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Survey team on: conceptualisation of the role of competencies, knowing and knowledge in mathematics education research

Mogens Niss · Regina Bruder · Núria Planas · Ross Turner ·

Jhony Alexander Villa‐ Ochoa

Abstract This paper presents the outcomes of the work of the ICME 13 Survey Team on 'Conceptualisation and the role of competencies, knowing and knowledge in mathematics education research'. It surveys a variety of historical and contemporary views and conceptualisations of what it means to master mathematics, focusing on notions such as mathematical competence and competencies, mathematical proficiency, and mathematical practices, amongst others. The paper provides theoretical analyses of these notions— under the generic heading of mathematical competencies—and gives an overview of selected research on and by means of them. Furthermore, an account of the introduction and implementation of competency notions in the curricula in various countries and regions is given, and pertinent issues are reviewed. The paper is concluded with a set of reflections on current trends and challenges concerning mathematical competencies.

Keywords Mathematical competence · Mathematical competency · Mastering mathematics · Mathematical proficiency · Mathematical literacy · Bildungsstandards

1 Introduction

This paper presents the outcomes of the work of the ICME 13 Survey Team on 'Conceptualisation and the role of competencies, knowing and knowledge in mathematics education research'. The point of departure for the work is the question 'what does it mean to master mathematics?' This question has a number of related questions, such as 'what does it mean to possess *knowledge* of mathematics?'; '…to *know* mathematics?'; '…to have *insight* in mathematics?'; '…to be able *to do* mathematics?'; '…to possess *mathematical competence* (or *proficiency*)?'; and '…to be well versed in *mathematical practices*?'.

At a first glance, these related questions may seem to be roughly equivalent versions of the same question. However, they are, in fact, not (all) equivalent. The first three questions tend to focus on what we might think of as the container of mathematical products, i.e. the mathematical concepts, definitions, rules, theorems, algorithms, formulae, methods, historical and other facts, and so on, which have accumulated in the mind of "the knower". Thus, these questions deal with the receptive side of mathematical knowledge and with the nature of the specific content of the mathematics container and with the nature of "the knower's" ability to make statements about and relate to this content, for example by stating and explaining definitions, citing theorems and formulae—including the conditions and assumptions on which they rest—and relationships between these elements and the networks they are part of. The educational aspects of these issues are to do with what it takes for a learner of mathematics to become a knower of mathematics. In contrast, the latter three questions all tend to focus on the enactment of mathematics—its constructive side, so to speak—i.e. what it means, in specific terms, to engage in carrying out different kinds of characteristic mathematical processes. Here, the corresponding educational preoccupation is on what it takes for a learner of mathematics to become "a doer" of mathematics. Whilst it is pretty clear that knowing mathematics and doing mathematics are not quite the same thing, analytically speaking, it is also clear that there has to be an intimate relationship between the two. But what is the exact nature of this relationship? There is a plurality of answers to this question, which is indicative of the variety of approaches to the conceptualisation and roles of competencies, knowing and knowledge of mathematics.

It may be interesting to first note that the linguistic and connotational relationship between knowing and doing is different in different languages. In Scandinavian languages, for example in Danish, the verb "at vide"—"to know"—is basically intransitive. Typically, this verb requires a specification of what is known in the form of *a sentence* "I know *that* such and such is the case", but in contrast to what holds for English one cannot use "know" directly together with a noun object such as, say, mathematics or science. If you want a verb corresponding to "to know" to take a direct object, you have to use another verb, "at kende", but this implies a slight distortion of the meaning, corresponding to "know of" or "know about" (something) in English. However, in Danish (and in the other Scandinavian languages) we also have the verbs "at kunne", the basic meaning of which is "to be able to (do)". It is perfectly normal to say in Danish "hun kan matematik", formally translated into "she can do mathematics". However, by containing quite an amount of "knowing", the scope of these verbs is much wider than that of its English counterpart. So, a more adequate translation would be "she both knows and can do mathematics". In other words, by saying in Danish that someone "kan matematik" it is not really clear what exact balance has been struck between "knowing" and "being able to do" mathematics. It seems that the French verb "savoir" and the Spanish verb "saber" (both meaning "to know") contains the same duality of "knowing" and "being able to do", and that the distinction in German between "wissen" ("to know") and "können" ("to be able to (do)") is rather similar to the distinction found in Danish.

This linguistic detour suggests that the degree of involvement of the enactment of mathematics in the receptive aspect of mathematical knowledge, on the one hand, and of the receptive aspects of knowledge of mathematics in its enactment, on the other hand, gives rise to delicate issues, and that the intrinsic balance between these two aspects may take different shapes in different socio-cultural and linguistic environments. Whilst it may be possible—at least in principle—to possess an entirely receptive knowledge of mathematics without being able to engage in its enactment, this would be very difficult in practice. Not being able to enact mathematics, at least at some very basic level, seems to exclude significant parts of the receptive knowledge of mathematics. Conversely, it is almost a contradiction in terms to think that one might be able to enact mathematics without possessing any receptive knowledge about it.

Varying with time and place, different answers have been offered to all the questions above by people and agencies with different perspectives, points of view, sorts of backgrounds and positions (see, e.g. Bruder et al. 2015 concerning the question of what "basic knowledge" can mean at the upper secondary level). We shall provide examples in later sections of this paper. It is worth remarking here, though, that oftentimes neither the questions nor answers offered to them are stated explicitly. This does not mean that they are absent, only that both questions and answers tend to be taken for granted within a given context. In other words, they seem to remain on the level of tacit knowledge in corresponding quarters of current mathematics education research. Moreover, as already hinted at, below the surface things are more complicated and answers to the questions are much less uniform than one might think.

However, before attempting to answer these questions, we should consider another question: 'why are the initial questions significant?' Does it really matter what answers we give to the questions posed above? Are not the important issues to do with the actual content of mathematics teaching and learning, with the concrete activities students are supposed to engage in, with the tasks they are supposed to undertake, with the textbooks and other teaching materials made available to them, with the assessment of their achievements, and so on? Yes, indeed, but on what grounds have all these components been designed, selected, and composed? Whether explicit or implicit, answers to the initial questions determine at least three key components in mathematics education, including the ultimate purposes and the specific goals of mathematics education ('what do "we" wish to accomplish?'), the ensuing criteria for success of mathematics teaching and learning ('how, and when, do we know whether we have accomplished what we aim at?' and 'what means of assessment are suitable for generating valid information about the outcomes of mathematics education?'), and the structure and organisation of mathematics teaching ('what is going to happen inside and outside the mathematics classroom?'; 'what activities are teachers and students supposed to be engaged in?'; and 'what materials for teaching and learning are (should be) available to teachers and students?'). If answers to these questions and to the ones posed in the beginning differ in substantive ways, they will give rise to very different kinds of mathematics teaching and learning. In fact, one may well argue that one of the most important reasons why mathematics education around the world is, after all, so diverse is the very diversity of answers to this set of rather fundamental questions. Another aspect of the issue of significance of the questions at issue is to do with the utility of these questions for the progress of research in the field. How do answers to the initial questions relate to research on other topics and themes of mathematics education?

Answers to the questions above may be utilised in two different kinds of ways, in prescriptive/normative or in descriptive/analytic ways. The *prescriptive/normative* use focuses on *what ought to be* the case, for instance in specifying the goals and aims of mathematics education, in defining and designing curricula and teaching–learning activities, or in designing modes and instruments of formative or summative assessment, including tests and exams, to mention just a few. In contrast, *descriptive* and *analytic* uses focus on *what is actually* the case, for instance by uncovering what is on the agenda in various curricula, what is actually happening in mathematics teaching and learning in different settings and contexts, what the outcomes of this actually are, how students progress through the stages of mathematics education, and how well they "survive" the transition from stage to stage, or from one type of institution to another, within the education system. Such uses may also deal with judging whether some ways of orchestrating teaching and learning are superior to others when it comes to pursuing the goals and meeting the criteria for success in mathematics education. These kinds of use typically require a non-negligible amount of research and development. It should be kept in mind that "what is the case" is not a matter of universally valid facts. Rather it is context dependent, so that different answers are likely to emerge from the different contexts in which the corresponding questions are posed.

2 What does it mean to master mathematics, then? First, a brief historical outline

So far we have been considering the meaning and importance of dealing with the *question* of what it means to master mathematics, whether from an academic/intellectual point of view or from a policy or practice oriented point of view. In this section we shall take a closer look at *answers* actually given to these questions in the years 1935–1985 by different people and agencies in different places and contexts.

