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PERIODS OF HOMEOMORPHISMS ON SOME COMPACT SPACES

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Abstract. The objective of the present paper is to provide information on the set of periodic points of a homeomorphism defined in the following compact spaces: $\mathbb{S}^n$ (the n–dimensional sphere), $\mathbb{S}^n \times \mathbb{S}^m$ (the product space of the n–dimensional with the m–dimensional spheres), $\mathbb{C}P^n$ (the n–dimensional complex projective space) and $\mathbb{H}P^n$ (the n–dimensional quaternion projective space). We use as main tool the action of the homeomorphism on the homology groups of these compact spaces.

1. Introduction

Let $f : \mathbb{X} \to \mathbb{X}$ be a homeomorphism on a compact space $\mathbb{X}$. A point $x \in \mathbb{X}$ is periodic of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \ldots, n-1$. We denote by $\mathrm{Per}(f)$ the set of periods of all periodic points of f. The aim of the present paper is to provide some information on $\mathrm{Per}(f)$ for some compact spaces. More precisely, we shall present results for the cases $\mathbb{X} \in \Delta$, where Δ is the set formed by the spaces: $\mathbb{S}^n$ (the n–dimensional sphere), $\mathbb{S}^n \times \mathbb{S}^m$ (the product space of the n–dimensional with the m–dimensional spheres), $\mathbb{C}P^n$ (the n–dimensional complex projective space) and $\mathbb{H}P^n$ (the n–dimensional quaternion projective space).

Our first result is the following.

Theorem 1. *Let $\mathbb{X}$ be a compact space and let f be a self–homeomorphism of $\mathbb{X}$. Then the following statements hold.*

(a) *If $\mathbb{X} = \mathbb{S}^n$ with n even, then* $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.
(b) *If $\mathbb{X} = \mathbb{S}^n \times \mathbb{S}^n$ with n even, then* $\mathrm{Per}(f) \cap \{1, 2, 3, 4\} \neq \emptyset$.
(c) *If $\mathbb{X} = \mathbb{S}^n \times \mathbb{S}^m$ with $n \neq m$ and both even, then* $\mathrm{Per}(f) \cap \{1, 2, 3, 4\} \neq \emptyset$.
(d) *If $\mathbb{X}$ is either $\mathbb{C}P^n$ or $\mathbb{H}P^n$, then* $\mathrm{Per}(f) \cap \{1, 2, ..., n+1\} \neq \emptyset$.

The proof of Theorem 1 is done in section 2. The main tool for proving it is a result due to Fuller [5].

The results of Theorem 1 restricted to the orientable and non–orientable closed surfaces with and without boundary were already obtained respectively by Franks and Llibre [4] and Guirao and Llibre [7].

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The objective of the rest of the paper is to improve the information provided in Theorem 1 using as a main tool the Lefschetz fixed point theory.

Our main results are stated in the following three theorems proved in Section 3, and these results improve the ones given in Theorem 1.

We note that if $f\colon \mathbb{S}^n \to \mathbb{S}^n$ is a homeomorphism and either n is even and f is orientation preserving or n is odd and f is orientation reversing, then $L(f) = 2$ so f has a fixed point.

Theorem 2. *Let $f : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n \times \mathbb{S}^n$ be a homeomorphism, and let $f_{*n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$, the action of f on the n–th homology group $H_n(\mathbb{S}^n \times \mathbb{S}^n, \mathbb{Q}) \approx \mathbb{Q} \oplus \mathbb{Q}$. Then the following statements hold.*

(a) *Suppose that f is orientation preserving and n is odd. If the two equations $bc + (d-1)^2 = 0$ and $a + d = 2$ are not satisfied, then the sets $\{1, 2\}$, $\{1, 3\}$ and $\{1, 2, 4\}$ intersect $\mathrm{Per}(f)$.*

(b) *If f is orientation preserving and n is even, then $\mathrm{Per}(f) \cap \{1, 2, 3\} \neq \emptyset$ and $\mathrm{Per}(f) \cap \{1, 2, 4\} \neq \emptyset$.*

(c) *Suppose that f is orientation reversing and n is odd.*

(c.1) *If the equation $a + d = 0$ is not satisfied, then $\mathrm{Per}(f) \cap \{1, 3\} \neq \emptyset$.*

(c.2) *If the two equations $bc + d^2 = 1$ and $a + d = 0$ are not satisfied, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*

(c.3) *If the three equations $bc + d^2 = \pm 1$ and $a + d = 0$ are not satisfied, then $\mathrm{Per}(f) \cap \{1, 2, 4\} \neq \emptyset$.*

(d) *Suppose that f is orientation reversing and n is even.*

(d.1) *We have that $\mathrm{Per}(f) \cap \{1, 2, 4\} \neq \emptyset$.*

(d.2) *If the equation $a + d = 0$ is not satisfied, then $\mathrm{Per}(f) \cap \{1, 3\} \neq \emptyset$.*

