

## POLYNOMIAL FIRST INTEGRALS FOR WEIGHT-HOMOGENEOUS PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS OF WEIGHT DEGREE 4

JAUME LLIBRE AND CLÀUDIA VALLS

ABSTRACT. We classify all the weight-homogeneous planar polynomial differential systems of weight degree 4 having a polynomial first integral.

### 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this paper we deal with polynomial differential systems of the form

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \quad \mathbf{x} = (x, y) \in \mathbb{C}^2, \quad (1)$$

with  $\mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), P_2(\mathbf{x}))$  and  $P_i \in \mathbb{C}[x, y]$  for  $i = 1, 2$ . As usual  $\mathbb{Q}^+$ ,  $\mathbb{R}$  and  $\mathbb{C}$  will denote the sets of non-negative rational, real and complex numbers, respectively; and  $\mathbb{C}[x, y]$  denotes the polynomial ring over  $\mathbb{C}$  in the variables  $x, y$ . Here,  $t$  is real or complex.

We say that system (1) is *weight homogeneous* or *quasi-homogeneous* if there exist  $\mathbf{s} = (s_1, s_2) \in \mathbb{N}^2$  and  $d \in \mathbb{N}$  such that for arbitrary  $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R}, a > 0\}$ ,

$$P_i(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_i-1+d}P_i(x, y), \quad (2)$$

for  $i = 1, 2$ . We call  $\mathbf{s} = (s_1, s_2)$  the *weight exponent* of system (1), and  $d$  the *weight degree* with respect to the weight exponent  $\mathbf{s}$ . In the particular case that  $\mathbf{s} = (1, 1)$  system (1) is called a *homogeneous polynomial differential system of degree  $d$* .

Recently such systems have been investigated by several authors. Labrunie [12] and Moulin Ollagnier [15] characterize all polynomial first integrals of the three dimensional  $(a, b, c)$  Lotka–Volterra systems. Maciejewski and Strelcyn [14] proved that the so-called Halphen system has no algebraic first integrals. But some of the best results for general weight homogeneous polynomial differential systems have been provided by Furta [10] and Goriely [11], and additionally for quadratic homogeneous polynomial differential systems by Tsygvintsev [16], and Llibre and Zhang [13]. See also the works of Algaba, Freire, Fuentes, Garcia and Teixeira [1]–[5].

A non-constant function  $H(x, y)$  is a *first integral* of system (1) if it is constant on all solution curves  $(x(t), y(t))$  of system (1); i.e.  $H(x(t), y(t)) =$

---

2010 *Mathematics Subject Classification*. Primary: 34A05, 34A34, 34C14.

*Key words and phrases*. Polynomial first integrals, weight-homogeneous polynomial differential systems.

The first author is supported by the grants MICIIN/FEDER MTM 2008–03437, AGAUR 2009SGR410, and ICREA Academia. The third author is partially supported by FCT through CAMGDS, Lisbon.

constant for all values of  $t$  for which the solution  $(x(t), y(t))$  is defined. If  $H$  is  $C^1$ , then  $H$  is a first integral of system (1) if and only if

$$P_1 \frac{\partial H}{\partial x} + P_2 \frac{\partial H}{\partial y} = 0. \quad (3)$$

The function  $H(x, y)$  is *weight homogeneous of weight degree  $m$  with respect to the weight exponent  $\mathbf{s}$*  if it satisfies  $H(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^m H(x, y)$ , for all  $\alpha \in \mathbb{R}^+$ .

Given  $H \in \mathbb{C}[x, y]$  we can split it into the form  $H = H_m + H_{m+1} + \dots + H_{m+l}$ , where  $H_{m+i}$  is a *weight homogeneous* polynomial of weight degree  $m+i$  with respect to the weight exponent  $\mathbf{s}$ ; i.e.  $H_{m+i}(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{m+i}H_{m+i}(x, y)$ . The following well-known proposition (see [13] for a proof) reduces the study of the existence of analytic first integrals of a weight-homogeneous polynomial differential system (1) to the study of the existence of a weight-homogeneous polynomial first integral.

**Proposition 1.** *Let  $H$  be an analytic function and let  $H = \sum_i H_i$  be its decomposition into weight-homogeneous polynomials of weight degree  $i$  with respect to the weight exponent  $\mathbf{s}$ . Then  $H$  is an analytic first integral of the weight-homogeneous polynomial differential system (1) if and only if each weight-homogeneous part  $H_i$  is a first integral of system (1) for all  $i$ .*

The main goal of this paper is to classify all analytic first integrals of the weight-homogenous planar polynomial differential systems of weight degree 4. In view of Proposition 1 we only need to classify all the polynomial first integrals of the weight-homogenous planar polynomial differential systems of weight degree 4. The classification of all polynomial first integrals (and hence of all analytic first integrals) of the weight-homogenous planar polynomial differential systems of weight degree 1 is straightforward and trivial. The classification of all polynomial first integrals (and hence of all analytic first integrals) of the weight-homogenous planar polynomial differential systems of weight degree 2 was given in [13, 8] and for systems of weight degree 3 was given in [6, 8].

In the classification of all polynomial first integrals for weight-homogenous planar polynomial differential systems of weight degree 2 and 3 the authors use the Kowalevskaya exponents, but as it was shown in Theorems 4 of [6] these exponents are useless for classifying the polynomial first integrals for weight-homogenous planar polynomial differential systems of weight degree larger than 3.

**Proposition 2.** *In  $\mathbb{C}^2$  the systems with weight degree 4 can be written as the following ones with their corresponding values of  $\mathbf{s}$ :*

$$\mathbf{s} = (1, 1) : \begin{aligned} \dot{x} &= a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4, \\ \dot{y} &= b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4; \end{aligned}$$

$$\mathbf{s} = (1, 2) : \begin{aligned} \dot{x} &= a_{40}x^4 + a_{21}x^2y + a_{02}y^2, \\ \dot{y} &= b_{50}x^5 + b_{31}x^3y + b_{12}xy^2; \end{aligned}$$

$$\mathbf{s} = (1, 3) : \begin{aligned} \dot{x} &= a_{40}x^4 + a_{11}xy, \\ \dot{y} &= b_{60}x^6 + b_{31}x^3y + b_{02}y^2; \end{aligned}$$

$$\mathbf{s} = (1, 4) : \begin{aligned} \dot{x} &= a_{40}x^4 + a_{01}y, \\ \dot{y} &= b_{70}x^7 + b_{31}x^3y; \end{aligned}$$

$$\mathbf{s} = (2, 3) : \begin{aligned} \dot{x} &= a_{11}xy \\ \dot{y} &= b_{30}x^3 + b_{02}y^2; \end{aligned}$$

$$\mathbf{s} = (2, 5) : \begin{aligned} \dot{x} &= a_{01}y, \\ \dot{y} &= b_{40}x^4; \end{aligned}$$

$$\mathbf{s} = (3, 3) : \begin{aligned} \dot{x} &= a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} &= b_{20}x^2 + b_{11}xy + b_{02}y^2; \end{aligned}$$

$$\mathbf{s} = (6, 9) : \begin{aligned} \dot{x} &= a_{01}y, \\ \dot{y} &= b_{20}x^2. \end{aligned}$$

Proposition 2 is proved in section 2.

In what follows we state our main results, i.e. we classify when the systems of Proposition 2 exhibit a polynomial first integral. The systems with weight exponent (1, 1) having a polynomial first integral are given in section 3, because its classification is very long. The systems with weight exponent (3, 3) having a polynomial first integral are studied inside the systems with weight exponent (1, 2). For the other systems of Proposition 2 we provide in this introduction their polynomial first integrals.

We introduce the change  $(X, Y) = (x^2, y)$  in the planar weight homogeneous polynomial differential systems (1) of weight degree 4 with weight exponent (1, 2). With these new variables  $(X, Y)$  the systems with weight exponent (1, 2) becomes, after introducing the new independent variable  $d\tau = x dt$ ,

$$X' = 2a_{40}X^2 + 2a_{21}XY + 2a_{02}Y^2, \quad Y' = b_{50}X^2 + b_{31}XY + b_{12}Y^2, \quad (4)$$

where the prime denotes derivative with respect to  $\tau$ .

