimal valuation is more general, see [11], but we will not need it here). We recall from [11] the following:

Conjecture 3.13 ([11, Conjecture 5.11]). If C is sufficiently general (in a sense which is made precise in l.c.) and $s \geq 8 + \frac{1}{36}$, then $\hat\mu(C, s) = \sqrt{s}$.

Remark 3.14. According to [11, Proposition 5.4], this Conjecture (actually a weaker form of it, considering only $s \geq 9$ and C any curve), implies Nagata's Conjecture.

Remark 3.15. We recall from [11] some known values of $\hat\mu(C, s)$.

- If C is a line, then
$$\hat\mu(C, s) = \begin{cases} s & \text{if } 1 \leq s \leq 2 \\ 2 & \text{if } 2 \leq s \end{cases}$$
- If C is a conic, then
$$\hat\mu(C, s) = \begin{cases} s & \text{if } 1 \leq s \leq 2 \\ 2 & \text{if } 2 \leq s \leq 4 \\ \frac{s}{2} & \text{if } 4 \leq s \leq 5 \\ 5/2 & \text{if } 5 \leq s \end{cases}$$
- If $s \leq 7 + 1/9$ and $\deg(C) \geq 3$, then
$$\hat\mu(C, s) = \begin{cases} \frac{F_{i-2}}{F_i} s & \text{if } \frac{F_i^2}{F_{i-2}^2} \leq s \leq \frac{F_{i+2}}{F_{i-2}}, \quad i \geq 1 \text{ odd}, \\ \frac{F_{i+2}}{F_i} & \text{if } \frac{F_{i+2}}{F_{i-2}} \leq s \leq \frac{F_{i+2}^2}{F_i^2}, \quad i \geq 1 \text{ odd}, \\ \frac{1+s}{3} & \text{if } \varphi^4 \leq s \leq 7, \\ \frac{8}{3} & \text{if } 7 \leq s \leq 7 + \frac{1}{9}, \end{cases}$$
where $F_{-1} = 1$, $F_0 = 0$ and $F_{i+1} = F_i + F_{i-1}$ are the Fibonacci numbers, and
$$\varphi = \frac{1+\sqrt{5}}{2} = \lim_{i\to\infty} \frac{F_{i+1}}{F_i}$$
is the golden ratio.

  The values of $\hat\mu$ above are computed using the series of Orevkov rational cuspidal curves (see [26] and Proposition 5.25 below). There are a few more sporadic values of s in the range $[7 + 1/9, 9]$ where the value of $\hat\mu$ is known, see [11] for details.
- If s is an integer square and C is a general curve of degree at least $\sqrt{s}$, then one has $\hat\mu(C, s) = \sqrt{s}$.

4. Cluster of centres and associated flags

In this section the main goal is to introduce the geometric structures related to valuations $v_1(C, s)$ and $v_\pm(C, s)$. We give a full description of how to find the birational model of X (the cluster of centres together with their weights) on which these two valuations are equivalent to a flag valuation on this model.

4.1. Weighted cluster of centres.
As usual, we will refer to the case
$$x = O \in \mathbb{A}^2 \subset \mathbb{P}^2 := X_0 .$$

Each valuation v with centre $O \in \mathbb{P}^2$ determines a cluster of centres as follows. Let $P_1 = \text{centre}_{X_0}(v) = O$. Consider the blowup $\pi_1 : X_1 \to X_0$ of P_1 and let $E_1 \subset X_1$ be the corresponding exceptional divisor. Then $\text{centre}_{X_1}(v)$ may either be E_1 or a point $P_2 \in E_1$. Iteratively blowing up the centres $P_1, P_2, \ldots$ of v we may end up, after $k > 1$ steps, with a surface X_k dominating $\mathbb{P}^2$, where the centre of v is the exceptional divisor E_k. In this case v is discrete of rank 1, given by the order of vanishing along E_k, by Remark 2.5. Otherwise, this process goes on indefinitely. In particular, for quasimonomial valuations $v_1(C, s)$, the process terminates if and only if the characteristic exponent s is rational.

Let $v = (v_1, v_2)$ be now a rank 2 valuation whose truncation v_1 is quasimonomial. From Abhyankar's inequalities, [12, p. 12], one concludes that v_1 has rational rank 1. Hence, by Remark 3.3, we have $v_1 = v_1(C, s)$ for some $s \in \mathbb{Q}$. By the above then, the sequence of centres of v is infinite, whereas the sequence of 1-dimensional homes (centres of v_1) terminates at a blowup X_k where $\mathrm{centre}_{X_k}(v_1) = E_k$ is an exceptional divisor. In particular, v is equivalent to the valuation $v_{Y_\bullet}$, defined by the flag

$$Y_\bullet : X_k \supset E_k \supset \mathrm{centre}_{X_k}(v) = P_{k+1}.$$

The punchline of all this is that the process of blowing up all 0-dimensional centres of the truncation provides an effective method to find a model where a given rank 2 valuation becomes a flag valuation. By Theorem 2.9, such a model exists for every valuation of maximal rank on a projective variety. The above method works for any valuation of rank 2 on any projective surface (i.e., not necessarily $\mathbb{P}^2$).

For each centre P_i of a valuation v, general curves on X_{i-1} through P_i and smooth at P_i have the same value $e_i = v(E_i)$, which we call the weight of P_i for v. Following [9, Chapter 4], we call the (possibly infinite) sequence $K_v = (P_1^{e_1}, P_2^{e_2}, \ldots)$ the weighted cluster of centres of v. In general a sequence like $K = (P_1^{e_1}, P_2^{e_2}, \ldots)$ is called a weighted cluster of points and $\mathrm{supp}(K) = (P_1, P_2, \ldots)$ is called its support.

If v is a valuation with centre at O, then its weighted cluster of centres completely determines v. Indeed, for every effective divisor Z on $\mathbb{P}^2$, one has

$$v(Z) = \sum_i e_i \cdot \mathrm{mult}_{P_i}(\tilde{Z}_i), \tag{9}$$

where $\tilde{Z}_i$ is the proper transform of Z on X_i, whenever the sum on the right has finitely many non-zero terms. This is always the case unless v is a rank 2 valuation with home at a curve through O and Z contains this curve; in particular, for valuations of rank 1, such as $v_1(C, s)$, formula (9) always computes $v(Z)$ [9, §8.2].

As usual, with the above notation, we say that a curve Z passes through an infinitely near point $P_i \in X_i$ if its proper transform $\tilde{Z}_i$ on X_i contains P_i.

4.2. The cluster associated to $v_1(C, s)$.

The description of the cluster $K_{(C,s)} := K_{v_1(C,s)}$ is classical and we refer for complete proofs to [9]. Here, we merely focus on the construction of the cluster of centres for $v_1(C, s)$ and its main properties that will be used in the next section. The cluster $K_{(C,s)}$ is a very specific one, and we will need the following definition to make things more clear.

Definition 4.1. With notation as above, the centre $P_i \in X_{i-1}$ is called proximate to $P_j \in X_j$, for $1 \leqslant j < i \leqslant k$, (and one writes $P_i \to P_j$) if P_i belongs to the proper transform $E_{i-1,j}$ on X_{i-1} of the exceptional divisor $E_{j+1} := E_{j+1,j}$ over $P_j \in X_{j-1}$. For the cluster $K_{(C,s)}$, each P_i, with $i > 2$, is proximate to P_{i-1} and to at most one other centre P_j, with $1 \leqslant j < i-1$; in this case $P_i = E_{i-1,j} \cap E_{i-1}$ and P_i is called a satellite point. A point which is not satellite is called free.

We know that the support of the cluster $K_{(C,s)} = K_{v_1(C,s)}$ is determined by the continued fraction expansion

$$s = \frac{p}{q} = [n_1; n_2, \dots, n_r] = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots + \frac{1}{n_r}}}} ,$$

where p, q are coprime and $r \in \mathbb{Z}_{>0}$. Before moving forward, let's fix some notation. Let $k_i = n_1 + \dots + n_i$ and $k = k_r$. We denote by

$$s_i = \frac{p_i}{q_i} = [n_1; n_2, n_3, \dots, n_i], \quad \text{for } i = 1, \dots, r$$

the partial fractions of s, where p_i, q_i are coprime positive integers.