Classically, the focus of attention has been the *knowledge* of mathematical *facts* ("knowing what", concerning concepts, terms, results, rules, methods) as well as procedural *skills*, i.e. the ability to carry out well-delineated and well-rehearsed rule-based operations and routines fast and without errors ("knowing how"). For one illustration of this point of view—of course, others might equally well have been chosen—let us look at the case of Denmark, more specifically the royal decrees and the departmental order of 1935 (Undervisningsministeriet 1935a, b, c), issued by the Ministry of Education, concerning mathematics in the mathematics and science stream of upper secondary school, grades 10–12, which was allotted six lessons per week during all 3 years (all translations from Danish are by MN):

> "The aim of the teaching is to provide students with knowledge about the real numbers and their application in the description of functions, and knowledge about simple figures in the plane as well as in space [sic! the word "figure" is used for 3-dimensional objects]. The students should learn to operate with the apparatus of mathematical formulae and to acquire certainty and skill in numerical computation. Teaching will encompass the following topics:
>
> *a. Arithmetic and plane geometry"* (p. 92)
>
> [A total of 31 topics, including:]"1. Real numbers, sequences, limits" (p. 92)
>
> "3. The concept of function (including its graphical representation)." (p. 92)
>
> "5. General theory of similarity, including circles' corresponding points, definition and determination of the length of circle arcs." (p. 92)
>
> "8. Investigation of special functions (linear and affine functions, the function of inverse proportionality, quadratic functions, power functions with rational exponents, the exponential functions in base e and in base 10, base-ten logarithms)." (p. 92)
>
> "9. Trigonometric functions (sine, cosine, tangent and cotangent) of arbitrary angles and their interrelations (formulae for trigonometric functions of sums and differences of angles, computation of chord lengths)." (p. 92)
>
> "12. Continuous and differentiable functions. Differentiation of sums, products and quotients of functions, of composite and inverse functions […]; the
>
> mean value theorem; maximum and minimum."
>
> (p. 93)
>
> "13. Definite and indefinite integrals. […] integration by substitution; integration by parts; applications to the determination of areas and volumes of solids of revolution." (p. 93)
>
> "19. Ellipses and hyperbolas with their axes of symmetry as coordinate axes […]" (p. 94)
>
> "24 [Geometric] constructions, based on (i) loci known from middle school, (ii) the locus of those points whose distances to two given points have a given ratio, (iii) the locus of those points whose distances to two given lines have a given ratio. Constructions based on the theory of similar figures.
>
> 25. Complex numbers; the binomial equation; solution of quadratic equations in complex numbers." (p. 94)
>
> "28. Finite arithmetic and geometric series; examples of convergent and divergent series; infinite geometric series." (p. 95)
>
> "30 The induction proof." (p. 95)
>
> *"b. Stereometry"*(p. 95)

[A total of seven topics, including:]

"33. Congruence, symmetry and similarity." (p. 95)

"35. The sphere. Spherical triangles. The cosine and sine theorem with simple applications to—among other things—astronomical and geographical problems. (p. 95)

38. Determination of plane sections in cylinders and cones of revolution." (p. 95) (Undervisningsministeriet 1935a)

As is evident, this is an excerpt of a comprehensive and detailed syllabus—addressing a highly select and elite group of students, as was the case for upper secondary mathematics education many places in those days—which little doubt is left concerning the subject matter that had to be covered during the 3 years of upper secondary mathematics education in the mathematics and science stream. When it comes to teaching, the departmental order (Undervisningsministeriet 1935b) had the following—and nothing else—to say:

"As much as possible, teaching should pursue coherence across the different domains of the subject matter, thus putting the concept of function in the foreground in a natural manner.

In the theory of constructions one should refrain from dealing with too complicated problems; emphasis should be placed on clear and exhaustive explanations as well as on transparent and accurate figures.

Moreover, emphasis should put on developing students' sense of space (possibly by means of orthonormal projections).

Students should master mathematical formalism so that they can carry out simple computations. To this end the use of four-digit tables of logarithms, of trigonometric functions and their logarithms, and tables of quadratic numbers and of interests should be drilled.

Collaboration with those subjects, especially physics, to which mathematics may be applied, should be pursued. In the planning of teaching, attention should therefore be paid to bringing such collaboration to fruition." (p. 127)

Another royal decree (Undervisningsministeriet 1935c) specified the mandatory subject matter selection and examination requirements for the final national exam leading to the higher certificate of secondary education (baccalaureate):

"11. The test is written and oral.

1. At the written test candidates sit 2 sets of problems. At least half of the problems will be immediate applications of the subject matter studied. One of the problems may consist in giving a proof of a theorem in the subject matter selected for examination at the oral test. At least one problem will allow for assessment of candidates' skill and certainty in numerical computations. The time allotted for each set of problems is 4 h.

2. At the oral test candidates are assessed in subject matter roughly corresponding to half of the total amount of subject matter studied. The ministerial inspector informs each school of the subject matter selected for examination before the end of January.

The oral test is meant to particularly serve the purpose of examining whether the candidate has obtained both a thorough understanding and a general overview of the subject [mathematics]. One should not, therefore, restrict oneself to examining [the student in] a too narrowly delineated section [of the subject]." (pp. 648–649)

We have presented this rather extensive and detailed excerpt to give the reader an opportunity to take an "authentic look" at the way curricula were, in many places, formulated in the past. What can we infer from this about how the Danish Ministry of Education would, in 1935, answer the question "what does it mean to master mathematics?" (for upper secondary mathematics and science stream students, that is)? Well, first of all the opening statement concerning aims expresses an emphasis on factual knowledge and computational skill. Next, the 38 topics are all formulated in terms of concepts and results to be learnt and particular skills to be acquired, such as differentiating sums, products, and quotients of functions, calculating integrals, and carrying out certain geometric constructions. Finally, the guidelines for teaching focus on an integrative treatment of all the topics with the concept of function as an integrating factor. In other words, the predominant focus is on content, but with a derived focus on students' ability to provide careful and exhaustive explanations and produce accurate figures pertaining to the geometric constructions they are required to carry out. In the examination requirements, too, the Ministry emphasises the solving of problems involving immediate applications of the subject matter studied and—once again— skill and certainty in (numerical) computations. The only point at which the Ministry uses the verb "master" regards mathematical formalism and its application to (simple) computations. Whilst there is an evident emphasis on subject matter knowledge and procedural skill, it is interesting to notice that the Ministry also wants a different kind of learning outcome, namely sense of space, albeit by a very particular (if not peculiar) means: orthogonal projections. It should be mentioned that even though the 38 topics did indeed provide a tight and comprehensive syllabus, teachers and text book authors enjoyed a high degree of freedom to orchestrate their teaching or writing as they wished, as long as they observed the ministerial requirements and guidelines, and as long as not too many of their students were failed at the final national written and oral exams, organised by the Ministry. This implies that the teachers might well hold other views of what it means to master mathematics than those expressed in the ministerial documents (Niss 2016).

The sorts of conceptions of what mathematics education is all about, inherent in the above excerpts, became challenged from the end of the 1930s onward. The much quoted so-called Spens Report (Board of Education 1938) in the UK had the following to say about mathematics:

> "35. No school subject, except perhaps Classics, has suffered more than Mathematics from the tendency to stress secondary rather than primary aims, and to emphasise extraneous rather than intrinsic values. As taught in the past, it has been informed too little by general ideas, and instead of giving broad views has concentrated too much upon the kind of methods and problems that have been sometimes stigmatised as 'low cunning'. It is sometimes utilitarian, even crudely so, but it ignores considerable truths in which actual Mathematics subserves important activities and adventures of civilized man. It is sometimes logical, but the type and 'rigour' of the logic have not been properly adjusted to the natural growth of young minds. These defects are largely due to an imperfect synthesis between the idea that some parts of Mathematics are useful to the ordinary citizen or to certain widely followed vocations, and should therefore be taught to everybody, and the old idea that, when Mathematics is not directly useful, it has indirect utility in strengthening the powers of reasoning or in inducing a general accuracy of mind. *We believe that school Mathematics will be put on a sound footing only when teachers agree that it should be taught as art and music and Physical Science should be taught, because that it is one of the main lines which the creative spirit of man has followed in its development. If it is taught in this way we believe that it will no longer be true to say that 'the study of Mathematics is apt to commence in disappointment' […], and that it will no longer be necessary to give the number of hours to the subject that are now generally assumed to be necessary* [italics in the original]." (pp. 176–177).

In its somewhat ornate language this quotation proposes a change to what the Committee considered to be the traditional, superficial, low cunning approach to the teaching of mathematics which fails to pay attention to general ideas and broad views and to the truth by which mathematics has always underpinned civilised man's activities, adventures and creative spirit, an approach based on an unsatisfactory synthesis of mathematics as a subject permeated by reasoning and mathematics as a utilitarian and applicational subject. If such a change were instigated, the Committee believed that mathematics could do with fewer hours than those allocated to it in the late 1930s.

Already in the 1940s mathematicians and mathematics educators went on to point to other significant aspects of mastery of mathematics than just factual knowledge and procedural and computational skill. In the preface written in 1944 to the first edition of his soon famous book "How to Solve It", George Pólya (1945) wrote:

> "…a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of independent thinking."
> [Quoted from the 1957 (2nd) edition, p. v.]

Furthermore, later in the preface:

> "Studying the methods of solving problems, we perceive another face of mathematics. Yes, mathematics has two faces; it is the rigorous science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a systematic deductive science; but mathematics in the making appears as an experimental inductive science. Both aspects are as old as the science of mathematics itself. But the second aspect is new in one respect; mathematics 'in statu nascendi', in the process of being invented, has never before been presented in quite this manner to the student, or to the teacher himself, or to the general public."
> [Quoted from the 1957 (2nd) edition, p. vii.]

This preface speaks for itself. In addition to inaugurating problem solving as that key component in the teaching, learning and mastering of mathematics it became since the

1950s, the preface makes a more general plea for taking process oriented aspects of mathematics into consideration, including mathematics in the making.

Other process-oriented aspects entered the stage from the late 1950s on. Thus, as related in Barry Cooper's book *Renegotiating Secondary Mathematics: A Study of Curriculum Change and Stability* (Cooper 1985), in the UK there was a heated debate about the point made by industrialists that people with a university degree in mathematics far too often were unable to engage in putting their theoretical knowledge to use in dealing with extra-mathematical problems for purposes of application, and that mastery of mathematics, according to those industrialist, should therefore encompass the ability to undertake mathematical modelling and applied problem solving.