System (4) is a homogeneous quadratic planar polynomial system (with  $\mathbf{s} = (1, 1)$ ). It is well-known, see [9], that for each quadratic homogeneous system there exists some linear transformation and a rescaling of time which transforms system (4) into systems in (5).

$$\begin{aligned}
\dot{x} &= -2xy + \frac{2}{3}x(p_1x + p_2y), & \dot{y} &= -x^2 + y^2 + \frac{2}{3}y(p_1x + p_2y), \\
\dot{x} &= -2xy + \frac{2}{3}x(p_1x + p_2y), & \dot{y} &= x^2 + y^2 + \frac{2}{3}y(p_1x + p_2y), \\
\dot{x} &= -x^2 + \frac{2}{3}x(p_1x + p_2y), & \dot{y} &= 2xy + \frac{2}{3}y(p_1x + p_2y), \\
\dot{x} &= \frac{2}{3}x(p_1x + p_2y), & \dot{y} &= x^2 + \frac{2}{3}y(p_1x + p_2y), \\
\dot{x} &= \frac{2}{3}x(p_1x + p_2y), & \dot{y} &= \frac{2}{3}y(p_1x + p_2y).
\end{aligned} \tag{5}$$

We prove the following theorem that characterizes all the polynomial first integrals for the systems in (5).

**Theorem 1.** *The homogeneous polynomial systems in (5) have a polynomial first integral  $H$  if and only if one of the following conditions hold.*

- (a) *The first system in (5) with  $p_1 = 0$ ,  $p_2 = 3(1 - q)/(1 + 2q)$  with  $q = n/m \in \mathbb{Q}^+$ , and in this case  $H = x^m(3y^2 - x^2)^n$ .*
- (b) *The second system in (5) with  $p_1 = 0$ ,  $p_2 = 3(1 - q)/(1 + 2q)$  with  $q = n/m \in \mathbb{Q}^+$ , and in this case  $H = x^m(3y^2 + x^2)^n$ .*
- (c) *The third system in (5) with  $p_1 = 0$  and  $p_2 = 3(1 - 2q)/(2(1 + q))$  with  $q = n/m \in \mathbb{Q}^+$ , and in this case  $H = x^m y^n$ .*
- (d) *The fourth system in (5) with  $p_1 = p_2 = 0$ , and in this case  $H = x$ .*

We note that systems with weight exponent (3, 3) coincide with the systems (4) and hence it can be written into systems in (5). Therefore, Theorem 1 applies to those systems. The proof of Theorem 1 is given in section 4.

**Theorem 2.** *The weight homogeneous polynomial differential systems with weight exponent (1, 3) and weight degree 4 have a polynomial first integral  $H$  if and only if the following conditions hold.*

- (a)  $a_{11} = a_{40} = 0$  with  $H = x$ .
- (b)  $b_{60} = b_{31} = b_{02} = 0$  with  $H = y$ .
- (c)  $(3a_{11} - b_{02})(3a_{11} - 2b_{02}) \neq 0$ ,  $a_{40} = -a_{11}b_{31}/(3a_{11} - 2b_{02})$ ,  $3a_{11}/(6a_{11} - 2b_{02}) = m/n \in \mathbb{Q}^+$  and  $m/n < 1$  with

$$H = x^{3(n-m)}((3a_{11} - 2b_{02})b_{60}x^6 + 2(3a_{11} - b_{02})b_{31}x^3y - (9a_{11}^2 - 9a_{11}b_{02} + 2b_{02}^2)y^2)^m.$$

- (d)  $b_{31}/a_{40} = -m/n$  and  $m/n \in \mathbb{Q}^+$  with  $H = x^{3m}(b_{60}x^3 + (b_{31} - 3a_{40})y)^{3n}$ .
- (e)  $b_{02} = 0$ ,  $a_{11} \neq 0$ ,  $a_{40} = -b_{31}/3$  and  $b_{06} = -(3a_{40} - b_{31})^2/(12a_{11})$  with  $H = b_{31}x^3 - 3a_{11}y$ .
- (f)  $(3a_{11} - b_{02})(3a_{11} - 2b_{02}) \neq 0$ ,  $a_{40} = -a_{11}b_{31}/(3a_{11} - 2b_{02})$ ,  $b_{06} = -(3a_{40} - b_{31})^2/(4(3a_{11} - b_{02}))$ ,  $b_{02} \neq 0$  and  $-3a_{11}/b_{02} = n/m \in \mathbb{Q}^+$  with

$$H = x^{3n}(b_{31}x^3 + (2b_{02} - 3a_{11})y)^m.$$

The proof of Theorem 2 is given in section 5.

**Theorem 3.** *The weight homogeneous polynomial differential systems with weight exponent (1, 4) and weight degree 4 have a polynomial first integral  $H$  if and only if the following conditions hold.*

- (a)  $a_{40} = a_{01} = 0$  and  $H = x$ .
- (b)  $b_{70} = b_{31} = 0$  and  $H = y$ .

- (c)  $b_{31} = -4a_{40}$ , and  $4a_{40}^2 + a_{01}b_{70} \neq 0$  with  $H = -b_{70}x^8 + 8a_{40}x^4y + 4a_{01}y^2$ .
- (d)  $b_{31} = -4a_{40}$ ,  $a_{40}b_{70} \neq 0$  and  $4a_{40}^2 + a_{01}b_{70} = 0$  with  $H = b_{70}x^4 - 4a_{40}y$ .

Theorem 3 is proved in section 6.

**Theorem 4.** *The weight homogeneous polynomial differential systems with weight exponent  $(2, 3)$  and weight degree 4 have a polynomial first integral  $H$  if and only if  $a_{11} = 0$  in which case  $H = x$ , or  $b_{30} = b_{02} = 0$  in which case  $H = y$ , or  $a_{11}(3a_{11} - 2b_{02}) \neq 0$  and  $-2b_{02}/a_{11} = n/m \in \mathbb{Q}^+$ , in which case*

$$H = x^n(2b_{30}x^3 - 3a_{11}y^2 + 2b_{02}y^2)^m.$$

The proof of Theorem 4 is given in section 7.

**Theorem 5.** *The weight homogeneous polynomial differential systems with weight exponent  $(2, 5)$  and weight degree 4 have the polynomial first integral  $H = 2b_{40}x^5 - 5a_{01}y^2$ .*

The proof of Theorem 5 is given in section 8.

**Theorem 6.** *The weight homogeneous polynomial differential systems with weight exponent  $(6, 9)$  and weight degree 4 have the polynomial first integral  $H = 2b_{20}x^3 - 3a_{01}y^2$ .*

The proof of Theorem 6 is given in section 9.

## 2. PROOF OF PROPOSITION 2

From the definition of weight homogeneous polynomial differential systems (1) with weight degree 4, the exponents  $u_i$  and  $v_i$  of any monomial  $x^{u_i}y^{v_i}$  of  $P_i$  for  $i = 1, 2$ , are such that they satisfy respectively the relations

$$s_1u_1 + s_2v_1 = s_1 + 3, \quad \text{and} \quad s_1u_2 + s_2v_2 = s_2 + 3. \quad (6)$$

Moreover, we can assume that  $P_1$  and  $P_2$  are coprime, and without loss of generality we can also assume that  $s_1 \leq s_2$ . We consider different values of  $s_1$ .

*Case  $s_1 = 1$ .* If  $s_2 = 1$  then in view of (6) we must have  $u_1 + v_1 = 4$  and  $u_2 + v_2 = 4$ , that is,  $(u_i, v_i) = (0, 4)$ ,  $(u_i, v_i) = (1, 3)$ ,  $(u_i, v_i) = (2, 2)$ ,  $(u_i, v_i) = (3, 1)$  and  $(u_i, v_i) = (4, 0)$  for  $i = 1, 2$ .

If  $s_2 = 2$  then in view of (6) we must have  $u_1 + 2v_1 = 4$  and  $u_2 + 2v_2 = 5$ , that is,  $(u_1, v_1) = (0, 2)$ ,  $(u_1, v_1) = (2, 1)$  and  $(u_1, v_1) = (4, 0)$ , while  $(u_2, v_2) = (1, 2)$ ,  $(u_2, v_2) = (3, 1)$  and finally  $(u_2, v_2) = (5, 0)$ .

If  $s_2 = 3$  then in view of (6) we must have  $u_1 + 3v_1 = 4$  and  $u_2 + 3v_2 = 6$ , that is,  $(u_1, v_1) = (1, 1)$ ,  $(u_1, v_1) = (4, 0)$ , while  $(u_2, v_2) = (0, 2)$ ,  $(u_2, v_2) = (3, 1)$  and finally  $(u_2, v_2) = (6, 0)$ .