First, the cluster $K_{(C,s)}$ consists of $k = \sum n_i$ centres (if s is irrational there are infinitely many centres). Set $K = K_{(C,s)}$ and for each $i = 0, \dots, k-1$ let $\pi_i : X_{i+1} \to X_i$ be the blow-up of X_i at the centre P_{i+1} with exceptional divisor E_{i+1}. As usual we start with $X_0 := \mathbb{P}^2$. Denote $X_K := X_k$ and let $\pi : X_R \to X$ be the composition of the k blowups.

With this in hand, we explain the algorithm for the construction of K. If $s = n_1$ (so that $r = 1$), then the centre P_{i+1} is the point of intersection of the proper transform of C through the map $X_i \to X_0$ and the exceptional divisor E_i of π_{i-1}, for each $i = 1, \dots, n_1 - 1$. When $r > 1$, then the first $n_1 + 1$ (including P_1) centres of K are obtained as in the case when s was integral, i.e. these points are chosen to be free. The rest are satellites: starting from P_{n_1+1} there are $n_2 + 1$ points proximate to P_{n_1}, i.e. each P_j is the point of intersection of the proper transform of E_{n_1} and the exceptional divisor E_{j-1}. Thus, E_{n_1} plays the same role for these centres as C did in the first step. Then, one chooses $n_3 + 1$ points proximate to $P_{n_1+n_2}$ and so on. Since $r < \infty$, then the last n_r points (not $n_r + 1$) are proximate to $P_{n_1+\dots+n_{r-1}}$. The final space X_K is where $v_1(C, s)$ becomes a divisorial valuation, defined by the order of vanishing along the exceptional divisor $E_k \subseteq X_K$. Finally note that C plays a role only in the choice of the first n_1 centres. This is due to Remark 3.4, saying that the valuation $v_1(C, s)$ depends only on the bsc-th jet of C.

The weights in $K_{(C,s)}$ are proportional to the multiplicities of the curve with Puiseux series $y = \xi(x) + \theta x^s$ at the points of $\mathrm{supp}(K_{(C,s)})$. These and the continued fraction expansion are computed as follows. Consider the euclidean divisions

$$m_i = n_{i+1} m_{i+1} + m_{i+2} \quad \text{of} \quad m_i \quad \text{by} \quad m_{i+1}, \quad \text{for} \quad i = 0, \dots, r-1 ,$$

where $m_0 := p, m_1 := q$. Then the first n_1 points of $K_{(C,s)}$ have weight

$$e_1 = e_2 = \dots = e_{n_1} = \frac{m_1}{q} = 1,$$

the subsequent n_2 points have weight m_2/q, ..., the final n_r points have weight $m_r/q = 1/q$. Therefore the proximity equality

$$e_j = \sum_{P_i \to P_j} e_i \tag{10}$$

holds for all $j = 0, \dots, k-1$. Conversely, for every weighted cluster K with finite support, in which every point is infinitely near the previous one, no satellite point precedes a free point, and the proximity equality holds, there exist a smooth curve through O and a rational number s such that $K = K_{(C,s)}$.

4.3. $v_\pm(C, s)$ and the associated flag valuation. In order to describe the flag valuation associated to $v_\pm(C, s)$, it is necessary to understand first the intersection theory of all the proper and total transforms of the exceptional divisors on X_K.

To ease notation, let A_i (resp. B_i) be the proper (resp. total) transform of $E_i \subset X_i$ on X_K, for $i \in \{0, \dots, k-1\}$. Then:

Lemma 4.2. (i) $A_k = E_k$ is the only curve with $A_i^2 = -1$ for any $i = 1, \dots, k$;
(ii) $A_{k_i} = B_{k_i} - B_{k_i+1} - \dots - B_{k_{i+1}+1}$ and $A_{k_i}^2 = -2 - n_{i+1}$; for each $1 \leqslant i < r-1$;
(iii) $A_{k_{r-1}} = B_{k_{r-1}} - B_{k_{r-1}+1} - \dots - B_k$ and $A_{k_{r-1}}^2 = -1 - n_r$;
(iv) $A_j = B_j - B_{j+1}$ and $A_j^2 = -2$ for every $j \in \{1, \dots, k\} \setminus \{k_1, \dots, k_r\}$.

The sheaf $\pi^*(\mathcal{I}_{B_1|\mathbb{P}^2})$ is invertible on X_K and defines the fundamental cycle E of π. Write $E = \sum_{i=1}^{k} a_i A_i$. Then, making use of Lemma 4.2, the multiplicities a_i can be easily computed as follows:

Lemma 4.3. If one assumes $k_0 = a_0 = 0$ and $a_1 = 1$, then the multiplicities of the fundamental cycle E are computed by the following formula

$$a_i = a_{k_{j-1}}(i - k_{j-1} - 1) + a_{k_{j-1}+1}, \text{ for } k_{j-1} + 2 \leqslant i \leqslant k_j + \varepsilon \text{ and } 1 \leqslant j \leqslant r$$

where $\varepsilon = 0$ if $j = r$ and $\varepsilon = 1$ otherwise. In particular, $a_{k_j+1} = a_{k_j} + a_{k_{j-1}}$ for $1 \leqslant j \leqslant r-1$.

Remark 4.4. By Lemma 4.3, one has $a_{k_j} = n_j a_{k_{j-1}} + a_{k_{j-2}}$ for any $j = 2, \dots, r$, where $a_0 = 0$ and $a_{k_1} = 1$.

On the other hand, using the partial fractions $s_i = \frac{p_i}{q_i}$ of $s = \frac{p}{q}$, one has the same recursive relations $q_j = n_j q_{j-1} + q_{j-2}$ for $j = 2, \dots, r$, with $q_0 = 0$ and $q_1 = 1$. Thus, we get that $a_{k_j} = q_j$ for any $j = 0, \dots, r$. In particular, we have $a_{k_r} = q$.

In the following the pair (p_{r-1}, q_{r-1}) of the partial fraction $s_{r-1} = \frac{p_{r-1}}{q_{r-1}}$ will play an important role, so we fix some notation. When s is not an integer (i.e. $r > 2$), we set

$$p' = p_{r-1}, \quad q' = q_{r-1} \text{ so that } s_{r-1} = \frac{p'}{q'}$$

If s is an integer, i.e., $r = 1$, then we set $p' = q' = 1$.

In order to find the flags on X_K associated to $v_\pm(C, s)$, we need to have a better understanding of the cycle E through its dual graph. The dual graph of E is a chain, i.e. a tree with only two end points, corresponding to A_1 and A_{n_1+1}. If A is the proper transform of C on X_K, then A intersects E only at one point on A_{n_1+1}. Thus the dual graph of $A + E$ is also a chain, with end points corresponding to A_1 and A. The curve A_k intersects exactly two other components of $A+E$, precisely:

(a) if s is not an integer (so that $r > 2$), then A_k intersects A_{k-1} and $A_{k_{r-1}}$, whose multiplicities in the cycle $A + E$ are $a_{k-1} = q - q'$ and $a_{k_{r-1}} = q'$;

(b) if s is an integer (so that $s = k = n_1$), then A_k intersects A_{k-1} and A, both having multiplicity one.

Note that $A + E - A_k$ has two connected components, only one containing A. We denote this component by A_+ and the other by A_-. We will denote by $x_\pm$ the intersection point of A_k with $A_\pm$, and by x the general point of A_k.

The total transform C^* on X_K of C has the same support as $A + E$, but the multiplicities are different. In particular, denoting

$$p'' = \begin{cases} p' & \text{if } r \text{ is odd} \\ p - p' & \text{if } r \text{ is even} \end{cases} \qquad q'' = \begin{cases} q' & \text{if } r \text{ is odd} \\ q - q' & \text{if } r \text{ is even} \end{cases}$$

Lemma 4.5. (i) The divisor C^* contains A_k with multiplicity p and $C^* - pA_k$ passes through x_+ (resp. x_-) with multiplicity p''(resp. $p - p''$).
(ii) The total transform L of the line $x = 0$ on X_K contains A_k with multiplicity q and $L - qA_k$ passes through x_+ (resp. x_-) with multiplicity q'' (resp. $q - q''$).

Proof. We prove only (i), the proof of (ii) being analogous.

When s is an integer the assertion is trivial. So, assume that s is not an integer (i.e. $r > 2$). We first show that the multiplicity of A_k in C^* is equal to p. This is done inductively on $k = n_1 + \ldots + n_r$. From the standard properties of continuous fractions it is worth to note that the numerator of $[n_1; n_2, \ldots, n_r - 1]$ is equal to $p - p_{r-1}$, where p_{r-1} is the numerator of the continued fraction

$$s_{r-1} = \frac{p_{r-1}}{q_{r-1}} = [n_1; n_2, \ldots, n_{r-1}] .$$

The multiplicity of A_k in C^* is the same as the multiplicity of A_k in $B_1 + \ldots + B_{n_1+1}$. So, using Lemma 4.2 repeatedly, the statement follows easily.