When conducting its First International Mathematics Study (FIMS), in the early 1960s, the IEA (the International Association for the Evaluation of Educational Achievement, which later also conducted the well-known TIMSS studies), identified five "cognitive behaviour levels", which, along with a number of traditional mathematical topics defined in terms of content, are involved in mathematics achievement. These are: "(a) knowledge and information: recall of definitions, notation, concepts; (b) techniques and skills: solutions; (c) translation of data into symbols or schema or vice versa; (d) comprehension: capacity to analyze problems, to follow reasoning; and (e) inventiveness: reasoning creatively in mathematics." (Husén 1967). Whilst (a) and (b) can be seen to just cast content knowledge and procedural skills in terms of cognitive behaviours, items (c), (d) and (e) point to overarching mathematical processes of a different nature.

In a paper with the telling title *Teaching Children to be Mathematicians vs. Teaching Children About Mathematics*, Seymour Papert (Papert 1972), the inventor and designer of the education software Logo, made a series of striking comments and suggestions concerning what it means to master mathematics (even though he did not use that word):

> "Being a mathematician is no more definable as 'knowing' a set of mathematical facts than being a poet is definable as knowing a set of linguistic facts. Some modern mathematical education reformers will give this statement a too easy assent with the comment: "Yes, they must understand, not merely know." But this misses the capital point that being a mathematician, again like being a poet, or a composer or an engineer, means *doing*, rather than knowing or understanding. This essay is an attempt to explore some ways in which one might be able to put children in a better position to *do* mathematics rather than merely to *learn about* it [italics in the original]." (p. 249) And later:

> "In becoming a mathematician does one learn something other and more general than the specific content of particular mathematical topics? Is there such a thing as a Mathematical Way of Thinking? Can this be learnt and taught? Once one has acquired it, does it then become quite easy to learn *particular* topics— like the ones that obsess our elitist and practical critics? [italics in the original]." (p. 250)

The cases and quotations presented above suffice to show that rather different answers to the question of what it means to master mathematics have been offered not only recently but also in the past, and that some of these answers point to aspects that go (far) beyond the knowledge of mathematical facts and acquisition of procedural skills. In general terms these answers pay attention to what is involved in the *enactment* of mathematics, i.e. working within and by means of mathematics in intra- and extramathematical contexts. The emphasis given to such aspects are based on one or more of the following views of mathematics. 'Mathematics is what professional mathematicians do'; 'mathematics is what users of mathematics do in their workplace'; 'mathematics is what ordinary citizens do in their private, social and societal lives'; and 'mathematics is what mathematics teachers do'.

Since the early 1990s much work has been done to develop notions such as mathematical competenc*e* and competenc*ies*, fundamental mathematical capabilities (PISA 2012), mathematical proficiency, and mathematical practices, in addition to their slightly more distant relatives: mathematical literacy, numeracy and quantitative literacy. One might say that the increasing attention being paid to these notions almost constitutes a "turn" in parts of mathematics education. The sections to follow provide a more systematics accounts of these notions and their role in mathematics education research and practice in various parts of the world.

3 Significant NCTM reports (USA) 1980–2000

Since the early 1950s, developments of mathematics education in the United States of America have exerted considerable influence on mathematics education discourses and practices throughout the world. It therefore seems warranted to take a closer look at those developments. One of the first systematic attempts to capture significant aspects of mastery of mathematics was made in the USA by the National Council of Teachers of Mathematics (NCTM). Already in 1980, as a response to the back-to-basics movement in the USA (which in turn was meant to counteract the negative consequences of the set theory based New Mathematics approach to mathematics education), the NCTM published a pamphlet called *An Agenda for Action: Recommendations for School Mathematics of the 1980s* (NCTM 1980), which insisted that also aspects that go beyond factual knowledge and procedural skills ought to be considered basic, above all problem solving. Of the eight recommendations put forward by the Board of Directors, the three crucial ones in relation to our context read as follows (p. 1):

> "The National Council of Teachers of Mathematics recommends that
>
> 1. problem solving be the focus of school mathematics in the 1980s;
>
> 2. basic skills in mathematics be defined to encompass more than computational facility;
>
> 5. the success of mathematics programs and student learning be evaluated by a wider range of measures than conventional testing;"

Following up on *An Agenda for Action*, the next main step taken by the NCTM was to establish, in the second half of the 1980s, a proposed set of national standards for school mathematics. The highly influential publication *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) identified five ability or attitude oriented mathematics goals for all K-12 students: (1) that they learn to value mathematics, (2) that they become confident in their ability to do mathematics, (3) that they become mathematical problem solvers, (4) that they learn to communicate mathematically, and (5) that they learn to reason mathematically (op. cit., p. 5). Furthermore four overarching standards are put forward, the same for all grade levels, albeit specified differently when it comes to details. These standards are presented as different essential process aspects of mathematics that permeate the acquisition of the subject: Mathematics as Problem Solving, Mathematics as Communication, Mathematics as Reasoning, and Mathematical Connections.

The 1989 *Standards* soon gave rise to debates and controversies in the USA, culminating during the next decade in the so-called "Math Wars" between quarters that adhered to and supported the *Standards* and *Standards*-based approaches to mathematics education, and quarters that objected to the underlying philosophy as well as to actual curricular implementations of such approaches. These objections—many of which came from some research mathematicians in universities—were founded on views of what it means and takes to come to grips with mathematics that were seen to be at odds with those that prevailed in the *Standards* movement, especially as regards understanding of theoretical concepts, procedural skills and the actual or potential role of technology.

The "Math Wars" was one of a number of factors behind the revision of the *Standards* undertaken by the NCTM in the last years of the twentieth century. This resulted in the publication *Principles and Standards for School Mathematics* (NCTM 2000), which after having formulated six basic principles for school mathematics education pertaining to equity, curricula, teaching, learning, assessment, and technology, put forward six overarching so-called *process standards* for all grade levels: Problem Solving, Reasoning and Proof, Communication, Connections, and Representations. It is readily seen that this set of process standards is an extension of the process standards of 1989. It is worth noting that in contrast to what one finds in the 1989 Standards, attitudinal aspects of individual's relating to and dealing with mathematics are no longer present in 2000.

4 Australian initiatives

In Australia, curriculum documents and their associated practices have incorporated mathematical processes (variously conceived and described) for many years. The state of Victoria was a protagonist in this development since the 1980s, especially when it came to implementing the ideas in assessment schemes and practices. Even though education is largely a state and territory rather than a national responsibility in Australia, a "National Statement on Mathematics for Australian Schools" (Australian Education Council 1990) was endorsed by each State, Territory and Commonwealth Minister for Education. That statement positioned mathematics to involve observing, representing, and investigating patterns and relationships in social and physical phenomena and between mathematical objects. It gave emphasis to both mathematical *products* (a body of knowledge) and mathematical *processes* (ways of knowing) that included mathematical thinking skills enabling the products to be developed, applied and communicated. *Mathematical modelling* was explicitly presented as a key element of "choosing and using mathematics". This line of thinking was developed further in the later 1990s to focus on what it means to "work mathematically". The document "Mathematics—a curriculum profile for Australian schools", published in 1994 by the Australian Education Council (1994), specified outcomes for working mathematically in the areas of *investigating*, *conjecturing*, *using problem solving strategies*, *applying and verifying*, *using mathematical language*, and *working in context* (italics added). These ideas greatly influenced the curriculum development in several Australian states such as Western Australia, New South Wales and Victoria. Later developments are outlined in the section "Mathematical competencies and similar constructs in selected national curricula" below.

5 The Danish KOM project: competencies and the learning of mathematics

For a variety of reasons, work done in Denmark since the late 1990s has inspired developments in a number of other countries. One such reason is that this work has informed, in various ways, the mathematical frameworks underlying the PISA mathematics surveys 2000–2012. So, even if Denmark is just one country amongst hundreds, we have found it well justified to give a more detailed account of this work.

In the second half of the 1990s various Danish education authorities, including the Ministry of Education, asked the Chair of this Survey Team—Mogens Niss—to direct a project in order to rethink the fundamentals of Danish mathematics education. Part of the reason for this was a number of observed problems in the teaching and learning of mathematics. Students' outcomes of mathematics education seemed to be unsatisfactory at primary, secondary and tertiary levels; the progression—and progress—achieved within any given segment of the education system was perceived as insufficient; major problems occurred in the transition from primary to lower secondary school, from lower to upper secondary school and from upper secondary school to tertiary education; and the recruitment of students to mathematics-laden tertiary programmes in mathematics, science, engineering, economics, and ICT (information and communication technology) was weakening both in terms of quantity and quality.

Based on previous work by its director (e.g. Niss 1999), the KOM group decided to focus on what it means to master mathematics across educational levels and institutions and across mathematical topics. In so doing it was expected to be possible to highlight the fundamentals, the characteristics and the commonalities of mathematics in all its manifestations, regardless of institution and level.

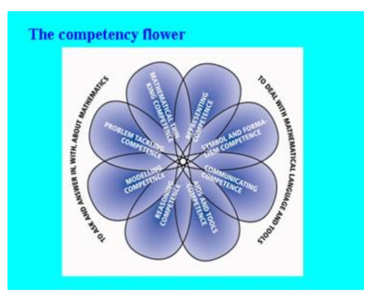
The essential point in this work was to define the notion of mathematical competence in terms of the ability to undertake mathematical *activity* in order deal with mathematical challenges of whichever kind (Niss and Jensen 2002, p. 43):

"*mathematical competence* means to have knowledge about, to understand, to exercise, to apply, and to relate to and judge mathematics and mathematical activity in a multitude of contexts which actually do involve, or potentially might involve, mathematics." [Translated from Danish by MN], and to identify the essential constituents of mathematical competence. The project identified eight such well-defined but overlapping constituents, named *mathematical competenc*ies (non-italics intended):

> "*a* mathematical competen*cy* is insight-based readiness to act purposefully in situations that pose *a particular kind* of mathematical challenge." [op. cit., p. 43, translated by MN. Italics added.] These eight mathematical competencies are

- mathematical *thinking* competency,
- *problem handling* competency,
- *modelling* competency,
- *reasoning* competency,
- *representations* competency,
- *symbols and formalism* competency, • *communication* competency, and
- *aids and tools* competency.