If  $s_2 = 4$  then in view of (6) we must have  $u_1 + 4v_1 = 4$  and  $u_2 + 4v_2 = 7$ , that is,  $(u_1, v_1) = (0, 1)$ ,  $(u_1, v_1) = (4, 0)$ , while  $(u_2, v_2) = (3, 1)$  and finally  $(u_2, v_2) = (7, 0)$ .

If  $s_2 = 4 + l$  with  $l \geq 1$ , then equation (6) becomes

$$u_1 + (4 + l)v_1 = 4 \quad \text{and} \quad u_2 + (4 + l)v_2 = 7 + l. \quad (7)$$

From the first equation of (7) we get  $v_1 = 0$  and  $u_1 = 4$ . By the second equation of (7) it follows that  $v_2 \in \{0, 1\}$ . If  $v_2 = 0$  then  $u_2 = 7 + l$ , and if  $v_2 = 1$  then  $u_2 = 3$ . In both cases  $P_1$  and  $P_2$  are not coprime. So this case is not considered.

*Case  $s_1 = 2$ .* Now we have  $s_2 \geq 2$ . If  $s_2 = 2$  then in view of (6) we must have  $2u_1 + 2v_1 = 5$  and  $2u_2 + 2v_2 = 5$ , which is not possible because 5 is not an even number.

If  $s_2 = 3$  then in view of (6) we must have  $2u_1 + 3v_1 = 5$  and  $2u_2 + 3v_2 = 6$ , that is,  $(u_1, v_1) = (1, 1)$ , while  $(u_2, v_2) = (0, 2)$  and  $(u_2, v_2) = (3, 0)$ .

If  $s_2 = 4$  then in view of (6) we must have  $2u_1 + 4v_1 = 5$  and  $2u_2 + 4v_2 = 7$ , which is not possible because 5 is not even.

If  $s_2 = 5$  then in view of (6) we must have  $2u_1 + 5v_1 = 5$  and  $2u_2 + 5v_2 = 8$ , that is,  $(u_1, v_1) = (0, 1)$  and  $(u_2, v_2) = (4, 0)$ .

If  $s_2 = 5 + l$  with  $l \geq 1$ , then equation (6) becomes

$$2u_1 + (5 + l)v_1 = 5 \quad \text{and} \quad 2u_2 + (5 + l)v_2 = 8 + l. \quad (8)$$

The first equation of (8) is not possible because 5 is not an even number,  $5 + l \geq 6$  and  $u_1, v_1$  are non-negative integers.

*Case  $s_1 = 3$ .* Now we have  $s_2 \geq 3$ . If  $s_2 = 3$  then in view of (6) we must have  $3u_1 + 3v_1 = 6$  and  $3u_2 + 3v_2 = 6$ , that is,  $(u_i, v_i) = (0, 2)$ ,  $(u_i, v_i) = (1, 1)$ ,  $(u_i, v_i) = (2, 0)$ , for  $i = 1, 2$ .

If  $s_2 = 3 + l$  with  $l \geq 1$ . In view of (6) we must have

$$3u_1 + (3 + l)v_1 = 6 \quad \text{and} \quad 3u_2 + (3 + l)v_2 = 6 + l. \quad (9)$$

From the first equation of (9) we have that

$$v_1 = \frac{6 - 3u_1}{3 + l} \leq \frac{6}{3 + l},$$

and using that  $l \geq 1$  then  $v_1 \in \{0, 1\}$ .

When  $v_1 = 0$  then  $3u_1 = 6$  and thus  $u_1 = 2$ . Then from the second equation of (9) we get that  $v_2 \in \{0, 1\}$ . If  $v_2 = 0$  then  $u_2 \geq 1$ , and if  $v_2 = 1$  then  $u_2 = 1$ . In both cases we have that  $P_1$  and  $P_2$  are not coprime.

When  $v_1 = 1$  then  $3u_1 = 3 - l$ , which is not possible since  $u_1$  is an integer and  $l \geq 1$ .

*Case  $s_1 = 3 + l$  with  $l \geq 1$ .* Now we have  $s_2 \geq 3 + l$  with  $l \geq 1$  and equation (6) becomes

$$(3 + l)u_1 + s_2v_1 = 6 + l = (3 + l) + 3 \quad \text{and} \quad (3 + l)u_2 + s_2v_2 = 3 + s_2. \quad (10)$$

From the first equation of (10) and taking into account that  $l \geq 1$ , we get that  $u_1 \in \{0, 1\}$ .

When  $u_1 = 0$  we must have  $s_2v_1 = 6 + l$ , and since  $s_2 \geq 3 + l$  we get

$$v_1 = \frac{6 + l}{s_2} \leq \frac{(3 + l) + 3}{3 + l} = 1 + \frac{3}{3 + l}.$$

Since  $v_1 \neq 0$  and  $l \geq 1$  we must have  $v_1 = 1$ , and then  $s_2 = 6 + l$ . Now the second equation of (10) becomes

$$(3 + l)u_2 + (6 + l)v_2 = (6 + l) + 3. \quad (11)$$

Then  $v_2 = 0$ . From (11) we have  $u_2 = 1 + \frac{6}{l+3}$ . Thus  $l = 3$  and  $u_2 = 2$ . So we get the systems with weight exponent  $(6, 9)$ .

When  $u_1 = 1$  we must have  $s_2 v_1 = 3$  and since  $s_2 \geq 3+l$  we get  $(3+l)v_2 \leq 3$ , which is not possible because  $l \geq 1$ . This concludes the proof of the proposition.

### 3. WEIGHT EXPONENT $\mathbf{s} = (1, 1)$

A weight homogeneous polynomial system

$$\dot{x} = P_1(x, y); \quad \dot{y} = P_2(x, y),$$

with weight exponent  $(1, 1)$  and weight degree  $d$  is integrable and its inverse integrating factor is  $V(x, y) = xP_2(x, y) - yP_1(x, y)$ . See [7] for more details.

As  $P_1(x, y)$ ,  $P_2(x, y)$  and  $V(x, y)$  are homogeneous polynomials, if the degree of  $P_1(x, y)$  and  $P_2(x, y)$  is  $d$ , then of course, the degree of  $V(x, y)$  is  $d+1$ . Thus, for  $d = 4$  we can write the homogeneous polynomials as follows:

$$\begin{aligned} P_1(x, y) &= (p_1 - a_1)x^4 + (p_2 - 4a_2)x^3y + (p_3 - 6a_3)x^2y^2 + \\ &\quad (p_4 - 4a_4)xy^3 - a_5y^4, \\ P_2(x, y) &= a_0x^4 + (4a_1 + p_1)x^3y + (6a_2 + p_2)x^2y^2 + \\ &\quad (4a_3 + p_3)xy^3 + (a_4 + p_4)y^4, \end{aligned} \tag{12}$$

and

$$V(x, y) = a_0x^5 + 5a_1x^4y + 10a_2x^3y^2 + 10a_3x^2y^3 + 5a_4xy^4 + a_5y^5.$$

So, the first integral is  $H(x, y) = \int (P_1(x, y)/V(x, y))dy + g(x)$ , satisfying  $\partial H/\partial x = -P_2/V$ . The canonical forms appear in the factorization of  $V$ . Assume that  $V(x, y)$  factorizes as

- 1) 5 simple real roots:  $a_0(x - r_1y)(x - r_2y)(x - r_3y)(x - r_4y)(x - r_5y)$ ,
- 2) 1 double and 3 simple real roots:  $a_0(x - r_1y)^2(x - r_2y)(x - r_3y)(x - r_4y)$ ,
- 3) 2 double roots and 1 simple real roots:  $a_0(x - r_1y)^2(x - r_2y)^2(x - r_3y)$ ,
- 4) 1 triple and 2 simple real roots:  $a_0(x - r_1y)^3(x - r_2y)(x - r_3y)$ ,
- 5) 1 triple and 1 double real roots:  $a_0(x - r_1y)^3(x - r_2y)^2$ ,
- 6) 1 quadruple and 1 simple real roots:  $a_0(x - r_1y)^4(x - r_2y)$ ,
- 7) 1 quintuple real root:  $a_0(x - ry)^5$ ,
- 8) 3 real and 1 couple of conjugate complex roots:  $a_0(x - r_1y)(x - r_2y)(x - r_3y)(x^2 + bxy + cy^2)$  with  $b^2 - 4c < 0$ ,
- 9) 1 double, 1 simple real and 1 couple of conjugate complex roots:  $a_0(x - r_1y)^2(x - r_2y)(x^2 + bxy + cy^2)$  with  $b^2 - 4c < 0$ ,
- 10) 1 triple real and 1 couple of conjugate complex roots:  $a_0(x - r_1y)^3(x^2 + bxy + cy^2)$  with  $b^2 - 4c < 0$ ,
- 11) 1 simple real and 2 couples of conjugate complex roots:  $a_0(x - ry)(x^2 + b_1xy + c_1y^2)(x^2 + b_2xy + c_2y^2)$  with  $b_1^2 - 4c_1 < 0, b_2^2 - 4c_2 < 0$ ,
- 12) 1 simple real and 1 double couple of conjugate complex roots:  $a_0(x - ry)(x^2 + bxy + cy^2)^2$  with  $b^2 - 4c < 0$ .