The multiplicities of $C^* - pA_k$ at x_+ and x_- equal the multiplicities in C^* of $A_{k_{r-1}}$ and of A_{k-1} respectively in this order if r is odd, and reversed if r is even (as $r > 2$). Arguing as before, one deduces easily also these statements.

Example 4.6. Consider $s = 48/7$. Its continued fraction is $[6; 1, 6] = 6 + \frac{1}{1+1/6}$. Therefore the cluster of centres of $v_1(C, s)$ consists of 7 free points on C followed
by six satellites; of these, P_8 is proximate to P_6 and P_7, and each of $P_9, \ldots, P_{13}$ is proximate to its predecessor and to P_7. See Figure 2, where the weights e_i are printed in boldface: $e_1 = 1$, $e_2 = 6/7$, $e_3 = 1/7$.

The proximities mean that the exceptional components are $A_6 = B_6 - (B_7 + B_8)$, $A_7 = B_7 - (B_8 + \ldots + B_{13})$, $A_{13} = B_{13}$ and, for all $i = 6, 7, 13$, $A_i = B_i - B_{i+1}$. Solving for $B_1 = E$ one gets the fundamental cycle

$$E = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + 2A_8 + 3A_9 + 4A_{10} + 5A_{11} + 6A_{12} + 7A_{13} .$$

Since C goes through $P_1, \ldots, P_7$ with multiplicity 1, its total transform on X_K is

$$\begin{aligned} C^* = \tilde{C} + B_1 + \ldots + B_7 = \\ \tilde{C} + A_1 + 2A_2 + 3A_3 + 4A_4 + 5A_5 + 6A_6 + 7A_7 + \\ 13A_8 + 20A_9 + 27A_{10} + 34A_{11} + 41A_{12} + 48A_{13} . \end{aligned}$$

Clusters are often represented by means of Enriques diagrams (see [9, p.98]) as explained in Figure 2 illustrating this example.

Proposition 4.7. In the above setting, the flags associated to the rank 2 valuations $v_-(C, s)$ and $v_+(C, s)$ are

$$Y_- : \quad X_K \supset A_k \supset x_- \quad \text{and} \quad Y_+ : \quad X_K \supset A_k \supset x_+$$

respectively.

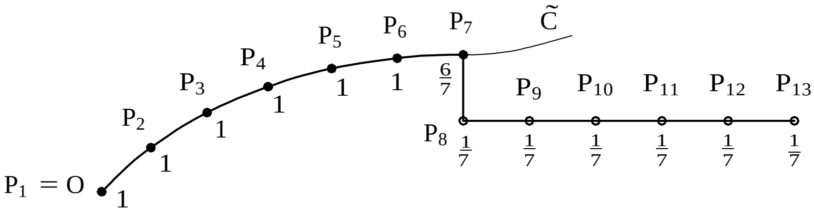

Figure 2. The Enriques diagram [9, 3.9] of the cluster of centres of Example 4.6. Each vertex in the diagram corresponds to one of the points, with each vertex joined to its immediate prececessor by an edge; edges are curved for free points, and straight segments for satellites, to represent the rigidity of their position. The segments joining a sequence of satellites proximate to the same point lie on the same line, orthogonal to the immediately preceding edge.

Proof. The above discussion makes it clear that A_k is the centre of $v_1(C, s)$. It remains to prove that $x_\pm$ are the centres of $v_\pm(C, s)$. Let $\eta = 0$ be a local equation of A_k on X_K around x_+. Consider $f_1 = f_0^q/x^p$ as in the proof of Proposition 3.10. By Lemma 4.5, the pull-back of f_1 to X_K is not divisible by η. Again by Lemma 4.5, it vanishes at x_+ with multiplicity p''. Furthermore, by Proposition 3.10, one has $v_+(C, s; f_1) > 0$. By the same token, f_1^{-1} is not divisible by η, it vanishes at x_- and has $v_-(C, s; f_1^{-1}) > 0$, proving the assertion.

Remark 4.8. Unless s is an integer and the sign $+$ holds, the valuations $v_\pm(C, s)$ are not equal to the evalutations associated to the flags $Y_\pm$ (see Remark 2.7), but they are equivalent to them.

Let $f_0 = 0$ be an equation of C (which we may assume to be algebraic, see the proof of Proposition 3.10) and note that $K[[x, y]] \cong K[[x, f_0]]$. One has

$$v_\pm(C, s; x) = (1, 0), \quad v_\pm(C, s; f_0) = (s, \pm 1) \,, \tag{11}$$

by the proof of Proposition 3.10. By Lemma 4.5, one has

$$\begin{aligned} v_{Y_+}(x) &= (q, q''), & v_{Y_+}(f_0) &= (p, p'') \\ v_{Y_-}(x) &= (q, q - q''), & v_{Y_-}(f_0) &= (p, p - p'') \,. \end{aligned}$$

By standard properties of continued fractions, one has $pq' - qp' = (-1)^r$. Thus

$$v_+(C, s) = \begin{pmatrix} \frac{1}{q} & 0 \\ -q'' & q \end{pmatrix} v_{Y_+},$$

$$v_-(C, s) = \begin{pmatrix} \frac{1}{q} & 0 \\ q'' - q & q \end{pmatrix} v_{Y_-}.$$

Remark 4.9. The same relations, given in Remark 4.8, hold for the corresponding Newton–Okounkov bodies. It is worth to note that both 2×2 matrices transform vertical line into vertical lines. Furthermore, any vertical segment in $\Delta_{Y_\pm}(D)$ is translated into a vertical segment in $\Delta_{C,s_\pm}$ whose length is multiplied by a factor of q with respect to the initial one, where D is the class of a line.

4.4. Zariski decomposition of valuative divisors. In this subsection we will describe, with few details, some of the properties of the valuation v_1 that will be used in the next section. As before let $s = p/q > 1$ be a rational number and K the cluster of centres associated to the rank 1 valuation $v_1(C, s)$, with $\pi : X_K \to \mathbb{P}^2$ the sequence of blow-ups constructed in the previous section where the valuation $v_1(C, s)$ becomes equivalent to a valuation given by the order of vanishing along an exceptional curve on X_K. We will denote by

$$B_s \stackrel{\text{def}}{=} e_1 B_1 + \ldots + e_k B_k\ ,$$

where as usual B_i is the total transform of the exceptional divisor E_i on X_K and e_i is the weight of the center P_i, whose blow-up is the curve E_i (whereas A_i is the strict transform in X_K of E_i). Note that the proximity equalities (10) mean that $B_s \cdot A_i = 0$ for all $1 \leq i \leq k-1$, and that the weights are also determined by these equalities and $e_k = 1/q$ (see section 8.2 in [9]). Knowing this divisor B_s we usually know almost everything about the valuation v_1. Using (9) one deduces the following:

Lemma 4.10. *For a divisor Z on X_K not containing any of the exceptional curves A_i, one has $v_1(C, s; \pi_*(Z)) = B_s \cdot Z$.*

For the computation of Newton–Okounkov bodies, the following properties of B_s will also be useful.

Lemma 4.11. *(i) $B_s^2 = -s$, $\mathrm{ord}_{A_k}(B_s) = p$, $(B_s \cdot A_i) = 0$ for any $i = 1, \ldots, k-1$, $A_k) = -1/q$.*
(ii) For every positive $x \in \mathbb{Q}$ such that the $\mathbb{Q}$-divisor $D_x = D - xB_k$ is pseudo-effective (where D is the class of a line as usual), the Zariski decomposition of D_x contains $\frac{x}{p}B_s - xB_k$ in its negative part.

Proof. The proof of (i) is done inductively using the description of the cluster of centres obtained previously, and we leave the details to the reader.

Let us prove (ii). If $k = 1$ then $s = p = 1$, $B_s = B_k = B_1$ and there is nothing to prove; so assume $k > 1$. Since the intersection matrix of the collection $\{A_1, \ldots, A_k\}$ is negative definite, there exists a unique effective $\mathbb{Q}$-divisor $N_\pi = \sum \nu_i A_i$ with

(a) $(D_x - N_\pi) \cdot A_i \geq 0$ for all $1 \leq i \leq k$,
(b) $(D_x - N_\pi) \cdot A_i = 0$ for all i with $\nu_i = 0$.