(d.3) *If the two equations $bc + d^2 = -1$ and $a + d = 0$ are not satisfied, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*

Theorem 3. *Let $f : \mathbb{S}^n \times \mathbb{S}^m \to \mathbb{S}^n \times \mathbb{S}^m$ with $n \neq m$ be a homeomorphism, $f_{*n} = (a)$, $f_{*m} = (b)$ with $a, b \in \mathbb{Z}$. Here for $k = n, m$, f_{*k} denotes the action on the k–th homology group $H_k(\mathbb{S}^n \times \mathbb{S}^m, \mathbb{Q}) \approx \mathbb{Q}$. Then the following statements hold.*

(a) *Suppose that f is orientation preserving, n and m are even.*

(a.1) $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.

(a.2) *If $(a, b) \neq (-1, -1)$, then $\mathrm{Per}(f) \cap \{1, 3\} \neq \emptyset$.*

(b) *Suppose that f is orientation reversing, n and m are even. Then the sets $\{1, 2, 3\}$ and $\{1, 2, 4\}$ intersect $\mathrm{Per}(f)$.*

(c) *Suppose that f is orientation reversing and n is even and m is odd. If $(a, b) \neq (-1, 1)$, then the sets $\{1, 2\}$ and $\{1, 3\}$ intersect $\mathrm{Per}(f)$.*

(d) *Suppose that f is orientation reversing and n is odd and m is even. If $(a, b) \neq (1, -1)$, then the sets $\{1, 2\}$ and $\{1, 3\}$ intersect $\mathrm{Per}(f)$.*

(e) *Suppose that f is orientation preserving, n and m are odd. If $(a,b) \neq (1,1)$, then the sets $\{1,2\}$ and $\{1,3\}$ intersect* $\mathrm{Per}(f)$.

Theorem 4. *Let $f : \mathbb{X} \to \mathbb{X}$ be a homeomorphism and let $\mathbb{X}$ be either $\mathbb{C}P^n$ or $\mathbb{H}P^n$. If f_{*k} denotes the action on the k–th homology group $H_k(\mathbb{X}, \mathbb{Q}) \approx \mathbb{Q}$, with $k = 2n$ if $\mathbb{X} = \mathbb{C}P^n$, and $k = 4n$ if $\mathbb{X} = \mathbb{H}P^n$. Then $a = \pm 1$, $f_{*2n} = f_{*4n} = (a^n)$ and $1 \in \mathrm{Per}(f)$ if $a = 1$, or $a = -1$ and n is odd, and $\{1,2\} \cap \mathrm{Per}(f)$ if $a = -1$ and n is even.*

The results of this theorem are related to the problem 3 in Section 2C; and to the problem 4 in Section 3.2 of [9].

Studies of the periods of $\mathcal{C}^1$ self–maps using different techniques but also based in the Lefschetz fixed point theory can be found in [6, 8].

2. Proof of Theorem 1

The proof of Theorem 1 is a consequence of a general result about polyhedron homeomorphisms proved in [5], see also Halpern [10] and Brown [1] for more details on it.

Theorem 5 (Fuller's Theorem)**.** *Let f be a homeomorphism of a compact polyhedron X onto itself. If the Euler characteristic of X is not zero, then f has a periodic point with period not greater that the maximum of $\sum_{k\, odd} B_k(X)$ and $\sum_{k\, even} B_k(X)$, where $B_k(X)$ denotes the k–th Betti number of X.*

Proof of Theorem 1. Assume that $\mathbb{X} = \mathbb{S}n$ with n even. It is well known that the Euler characteristic $\mathcal{X}(\mathbb{S}^n) = B_0(\mathbb{S}^n) + (-1)^n B_n(\mathbb{S}^n) = 2 \neq 0$.

Since

$$\max\left\{\sum_{k\, even} B_k(\mathbb{S}^n) = 2, \sum_{k\, odd} B_k(\mathbb{S}^n) = 0\right\} = 2,$$

by Theorem 5 we have that $\mathrm{Per}(f) \cap \{1,2\} \neq \emptyset$, proving statement (a).

Assume now that $\mathbb{X} = \mathbb{S}^n \times \mathbb{S}^n$ with n even. It is known that the Euler characteristic $\mathcal{X}(\mathbb{S}^n \times \mathbb{S}^n) = B_0(\mathbb{S}^n \times \mathbb{S}^n) + (-1)^n B_n(\mathbb{S}^n \times \mathbb{S}^n) + (-1)^{2n} B_{2n}(\mathbb{S}^n \times \mathbb{S}^n) = 4 \neq 0$.