Now we shall compute for each case the first integral and obtain the conditions in order that it is a polynomial.

We define the function

$$f(r) = 5(p_4 + p_3r + p_2r^2 + p_1r^3).$$

*Case 1):* A first integral  $H$  is

$$(x - r_1y)^{\gamma_1}(x - r_2y)^{\gamma_2}(x - r_3y)^{\gamma_3}(x - r_4y)^{\gamma_4}(x - r_5y)^{\gamma_5},$$

where

$$\gamma_i = \frac{f(r_i) + a_0 \prod_{j=1, j \neq i}^5 (r_i - r_j)}{\prod_{j=1, j \neq i}^5 (r_i - r_j)}.$$

We note that an integer power of  $H$  is a polynomial if and only if  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2, 3, 4, 5$  and they all have the same sign.

*Case 2):* A first integral  $H$  is

$$(x - r_1y)^{\gamma_1}(x - r_2y)^{\gamma_2}(x - r_3y)^{\gamma_3}(x - r_4y)^{\gamma_4} \exp\left(\frac{f(r_1)x}{r_1(r_1 - r_2)(r_1 - r_3)(r_1 - r_4)(x - r_1y)}\right)$$

with  $\gamma_1 = A_1/B_1 = A_1/[(r_1 - r_2)^2(r_1 - r_3)^2(r_1 - r_4)^2]$  and

$$\begin{aligned} A_1 = & 5p_1((r_2 + r_3 + r_4)r_1^2 - 2(r_3r_4 + r_2(r_3 + r_4))r_1 + 3r_2r_3r_4)r_1^2 + \\ & 5p_2(r_1^3 - (r_3r_4 + r_2(r_3 + r_4))r_1 + 2r_2r_3r_4)r_1 + \\ & 5p_3(2r_1^3 - (r_2 + r_3 + r_4)r_1^2 + r_2r_3r_4) + \\ & 5p_4(3r_1^2 - 2(r_2 + r_3 + r_4)r_1 + r_3r_4 + r_2(r_3 + r_4)) - 2a_0B_1, \end{aligned}$$

while for  $i = 2, \dots, 4$  and

$$\gamma_i = \frac{A_i}{B_i} = \frac{-(f(r_i) + a_0B_i)}{(r_1 - r_i)^2 \prod_{j=2; j \neq i}^4 (r_i - r_j)}.$$

We note that an integer power of  $H$  is a polynomial if and only if  $f(r_1) = 0$  and  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2, 3, 4$  and they all have the same sign.

*Case 3):* A first integral  $H$  is

$$(x - r_1y)^{\gamma_1}(x - r_2y)^{\gamma_2}(x - r_3y)^{\gamma_3} \exp\left(-\sum_{i=1}^2 \frac{f(r_i)x}{r_i(r_1 - r_2)^2(r_i - r_3)(r_1y - x)}\right),$$

with  $\gamma_i = A_i/B_i = A_i/[(r_1 - r_2)^3(r_i - r_3)^2]$  for  $i = 1, 2$ ,  $\gamma_3 = A_3/B_3 = -(f(r_3) + a_0B_3)/[(r_1 - r_3)^2(r_2 - r_3)^2]$ ,

$$\begin{aligned} A_i = & -2a_0B_i + (-1)^{i+1}(5p_4(3r_i - r_j - 2r_3) - 5p_3((r_1 + r_2)r_3 - 2r_i^2) + \\ & 5p_2r_i(r_i(r_1 + r_2) - 2r_jr_3) + 5p_1r_i^2(-3r_jr_3 + r_i(2r_j + r_3))), \end{aligned}$$

for  $i, j = 1, 2$  and  $i \neq j$ . We note that an integer power of  $H$  is a polynomial if and only if  $f(r_i) = 0$  for  $i = 1, 2$  and  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2, 3$  and they all have the same sign.

*Case 4):* A first integral  $H$  is

$$(x - r_1y)^{\gamma_1}(x - r_2y)^{\gamma_2}(x - r_3y)^{\gamma_3} \exp\left(\frac{5\beta x}{2r_1^2(r_1 - r_2)^2(r_1 - r_3)^2(r_1y - x)^2}\right),$$



with

$$\begin{aligned} \beta = & (p_1 (r_1^2 - 3r_2r_1 - 3r_3r_1 + 5r_2r_3) r_1^3 - p_2 (r_1^2 + r_2r_1 + r_3r_1 - 3r_2r_3) r_1^2 \\ & - p_3 (3r_1^2 - r_2r_1 - r_3r_1 - r_2r_3) r_1 + p_4 (-5r_1^2 + 3r_2r_1 + 3r_3r_1 - r_2r_3)) x \\ & + 2r_1 (p_1 (r_1r_2 - 2r_3r_2 + r_1r_3) r_1^3 + p_3 (2r_1 - r_2 - r_3) r_1^2 \\ & + p_2 (r_1^2 - r_2r_3) r_1^2 + p_4 (3r_1^2 - 2r_2r_1 - 2r_3r_1 + r_2r_3)) y, \end{aligned}$$

$$\gamma_1 = A_1/B_1 = A_1/[(r_1 - r_2)^3(r_1 - r_3)^3],$$

$$\begin{aligned} A_1 = & -3a_0B_1 - 5(r_2(p_2 + p_1r_2)r_1^3 + (r_1 - 3r_2)(p_2 + p_1r_2)r_3r_1^2 \\ & + (p_2r_2^2 + p_1r_1(r_1^2 - 3r_2r_1 + 3r_2^2))r_3^2) \\ & + 5p_3(r_1^3 - 3r_2r_3r_1 + r_2r_3(r_2 + r_3)) \\ & + 5p_4(3r_1^2 - 3(r_2 + r_3)r_1 + r_2^2 + r_3^2 + r_2r_3), \end{aligned}$$

$\gamma_i = A_i/B_i = (-1)^i(f(r_i) + a_0B_i)/[(r_1 - r_i)^3(r_2 - r_3)]$  for  $i = 2, 3$ . We note that an integer power of  $H$  is a polynomial if and only if  $\beta = 0$  and  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2, 3$  and they all have the same sign.

*Case 5):* A first integral  $H$  is

$$(x - r_1y)^{\gamma_1}(x - r_2y)^{\gamma_2} \exp(\beta),$$

where

$$\begin{aligned} \beta = & \frac{2(r_1 - r_2)xf(r_2)r_1^2}{r_2(r_2y - x)} + \frac{(r_1 - r_2)^2x^2f(r_1)}{(x - r_1y)^2} + \\ & \frac{10(r_1 - r_2)((2p_3 + 2p_1r_1r_2 + p_2(r_1 + r_2))r_1^2 + p_4(3r_1 - r_2))x}{r_1y - x}, \end{aligned}$$

and

$$\begin{aligned} \gamma_1 = & -2r_1^2(3a_0(r_1 - r_2)^4 + 15p_4 + 5p_3(r_1 + 2r_2) \\ & + 5r_2(3p_1r_1r_2 + p_2(2r_1 + r_2))), \\ \gamma_2 = & -2r_1^2(2a_0(r_1 - r_2)^4 - 15p_4 - 5p_3(r_1 + 2r_2) \\ & - 5r_2(3p_1r_1r_2 + p_2(2r_1 + r_2))). \end{aligned}$$

We note that an integer power of  $H$  is a polynomial if and only if  $\beta = 0$  and  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2$  and they all have the same sign.