The Zariski decomposition of D_x relative to π is $D_x = P_\pi + N_\pi$ (see [10, §8]). It satisfies $H^0(X_K, \mathcal{O}_{X_K}(mD_x)) \cong H^0(X_K, \mathcal{O}_{X_K}(mP_\pi))$ for all m such that mD_x is a Weil divisor, and the negative part of this relative Zariski decomposition is a part of the full Zariski decomposition: $N_\pi \leq N$.

We claim that $N_\pi = \frac{x}{p}B_s - xB_k$; to prove it, we need to show that $\frac{x}{p}B_s - xB_k$ is an effective divisor satisfying (a) and (b). A direct computation shows that the coefficient ν_i of A_i in $\frac{x}{p}B_s - xB_k$ is positive for $i = 1, \ldots, k-1$ and zero for $i = k$, so it is an effective divisor. On the other hand, $(D_x - \frac{x}{p}B_s + xB_k) \cdot A_i = (D - \frac{x}{p}B_s) \cdot A_i = -\frac{x}{p}B_s \cdot A_i$, which using (i) gives $(D_x - \frac{x}{p}B_s + xB_k) \cdot A_i = 0$ for $i = 1, \ldots, k-1$, and $(D_x - \frac{x}{p}B_s + xB_k) \cdot A_k > 0$, as wanted.

5. Newton–Okounkov bodies on the tree QM

From now on we will mainly concentrate on the study of Δ_{C,s_+} when s varies in $[1, +\infty)$. The case of Δ_{C,s_-} is not conceptually different and will be often left to the reader.

5.1. General facts.

Corollary 5.1. Let $C \subseteq \mathbb{P}^2$ be a curve of degree d. For any $s > 1$, one has the following inclusions

$$\Delta_{1,\frac{s}{d},\pm\frac{1}{d}} \subseteq \Delta_{C,s_\pm} \subseteq \Delta_{\hat\mu,\hat\mu,\pm\frac{\hat\mu}{s}} ,$$

$\hat\mu = \hat\mu(C, s)$. Equality for the first inclusion holds if and only if $d = 1$. Equality for the second one takes place if and only if $\hat\mu(C, s) = \sqrt{s}$.

Proof. For the first inclusion, note that by the proof of Proposition 3.10 evaluating an equation of C and the variable x, forces both points $(1, 0)$ and $(\frac{s}{d}, \pm\frac{1}{d})$ to be contained in $\Delta_{C,s_\pm}$. The origin is also contained in $\Delta_{C,s_\pm}$ since it is the valuation of any line not passing through the centre of the valuation. For the equality statement one uses that the area $\Delta_{C,s_\pm}$ is $\frac{1}{2}$, by Theorem 2.13 and Remark 3.12.

For the second inclusion notice first that by definition of $\hat\mu(C, s)$ from §3.4 one has that the convex sets $\Delta_{C,s_\pm}$ sit to the left of the vertical line $t = \hat\mu$. To prove that Δ_{C,s_+} also lies above the t-axis and below the line $t = su$, we need to show

$$v_1(C, s; f) > s \cdot \partial_+ v_1(C, s; f) > 0 , \forall f \in K[x, y] \setminus \{0\} .$$

Assuming (5) holds, this follows from (6) and (8), as $i + sj > sj$. The equality statement is again implied by the fact that the area of Δ_{C,s_+} is equal to $\frac{1}{2}$. The analogous facts for Δ_{C,s_-} are left to the reader.

Remark 5.2. As a consequence of the above, then Δ_{C,s_+} sits above the t axis and below the line with equation $su = t$ in the (t, u) plane. Also, notice that $(0, 0)$ and $(\frac{s}{d}, \frac{1}{d})$ are valuative points, where the latter is given by the valuation of a local equation of C by Remark (2.12). Thus, every point with rational coordinates on the line $su = t$, lying between the origin and the point $(s/d, 1/d)$, is valuative. The corresponding picture also holds for Δ_{C,s_-}.

Remark 5.3. The valuation v_{gen} associated to the generic flag

$$Y_{gen} : \quad X_K \supset A_k \supset x$$

has nothing to do with C. On X_K there is a smooth curve Γ transversally intersecting A_k at x. Its image on X has local equation $\varphi = 0$ at P_1. Assume $X = \mathbb{P}^2$ and $\deg(\varphi) = d$. Then for $f = 0$ a general line through P_1 one has $v_{gen}(f) = (q, 0)$ and $v_{gen}(\varphi) = (\frac{q}{d}, \frac{1}{d})$. Thus $\Delta_{v_{Y_{gen}}}(D)$, with $D \in |\mathcal{O}_{\mathbb{P}^2}(1)|$, contains $\Delta_{q,\frac{q}{d},\frac{1}{d}}$. Since in general $d > q$ (equality may hold only if $s = n_1$), then $\Delta_{v_{Y_{gen}}}(D)$ is strictly larger than this triangle by Theorem 2.13.

Remark 5.4. By Corollary 5.1, we see that Conjecture 3.13 is equivalent to asking whether for all $s > 8 + \frac{1}{36}$ and C general enough, one has

$$\Delta_{C,s_+} = \Delta_{\sqrt{s},\sqrt{s},\frac{1}{\sqrt{s}}} .$$

In particular, this implies Nagata's Conjecture and it shows how difficult it is to compute Newton–Okounkov bodies.

Corollary 5.5. Let C be a plane curve of degree d. Then $\Delta_{C,d^2_\pm} = \Delta_{d,d,\pm\frac{1}{d}}$.

Proof. The cluster of centres of $v_1(C, d^2)$ consists of $P_1 = O$ and the next $d^2 - 1$ points on C infinitely near O, i.e., $P_i = E_{i-1} \cap \tilde{C}$ for $i = 2, \ldots, d^2$. If A is the strict transform of C on X_{d^2}, then

$$A \equiv dD - E_1 - \ldots - E_{d^2} = dD - \sum_{i=1}^{d^2} iA_i$$

where D is the pull back to X_{d^2} of a line. Let $Z := D - dA_{d^2}$, which can be written

$$Z = \frac{A}{d} + \sum_{i=1}^{d^2-1} \frac{i}{d} A_i .$$

Remark that this is actually the Zariski decomposition of Z, because A is nef, as A is irreducible and $A^2 = 0$, and $\sum_{i=1}^{d^2-1} \frac{i}{d} A_i$ has clearly a negative definite intersection form. Also, Z sits on the boundary of the pseudo-effective cone, as $A^2 = 0$. Thus, $Z - tA_{d^2}$ is not pseudo-effective for $t > 0$.

Now the proof follows easily using Theorem 2.17. Alternatively, by Remark 5.4, it suffices to prove that $\mu(C, d^2) = d$. Since $v_1 = \mathrm{ord}_{E_{d^2}}$ as valuations (by Remark 4.8, noting that s is an integer) one gets from the above paragraph that $\mu(C, d^2) \leq d$. The opposite inequality follows from Lemma 5.1.

Corollary 5.6. Let C be a plane curve of degree d. For every > 0, there exists a non-zero $f \in K[x, y]$ whose C-expansion $f(x, y) = \sum a_{ij} x^i (y - \xi(x))^j$ satisfies:
(i) $v_1(C, d^2; f) = \min\{i + d^2 j \,|\, a_{ij} = 0\} > \deg(f) \cdot (d - \ )$,
(ii) $\partial_+ v_1(C, d^2; f) = \min\{j \,|\, \exists i : a_{ij} = 0 \ , i + d^2 j = v_1(C, s; f)\} \leq \deg(f) \cdot \ $.
A similar statement holds for $\partial_- v_1(C, d^2; f)$.

By Corollary 5.1, there exists a real number $\lambda > 0$ such that

$$\Delta_{\lambda,\lambda,\pm\frac{\lambda}{s}} \subset \Delta_{C,s_\pm}. \tag{12}$$

This can be seen as an infinitesimal counter-part of Theorem 2.22 for $X = \mathbb{P}^2$. When $s = 1$ and X is any smooth projective surface, these ideas were also developed in [19] along with Theorem 2.22. The largest λ turned out to be the Seshadri constant of the divisor. This connection can be seen clearly in the following proposition, where the notation comes from §4.4.