Since

$$\max\left\{\sum_{k\, even} B_k(\mathbb{S}^n \times \mathbb{S}^n) = 4, \sum_{k\, odd} B_k(\mathbb{S}^n \times \mathbb{S}^n) = 0\right\} = 4,$$

by Theorem 5 we obtain that $\mathrm{Per}(f) \cap \{1,2,3,4\} \neq \emptyset$, obtaining statement (b).

Let now $\mathbb{X}$ be equal to $\mathbb{S}^n \times \mathbb{S}^m$ with $n \neq m$, both even. It is known that the Euler characteristic $\mathcal{X}(\mathbb{S}^n \times \mathbb{S}^m) = B_0(\mathbb{S}^n \times \mathbb{S}^m) + (-1)^n B_n(\mathbb{S}^n \times \mathbb{S}^m) + (-1)^m B_m(\mathbb{S}^n \times \mathbb{S}^m) + (-1)^{n+m} B_{n+m}(\mathbb{S}^n \times \mathbb{S}^m) = 4 \neq 0$.

Since

$$\max\left\{\sum_{k\, even} B_k(\mathbb{S}^n \times \mathbb{S}^n) = 4, \sum_{k\, odd} B_k(\mathbb{S}^n \times \mathbb{S}^n) = 0\right\} = 4,$$

by Theorem 5 we get that $\text{Per}(f) \cap \{1,2,3,4\} \neq \emptyset$. Hence statement (c) follows.

Finally, let $\mathbb{X}$ be either equal to $\mathbb{C}P^n$ or $\mathbb{H}P^n$. We have that the Euler characteristic $\mathcal{X}(\mathbb{C}P^n) = B_0(\mathbb{C}P^n) + (-1)^2 B_2(\mathbb{C}P^n) + ... + (-1)^{2n} B_{2n}(\mathbb{C}P^n) = n+1 \neq 0$. Analogously, $\mathcal{X}(\mathbb{H}P^n) = B_0(\mathbb{H}P^n) + (-1)^4 B_4(\mathbb{H}P^n) + ... + (-1)^{4n} B_{2n}(\mathbb{H}P^n) = n+1 \neq 0$.

Since

$$\max\left\{\sum_{k\,even} B_k(\mathbb{X}) = n+1, \sum_{k\,odd} B_k(\mathbb{X}) = 0\right\} = n+1,$$

with $\mathbb{X} \in \{\mathbb{C}P^n, \mathbb{H}P^n\}$, then by Theorem 5 we have that $\text{Per}(f) \cap \{1,2,...,n+1\} \neq \emptyset$. Hence statement (d) is proved. This completes the proof of the theorem. □

3. Proofs of Theorems 2, 3 and 4

Assume that $\mathbb{X} \in \Delta$ with dimension n and let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map, there exist $n+1$ induced linear maps $f_{*k} : H_k(\mathbb{X}, \mathbb{Q}) \to H_k(\mathbb{X}, \mathbb{Q})$ for $k = 0, 1, \ldots, n$ by f. Every linear map f_{*k} is given by an $n_k \times n_k$ matrix with integer entries, where n_k is the dimension of $H_k(\mathbb{X}, \mathbb{Q})$.

In this setting is defined the *Lefschetz number* $L(f)$ as

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}). \tag{1}$$

The importance of this notion is given by the existence of a result connecting the algebraic topology with the fixed point theory called the *Lefschetz Fixed Point Theorem* which establishes the existence of a fixed point if $L(f) \neq 0$, see for instance [1].

Since our aim is to obtain information on the set of periods of f for self–homeomorphisms of Δ, it is useful to have information on the whole sequence $\{L(f^m)\}_{m=0}^{\infty}$ of the Lefschetz numbers of all iterates of f. Thus we define the *Lefschetz zeta function* of f as

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m\right). \tag{2}$$

This function generates the whole sequence of Lefschetz numbers, and it may be independently computed through

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}}, \tag{3}$$

where I_{n_k} is the $n_k \times n_k$ identity matrix, and we take $\det(I_{n_k} - t f_{*k}) = 1$ if $n_k = 0$. Note that the expression (3) is a rational function in t. So the information on the infinite sequence of integers $\{L(f^m)\}_{m=0}^{\infty}$ is contained in two polynomials with integer coefficients, for more details see [3].

Proof of Theorem 2. Let f be a continuous self–map of $\mathbb{S}^n \times \mathbb{S}^n$. We know that the induced linear maps are $f_{*0} = (1)$, $f_{*n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$, $f_{*2n} = (D)$ where D is the degree of the map f and $f_{*i} = (0)$ for $i \in \{0, ..., 2n\}$, $i \neq 0, n, 2n$ (see for more details [2]). From (3) the Lefschetz zeta function of f is

$$\mathcal{Z}_f(t) = \frac{p(t)^{(-1)^{n+1}}}{(1-t)(1-Dt)}. \tag{4}$$

where $p(t) = 1 - (a + d)t + (ad - bc)t^2$.