*Case 6):* A first integral  $H$  is

$$(x - r_1y)^{\gamma_1}(x - r_2y)^{\gamma_2} \exp(\beta),$$

where

$$\begin{aligned} \beta = & \frac{2(r_1 - r_2)^3f(r_1)x^3}{r_1^3(r_1y - x)^3} + \\ & \frac{30(r_1 - r_2)((p_3 + r_2(p_2 + p_1r_2))r_1^3 + p_4(3r_1^2 - 3r_2r_1 + r_2^2))x}{r_1^3(r_1y - x)} + \\ & \frac{15(r_1 - r_2)^2(p_4(3r_1 - 2r_2) + r_1((p_2 + p_1r_2)r_1^2 + p_3(2r_1 - r_2)))x^2}{r_1^3(x - r_1y)^2}, \end{aligned}$$

and

$$\begin{aligned} \gamma_1 = & -6(-4a_0(r_1 - r_2)^4 + 5p_4 + 5r_2(p_3 + r_2(p_2 + p_1r_2))), \\ \gamma_2 = & 6(a_0(r_1 - r_2)^4 + 5p_4 + 5r_2(p_3 + r_2(p_2 + p_1r_2))). \end{aligned}$$

We note that an integer power of  $H$  is a polynomial if and only if  $\beta = 0$  and  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2$  and they all have the same sign.

*Case 7):* A first integral  $H$  is

$$(x - ry)^{\gamma_1} \exp\left(\frac{\beta x}{(x - ry)^4}\right),$$

where

$$\beta = \begin{aligned} & rx(rx(-p_2x + 3p_1rx + 4p_2ry) + p_3(x^2 - 4ryx + 6r^2y^2)) \\ & - 3p_4(x - 2ry)(x^2 - 2ryx + 2r^2y^2), \end{aligned}$$

and  $\gamma_1 = -12a_0r^4$ . We note that  $x - ry$  is a polynomial first integral if and only if  $\beta = 0$ .

*Case 8):* A first integral  $H$  is

$$\begin{aligned} & (x - r_1y)^{\gamma_1}(x - r_2y)^{\gamma_2}(x - r_3y)^{\gamma_3}(x^2 + bxy + cy^2)^{\gamma_4} \\ & \exp\left(\frac{\beta x}{\prod_{i=1}^3(c + br_i + r_i^2)\sqrt{(4c - b^2)x^2}} \arctan\left(\frac{bx + 2cy}{\sqrt{(4c - b^2)x^2}}\right)\right), \end{aligned}$$

where

$$\begin{aligned} \beta = & 5(2p_1c^3 - (b(p_2 - p_1(r_1 + r_2 + r_3)) + 2(p_3 + p_2(r_1 + r_2 + r_3) + \\ & p_1(r_2r_3 + r_1(r_2 + r_3))))c^2 + ((p_3 + p_1r_1r_2 + p_1(r_1 + r_2)r_3)b^2 + \\ & (3p_4 - 3p_1r_1r_2r_3 + p_3(r_1 + r_2 + r_3) - p_2(r_2r_3 + r_1(r_2 + r_3))))b + \\ & 2(p_2r_1r_2r_3 + p_4(r_1 + r_2 + r_3) + p_3(r_2r_3 + r_1(r_2 + r_3))))c - \\ & bp_4(b + r_1)(b + r_2) + ((p_1r_1b^3 - p_2r_1b^2 - p_4b + p_3r_1b - 2p_4r_1)r_2 - \\ & bp_4(b + r_1))r_3), \end{aligned}$$

and  $\gamma_i = A_i/B_i = -(f(r_i) + a_0B_i)/[(c + br_i + r_i^2)\prod_{j=1, j \neq i}^3(r_i - r_j)]$  for  $i = 1, 2, 3$ ,  $\gamma_4 = A_4/B_4 = A_4/[2\prod_{i=1}^3(c + br_i + r_i^2)]$

$$\begin{aligned} A_4 = & -a_0B_4 + 5((p_2 + p_1(r_1 + r_2 + r_3))c^2 - (p_4 + p_1r_1r_2r_3 + \\ & p_3(r_1 + r_2 + r_3) + p_2(r_2r_3 + r_1(r_2 + r_3)) + \\ & b(p_3 - p_1(r_2r_3 + r_1(r_2 + r_3))))c + p_4(b + r_1)(b + r_2) + \\ & (p_4(b + r_1) + (p_4 + (p_1b^2 - p_2b + p_3)r_1)r_2)r_3). \end{aligned}$$

We note that an integer power of  $H$  is a polynomial if and only if  $\beta = 0$  and  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2, 3, 4$  and they all have the same sign.

*Case 9):* A first integral  $H$  is

$$\begin{aligned} & (x - r_1y)^{\gamma_1}(x - r_2y)^{\gamma_2}(x^2 + bxy + cy^2)^{\gamma_3} \\ & \exp\left(-\frac{f(r_1)x}{r_1(r_1 - r_2)(c + r_1b + r_1^2)(r_1y - x)} + \right. \\ & \left. \frac{\beta x}{\prod_{i=1}^2(c + br_i + r_i^2)^{3-i}\sqrt{(4c - b^2)x^2}} \arctan\left(\frac{bx + 2cy}{\sqrt{(4c - b^2)x^2}}\right)\right), \end{aligned}$$

where

$$\begin{aligned} \beta = & 5(2p_1c^3 - (b(p_2 - p_1(2r_1 + r_2)) + 2(p_3 + p_2(2r_1 + r_2) + \\ & p_1r_1(r_1 + 2r_2)))c^2 + ((p_3 + p_1r_1(r_1 + 2r_2))b^2 + \\ & (3p_4 + p_3(2r_1 + r_2) - r_1(3p_1r_1r_2 + p_2(r_1 + 2r_2)))b + \\ & 2(p_4(2r_1 + r_2) + r_1(p_2r_1r_2 + p_3(r_1 + 2r_2))))c - \\ & bp_4(b + r_1)^2 + (-p_4b^2 - 2p_4r_1b + (b(p_1b^2 - p_2b + p_3) - 2p_4)r_1^2)r_2), \end{aligned}$$

$$\gamma_1 = -\frac{A_1}{B_1} = -\frac{A_1}{(r_1 - r_2)^2(c + br_1 + r_1^2)^2},$$

$$\begin{aligned} A_1 = & 2a_0c^2(r_1 - r_2)^2 + 2a_0b^2r_1^2(r_1 - r_2)^2 + \\ & c(-5p_4 + r_1(4a_0(b + r_1)(r_1 - r_2)^2 + 5p_1r_1(2r_1 - 3r_2) + \\ & 5p_2(r_1 - 2r_2)) - 5p_3r_2) + b((4a_0r_1(r_1 - r_2)^2 - 5p_3 + \\ & 5p_1r_1(r_1 - 2r_2) - 5p_2r_2)r_1^2 + 5p_4(r_2 - 2r_1)) + r_1(5p_4(2r_2 - 3r_1) + \\ & r_1((2a_0(r_1 - r_2)^2 - 5p_2 - 5p_1r_2)r_1^2 + 5p_3(r_2 - 2r_1))), \end{aligned}$$

$$\gamma_2 = \frac{A_2}{B_2} = \frac{-f(r_2) + a_0B_2}{(r_1 - r_2)^2(c + br_2 + r_2^2)},$$

$$\gamma_3 = \frac{A_3}{B_3} = \frac{A_3}{2(c + br_1 + r_1^2)^2(c + br_2 + r_2^2)},$$

$$\begin{aligned} A_3 = & -a_0B_3 + 5((p_2 + p_1(2r_1 + r_2))c^2 - (p_4 + p_3(2r_1 + r_2) + \\ & r_1(p_1r_1r_2 + p_2(r_1 + 2r_2)) + b(p_3 - p_1r_1(r_1 + 2r_2)))c + \\ & p_4(b + r_1)^2 + ((p_1b^2 - p_2b + p_3)r_1^2 + 2p_4r_1 + bp_4)r_2). \end{aligned}$$

We note that an integer power of  $H$  is a polynomial if and only if  $f(r_1) = 0$ ,  $\beta = 0$  and  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2, 3$  and they all have the same sign.