(For more detailed accounts of these competencies, including their definitions, nature and roles, see Niss and Højgaard 2011; Niss 2015a).The first four competencies primarily deal with posing and answering questions in, with and about mathematics, whereas the last four ones deal with the language and tools of mathematics. The competencies can be illustrated by the so-called *competency flower.* Each petal has a well-defined identity, the colour being more intense at its centre and gradually fading away towards to edge. The set of petals have a non-empty intersection, which suggests that whilst mutually distinct they all overlap.



Whilst the mathematical competencies all deal with the enactment of mathematics in situations involving particular kinds of mathematical challenges, it goes without saying that this enactment cannot take place without mathematical content knowledge and skills. However, the position taken in the project is that knowledge and skills are fuel to the enactment of the competencies in the same way as vocabulary and grammar are indeed necessary, yet highly insufficient, for the mastery of a given language in speech and writing.

The KOM project, in addition to mathematical competence and the eight competencies, all of which pertain to actually or potentially mathematics-laden *situations*, also identified three kinds of *overview and judgment* concerning mathematics as a *discipline*, the *actual application of mathematics in other fields and areas of practice*, the *historical development of mathematics* and *the specific nature of mathematics as a discipline and a subject*. Whilst certainly informed by mathematical competencies, these three forms of overview and judgment are not resulting automatically from the possession of the competencies but have to be cultivated separately in order to become part of the educational luggage of a mathematically competent person.

The KOM project has had considerable impact on a number of mathematics education undertakings in different parts of the world, partly directly and partly indirectly (through PISA), whether in curriculum reform, education and professional development of mathematics teachers, practices of mathematics teaching, or in national and international assessment schemes or programmes. Some of these influences will be subject of consideration in subsequent sections of this report. Suffice it, here, to be mentioned that the mathematical competencies have markedly informed the development of OECD's Programme of International Student Achievement (PISA) and its key construct mathematical literacy in a variety of different and sometimes complicated ways, the details of which can be found in (Stacey and Turner 2015; Niss 2015a).

6 Three high impact reports from the USA

Almost concurrently with the NCTM's publication of *Principles and Standards*, the National Research Council in the USA published the book *Adding It Up: Helping Children Learn Mathematics*, an outcome of the work of the Mathematics Learning Study Committee appointed by the Council (National Research Council 2001). After having looked at various terms the Committee decided to use the term mathematical proficiency to "capture what we believe is necessary for anyone to learn mathematics successfully". The Committee moved on to write

- "Mathematical proficiency, as we see it, has five components, or *strands*:*conceptual understanding*—comprehension of mathematical concepts, operations, and relations
- *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *strategic competence*—ability to formulate, represent, and solve mathematical problems
- *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
- *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

> These strands are not independent. They represent different aspects of a complex whole." (National Research Council 2001, p. 116)

To illustrate the last point, the report presents (p. 5) a metaphorical picture of the notion of mathematical proficiency by means of a cut-out section of a braided rope composed of five intertwined threads. It is interesting to note that *Adding It Up* puts an attitudinal component back into the picture (productive disposition), in addition to a general, i.e. not mathematics specific, mental capacity—adaptive reasoning.

At the same time as *Adding It Up* was being prepared, another influential group of prominent mathematics researchers in the USA—some of whom were also members of the *Adding It Up* group—prepared a publication for the RAND Corporation (RAND Mathematics Study Panel, 2003). This group, called the RAND Mathematics Study Panel was chaired by Deborah Loewenberg Ball and took its point of departure in the very same set of five intertwined mathematical proficiencies as did *Adding It Up*. The panel referred to these proficiencies as forming the "conception of what it means to be competent in mathematics" (p. 9). Chapter 3 of the RAND report is devoted to what the panel denoted *mathematical practices*, which are introduced as follows:

> "Because expertise in mathematics, like expertise in any field, involves more than just possessing certain kinds of knowledge, we recommend that […] the proposed research and development program focus explicitly on mathematical know-how—what successful mathematicians and mathematics users *do*. We refer to the things that they do as mathematical practices. Being able to justify mathematical claims, use symbolic notation efficiently, and make mathematical generalizations are examples of mathematical practices. Such practices are important in both learning and doing mathematics, and the lack of them can hamper the development of mathematical proficiency.
>
> […] While some students develop mathematical knowledge and skill, many do not, and those who do

> acquire mathematical knowledge are often unable to use that knowledge proficiently.” (op. cit., p. 29)

Without undertaking a systematic charting of mathematical practices, the panel gives further examples such as “mathematical representation, attentive use of mathematical language and definitions, articulated and reasoned claims, rationally negotiated disagreement, generalizing ideas and recognizing patterns” (op. cit. p. 32) and “problem solving” and “communication” (op. cit. p. 33). The panel perceives the role of these and other mathematical practices as underpinning mathematical proficiency (op. cit. p. 33), rather than as constituting it.

The term mathematical practices was also the one adopted in the second decade of the twenty-first century by the US *Common Core State Standards Initiative* (CCSSI) which was established in order to provide a platform for states in the US to join forces in basing their state curricula on an elaborate and detailed set of standards that they might decide to adopt if they so wanted. Interestingly enough, CCSSI-Mathematics combine standards and practices and speak of eight “Standards for Mathematical Practice” addressing all school levels (pp. 1–2):

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

7 Mathematical literacy, numeracy and quantitative literacy

Along with the notions of mathematical competence and competency, proficiency and practices other related notions have gained momentum around the world. The most prominent and widespread one is *mathematical literacy*, which has been the key construct in the PISA surveys since the very beginning. This is not the place to provide a substantive account of mathematical literacy and PISA (for an extensive exposition, see— especially—Stacey and Turner (2015); see also Jablonka and Niss (2014) and Niss 2015b). The most important point in our context is to clarify the complex relationship between mathematical literacy and mathematical competence/ies. The thrust of mathematical literacy is the ability to put mathematics to functional use in dealing with mathematics-laden aspects of the everyday, social and societal world in which an ordinary citizen lives. This certainly requires and involves mathematical competence and competencies but it does not exhaust them. Mathematical competencies deal with all aspects of the enactment of mathematics in all its manifestations, be this enactment functionally related to living in the world or not. In other words, there are several aspects of mathematical competencies that are not activated in exercising mathematical literacy. However, in the conceptualisation and analysis of mathematical literacy in PISA the mathematical competencies play an essential part in *underpinning* them. This is particularly true of PISA 2012 (OECD Organisation of Economic Co-Operation and Development (OECD) 2013), in which a condensed and “disjointified” version of the KOM Project competencies consisting of six fundamental mathematical capabilities forms the basis of the main components of the construct (for details, see Niss 2015a).

It can be debated whether *numeracy* and *quantitative literacy* are different names for mathematical literacy, just having emerged in other national contexts (the United Kingdom and the USA, respectively, Jablonka and Niss (2014, p. 392), or whether they stand for different notions and constructs. It does seem, though, that numeracy and quantitative literacy, as the terms suggest, are more to do with numerical aspects of dealing with real world magnitudes and with analysing and interpreting real life quantitative data than with aspects involving, say, geometry, algebra and functions which are included in mathematical literacy Common Core State Standards Initiative (2012).

8 Briefly summing up

Up till now this paper has addressed the scope of what in some contexts has been taken to constitute notions of mathematical mastery within mathematics education, as well as how such notions have changed over time. It is clear from the above brief accounts of a number of different initiatives in Australia, the USA and Denmark that mathematics educators in different parts of the world, in their struggle to propose theoretical concepts and constructs that can capture—beyond content knowledge and procedural skills— what it means to master mathematics, to be mathematically competent or proficient, to work mathematically or to be able to undertake mathematical practices, have identified remarkably similar foci and notions, even though they have worked somewhat independently of each other, at least initially, and even if the actual wording adopted varies from place to place.

Against this background, in the following sections of this report we shall, for brevity, use the term *mathematical competence* and *mathematical competencies* as the generic terms for all the constructs just mentioned without implying that the constructs are actually identical.

9 Research concerning mathematical competencies

When it comes to research concerning mathematical competencies, two types of research are of importance. The first type includes research in which the very construct of competence and competency is itself the *object* of theoretical or empirical investigation. The second type includes research in which mathematical competencies constitute a *means* of research for some other purpose. These differences are of an analytical nature, of course. The two types of research are not in conflict with one another, and a given piece of research may well combine the two types.

It is in no way possible, in this paper, to do justice to the existing body of research concerning mathematical competencies, which is already rather massive. We will have to confine ourselves to identifying and presenting some main lines of research and a few selected contributions within each such line.