Proposition 5.7. Let $C \subset \mathbb{P}^2$ be a curve and $s = p/q > 1$. Let $\alpha \in \mathbb{Q}$ be such that the $\mathbb{Q}$-divisor $\alpha D - B_s$ is nef. Then $\Delta_{\frac{s}{\alpha},\frac{s}{\alpha},\frac{1}{\alpha}} \subseteq \Delta_{C,s_+}$.

Proof. Let's check first that $(\frac{s}{\alpha}, 0) \in \Delta_{C,s_+}$. By Remark 4.8, this is equivalent to showing that $(\frac{p}{\alpha}, 0) \in \Delta_{Y_+}(D)$. Since $\alpha D - B_s$ is nef, then there exists a sequence of effective ample divisors $H_n, n > 1$, where $x_+ \notin \mathrm{Supp}(H_n)$, so that D is the limit of $\frac{1}{\alpha} B_s + H_n$. So, the point $(\frac{p}{\alpha}, 0)$ is contained in $\Delta_{Y_+}(D)$, as $\mathrm{ord}_{A_k}(B_s) = p$.

By Remark 4.9, it remains to show that the height of the slice of $\Delta_{Y_+}(D)$ with first coordinate $t = \frac{p}{\alpha}$ is equal to $\frac{1}{q\alpha}$. For this, we apply Theorem 2.17 for $t = \frac{p}{\alpha}$. Let $N_t + P_t$ be the Zariski decomposition of $D - tB_k$. By Lemma 4.11, we know that

$$N_t - \frac{1}{\alpha} B_s + \frac{p}{\alpha} B_k \text{ is effective.}$$

Thus, one has $P_t \leq D - (1/\alpha)B_s$; but the latter $\mathbb{Q}$-divisor is nef by hypothesis, therefore $N_t = (1/\alpha)(B_s - pB_k)$ and $P_t = D - (1/\alpha)B_s$. In particular, the height of Δ_{Y_+} at $t = \frac{p}{\alpha}$ is equal to $(1/\alpha)P_t \cdot B_k = (1/\alpha)e_k = 1/(q\alpha)$.

Based on the previous statement, it is natural to introduce the following constant

$$\lambda(C, s) \stackrel{\text{def}}{=} \max\{\lambda > 0 \mid \Delta_{\lambda,\lambda,\frac{\lambda}{s}} \subset \Delta_{C,s_+}\} .$$

As mentioned before, when $s = 1$, the constant $\lambda(C, s)$ is nothing else than the Seshadri constant of D, the class of a line, at the origin O. So, one expects $\lambda(C, s)$ to encode plenty of geometry also for $s > 1$. Note that we have the inequalities

$$\lambda(C, s) \leqslant \sqrt{s} \leqslant \hat{\mu}(C, s) ,$$

where the left-hand side is an equality if and only if the right-hand side is also such. From Conjecture 3.13 this is expected to happen when s is large enough and C a sufficiently generic choice of a curve. Thus it becomes natural to ask about the shape of the convex set Δ_{C,s_+} when $\hat{\mu}(C, s) = \lambda(C, s) = \sqrt{s}$ does not happen.

Corollary 5.8. Under the assumptions above, one of the following happens:
(i) $\hat{\mu}(C, s) = \lambda(C, s) = \sqrt{s}$, in which case $\Delta_{C,s_+} = \Delta_{\hat{\mu},\hat{\mu},\frac{\hat{\mu}}{s}}$, where $\hat{\mu} = \hat{\mu}(C, s)$.
(ii) $\hat{\mu}(C, s) > \sqrt{s} > \lambda(C, s)$, in which case $\lambda(C, s) = s/\hat{\mu}(C,s)$ and the convex polygon Δ_{C,s_+} is the quadrilateral $OABE$, where

$$O = (0, 0), \quad A = (\lambda(C, s), 0), \quad B = (\lambda(C, s), \lambda(C, s)/s), \quad E = (\hat{\mu}(C, s), c) ,$$

for some $c \in [0, \frac{\mu(C,s)}{s}]$. Hence, Δ_{C,s_+} is a triangle if and only if $c = 0$ or $c = \frac{\hat{\mu}(C,s)}{s}$.

Proof. By definition of $\hat{\mu}(C, s)$ and Lemma 4.10, for any effective divisor Z in X_K, not containing in its support any of the exceptional curves A_i, one has

$$\hat{\mu}(C, s) > \frac{v_1(C,s)(Z)}{Z \cdot D} = \frac{Z \cdot B_s}{Z \cdot D} .$$

So, $(\mu(C, s)D - B_s) \cdot Z > 0$ for any such cycle Z. By Lemma 4.11, we already know $\mu(C, s)D - B_s) \cdot A_i > 0$ for any $i = 1, \ldots, k$. Thus, the divisor $\mu(C, s)D - B_s$ $\mu(C, s)$. When $\hat{\mu}(C, s) = \sqrt{s}$ we land in case (i). Otherwise, if $(\hat{\mu}(C, s), c) \in \Delta_{C,s_+}$ for some $c > 0$, then this latter condition implies that Δ_{C,s_+} contains the convex hull of the points

$$(0, 0), \quad \left(\frac{s}{b}, 0\right), \quad \left(\frac{s}{\mu(C, s)}, \frac{1}{\hat{\mu}(C, s)}\right), \quad (\hat{\mu}(C, s), c) .$$

Note that such a c must exist, since the projection of Δ_{C,s_+} to the first axis is $[0, \hat{\mu}(C, s)]$ as noted before. Since the area of this convex hull is $1/2$, it coincides with Δ_{C,s_+}, and one has $\lambda(C, s) = s/\hat{\mu}(C, s)$.

Remark 5.9. It is worth to note that Corollary 5.8 takes place only because our ambient space is P^2, especially due to properties like the Picard group of P^2 is generated by a single class, whose associated line bundle is globally generated with self-intersection equal to 1. One does not expect these phenomena to happen when we consider the valuations $v_\pm$ on any smooth projective surface X. But we do expect that some parts of the considerations about the infinitesimal picture developed in [19] to be true in this more general setup. For example, the constant $\lambda(C, s)$ should in some ways encode many interesting local positivity properties of the divisor class we are studying, as partially seen in Proposition 5.7.

Example 5.10. Continuing with $s = 48/7$ as in Example 4.6, and assuming O is a general point of a curve C of degree $d > 3$, we know from [11, Theorem C] and Remark 3.15 that $\hat{\mu}(C, 48/7) = (1 + 48/7)/3 = 55/21$ and there is a unique curve V with $v_1(C, 48/7; V)/\deg V = 55/21$, namely the unique cubic nodal at O which has one of the branches γ at the node satisfying $(C,\gamma)_O = 7$. Indeed, V has multiplicity 2 at O, and (its strict transform) multiplicity 1 at each centre $P_2,\dots,P_7$, whereas it does not pass through any of the remaining centres $P_8,\dots,P_{13}$. So by (9) one has $v_1(C, 48/7; V) = 2 + 5 + 6/7 = 55/7$, which divided by $\deg(V) = 3$ gives $\mathfrak{b}(C, 48/7) = 55/21$. The Newton polygon of V with respect to C has three vertices, namely $(0, 2)$, $(1, 1)$ and $(8, 0)$, showing that $v_1(C, s; V) = 1 + s$ for $s < 7$, and so $v_+(C, 48/7; V) = (55/7, 1)$. Therefore, the rightmost point of $\Delta_{C,48/7_+}$ is the valuative point $(55/21, 1/3)$. Alternatively, $v_+(C, 48/7; V)$ can be computed from the pullback of V to X_K, which is

$$V^* = \tilde{V} + 2B_1 + B_2 + \dots + B_7 = A + 2A_1 + 3A_2 + 4A_3 + 5A_4 + 6A_5 + 7A_6 + 8A_7 + 15A_8 + 22A_9 + 31A_{10} + 39A_{11} + 47A_{12} + 55A_{13}.$$

Therefore the flag valuation applied to V is $\nu_{Y_+}(V) = (55, 8)$ (recall that one has $x_+ = A_7 \cap A_{13}$) and the computation from Remark 4.8 gives

$$v_+(C, 48/7)(V) = \begin{pmatrix} \frac{1}{q} & 0 \\ -q' & q \end{pmatrix} \nu_{Y_+}(V) = \begin{pmatrix} \frac{1}{7} & 0 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 55 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{55}{7} \\ 1 \end{pmatrix},$$

consistent with the computation using the Newton polygon. By Corollary 5.8, the remaining vertices of $\Delta_{C,48/7_+}$ are $(0, 0)$, $(144/55, 0)$ and $(144/55, 21/55)$.