*Case 10):* A first integral  $H$  is

$$\begin{aligned} & (x - r_1y)^{\gamma_1}(x^2 + bxy + cy^2)^{\gamma_2} \exp\left(\frac{f(r_1)(c + br_1 + r_1^2)^2x^2}{r_1^2(x - r_1y)^2}\right. \\ & \left. - \frac{\beta_1(c + br_1 + r_1^2)x}{r_1^2(x - r_1y)} + \frac{\beta_2x}{\sqrt{(4c - b^2)x^2}} \arctan\left(\frac{bx + 2cy}{\sqrt{(4c - b^2)x^2}}\right)\right), \end{aligned}$$

where

$$\beta_1 = 10((p_2 - bp_1)r_1^4 + 2(p_3 - cp_1)r_1^3 + (-cp_2 + bp_3 + 3p_4)r_1^2 + 2bp_4r_1 + cp_4),$$

$$\begin{aligned} \beta_2 = & 10((p_1r_1^3 - p_4)b^3 - r_1(p_2r_1^2 + 3p_4)b^2 + r_1^2(p_3r_1 - 3p_4)b - \\ & 2p_4r_1^3 + 2c^3p_1 - c^2(2p_3 + b(p_2 - 3p_1r_1) + 6r_1(p_2 + p_1r_1)) + \\ & c((3p_1r_1^2 + p_3)b^2 + 3(p_4 + r_1(p_3 - r_1(p_2 + p_1r_1)))b + \\ & 2r_1(3p_4 + r_1(3p_3 + p_2r_1))), \end{aligned}$$

and

$$\begin{aligned} \gamma_1 = & -2(3a_0(c + r_1(b + r_1))^3 + 5((p_1b^2 - p_2b + p_3)r_1^3 + 3p_4r_1^2 + \\ & 3bp_4r_1 + b^2p_4 + c^2(p_2 + 3p_1r_1) - c(p_4 + b(p_3 - 3p_1r_1^2) + \\ & r_1(3p_3 + r_1(3p_2 + p_1r_1))))), \end{aligned}$$

$$\begin{aligned} \gamma_2 = & 5((p_1b^2 - p_2b + p_3)r_1^3 + 3p_4r_1^2 + 3bp_4r_1 + b^2p_4 + c^2(p_2 + 3p_1r_1) \\ & - c(p_4 + b(p_3 - 3p_1r_1^2) + r_1(3p_3 + r_1(3p_2 + p_1r_1))) \\ & - 2a_0(c + r_1(b + r_1))^3) \end{aligned}$$

We note that an integer power  $H$  is a polynomial if and only if  $f(r_1) = 0$ ,  $\beta_1 = \beta_2 = 0$  and  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2$  and they all have the same sign.

*Case 11):* A first integral  $H$  is

$$(x^2 + b_1xy + c_1y^2)^{\gamma_1} (x^2 + b_2xy + c_2y^2)^{\gamma_2} (x - r_1y)^{\gamma_3} \exp \left( \sum_{i=1}^2 \frac{\beta_i x}{\sqrt{(4c_i - b_i^2)x^2}} \arctan \left( \frac{b_i x + 2c_i y}{\sqrt{(4c_i - b_i^2)x^2}} \right) \right),$$

with  $\beta_i = \alpha_i/\delta_i$  for  $i = 1, 2$ , where

$$\begin{aligned} \alpha_i = & 5(-b_i(c_i^2 p_2 + c_i c_j p_2 + b_j c_i(c_i p_1 + p_3) - 3c_i p_4 + c_j p_4) + \\ & b_i(c_i^2 p_1 + c_j p_3 + c_i(-3c_j p_1 + b_j p_2 + p_3) + b_j p_4) r_1 + \\ & b_i^3(-p_4 + c_j p_1 r_1) + b_i^2(c_i c_j p_1 + c_i p_3 + b_j p_4 - (b_j c_i p_1 + c_j p_2 + p_4) r_1) + \\ & 2(c_i^3 p_1 - c_j p_4 r_1 - c_i^2(c_j p_1 + p_3 + p_2 r_1 - b_j(p_2 + p_1 r_1)) + \\ & c_i(p_4 r_1 + c_j(p_3 + p_2 r_1) - b_j(p_4 + p_3 r_1))), \end{aligned}$$

$$\delta_i = ((b_2^2 c_1 + (c_1 - c_2)^2 + b_1^2 c_2 - b_1 b_2(c_1 + c_2))(c_i + b_i r_1 + r_1^2)),$$

for  $i, j = 1, 2$  and  $i \neq j$

$$\begin{aligned} \gamma_i = & -2a_0(b_j^2 c_i + (c_i - c_j)^2 + b_i^2 c_j - b_i b_j(c_i + c_j))(c_i + r_1(b_i + r_1)) + \\ & 5(b_i^2(p_4 + c_j p_1 r_1) + b_i(c_i c_j p_1 - c_i p_3 - c_j p_2 r_1 + p_4 r_1) + \\ & (c_i - c_j)(-p_4 - p_3 r_1 + c_i(p_2 + p_1 r_1)) - b_j(c_i^2 p_1 + p_4(b_i + r_1) - \\ & c_i(p_3 + (-b_i p_1 + p_2) r_1))), \end{aligned}$$

for  $i, j = 1, 2$  and  $i \neq j$ . Finally

$$\gamma_3 = \frac{A_3}{B_3} = \frac{-(f(r_1) + a_0 B_3)}{\prod_{i=1}^2 (c_i + b_i r_1 + r_1^2)}.$$

We note that an integer power of  $H$  is a polynomial if and only if  $\beta_1 = \beta_2 = 0$  and  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2, 3$  and they all have the same sign.

*Case 12):* A first integral  $H$  is

$$(x - ry)^{\gamma_1} (x^2 + bxy + cy^2)^{\gamma_2} \exp \left( \frac{\beta_1 x^3}{[(4c - b^2)x^2]^{3/2}} \arctan \left( \frac{bx + 2cy}{\sqrt{(4c - b^2)x^2}} \right) + \frac{10\beta_2(c + br + r^2)x}{(b^2 - 4c)c(x^2 + bxy + cy^2)} \right),$$

where

$$\begin{aligned} \beta_1 = & -10((p_4 + r(p_3 - r(p_2 + p_1 r)))b^3 + 4p_3 r^2 b^2 + 2r^2(p_3 r - 3p_4)b - \\ & 4p_4 r^3 + 4c^3 p_1 + c^2(4(p_3 + r(p_2 + 3p_1 r)) - 2b(p_2 - 3p_1 r)) - \\ & 2c(2p_2 r b^2 + (3p_4 - r(p_3 + r(3p_1 r - p_2)))b + 2r(3p_4 + r(p_3 + p_2 r))), \end{aligned}$$

$$\begin{aligned} \beta_2 = & -p_4 y b^3 + (c(p_1 r x + p_3 y) - p_4(x + ry))b^2 + ((p_1(x + ry) - p_2 y)c^2 + \\ & (-p_2 r x + 3p_4 y + p_3(x + ry))c - p_4 r x)b + \\ & 2c(p_1 y c^2 - (p_1 r x + p_3 y + p_2(x + ry))c + p_3 r x + p_4(x + ry)), \end{aligned}$$

$$\gamma_1 = 2(a_0(c + br + r^2)^2 + f(r)),$$

$$\gamma_2 = 4a_0(c + br + r^2)^2 - f(r).$$

We note that an integer power of  $H$  is a polynomial if and only if  $\beta_1 = \beta_2 = 0$  and  $\gamma_i \in \mathbb{Q}$  for  $i = 1, 2$  and they all have the same sign.

4. WEIGHT EXPONENT  $\mathbf{s} = (1, 2)$ 

We prove Theorem 1. Since systems in (5) are homogeneous, we know that they are integrable because they have the inverse integrating factor  $V = xy - y\dot{x}$ . The strategy will be to obtain such first integrals and determine whose of them are polynomials. Denoting systems in (5) by  $\dot{x} = P(x, y)$ ,  $\dot{y} = Q(x, y)$  the first integral is  $H(x, y) = \int (P(x, y)/V(x, y)) dy + g(x)$ , satisfying  $\partial H/\partial x = -Q(x, y)/V(x, y)$ .

The first system in (5) has the first integral

$$H = x^{-3-2p_2}(3y^2 - x^2)^{-3+p_2} \exp\left(-2\sqrt{3}p_1 \operatorname{arctanh}\left(x/(\sqrt{3}y)\right)\right).$$

Note that an integer power of  $H$  is a polynomial if and only if  $p_1 = 0$  and  $p_2 = 3(1 - q)/(1 + 2q)$  with  $q \in \mathbb{Q}^+$ .