9.1 Mathematical competencies as an object of research: Theoretical perspectives

A non-negligible amount of research on mathematical competencies and their nearer or more distant relatives has attempted to come to theoretical grips with the conceptual aspects of these notions. ‘What are the core constituents of these notions?’ and ‘what are the similarities and differences between them?’ are key questions to such research, as is the question of the extent to which differences can be explained by contextual, cultural or linguistic differences. Whichever specific notion we consider within the family of competency-oriented notions, it has first arisen as a theoretical construct defined and proposed by individual researchers, a group of researchers, a committee of agents appointed by and working on behalf of some organisation, or a task force operating on behalf of some official politico-administrative authority, for instance a local or national ministry of education. In other words, the construct does not grow out of systematic empirical, let alone experimental, work. Rather it grows out of several years of *reflective experience* accumulated and integrated in the minds of the proponents as a consequence of engaging in, observing, reflecting on and discussing situations, phenomena and traits in mathematics teaching and learning. Thus, the authors of the RAND report even called their notion of mathematical practices “speculative” (Rand Mathematics Study Panel 2003, p. 29): “After much deliberation, we chose it because we hypothesize that a focus on understanding these practices and how they are learned would greatly enhance our capacity to create significant gains in student achievement.” It is further characteristic of any competency construct that it involves a number of distinctions between different instances of the construct or between different sub-constructs or strands. As a matter of fact, these distinctions are in and of themselves essential components of the construct.

In the same way as it doesn’t make sense to claim that a proposed new definition is correct or incorrect, it doesn’t make sense to say that a proposed competency notion is right or wrong. Thus, by its very nature—*qua* definition—a particular competency definition is in the first instance a prescriptive construct, in that it introduces a certain term and specifies the conditions under which it can be used. Instead of discussing whether it is right or wrong one may well discuss whether it serves the purpose it was designed to serve, whether it contains all the significant features considered relevant and excludes the ones considered irrelevant, whether the level of aggregation of the categories involved is well balanced or, on the contrary, too coarse or too detailed, whether its range and scope are suitable, and what the consequences of adopting the notion are likely to be. Such discussions are indeed pertinent to the competency notions dealt with in this report and are reflected in the theoretical literature about them. We shall take a closer look at some of the issues.

First, there is an issue of whether it makes sense to derive the notions of mathematical competence or competencies from general notions of competence and competency that do not refer to any particular subject. In much of the German work on mathematical competencies done in the twenty-first century (see below), a general definition of competencies put forward by the German psychologist Franz Emanuel Weinert (2001, p. 27) has been taken as a guideline for subject specific competency definitions (Blum et al. 2006, p. 15). According to Weinert, competencies are: “the cognitive capabilities and skills available to or learnable by individuals in order to solve certain problems, as well as the associated motivational, volitional and social readiness and capability to successfully and responsibly utilise the respective problem solutions in various situations.” [Translated from German by RB and MN.]

Amongst the crucial words in this quotation are “certain problems” (“bestimmte Probleme” in German). Due to the lack of specification of the domain(s) in which the competencies are supposed to operate, the nature and grain size of the problems at issue remain unclear as does the associated notion of "solving" these problems. This is one reason why some authors are skeptical towards adopting a general but non-trivial notion of competence and competencies across disciplines and subjects (Niss and Jensen 2002, p. 66, Niss and Højgaard 2011, pp. 73-74).

Also in Latin America researchers have proposed overarching notions of competencies. Thus Tobón et al. (2010) have put forward a general proposal for education in competencies, including mathematical competencies, founded on the notion of problems. According to Tobón and his colleagues, competencies are integral actions undertaken to identify, analyse and solve problems in scenarios that include the issues of "know-why, "know-what" and "knowhow". Here, competencies are located with respect to the social contexts in which the problems occur, and content is closely related to the social contexts and problems under consideration.

The quotation from Weinert also contains another significant component, namely motivational, volitional and attitudinal elements in addition to the first-mentioned cognitive component. Similar elements are present in, say, the 1989 *NCTM Standards*, in *Adding It Up* and partly in the *CCSSI-mathematics*, whereas they are absent in the *KOM Project* and the *RAND Report*. Latin American researchers, too, amongst others, favour notions of competency that go beyond cognitive components and propose to include also dispositional and affective features such as attitudes, emotions, sensitivity and will in the constructs. This is for example true of D'Amore et al. (2008), Vasco (2012) and García et al. (2013).

In view of the remarks made in the beginning of this section, it cannot be determined on objective grounds whether a competency notion should be of a purely cognitive nature, or whether dispositional and affective elements should be included as well. This is simply a matter of choice with respect to the intended purpose of adopting the construct. From a research point of view, the most important thing, however, is not which choice has been made, but that those who make it recognise that the cognitive and the dispositional and affective elements, respectively, belong to different analytical categories and hence should not be mixed up, regardless of which stance one may take towards including both of them or not in competency notions. One consequence of this is that cognitive mathematical competencies are more determined by mathematical practices of humankind at large than are the dispositional and affective mathematical competencies which are much more closely tied to individuals or groups of individuals, and are likely to vary with time for these individuals or groups, a fact which has to be reflected in actual research on and by way of mathematical competencies.

The latter point touches upon an issue which in some parts of the mathematics education community is a subject of debate, sometimes of a controversial nature. If mathematical competencies are to do with mathematical practices, whose mathematical practices do we have in mind? To some, the focus should be on competencies related to universal mathematical practices, independent of technological, socioeconomic and cultural circumstances, as typically perceived by the international mathematics and mathematics education communities. To others—some of whom deny the existence of universal, context-free mathematical practices—the focus rather should be on competencies associated with mathematical practices that are closely related to the problems, contexts and conditions of local, national or regional communities and societies, and are perceived by their citizens as relevant to their culture and situation. The latter position is taken by researchers who adhere to what is called the socioepistemological theory of mathematics education (Cantoral 2013; Cantoral et al. 2014) or to the area of ethnomathematics (e.g. D'Ambrosio 2001). It should be kept in mind, though, that none of these researchers use the term competency but the term "situated mathematical knowledge".

9.2 Mathematical competencies as an object of research: empirical perspectives

The fact that mathematical competencies, in whatever specification we are talking about them, fundamentally are theoretical rather than empirical constructs certainly does not imply that, once defined, they are inaccessible to empirical investigation. On the contrary, they lend themselves to different kinds of empirical research.

The most basic question is whether some or all of the competencies can be detected and identified empirically in actual mathematical activities of people who are capable of "doing mathematics" to some degree or another. Although the general answer to this question is "yes", there are at least two important complications to consider.

The first one is that the competencies are not, in general, *defined* to be disjoint. On the contrary, as explicitly stated in *Adding It Up* and in the *RAND report*, they are *intertwined*. Even if each of them has a well-defined identity that makes it discernible from any other competency in theoretical terms, its execution will typically draw on some of the other competencies as well. In other words, the competencies are, in fact, more often than not, overlapping by definition, as is also acknowledged in the KOM report and visually represented by the intersecting petals in the flower metaphor. For example, the competency of posing and solving mathematical problems will necessarily involve at least some basic aspects of dealing with mathematical representations, mathematical symbols and formalism, or mathematical reasoning. If each of these three competencies were absent there would simply be no mathematical problem solving.

The second complication is that in cognitive terms mathematical competencies are neither developed nor possessed or enacted in isolation. They come together in large aggregate complexes. This is even more the case, if dispositional or affective components are taken into account as well. An individual who is able to justify his or her mathematical claims by way of some kind of mathematical reasoning will oftentimes also be able to communicate this reasoning to others, in some way or another, and probably also to support it by way of mathematical representations and some manipulation of symbolic expressions to help reach a conclusion.

These two kinds of complications imply that it is empirically demanding to disentangle the competencies from each other and especially to make a given one of them an object of study in isolation from the others. For this to be possible it is necessary to have a very clear and sharp definition of each competency and to have well thought-out research designs. We should not, however, exaggerate the difficulties involved in undertaking such research. Even though males and females, children and adults have many more physiological and biochemical features and properties in common than features and properties that separate them, we are certainly able to distinguish between members of these groups, and it is certainly possible to obtain valid and reliable physiological or biochemical research results on every one of them. Similarly, even though it is reasonable to expect socio-cultural differences in competency development, such differences can themselves be made objects of research investigations.

There is relatively little research designed to empirically investigate the existence of any entire *system* of mathematical competencies. Lithner and colleagues in Sweden have developed a modified version of the KOM Project competencies and have used it to study the fostering and development of these competencies in Swedish students (Lithner et al. 2010). Also Leuders (2014) has considered the entire set of competencies from a critical perspective. Otherwise, the research on an entire set of competencies has a different primary purpose. For example research done by members of the PISA mathematics expert group attempted to characterise, analyse, and explain the intrinsic difficulty of PISA items as well as their empirical item difficulty by means of such sets of mathematical competencies (Turner et al. 2013, 2015). The fact that these attempts turned out to be successful serves to empirically corroborate the existence and significance of the integral competency constructs involved. Also García et al. (2013) have employed PISAbased theoretical considerations to develop mathematical competencies with Colombian middle and high school students.

Lots of empirical research has been conducted on the individual competencies, in most cases long before competency-oriented notions were first coined. Such cases include *problem solving* on which masses of research has been carried out since the 1970s, initially inspired by the publication of the second edition in 1957 of Pólya's *How to Solve It* (Polya 1945). The same is true of *reasoning, proving and proof* which are processes and entities that, too, have formed the subject of a huge body of research since the 1970s. Another such case is *representations* and the transition and translation between them which have been studied intensively since the mid-1980s. As regards the ability to deal with *symbols and formalism*, research has tended to concentrate on algebraic manifestations of this competency. Actually, this is probably one of the areas of mathematics education on which research is most abundant and dates back the longest time, till the 1920s (Kieran 2007). Since the 1970s several thousands of papers, book chapters and books on algebraic symbolism and formalism have been published. But also more general aspects of symbols and symbols use have been studied, see e.g. Pimm (1995). Research on the ability to *communicate* mathematically dates back to the 1980s (Ellerton and Clarkson 1996; Pimm 1987) and has later grown considerably, see, e.g. Planas (2010). Research on the ability to deal with and use concrete *materials and technology* in mathematics education has a long history, gaining momentum from the 1970s (Szendrei 1996; Balacheff & Kaput 1996; Zbiek et al. 2007). As finally regards research on the *mathematical modelling* competency this is of a more recent date. Probably the first paper on this competency—in the paper called a metacognitive skill—was (Tanner & Jones 1995). The term "modelling competency" was introduced for the first time (in Danish) in a Master Thesis in 1996 (Hansen, Iversen & Troels-Smith 1996). Papers from the 2000s onwards made explicit use of this term, for example (Blomhøj & Jensen 2003), (Maass 2006) and (Böhm 2013). For a recent survey on what is empirically known about the learning and teaching of mathematical modelling, see Blum (2015).