The same computation applies to any $s \in (\varphi^4, 7)$, giving quadrilateral bodies Δ_{C,s_+} with vertices

$$(0,0), \quad \left(3 - \frac{3}{s+1}, 0\right), \quad \left(3 - \frac{3}{s+1}, \frac{3}{s+1}\right), \quad \left(\frac{s+1}{3}, \frac{1}{3}\right).$$

We leave the easy details to the reader.

5.2. Large s on curves of fixed degree.

As we have seen in Corollary 5.8, the convex set Δ_{C,s_+} can be either a rectangular triangle or a quadrilateral, for any rational number $s > 1$ and any degree $d = \deg(C)$. Interestingly enough, when s d one can say more. More precisely, we have the following theorem:

Theorem 5.11. Let $C \subset \mathbb{P}^2$ be a plane curve of degree d and $D \subseteq \mathbb{P}^2$ a line. Then $\Delta_{C,s_+} = \Delta_{d,\frac{s}{d},\frac{1}{d}}$ for any $s > d^2$. In particular, one has

$$\Delta_{(C,O)}(D) = \Delta_{\frac{1}{d},d} = \begin{pmatrix} 0 & 1 \\ 1 & -s \end{pmatrix} \Delta_{C,s_+} \tag{13}$$

Proof. First, the point $(0, 0)$ belongs to Δ_{C,s_+}, as it is given by the valuation of any polynomial not vanishing at O. Also, $(\frac{s}{d}, \frac{1}{d}) \in \Delta_{C,s_+}$, being the valuation of an equation of C, as we saw in the proof of Proposition 3.10 (see also Remark 4.8). It remains to show that $(d, 0) \in \Delta_{C,s_+}$. This will be enough, since we will have the containment $\Delta_{d,\frac{s}{d},\frac{1}{d}} \subseteq \Delta_{C,s_+}$ and the two coincide because both have area 1/2.

For the latter condition, fix any > 0 and let f be the polynomial as in Corollary 5.6, satisfying (i) and (ii). Consider the function $s \mapsto v_+(C, s; f) \in \mathbb{R}^2$ for $s > d^2$. By Proposition 3.7, the first coordinate of this function is non-decreasing and concave. Thus, the second coordinate is decreasing by concavity of the first. Now, these two remarks, together with the properties we know for $v_+(C,$ $s;$ f $)$ when

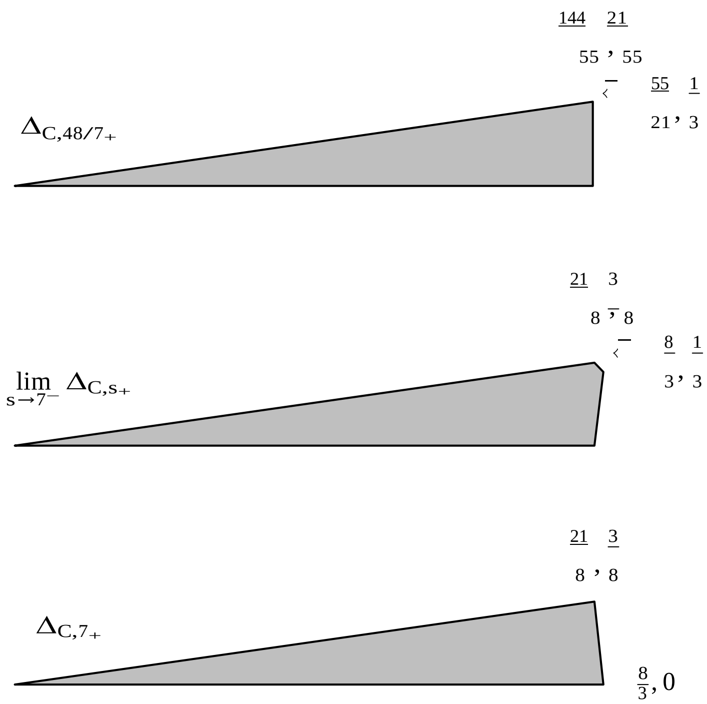

Figure 3. The Newton–Okounkov body computed in Example 5.10, on top. The difference $55/21 - 144/55 = 1/1155$ is so small that the quadrilateral looks like a triangle. As s grows in the interval $(\varphi^4, 7)$, the vertex $\left(\frac{s+1}{3}, \frac{1}{3}\right)$ sticks out further right from the two vertices with first coordinate $3 - \frac{3}{s+1}$, so that at the limit, the quadrilateral nature is clearly seen. At $s = 7$ the quadrilateral mutates into a triangle, shown in the bottom picture.

$s = d^2$ from Corollary 5.6, imply that the limit

$$\lim_{\to\infty} \frac{v_+(C,s;f)}{\deg(f)} = (t_s, 0), \text{ for any } s > d^2,$$

where $t_s > d$. In particular, $(t_s, 0) \in \Delta_{C,s+}$ and since the origin is contained in this convex set, then this implies that $(d, 0) \in \Delta_{C,s+}$.

By Example 2.21 we have $\Delta_{(C,O)}(D) = \Delta_{\frac{1}{d},d}$, thus the final assertion follows.

Remark 5.12. With similar arguments as in the proof of Theorem 5.11, one proves that $\Delta_{C,s-} = \Delta_{d,\frac{s}{d},-\frac{1}{d}}$ and

$$\Delta_{(C,O)} = \begin{pmatrix} 0 & -1 \\ 1 & s \end{pmatrix} \Delta_{C,s-}$$

Corollary 5.13. If C is a plane curve of degree d, then

$$\mu(C, s) = \frac{s}{d}, \quad \forall s > d^2.$$

Hence $v_1(C, s)$ is minimal (resp. not minimal) for $s = d^2$ (resp. for $s > d^2$) (see §3.4 for definition).

Remark 5.14. Corollary 5.13 seems to be in contrast with Conjecture 3.13, which in reality is not the case. Conjecture 3.13 applies only for a sufficiently general choice of C. Given s, this requires the degree of C to be large enough with respect to s. In other words, if $s > d^2$ then C is not sufficiently general.

Remark 5.15. Lemma 8 from [27] implies that the right-hand side of (13) converges to the left-hand side for $s \to +\infty$. So, Theorem 5.11 makes this statement more precise, i.e., in fact the two bodies are equal for $s > d^2$.

5.3. Mutations and supraminimal curves.

By fixing C, the goal of this section is to study Δ_{C,s_+} as a function of $s \in [1, +\infty)$, i.e. by walking along an arc of $\mathcal{QM}$ away from the root. By Theorem 5.11, the picture is well understood for $s > d^2$, so it remains to study the case when $s \in (1, d^2)$. Note that the origin $(0, 0)$ is always a vertex of Δ_{C,s_+}. The remaining vertices of Δ_{C,s_+} will be called proper and their behaviour is the focus of this subsection.

Definition 5.16. We say that Δ_{C,s_+} is continuous at $s_0 \in (1, d^2)$ if, for every $\ > 0$ there exists a $\delta > 0$ such that for all s with $|s - s_0| < \delta$, every vertex p of Δ_{C,s_+} is near the boundary of $\Delta_{C,s_{0+}}$, i.e. distance$(p, \partial\Delta_{C,s_{0+}}) <\ $.

If Δ_{C,s_+} is not continuous for some $s_0 \in (1, d^2)$, then we say that Δ_{C,s_+} presents a mutation at s_0 (or mutates at s_0). Also, Δ_{C,s_+} depends linearly on s in an interval $I \subseteq (1, d^2)$, if the number of proper vertices of Δ_{C,s_+} is the same for all $s \in I$ and the coordinates of the vertices of Δ_{C,s_+} are affine functions of s in I.

As we will see, a standard reason for mutation, taking place between intervals of linearity of Δ_{C,s_+}, is non-minimality of $v_1(C, s)$. Moreover mutations may behave differently according to whether O is sufficiently general on C or not.

Definition 5.17. We say that an irreducible curve V containing O, with equation $\mu(C, s)$ via $v_1(C, s)$ if $v_1(C, s; f) = \deg(f) \cdot \mu(C, s)$.

If $v_1(C, s)$ is non-minimal, there exists V computing $\mu(C, s)$ via $v_1(C, s)$. Hence

$$v_1(C, s; f) = \deg(f) \cdot \mu(C, s) > \deg(f) \cdot \sqrt{\ } s$$

(see [11, Lemma 5.1]). Such curves are called supraminimal for $v_1(C, s)$.