The second system in (5) has the first integral

$$H = x^{-3-2p_2}(3y^2 + x^2)^{-3+p_2} \exp\left(-2\sqrt{3}p_1 \operatorname{arctan}\left(x/(\sqrt{3}y)\right)\right).$$

Note that an integer power of  $H$  is a polynomial if and only if  $p_1 = 0$  and  $p_2 = 3(1 - q)/(1 + 2q)$  with  $q \in \mathbb{Q}^+$ .

The third system in (5) has the first integral

$$H = x^{-2(3+p_1)}y^{-3+2p_1} \exp(2p_2y/x).$$

Note that an integer power of  $H$  is a polynomial if and only if  $p_2 = 0$  and  $p_2 = 3(1 - 2q)/(2(1 + q))$  with  $q \in \mathbb{Q}^+$ .

The fourth system in (5) has the first integral

$$H = x \exp(-y(2p_1x + p_2y)/(3x^2)).$$

Note that an integer power of  $H$  is a polynomial if and only if  $p_1 = p_2 = 0$ .

The fifth system in (5) has the first integral  $x/y$  which is never a polynomial.

 5. WEIGHT EXPONENT  $\mathbf{s} = (1, 3)$ 

Doing the change of variables  $(X, Y) = (x^3, y)$  the planar weight homogeneous systems of weight degree 4 and weight exponent  $(1, 3)$  becomes

$$\dot{X} = 3a_{40}X^2 + 3a_{11}XY, \quad \dot{Y} = b_{60}X^2 + b_{31}XY + b_{02}Y^2. \quad (13)$$

Again we shall use the inverse integrating factor  $V = X\dot{Y} - Y\dot{X}$  for computing the first integrals of system (13).

It is clear that if  $a_{11} = a_{40} = 0$  then a polynomial first integral is  $X$ , and if  $b_{60} = b_{31} = b_{02} = 0$  then a polynomial first integral is  $Y$ . Now we consider the other cases.

*Case 1:*  $3a_{11} - b_{02} \neq 0$  and  $R = -(3a_{40} - b_{31})^2 + 4(-3a_{11} + b_{02})b_{60} \neq 0$ . In this case system (13) has the first integral

$$\begin{aligned} & \frac{6}{\sqrt{R}}(a_{11}(3a_{40} + b_{31}) - 2a_{40}b_{02}) \operatorname{arctan}\left(\frac{3a_{40}X - b_{31}X + 6a_{11}Y - 2b_{02}Y}{\sqrt{RX}}\right) + \\ & 2(3a_{11} - b_{02}) \log X + 3a_{11} \log\left(\frac{Y(3a_{40}X - b_{31}X + 3a_{11}Y - b_{02}Y)}{X^2} - b_{60}\right). \end{aligned}$$

Here  $\log A$  always means  $\log |A|$ , and as usual  $\log$  is the logarithm in base  $e$ . Since this first integral must be a polynomial we must have

$$a_{11}(3a_{40} + b_{31}) - 2a_{40}b_{02} = 0. \quad (14)$$

We consider different subcases.

If  $3a_{11} - 2b_{02} \neq 0$ . Then from (14) we get

$$a_{40} = -\frac{a_{11}b_{31}}{3a_{11} - 2b_{02}}.$$

Therefore, doing the exponential of the previous first integral we obtain that the first integral is

$$H = X^{1 - \frac{3a_{11}}{6a_{11} - 2b_{02}}} p(X, Y)^{\frac{3a_{11}}{6a_{11} - 2b_{02}}},$$

where

$$p(X, Y) = (3a_{11} - 2b_{02})b_{60}X^2 + 2(3a_{11} - b_{02})b_{31}XY - (9a_{11}^2 - 9a_{11}b_{02} + 2b_{02}^2)Y^2.$$

Note that since  $a_{11} - b_{02} \neq 0$  and  $3a_{11} - 2b_{02} \neq 0$  we have  $9a_{11}^2 - 9a_{11}b_{02} + 2b_{02}^2 \neq 0$ . Therefore an integer power of  $H$  is a polynomial first integral if and only if  $3a_{11}/(6a_{11} - 2b_{02}) = m/n \in \mathbb{Q}^+$ , and  $m/n < 1$ . In this case the first integral  $H$  is

$$X^{n-m} ((3a_{11} - 2b_{02})b_{60}X^2 + 2(3a_{11} - b_{02})b_{31}XY - (9a_{11}^2 - 9a_{11}b_{02} + 2b_{02}^2)Y^2)^m.$$

If  $3a_{11} - 2b_{02} = 0$ , that is,  $b_{02} = 3a_{11}/2$ . In this case from (14) we get  $a_{11}b_{31} = 0$ . Hence either  $a_{11} = 0$ , or  $b_{31} = 0$ . But if  $a_{11} = 0$ , then  $b_{02} = 0$  in contradiction with the fact that  $3a_{11} - 2b_{02} \neq 0$ . Therefore, this case is not possible and we must have  $b_{31} = 0$ . The first integral is then

$$H = \frac{-2b_{60}X^2 + 6a_{40}XY + 3a_{11}Y^2}{X},$$

which is never a polynomial.

*Case 2:*  $b_{02} = 3a_{11}$  and  $b_{31} - 3a_{40} \neq 0$ . In this case system (13) has the first integral

$$(b_{31} - 3a_{40})^2 \log X + \frac{3a_{11}(3a_{40} - b_{31})Y}{X} + 3(-3a_{40}^2 + b_{31}a_{40} - a_{11}b_{60}) \log \left( -\frac{b_{60}X - 3a_{40}Y + b_{31}Y}{X} \right).$$

In order that the first integral is a polynomial we must have  $a_{11}(3a_{40} - b_{31}) = 0$ , that is  $a_{11} = 0$  (and hence  $b_{02} = 0$ ). Then doing the exponential of the previous first integral we obtain the first integral

$$X^{1/3} \left( \frac{b_{60}X - 3a_{40}Y + b_{31}Y}{X} \right)^{\frac{a_{40}}{3a_{40} - b_{31}}}.$$

Then we must have  $b_{31}/a_{40} = -m/n$  with  $m/n \in \mathbb{Q}^+$ . In this case the previous first integral becomes  $H = X^m (b_{60}X - 3a_{40}Y + b_{31}Y)^{3n}$ .

*Case 3:*  $b_{02} = 3a_{11}$  and  $b_{31} = 3a_{40}$ . Then system (13) has the first integral

$$\frac{-2b_{60}X^2 \log X + 6a_{40}YX + 3a_{11}Y^2}{6X^2}.$$

Since the case  $a_{40} = a_{11} = 0$  has been studied, we have that in this case the first integral is never a polynomial.

*Case 4:*  $3a_{11} - b_{02} \neq 0$  and  $R = 0$ . Then

$$b_{06} = \frac{-(3a_{40} - b_{31})^2}{4(3a_{11} - b_{02})},$$

and system (13) has the first integral

$$\frac{3(-3a_{11}a_{40} + 2b_{02}a_{40} - a_{11}b_{31})X}{3a_{40}X - b_{31}X + 6a_{11}Y - 2b_{02}Y} + (3a_{11} - b_{02}) \log(36a_{11}X - 12b_{02}X) + 3a_{11} \log\left(-\frac{3a_{40}X - b_{31}X + 6a_{11}Y - 2b_{02}Y}{X}\right).$$

In order that it is a polynomial we must have

$$2a_{40}b_{02} - a_{11}(3a_{40} + b_{31}) = 0. \quad (15)$$

We consider two different subcases.

If  $3a_{11} \neq 2b_{02}$ . In this case condition (15) becomes

$$a_{40} = -\frac{a_{11}b_{31}}{3a_{11} - 2b_{02}}.$$

Then doing the exponential of the previous first integral we obtain the first integral

$$X^{\frac{1}{36a_{11} - 12b_{02}}} \left( \frac{b_{31}X - 3a_{11}Y + 2b_{02}Y}{X} \right)^{\frac{a_{11}}{4(b_{02} - 3a_{11})^2}}.$$

From this first integral we obtain the first integral

$$X(b_{31}X - 3a_{11}Y + 2b_{02}Y)^{-3a_{11}/b_{02}}.$$

So, if  $b_{02} \neq 0$  then  $3a_{11}/b_{02} = -m/n$  with  $m/n \in \mathbb{Q}^+$  and the polynomial first integral is

$$X^n(b_{31}X - 3a_{11}Y + 2b_{02}Y)^m.$$

If  $b_{02} = 0$  then  $H = b_{31}X - 3a_{11}Y$  is a polynomial first integral. This concludes the proof of the theorem.