As already mentioned, the majority of the research just referred to does not employ terms such as competencies, proficiency, practices or capabilities and hence cannot be claimed to be *designed* to subject these constructs and their sub-constructs to empirical investigation. However, the fact that their substantive aspects have been researched over several decades suggests that these constructs do indeed exist and have well-defined empirical content, in spite of conceptual or empirical overlaps amongst them. So, this research can be seen as a confirmation of the relevance and significance of the constructs. We might even hypothesise that the huge body of research on mathematical problem solving, reasoning, proving and proof, representations, symbols and formalism, communication, materials and (other) technology, and mathematical modelling constitutes the main source for these notions and their conceptualisations analysed in this report.

Since the introduction around the turn of the millennium of competency-oriented constructs in mathematics education, two issues have received particular research attention. The first issue is the *assessment of mathematical competencies*. It is no surprise that this has been a major focus of interest given the role of competencies in PISA and hence in subsequent attempts in various countries to instigate mathematics education reforms meant to increase student achievement in PISA terms. To such reforms, assessment of students' mathematical competencies become a primary priority. The possibilities and challenges involved in assessing students' possession and development of the competencies are in focus, both from a holistic and from an atomistic perspective, where a holistic perspective considers complexes of intertwined competencies in the enactment of mathematics, whereas an atomistic perspective zooms in on the assessment of the individual competency in contexts stripped, as much as possible, of the presence of other competencies. Thus, the book by Luis Rico and José Luis Lupiáñez (2008) in Spain devoted considerable attention to aspects of assessment of competencies. The impressive and massive large scale reform endeavours in Germany in response to the so-called 'PISA shock' gave rise to a large number of publications on the assessment of competencies, especially regarding different levels of competency possession. Examples include Siller et al. (2013) and Köller and Reiss (2013).

The second issue concerns *teachers'* coming to grips with the notion, interpretation and use of mathematical competencies, which, for obvious reasons, is seen as an essential factor in the dissemination and implementation of competency-oriented approaches to the teaching and learning of mathematics. As an example of such research we mention a rather large Swedish study by Boesen et al. (2014) investigating the impact of national reform in Sweden introducing mathematical competency goals. The study found that the teachers involved in the study are positive to the competency message

> "but the combination of using national curriculum documents and national tests to convey the message has not been sufficient for teachers to identify the meaning of the message. Thus, the teachers have not acquired the functional knowledge of the competence message required to modify their teaching in alignment with the reform." (p. 72)

The Danish KOM Project report (Niss and Jensen 2002, pp. 81–109, and Niss and Højgaard 2011, pp. 89–120) devoted an entire chapter to this *problématique* (see also Niss 2003). The IEA so-called Teacher Education and Development Study (TEDS-M) was focused on the readiness of primary and secondary mathematics teachers in 17 countries to teach mathematics (Tatto et al. 2012). It was also the main point of attention in the large long-term German development project *COACTIV: Professionswissen von Lehrkräften, kognitiv aktivierender Mathematikunterricht und die Entwicklung mathematischer Kompetenzen* (Professional knowledge of teachers, cognitively activating mathematics teaching and development of mathematical competencies), directed by the Max-Planck-Institut für Bildungsforschung (see Kunter et al. 2013).

9.3 Mathematical competencies as a means of research

In turning to research that has used mathematical competencies as a central vehicle, we shall confine ourselves to mentioning a few examples. The conceptualisation offered by mathematical competencies has been used in various ways to underpin theoretical and empirical research and development that does not have competencies as the primary focus. For example, the framework for designing a professional development course for upper secondary school mathematics teachers in Denmark to become mathematics counsellors is explicitly based on the notion of competencies (Jankvist & Niss 2015).

Jankvist & Misfeldt (2015) have used mathematical competencies in a study of the—sometimes problematic—effects of CAS use in upper secondary mathematics education in Denmark.

The mathematics working group of the European Society for Engineering Education (SEFI), as a result of indepth analysis and deliberations, has adopted a framework for mathematics curricula in engineering education which is based on the Danish KOM framework (Alpers et al. 2013).

Jaworski and her colleagues have used both frameworks as a means for identifying mathematical understanding with engineering students (Jaworski 2012).

As indicated in a previous section, mathematical competencies in PISA 2003 and fundamental mathematical capabilities in PISA 2012 were the crucial constructs adopted to theoretically analyse and empirically explain item difficulty in PISA (Turner et al. 2013, 2015).

10 Mathematical competencies and similar constructs in selected national curricula

This section deals with aspects of the state of the art as regards implementation of mathematical competencies and their relatives in practices of mathematics teaching and learning. The degree to which competencies have been put into practice varies greatly with place, educational context and educational level. So far, the implementation has primarily concerned curriculum planning and design, as well as pre-and in-service programmes for teachers—where it has been found to be challenging for teachers to come to grips with notions of mathematical competence/competencies and their relatives and, not the least, with their implementation. The same is true of the design and implementation of modes and instruments of assessment and evaluation of competencies. A general observation is that in most cases in which competencies or their relatives have been put to use in concrete contexts, the original notions and definitions have been modified or simply re-defined to suit the purposes and boundary conditions of that particular context. It also deserves to be mentioned that in some cases the introduction and implementation of competency-oriented notions in educational systems or sub-systems (i.e. particular segments—such as streams, levels or institutions—of an overarching educational system) have been of a rhetorical (i.e. 'lip service' like) rather than of a substantive nature.

It goes without saying that it is impossible in a journal article to chart the development and state of affairs regarding competency-oriented mathematics education in a large number of countries in the world. In what follows, the situation in selected countries is being outlined. These countries have been chosen because each of them brings important facets to the discussion of what it means to master mathematics. Other countries might have been chosen instead, so their absence in this report is not meant to suggest that they have less important contributions to offer to this discussion.

10.1 Australia and New Zealand

An Australian national curriculum (Australian Curriculum 2010) first came into being in 2010, and has been progressively introduced across Years K (kindergarten) to 10 in all Australian States since that time, according to different implementation timelines for each responsible State Education Authority. This curriculum is organised around the interaction of three content strands (Number and Algebra, Measurement and Geometry, and Statistics and Probability) and four *proficiency strands*, namely *understanding*, *fluency*, *problem solving* (which also makes reference to modelling of problem situations) and *reasoning*.

It is worth noting that these proficiency strands are almost identical—modulo wording—to the first four proficiency strands of *Adding It Up* and the *RAND report* in the USA.

Whilst the curriculum framework for mathematics in New Zealand is structured around three content strands, without explicit reference to mathematical competencies or processes, objectives for each of the defined levels refer to thinking *mathematically and statistically* and to the need in each content area to *solve problems* and *model situations*. Thus, the curriculum standards document states that while knowledge is critically important for mathematical understanding its primary role is to facilitate the student's solving of problems and modelling of situations. Just demonstrating knowledge—for example, by recalling basic facts—is not sufficient to meet a standard.

10.2 Germany, Austria and Switzerland

The unsatisfactory German mathematics results in the first PISA cycle in 2000 generated what was soon to be called "the PISA shock" in Germany. It was perceived as a national necessity to identify and implement serious measures to remedy the situation (for details, see Prenzel, Blum & Klieme 2015). Therefore, in 2003 the permanent congregation of the ministers of education and culture of the German "Länder" (states), the so-called *Kultusministerkonferenz* (abbreviated KMK), agreed to introduce a common set of binding educational standards, *Bildungsstandards*, in a number of key school subjects, including mathematics, across all 16 states, in the first step at the lower secondary level (Blum et al. 2006, p. 14 and http://www.iqb.huberling.de/bista), and in 2012 for the upper secondary level as well (Blum et al. 2015).

As far as mathematics is concerned, six general mathematical competencies formed what was termed the core of the mathematics standards (op. cit., p. 20): To reason mathematically ("mathematisch argumentieren"), to solve problems mathematically ("Probleme mathematisch lösen"), to do mathematical modelling ("mathematisch modellieren"), to use mathematical representations ("mathematische Darstellungen verwenden"), to deal with the symbolic/formal/technical aspects of mathematics ("mit Mathematik symbolisch/formal/technisch umgehen"), to communicate mathematically ("mathematisch kommunizieren") [Translated from German by MN]

According to the Standards, each of these competencies can be enacted at three different levels, briefly called "reproducing", "making connections" and "generalising and reflecting". It is readily seen that these competencies correspond closely to six of the eight competencies in the Danish KOM Project, to which explicit reference is made in the German framework.

As noted above, professional development programmes for teachers were undertaken to underpin the implementation of the *Bildungsstandards* in German schools.