Remark 5.18. The proof of [11, Lemma 5.1] shows that if V computes $\mu(C, s)$, in particular if it is supraminimal for $v_1(C, s)$, then there is no (other) supraminimal curve at s.

If V, with equation $f = 0$, computes $\mu(C, s)$, then the valuative points of $v_+(C, s)$ corresponding to f (i.e., the one-sided limits of $v_+(C, s)(f)$ with respect to s) are rightmost vertices of Δ_{C,s_+}. Note that there are two such vertices, if $\partial v_1(C, \cdot; f)$ is not defined at s, in which case there is a mutation of Δ_{C,s_+} at s, or there is only one such vertex.

Example 5.19. Corollary 5.13 tells us that C of degree d is supraminimal for all $v_1(C, s)$ with $s > d^2$: we consider this as a trivial case of supraminimality.

There is no supraminimal curve for the O-adic valuation $v_1(C, 1)$. So, given C, non-trivial supraminimal curves for $v_1(C, s)$ may occur only for $s \in (1, d^2)$.

Theorem 5.20. Let C be a smooth curve through O and let V be a curve, different from C, with equation $f = 0$. Then:

(i) the set of points $s \in [1, +\infty)$ such that V is supraminimal for $v_1(C, s)$ is open;

(ii) if V is supraminimal for $v_1(C, s)$ for all $s \in (a, b)$ but not at a and b, then

$$\Delta_{C,a_+} = \Delta_{\sqrt{\ }\overline{a},\sqrt{\ }\overline{a},1/\sqrt{\ }\overline{a}}, \quad \Delta_{C,b_+} = \Delta_{\sqrt{\ }\overline{b},\sqrt{\ }\overline{b},1/\sqrt{\ }\overline{b}},$$

and there is some point $\sigma \in (a, b)$ of discontinuity of the derivative of $v_1(C, s)$. Furthermore, there is some branch $\gamma \in \mathcal{B}(V)$ with Puiseux expansion starting as $y = ax^{\sigma} + \ldots$, such that $-\frac{1}{\sigma} = \mathrm{sl}(l)$ with $l = \phi_V(\gamma) \in E(f)$;
(iii) Δ_{C,s_+} mutates at the finitely many points $\sigma \in (a, b)$ as in (ii).

Proof. Part (i) follows from the continuity of $v_1(C; f)(s)$ and $\sqrt{s}$.

Let us prove (ii). There is no supraminimal curve for $v_1(C, a)$ and $v_1(C, b)$. Indeed, suppose W is supraminimal for $v_1(C, a)$ (the same argument works for $v_1(C, b)$). Then, by part (i), W is supraminimal for $v_1(C, s)$ with s in a neighborood of a. But V is also supraminimal for $v_1(C, s)$ with s in a right neighborood of a. By the uniqueness of supraminimal curves, we have $V = W$, against the assumption that V is not supraminimal for $v_1(C, a)$. Thus $\mu(C, s) = \sqrt{s}$ for $s = a, b$. Since $v_1(C; f)(s) > \deg(f) \cdot \sqrt{s}$ for $s \in (a, b)$ and $v_1(C; f)(s)$, as a function of $s \in [1, +\infty)$, is a tropical polynomial (see Proposition 3.7), certainly there is some point $\sigma \in (a, b)$ of discontinuity for its derivative. Moreover $v_1(C; f)(s) = \deg(f) \cdot \sqrt{s}$ for $s = a, b$. The rest of (ii) follows from the discussion in §3.2.

To show (iii) note that the mutation of Δ_{C,s_+} at the points $s \in (a, b)$ as in (ii) depends on the discontinuity of $\partial v_1(C; f)$ there (see Remark 5.18).

Proposition 5.21. *If $\mu(C, s_0) = \sqrt{s_0}$ then Δ_{C,s_+} is continuous at s_0.*

Proof. By Corollary 5.8, for every s there are inclusions

$$\Delta_{\frac{s}{\mu(C,s)},\frac{s}{\mu(C,s)},\frac{1}{\mu(C,s)}} \subseteq \Delta_{C,s_+} \subseteq \Delta_{\mu(C,s),\mu(C,s),\frac{\mu(C,s)}{s}}$$

Since $\mu(C, s)$ is a continuous function of s, it follows that for every $\epsilon > 0$ there is $\delta > 0$ such that for $|s - s_0| < \delta$,

$$\Delta_{\sqrt{s_0}-\epsilon,\sqrt{s_0}-\epsilon,\frac{\sqrt{s_0}-\epsilon}{s}} \subseteq \Delta_{C,s_+} \subseteq \Delta_{\sqrt{s_0}+\epsilon,\sqrt{s_0}+\epsilon,\frac{\sqrt{s_0}+\epsilon}{s}}$$

and the claim follows, since $\Delta_{C,s_0+} = \Delta_{\sqrt{s_0},\sqrt{s_0},1/\sqrt{s_0}}$.

The mutations described in Theorem 5.20(iii), can be called *supraminimality mutations*.

Corollary 5.22. *Any mutation of Δ_{C,s_+} is supraminimal.*

For general choices of O and C, all known supraminimal curves are (-1)-curves. It would be interesting to explore the behavior of $\Delta_{C,s_\pm}$ on surfaces different from $\mathbb{P}^2$, or for non-quasimonomial valuations (i.e., allowing singular C, and following arcs in the whole valuative tree $\mathcal{V}$ rather than only $\mathcal{QM}$). It is tempting to conjecture that mutations in general should be supraminimal and related to extremal rays in (some) Mori cone.

5.4. Explicit computations.

In this section we compute the Newton–Okounkov bodies in the range in which $\mu(C, s)$ is known (see [11] and §3.4).

5.4.1. The line case.

This case is an immediate consequence of Theorem 5.11, which we state here for completeness.

Proposition 5.23. *If C is a line, then for every $s > 1$ one has*

$$\Delta_{C,s_+} = \Delta_{1,s,1},$$

hence Δ_{C,s_+} depends linearly on s and there are no mutations.

5.4.2. The conic case.

Proposition 5.24. If C is a conic, then

$$\Delta_{C,s_+} = \begin{cases} \Delta_{1,s,1} & \text{if } 1 \leqslant s < 2 \\ \Delta_{2,\frac{s}{2},\frac{1}{2}} & \text{if } s \geqslant 2\ . \end{cases}$$

So, there is only one mutation at $s = 2$, and $v_1(C, s)$ is minimal only when $s = 1, 4$.

Proof. The case $s = 1$ is trivial. So, assume first that $s \in (1, 2) \cap \mathbb{Q}$. By Remark 4.8, we know that $(1, 0) \in \Delta_{C,s_+}$. It remains to show that $(s, 1) \in \Delta_{C,s_+}$. Note that the only free points in the cluster $K = K_{v_+(C,s)}$ are $P_1 = O$ and P_2. The line through P_1 and P_2, i.e., the tangent line to C, has equation $y = 0$ and behaves exactly like C with respect to K. Thus, again by Remark 4.8, we have

$$\frac{v_+(C,s;\ y)}{\deg(y)} = v_+(C, s; y) = (s, 1)\ ,$$

implying that $(s, 1) \in \Delta_{C,s_+}$.

Take now $s \in [2, 4) \cap \mathbb{Q}$ and use the notation of §4. By Remark 4.8, we have that $(\frac{s}{2}, \frac{1}{2}) \in \Delta_{C,s_+}$, given by the valuation of a local equation of C. It remains to show that $(2, 0) \in \Delta_{C,s_+}$. Let L on X_K be the total transform of the tangent line to C at O. Note that $L - B_1 - B_2$ does not contain any of the other exceptional curves. Thus, arguing as in Lemma 4.5(ii), it is easy to see that L contains A_k with multiplicity $2q$ and $L - 2qA_k$ passes through x_+ with multiplicity $2q''$. Hence $(2q, 2q') \in \Delta_{Y_+}$ and, by Remark 4.8, this implies that $(2, 0) \in \Delta_{C,s_+}$.

Finally, by Theorem 5.11, the assertion holds for $s \geqslant 4$.

5.4.3. The higher degree case. The case in which $\deg(C) \geqslant 3$ is more interesting, since it gives rise to infinitely many mutations of the Newton–Okounkov body. Recall the notation $\{F_i\}_{i\in\mathbb{Z}_{>0}\cup\{-1\}}$ for the sequence of the Fibonacci numbers, and φ for the golden ratio (see §3.4).