## 6. WEIGHT EXPONENT $\mathbf{s} = (1, 4)$

We introduce the change  $(X, Y) = (x^4, y)$  in the planar weight homogeneous polynomial differential systems (1) of weight degree 4 with weight exponent  $(1, 4)$ . In these new variables  $(X, Y)$  the systems with weight exponent  $(1, 4)$  becomes, after introducing the new independent variable  $d\tau = x^3 dt$ , as follows

$$X' = 4(a_{40}X + a_{01}Y), \quad Y' = b_{70}X + b_{31}Y, \quad (16)$$

where the prime denotes derivative with respect to  $\tau$ . If  $a_{40} = a_{01} = 0$  then a polynomial first integral is  $X$ , and if  $b_{70} = b_{31} = 0$  then a polynomial first integral is  $Y$ . Now we consider the other cases.

*Case 1:*  $R = -(4a_{40} - b_{31})^2 - 16a_{01}b_{70} \neq 0$ . Then a first integral of system (16) is

$$\frac{2(4a_{40} + b_{31})}{\sqrt{R}} \arctan\left(\frac{(4a_{40} - b_{31})X + 8a_{01}Y}{\sqrt{R}X}\right) + \log(-b_{70}X^2 + (4a_{40} - b_{31})XY + 4a_{01}Y^2).$$

Since it must be a polynomial we must have  $b_{31} = -4a_{40}$ , and we get the polynomial first integral is  $H = -b_{70}X^2 + 8a_{40}XY + 4a_{01}Y^2$ .

*Case 2:*  $R = 0$ . We consider different subcases.

We first study when  $b_{70} \neq 0$ . Then from  $R = 0$  we get

$$a_{01} = -\frac{(4a_{40} - b_{31})^2}{16b_{70}}. \quad (17)$$

If  $b_{31} - 4a_{40} \neq 0$  the first integral is

$$\frac{2(4a_{40} + b_{31})b_{70}X}{-2b_{70}X + (4a_{40} - b_{31})Y} + (4a_{40} - b_{31}) \log(2b_{70}X + (-4a_{40} + b_{31})Y),$$

which is a polynomial if and only if  $b_{31} = -4a_{40}$ . The polynomial first integral is  $b_{70}X - 4a_{40}Y$ .

If  $b_{31} = 4a_{40}$  then  $a_{40} \neq 0$  (otherwise  $b_{31} = 0$  and from (17) we also have  $a_{01} = 0$  and this case has been considered), and the first integral of (16) is

$$H = \frac{Y}{X} - \frac{b_{70} \log X}{4a_{40}},$$

which is never a polynomial.

If  $b_{70} = 0$  then from  $R = 0$  we get  $b_{31} = 4a_{40}$ . We only consider the case  $a_{40} \neq 0$ , and the first integral of (16) is

$$-\frac{a_{40}X}{Y} + a_{01} \log Y,$$

which is never a polynomial. This completes the proof of the theorem.

## 7. WEIGHT EXPONENT $s = (2, 3)$

We prove Theorem 4. The planar weight homogeneous polynomial differential systems (1) with weight degree 4 with weight-exponent  $(2, 3)$  are

$$\dot{x} = a_{11}xy, \quad \dot{y} = b_{30}x^3 + b_{02}y^2. \quad (18)$$

If  $a_{11}(3a_{11} - 2b_{02}) \neq 0$  then the first integral of system (18) is

$$H = x^{-\frac{2b_{02}}{a_{11}}} (2b_{30}x^3 - 3a_{11}y^2 + 2b_{02}y^2).$$

Then an integer power of  $H$  is a polynomial first integral if and only if  $-2b_{02}/a_{11} \in \mathbb{Q}^+$ .

If  $a_{11} = 0$  then  $H = x$  is a polynomial first integral of system (18).

If  $b_{30} = b_{02} = 0$  then  $H = y$  is a polynomial first integral of system (18).

If  $a_{11} \neq 0$  and  $3a_{11} = 2b_{02}$ , then the first integral of system (18) is

$$H = \frac{y^4}{x^3} - \frac{2b_{30}}{a_{11}} \log x,$$

which is never a polynomial. This completes the proof of the theorem.



8. WEIGHT EXPONENT  $\mathbf{s} = (2, 5)$ 

We prove Theorem 5. The planar weight homogeneous polynomial differential systems (1) of weight degree 4 with weight exponent  $(2, 5)$  are

$$\dot{x} = a_{01}y, \quad \dot{y} = b_{40}x^4.$$

It is straightforward to prove that  $H = 2b_{40}x^5 - 5a_{01}y^2$  is a polynomial first integral.

9. WEIGHT EXPONENT  $\mathbf{s} = (6, 9)$ 

We prove Theorem 6. The planar weight homogeneous polynomial differential systems (1) of weight degree 4 with weight exponent  $(6, 9)$  are

$$\dot{x} = a_{01}y, \quad \dot{y} = b_{20}x^2.$$

It is straightforward to prove that  $H = 2b_{20}x^3 - 3a_{01}y^2$  is a polynomial first integral.

## REFERENCES

- [1] A. ALGABA, N. FUENTES AND C. GARCÍA, *Centers of quasi-homogeneous polynomial planar systems*, Nonlinear Anal. Real World Appl. **13** (2012), 419–431.
- [2] A. ALGABA, C. GARCÍA AND M. REYES, *Integrability of two dimensional quasi-homogeneous polynomial differential systems*, Rocky Mountain J. Math. **41** (2011), 1–22.
- [3] A. ALGABA, C. GARCÍA AND M. REYES, *Rational integrability of two-dimensional quasi-homogeneous polynomial differential systems*, Nonlinear Anal. **73** (2010), 1318–1327.
- [4] A. ALGABA, C. GARCÍA AND M.A. TEIXEIRA, *Reversibility and quasi-homogeneous normal forms of vector fields*, Nonlinear Anal. **73** (2010), 510–525.
- [5] A. ALGABA, E. FREIRE, E. GAMERO AND C. GARCÍA, *Monodromy, center-focus and integrability problems for quasi-homogeneous polynomial systems*, Nonlinear Anal. **72** (2010), 1726–1736.
- [6] C. CAIRÓ AND J. LLIBRE, *Polynomial first integrals for weight-homogeneous planar polynomial differential systems of weight degree 3*, J. Math. Anal. Appl. **331** (2007), 1284–1298.
- [7] J. CHAVARRIGA, H. GIACOMINI, J. GINÉ AND J. LLIBRE, *Darboux integrability and the inverse integrating factor*, J. Differential Equations **194** (2003), 116–139.
- [8] A. CIMA AND J. LLIBRE, *Algebraic and topological classification of the homogeneous cubic vector fields in the plane*, J. Math. Anal. and Appl. **147** (1990), 420–448.
- [9] T. DATE AND M. LAI, *Canonical forms of real homogeneous quadratic transformations*, J. Math. Anal. Appl. **56** (1976), 650–682.
- [10] S. D. FURTA, *On non-integrability of general systems of differential equations*, Z. Angew Math. Phys. **47** (1996), 112–131.
- [11] A. GORIELY, *Integrability, partial integrability, and nonintegrability for systems of ordinary differential equations*, J. Math. Phys. **37** (1996), 1871–1893.
- [12] S. LABRUNIE, *On the polynomial first integrals of the  $(a, b, c)$  Lotka–Volterra system*, J. Math. Phys. **37** (1996), 5539–5550.
- [13] J. LLIBRE AND X. ZHANG, *Polynomial first integrals for quasi-homogeneous polynomial differential systems*, Nonlinearity, **15** (2002), 1269–1280.
- [14] A.J. MACIEJEWSKI AND J.M. STRELCYN, *On algebraic non-integrability of the Halphen system*, Phys. Lett. A **201** (1995), 161–166.
- [15] J. MOULIN–OLLAGNIER, *Polynomial first integrals of the Lotka–Volterra system*, Bull. Sci. Math. **121** (1997), 463–476.
- [16] A. TSYGVINTSEV, *On the existence of polynomial first integrals of quadratic homogeneous systems of ordinary differential equations*, J. Phys. A: Math. Gen. **34** (2001), 2185–2193.

DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193  
BELLATERRA, BARCELONA, CATALONIA, SPAIN  
*E-mail address:* `jllibre@mat.uab.cat`

DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, 1049-001 LISBOA,  
PORTUGAL  
*E-mail address:* `cvalls@math.ist.utl.pt`