Austria, too, in 2007, adopted *Bildungsstandards* in mathematics built on the notion of mathematical competencies, and developed these further in the years to come (Bundeskanzleramt [Austria] 2011; AEEC 2008). However, the notion of competency adopted is slightly different from and more complex than the ones found elsewhere. First of all the framework specifies three dimensions, called *mathematical action* ("mathematische Handlung"), *mathematical content* ("mathematischer Inhalt") and *complexity* ("Komplexität"). The dimension of mathematical action consists of four domains of action ("Handlungsbereiche"): H1: *Representing, building models* ("Darstellen, Modellbilden"), H2: *Computing, operating* ("Rechnen, Operieren"), H3: *Interpreting* ("Interpretieren") and H4: *Reasoning, justifying* ("Argumentieren, Begründen"), which is a category of the same kind as the one called mathematical competencies elsewhere. The four content domains are I1: "numbers and measures", I2; "variables, functional dependencies", I3: "geometrical figures and solids", and I4:"statistical representation and descriptors". The dimension of complexity looks at how involved the processes at issue are. There are three such levels: K1: "Activation of basic knowledge and skills", K2: "creating connections", and K3:"Activation of reflective knowledge, reflecting" which bear some resemblance with what in the PISA framework (OECD 2003) is called competency clusters. Altogether, this paves the way for defining the notion of *competency as a triple* (Hx, Iy, Kz) located in the three-dimensional space constituted by the three dimensions.

In Austria, a theoretically founded normative competency level model has been developed for the national school-leaving examinations (the so-called "Matura"), implemented as a nation-wide standard for the first time in 2015. Part of the purpose of this model was to provide a benchmark to ensure comparability of examination requirements over the next few years. Thus, Austria is one of the few countries to have consistently formulated basic mathematical competencies on the grounds of a specific education theoretical approach (developed by Roland Fischer and Günther Malle), seeking to verify these as far as possible within the framework of written tests (Siller et al. 2015).

In *Switzerland*, in 2007, EDK, the Swiss congregation of the education directors of the cantons, roughly corresponding to the German KMK, agreed on what was called the HarmoS-Konkordat, to instigate a harmonisation of compulsory school education across the cantons in the country. One outcome of this was the publication in 2011 (HarmoS 2011) of a set of Fundamental Competencies in Mathematics ("Grundkompetenzen für die Mathematik") for schools up to Year 11. Explicitly acknowledging inspiration from NCTM, PISA and KMK (the German "Bildungsstandards"), the Swiss standards identifies eight fundamental aspects of mathematical action: *knowing, realising and describing*; *operating and computing*; *employing instruments and tools*; *representing and communicating*; *mathematising and modelling*; *reasoning and justifying*; *interpreting and reflecting on results*; *investigating and exploring* (translated from German by MN), corresponding to what elsewhere is called competencies. These actions are placed as columns in a matrix, in which the rows—named "competency domains"—are five mathematical strands: number and variables; space and shape; magnitudes and measurement; functional relationships; and data and randomness. The framework then fills in the cells of the matrix for each of Year 4, Year 8 and Year 11. This gives rise to three levels of competence for each domain.

In a manner similar to the case of Austria, a competency is an entity with several different aspects. In this context, it is interesting to note that the Swiss framework, like the German one, also includes dispositional and volitional components:

> "Mathematical competence is not only manifested in knowing and doing ["Wissen und Können"] but also comprises interest, motivation and the ability and readiness for team work (non-cognitive dimensions). These dimensions belong to mathematical competence as well, but for the benefit of readability explicit formulations have been waived."

10.3 Asian countries

Classically, in East Asian mathematics curricula the emphasis has been on mathematical content, whilst the processes of *doing* mathematics have been seen as part of *learning* the content. However, recently revised mathematics curricula in East Asian countries tend to focus more on processes, which might be interpreted as versions of mathematical competencies. Thus, in the mathematics curriculum of 2011 in Korea, the Ministry of Education, Science and Technology stated:

> "Crucial capabilities required for members of a complex, specialized, and pluralistic future are believed to be fostered by learning and practicing mathematical processes, including mathematical problem solving, communication, and reasoning." (Ministry of Education, Science, and Technology 2011, p. 2)

As a matter of fact, problem solving, communication and reasoning had already been mentioned in the previous mathematics curriculum, but the 2011 curriculum put more emphasis on these processes/competencies and required them to be implemented in dealing with the content. For example, textbooks have to include them in each chapter. This curriculum "rejects learning by rote and emphasizes manipulation activities and the connection between mathematics and the real world. It particularly stresses self-directed problem-solving, reasoning, explanation and justification by utilizing students' intuitive understanding, knowledge and thinking skills" (Lew et al. 2012).

In paving the way for further new curriculum revisions in Korea, two more processes—now called core competencies—"creativity" and "information processing" have been added. This has given rise to discussions of whether "creativity" should be considered as being on the same level as other competencies or whether it is a higher order cognitive skill involving all these competencies. It is further being discussed whether "computational thinking" should be added to the set of competencies, and whether or not "mathematical modelling" is an independent competency. Some argue that a reasonably broad notion of "problem solving" naturally involves mathematical modelling, whereas others note that mathematical modelling has its own meaning and significance in the mathematics education community as well as in a number of other countries.

In a review conducted by the Australian Council for Educational Research (2016), on behalf of the South East Asian Ministers of Education Organization, of curriculum documents from several South East Asian countries, a number of interesting observations were made that bear on the role of mathematical competencies in those curricula, and on the ways in which mathematical proficiency is conceptualised and approached according to the formal curriculum statements. The report was based on an examination of mathematics curriculum documents for Brunei Darussalam, Cambodia, Indonesia, Lao PDR, Malaysia, Philippines, Singapore, Thailand, Timor Leste, and Vietnam. That examination revealed a high degree of consensus about the overarching purpose of education being to produce citizens who have skills and motivation to effectively apply their knowledge and skills in their everyday life. This conclusion is reflected in the definitions of mathematics used in most of these countries, which show a clear focus on connecting mathematical conceptual and procedural knowledge to usage in daily life and other ways of applying knowledge, but also in a variety of different ways in which those countries express broader goals of the mathematics curriculum. The countries continue to specify syllabi in a traditional way, but they are making clear moves towards acknowledging the importance of mathematical processes and competencies, albeit variously conceived and described. Most of the countries explicitly focus on mathematical thinking and reasoning, and on problem solving, and clearly identify highly valued learning outcomes that go beyond the narrow content-based skills inferable from a simple list of mathematical topics. The curriculum statements feature various ways of referencing different mathematical competencies.

For example, the curriculum statements of Singapore, from the Curriculum Planning and Development Division of the Ministry of Education,[1] articulate in detailed form a conception of the importance of competencies and an expression of mathematical knowledge as incorporating doing as an essential part of knowing. Singapore's curriculum is designed around the idea of mathematical problem solving and is underpinned by five inter-related components: skills, concepts, processes, attitudes and metacognition, all of which apply to all levels of the curriculum. It details three groups of mathematical processes: reasoning, communication and connections; applications and modelling; and thinking skills and heuristics. In particular, the curriculum documents for both primary and secondary level include a detailed presentation and discussion of the mathematical modelling process. Such a broad understanding of what mathematics should be for Singaporean schools is now very well established, having been cemented over the last several iterations of Singapore's regular curriculum review process.

[1] https//:www.moe.gov.sg/education/syllabuses/sciences

As seen in the ACER review referred to, several other neighbouring countries have inserted statements expressing similar perspectives in their revised curricula. Thus, the Indonesian curriculum documents now mention competencies including cognitive competencies, attitudes and skills, as well as the importance of being able to use mathematical concepts in solving problems that arise in daily life. Significant changes have been taking place in Indonesian mathematics education over a number of years, as described by Zulkardi in Chapter 15 of Stacey et al. (2015), with a move towards incorporating "reality" in mathematics education, the addition of PISA-like assessment tasks in the national assessment instruments, and through the introduction of a national competition that uses PISA-like tasks. Partly as a response to the poor performance of Indonesian students detected through programmes such as PISA, a new emphasis on competencies including reasoning, communication and solving contextualised problems is being pursued vigorously. Malaysia also explicitly refers to five process areas: communicating, reasoning, relating, problem solving and presenting. The curriculum document provides examples and strategies for developing these competencies. The Philippines curriculum also lists a number of highly valued process outcomes: knowing and understanding, estimating, computing and solving, visualising and modelling, representing and communicating, conjecturing, reasoning, proving and decision making, applying and connecting. Incorporation of these processes of mathematics learning is widespread amongst schools in the South East Asian region.

Another of the high–performing Asian countries is the Hong Kong Special Administrative Region of the People's Republic of China. The most recent curriculum statement from the government of Hong Kong [2] places strong emphasis on such generic skills as critical thinking, creativity, and the ability to communicate clearly and logically in mathematical language, as well as subject-specific knowledge and skills and, additionally, positive values and attitudes. Nine generic skills are specified in the curriculum that takes desired mathematical outcomes far beyond the mastery of specific mathematical content knowledge, into a realm that involves using that knowledge to deal with problems and challenges that come from all kinds of contexts.

10.4 Latin American countries

Since the late 1990s many Latin American countries, e.g. Brazil, Colombia and Chile, saw a development in mathematics curricula that focused on the fostering and development of mathematical thinking in diverse contexts, both as regards actual curriculum guidelines and curricula proposed by mathematics educators. From the beginning of the twenty-first century that development was taken considerably forward as new curriculum guidelines, much inspired by the PISA mathematics framework of 2003 (OECD 2003), introduced the notion of mathematical competency or similar constructs, especially in the context of national assessment schemes. In recent years, curriculum reforms along those lines have been carried out by the ministries of education in Colombia [Ministerio de Educación Nacional (MEN) 2006], Chile (Mineduc 2011), Mexico [Secretaría de Educación Pública (SEP) 2011], Costa Rica [Ministerio de Educación Pública (MEP) 2013] and the Dominican Republic [Ministerio de Educación (MINERD) 2014].