Proposition 5.25. For each odd integer $i \geqslant 5$, there exists a rational curve $C_i \subseteq \mathbb{P}^2$ of degree F_i with a single cuspidal (i.e., unibranch) singularity at O and characteristic exponent $\frac{F_{i+2}}{F_{i-2}} \in (6, 7)$, whose six free points infinitely near O are in general position. Let C_1 be a line (of degree F_1 with characteristic exponent $\frac{F_3}{F_{-1}} = 2$) and C_3 be a conic (of degree F_3 with characteristic exponent $\frac{F_5}{F_1} = 5$). All these curves are (-1)-curves in their embedded resolution (i.e., after blowing up the appropriate points of the cluster of centres determined by the characteristic exponent).

If C is a general curve with $\deg(C) \geqslant 3$ through the origin O, then the curve C_i, with equation $f_i = 0$, through O and the first six infinitely near points to O along C, satisfies

$$\mu(C, s) = \frac{v_1(C,s;\ f_i)}{\deg(f_i)} = \begin{cases} \frac{F_{i-2}}{F_i} s & \text{if } s \in \left[\frac{F_i^2}{F_{i-2}^2}, \frac{F_{i+2}}{F_{i-2}}\right], \\ \frac{F_{i+2}}{F_i} & \text{if } s \in \left[\frac{F_{i+2}}{F_{i-2}}, \frac{F_{i+2}^2}{F_i^2}\right]. \end{cases}$$

Thus C_i is supraminimal for $v_1(C, s)$ for $s \in \left[\frac{F_i^2}{F_{i-2}^2}, \frac{F_{i+2}^2}{F_i^2}\right]$ and any odd $i \geqslant 1$.

Proof. The existence result is in [26, Theorem C, (a) and (b)]. The rest of the assertion is [11, Proposition 5.5].

Proposition 5.26. If $\deg(C) > 3$ and $O \in C$ is a general point, then:
(i) one has

$$(14) \qquad \Delta_{C,s_+} = \begin{cases} \Delta_{1,s,1} & \text{if } 1 \leq s < 2 \\ \Delta_{2,\frac{s}{2},\frac{1}{2}} & \text{if } 2 \leq s \leq 5 \\ \Delta_{\frac{5}{2},\frac{2s}{5},\frac{2}{5}} & \text{if } 5 < s \leq 6 + \frac{1}{4}, \end{cases}$$

hence Δ_{C,s_+} mutates at $s = 2, s = 5$, and depends linearly on s between mutations;
(ii) for $s \in [6 + \frac{1}{4}, \varphi^4)$ one has

$$\Delta_{C,s_+} = \Delta_{\frac{F_i}{F_{i-2}},\frac{F_{i-2}}{F_i}s,\frac{F_{i-2}}{F_i}} \text{ if } s \in \left[\frac{F_i}{F_{i-4}}, \frac{F_{i+2}}{F_{i-2}}\right];$$

i.e., Δ_{C,s_+} mutates at $\frac{F_{i+2}}{F_{i-2}}$, for all odd integers $i > 5$ (these mutations agree with part (ii) of Theorem 5.20), and depends linearly on s between mutations;
(iii) for $s \in (\varphi^4, 7)$, the Newton–Okounkov body is the quadrilateral with vertices

$$(0,0), \left(\frac{3s}{1+s}, 0\right), \left(\frac{1+s}{3}, \frac{1}{3}\right), \left(\frac{3s}{1+s}, \frac{3}{1+s}\right);$$

(iv) for $s \in (7, 7 + \frac{1}{9})$ one has

$$\Delta_{C,s_+} = \Delta_{\frac{8}{3},\frac{3}{8}s,\frac{3}{8}}.$$

Accordingly, there is a mutation at $s = 7$.

Proof. All the claims follow from Corollary 5.8 taking into account the computations of $\hat{\mu}(C,s)$ from [11], see Remark 3.15. The only problem, when applying Corollary 5.8, is to know where the vertex farthest to the right of Δ_{C,s_+} lies on the line $t = \hat{\mu}(C,s)$, i.e., we have to compute the number c appearing in Corollary 5.8(ii). This is given by the valuation of the curve C_1 or C_3 in case (i), of the curve C_i for any $i > 5$ in case (ii) (where both examples of curves are retrieved from Proposition 5.25), and the cubic curve D_1 from [11, Table 5.1] for both (iii) and (iv).

Remark 5.27. If $\deg(C) = d$, Proposition 5.25 leaves an unkown interval $[7 + \frac{1}{9}, d^2)$ whereas for $s \in [d^2, +\infty)$ the Newton–Okounkov body is known by Theorem 5.11 and there are no mutations there. Conjecturally, the same should happen for

$$s > 8 + \frac{1}{36}$$

(see Conjecture 3.13) if O is a general point of C.

Corollary 5.28. Assume $\deg(C) > 3$ and O is a general point of C. Then for $s \leq 7$, the Newton–Okounkov body Δ_{C,s_+} lies in the half-plane $t + u \leq 3$. Moreover,

(i) If $s \in [1, \varphi^4) \cup [7, 7 + \frac{1}{9})$, then Δ_{C,s_+} is a triangle whose vertices are valuative.
(ii) If $s \in (\varphi^4, 7)$, then Δ_{C,s_+} is a quadrilateral with at least one vertex being a non-valuative point.

Proof. All claims follow from Proposition 5.26, except the fact that Δ_{C,s_+} has a non-valuative vertex when $s \in (\varphi^4, 7)$, and that $(\frac{3s}{8}, \frac{3}{8})$ is valuative when $s \in [7, \frac{64}{9})$.

First, assume that $(\frac{3s}{1+s}, \frac{3}{1+s})$ is valuative for some $s < 7$. This means that there is a polynomial f of degree d with

$$v_1(C, s; f) = \frac{3s}{1+s} d, \quad \text{and} \quad \partial_+ v_1(C; f)(s) = \frac{3}{1+s} d.$$

As $v_1(C; f)$ is piecewise linear as a function of s, then for small enough > 0,

$$v_1(C, s + \ ; f) = \frac{3s}{1+s}d + \frac{3}{1+s}d\ , \quad \text{and} \quad \partial_+ v_1(C; f)(s + \ ) = \frac{3}{1+s}d.$$

In particular, this implies that

$$\frac{3s}{1+s} + \frac{3}{1+s}, \frac{3}{1+s} \quad \in \Delta_{C,(s+\ )_+}.$$

But this contradicts that for $s + \ < 7$, the Newton–Okounkov body Δ_{C,s_+} lies in the half-plane $t + u \leqslant 3$. So, $(\frac{3s}{1+s}, \frac{3}{1+s})$ is a non-valuative vertex.

Finally, $(\frac{3s}{8}, \frac{3}{8})$ is valuative because there is a unique curve V of degree 24 whose Newton polygon with respect to C has vertices $(0, 9)$ and $(64, 0)$. Indeed, let K be the cluster of centres of $v_1(C, \frac{64}{9})$, which consists of 8 free points followed by 8 satellites, each proximate to its predecessor and to P_7 (the continued fraction of $\frac{64}{9}$ is $[7; 9]$). Then V has multiplicity 9 at each of $P_1, \dots, P_7$ and multiplicity 1 at $P_8, \dots, P_{16}$.

The curve V has genus 1 and is obtained in this way. Consider the Cremona transformation ω determined by the homaloidal system of curves of degree 8 with triple points at a cluster C of seven general infinitely near, free, base points (this Cremona transformation appears in the construction of the curves C_i in Proposition 5.25, see [26, proof of Theorem C]).

There is a unique cubic curve Γ with a double point at the first point of C and passing simply through the remaining six points of C. This curve is contracted to a point by ω.

Let $x \in \Gamma$ be a general point. There is a pencil P of cubics having intersection multiplicity 8 with Γ at x. Then P has 9 base points, 8 are given by the cluster formed by x and by the 7 points infinitely near to x along Γ, and there is a further base point $y \in \Gamma$. The general curve of P is irreducible, and its image via ω is the required curve V, which has genus 1.

Remark 5.29. It is somewhat mysterious that in the case (ii) of Corollary 5.28 one has a vertex of Δ_{C,s_+} that is not valuative, taking into account that for $s < 7$, $s \in \mathbb{Q}$, the Mori cone of X_K is polyhedral (see [11]).

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