A common thrust of these reform endeavours has been to focus on students' recognition of the social role of school mathematics, and above all of real world problem solving, in everyday, social and societal life.

The mathematics curricula in Costa Rica, Chile, the Dominican Republic, Mexico and Colombia have used different terms to focus the purpose of education, namely capabilities (Chile), competencies (Mexico, the Dominican Republic, Colombia) and abilities (Costa Rica). Their curriculum frameworks place an emphasis on developing *mathematical thinking* (about algebra, numbers, statistics and probability, measurement and geometry) through *processes* such as problem solving, communication, reasoning, and modelling. In Chile, the Dominican Republic and Mexico, explicit attention to (*mathematical*) *attitudes* is being paid in the curricula. In Colombia, mathematical, scientific and everyday *contexts* are highlighted. One premise within these perspectives, and of the Latin American research presented above, is the functional role of mathematics, meaning that *a* mathematics should be useful in society and culture. This does not mean, however, that there is a homogenous understanding across Latin American countries about the way to develop, produce or acquire such knowledge, nor about the way curricula should be structured or about the role of contexts and instruments in the constitution of such knowledge.

[2] http://www.edb.gov.hk/en/curriculum-development/kla/ma/curr/basic-education-2002.html.

10.5 Spain and Portugal

The dissemination by the Spanish *National Institute of Educational Assessment* (Instituto Nacional de Evaluación Educativa) of the PISA 2000 and 2003 mathematics results twice gave rise to shocks in the Spanish education community, because of the unexpected and increasing distance between Spain and other participating countries. At that time, the Spanish mathematics curriculum for compulsory education (primary school 6–12, and (lower) secondary 12–16) had incorporated the standards and processes taken from the 1989 NCTM Standards, which had been translated in full into Spanish and had had a decisive impact on Spanish mathematics education. It was then suggested that a renewed curriculum was needed to help overcome the poor performance of the students in the country. Building on the NCTM standards and processes and attending to the PISA 2003 framework (OECD 2003), the most recent curriculum reform of 2006 included the notion of mathematical competencies similarly to the way it was being used in the PISA framework. A major goal for mathematics in primary and secondary education was the development of mathematical competen*ce* (in the singular). The notion of mathematical competency in Spanish curricula is literally linked to the capacity to develop and reinforce particular abilities like analysing, reasoning, formulating, connecting, checking, communicating etc. mathematical ideas in a variety of situations. There is a tension, however, between the traditional focus in Spain on problem solving and the more recent emphasis on competencies. Specific knowledge and ability concerning mathematical competencies other than problem solving are viewed and treated as de facto and tacitly developable through problem solving. Whilst on paper all competencies are equally acknowledged, classroom practices are often guided by activities that subordinate reasoning modelling etc. to problem solving.

In Spain, it has proved a great problem that teachers are not provided with the professional competencies and didactico-pedagogical resources needed to create classroom cultures, in which regular work to develop students' mathematical competencies becomes the norm. This became a subject of intensive debates, even in the media, about the lack of guidelines and support for the teachers, who used to live under classroom traditions and teaching methods mostly oriented towards the acquisition of technical knowledge and procedural skills. So, at the national Spanish level the role of mathematical competencies appears strong on paper but remains weak in terms of actual implementation and practice.

The issue of the mathematics teacher as a user of a "de facto insufficient" curriculum is being addressed by the education community in the region of Catalonia and has been in focus of an institutional initiative Catalonian Department of Education (2013a):

> "The ARC [Application of Resources to the Curriculum] Project has been started in order to model, pilot and evaluate mathematical activities within a competency framework […]. Activities and orientations will help teachers meet the challenge to assist all learners in the development of mathematical competencies by providing validated classroom experiences and tasks." (p. 50). [Translated from Catalan by NP]

At the time of writing several professional development courses for mathematics teachers, funded by the Catalan Government as part of the ARC Project, were being offered to teachers. The courses, which have been influenced by successive PISA frameworks, the 1989 NCTM Standards, the Common Core State Standards Initiative and the Danish KOM Project, have three foci. (1) What the competencies are (such as reasoning mathematically; posing and solving mathematical problems; communicating in, with and about mathematics; modelling mathematically). (2) What the learner is supposed to acquire when developing them (such as the ability to understand a mathematical chain of reasoning, to formulate a question as a mathematical problem, to express oneself mathematically, and to deal with models set up by others), and (3) What and how teachers should/may teach in order to pursue the goals inherent in (1) and (2).

By means of such courses it is intended to "complete" the curriculum from the perspective of successful teaching and learning scenarios for the development of mathematical competen*ce*, as defined (Catalonian Department of Education 2013b) by the Catalonian *Centre of Resources for Mathematic Teaching and Learning* (Centre de Recursos per Ensenyar i Aprendre Matemàtiques), much in line with the definition provided in the KOM Report (Niss & Jensen 2002; Niss & Højgaard 2011): "Mathematical competencies, and mathematical competence as a whole, refer to the ability to understand, judge, do, and use mathematics in a diversity of situations where mathematics plays or can be imagined to play a role." (no pagination).[Translated from Catalan by NP.]

Portugal, since the 1970s has placed problem solving and problem posing at the heart of mathematics education. Initially, however, this was viewed as a skill across mathematical content areas rather than a mathematical ability to be developed with students. The paper (Abrantes

2001) represented a significant effort to introduce the idea of mathematical competence into the Portuguese national curriculum. In 2009, mathematical competencies were stated as educational goals for primary and secondary school. Portugal, too, has seen debates taking place regarding the need to introduce more detailed notions of the mathematical competencies in the curricula in order for teachers to better be able to deal with them in their teaching. However, teachers are still struggling with competency based teaching, perhaps because of a rather diverse terminology (basic content, basic skills, basic competencies, essential competencies, capacity etc.) adopted in different teacher education programmes. Recent curriculum developments in 2013—in which, by the way, the term “competency” is absent—show a tension between what is/should be considered as content and what as capacity, and a debate has arisen as to whether it is possible to reconcile the two dimensions. At the time of writing, there seems to be a tendency to focus on content first and to insist that the mathematical capacities should be seen as ways of dealing with specifically indicated content knowledge.

11 Conclusion and final remarks

The survey presented above shows that notions and constructs of mathematical competencies and their relatives have gained considerable momentum in research, development and practices of mathematics education during the last two decades. It is fair to claim that this reflects a growing need to free mathematics education from the traditional straightjacket of reducing mathematical mastery to possessing factual content knowledge and procedural skills, the significance of these notwithstanding. There evidently is agreement that “Something more”, and perhaps even more important, has to be added to package.

The survey also shows that there is an overwhelming terminological diversity—if not outright unclarity and confusion—at play when mathematics educators want to analyse, characterise and name mathematical mastery. So many different notions, constructs, terms and conceptualisations exist in different parts of the world that one has to pose two questions: To what extent are the terms encountered different names for the same entity, and to what extent is the same term used to designate notions and constructs, which actually turn out to be different at a closer analysis? Our survey shows that the answer to both questions is: to a remarkable extent! Whilst there is no central committee of mathematics education that can normatively decide which terms to use for what—which is not even true of mathematics as a science—and no one can claim ownership to a term, it would be favourable if more terminological clarity were sought and achieved. Of course this is not likely to be an easy thing to achieve, if only for the reason that people speak different languages in different parts of the world and because there is no one-to-one correspondence between, say, abstract English terms and abstract terms in other languages, especially of non-indoeuropean language families. Nevertheless, it does seem possible and indeed worthwhile to try to establish larger conceptual and terminological clarity in these matters than we currently see. In so doing, we should not only recognise differences in terminology about competencies but also differences in the associated epistemological views, which may call for a wider set of analytical approaches, strategies and methods of research compatible with these views.

Also socio-cultural and politico-administrative reasons are co-responsible for the diversity of notions, constructs and terms across countries. Thus, this diversity is a reflection of the very different boundary conditions, circumstances, traditions and priorities that exist in different countries. It is neither desirable nor possible to strive for international harmonisation of these characteristics and features—that would come close to socio-cultural and political imperialism. Every country has to find its own way whilst being informed and inspired by international work and trends.

Terminological issues aside, despite the fact that mathematical competency notions and constructs are here to stay there are four points that deserve further attention.

We still need much more empirical research on the system of competencies vis-à-vis each individual competency, and on the interdependencies amongst individual competencies.

This is closely related to—but not entirely the same as—the need for devising more varied as well as more focused modes and instruments of assessment of the competencies, both individually, in groups and in their entirety.

Fostering, developing and furthering mathematical competencies with students by way of teaching is a crucial and highly demanding current and future priority for the teaching and learning of mathematics in all countries. Certainly the philosophers’ stone for this hasn’t been found yet. There is a long way to go for all of us. Fortunately, more and more reports of progress by way of quality teaching are appearing. We now need to understand the specific nature of the contexts and other factors that help create such progress, so as to see to what “quality teaching” could mean and be, and extent these contexts and factors can be transferred and generalised to other settings.

Last but certainly not least, there is a huge task lying in front of us in making competency notions understood, embraced and owned by teachers and in empowering them to develop teaching approaches and instruments that allow for the implementation of conceptually and empirically sound versions of mathematical competencies and their relatives in mathematics teaching and learning all over the world.

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