The Poincaré duality (see [9]), or directly from the product structure of the cohomology ring, we have that $\det(f_{*n}) = \deg(f) = D$.

Let f be an orientation preserving homeomorphism of $\mathbb{S}^n \times \mathbb{S}^n$ and n odd. Therefore the degree D of f is 1. From (2) and (4) we have that

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m = \ & \log\left(\frac{1-(a+d)t+(ad-bc)t^2}{(1-t)^2}\right) \\
= \ & (2-a-d)t + \frac{1}{2}(2-a^2-2bc-d^2)t^2 \\
& + \frac{1}{3}(2-a^3-3abc-3bcd-d^3)t^3 \\
& + \frac{1}{4}(2-a^4-4a^2bc-2b^2c^2-4abcd-4bcd^2-d^4)t^4 + \dots \\
= \ & L(f)t + \frac{1}{2}L(f^2)t^2 + \frac{1}{3}L(f^3)t^3 + \frac{1}{4}L(f^4)t^4 + \dots
\end{aligned}$$

The solutions a, b, c, d of each one of the following three systems

$$\begin{aligned}
L(f) = L(f^2) = 0, \\
L(f) = L(f^3) = 0, \\
L(f) = L(f^4) = 0,
\end{aligned}$$

satisfy the two equations $bc + (d - 1)^2 = 0$ and $a + d = 2$. So, if a, b, c, d do not satisfy these two equation, by the Lefschetz Fixed Point Theorem, the statement (a) follows.

Let f be an orientation preserving homeomorphism of $\mathbb{S}^n \times \mathbb{S}^n$ and n even. Therefore the degree D of f is 1. By (2) and (4) we have that

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m &= \log\left(\frac{1}{(1-(a+d)t+(ad-bc)t^2)(1-t)2}\right) \\
&= (2+a+d)t + \frac{1}{2}(2+a^2+2bc+d^2)t^2 \\
&\quad + \frac{1}{3}(2a^3+3abc+3bcd+d^3)t^3 \\
&\quad + \frac{1}{4}(2+a^4+4a^2bc+2b^2c^2+4abcd+4bcd^2+d^4)t^4 + \dots \\
&= L(f)t + \frac{1}{2}L(f^2)t^2 + \frac{1}{3}L(f^3)t^3 + \frac{1}{4}L(f^4)t^4 + \dots
\end{aligned}$$

Since the systems $L(f) = L(f^2) = L(f^3) = 0$ and $L(f) = L(f^2) = L(f^4) = 0$ have no solutions in the variables a, b, c, d. This completes the proof of statement (b).

Let f be an orientation reversing homeomorphism of $\mathbb{S}^n \times \mathbb{S}^n$ and n odd. Therefore the degree D of f is -1. Again by (2) and (4) we have that

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m &= \log\left(\frac{1-(a+d)t+(ad-bc)t^2}{1-t^2}\right) \\
&= (-a-d)t + \frac{1}{2}(2-a^2-2bc-d^2)t^2 \\
&\quad - \frac{1}{3}(a+d)(a^2+3bc-ad+d^2)t^3 \\
&\quad + \frac{1}{4}(2-a^4-4a^2bc-2b^2c^2-4abcd-4bcd^2-d^4)t^4 + \dots \\
&= L(f)t + \frac{1}{2}L(f^2)t^2 + \frac{1}{3}L(f^3)t^3 + \frac{1}{4}L(f^4)t^4 + \dots
\end{aligned}$$

The solutions a, b, c, d of the system $L(f) = L(f^3) = 0$ are those satisfying the equation $a + d = 0$. Therefore statement (c.1) follows.

The solutions a, b, c, d of the system $L(f) = L(f^2) = 0$ are those satisfying the two equations $bc + d^2 = 1$ and $a + d = 0$. So, statement (c.2) is proved.

The solutions a, b, c, d of the system $L(f) = L(f^4) = 0$ are those satisfying the there equations $bc + d^2 = \pm 1$ and $a + d = 0$. So, statement (c.3) is proved.

Let f be an orientation reversing homeomorphism of $\mathbb{S}^n \times \mathbb{S}^n$ and n even. Therefore the degree D of f is -1. By (2) and (4) we have that

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m &= \log\left(\frac{1}{(1-(a+d)t+(ad-bc)t^2)(1-t^2)}\right) \\ &= (a+d)t + \frac{1}{2}(2+a^2+2bc+d^2)t^2 \\ &\quad + \frac{1}{3}(a^3+3abc+3bcd+d^3)t^3 \\ &\quad + \frac{1}{4}(2+a^4+4a^2bc+2b^2c^2+4abcd+4bcd^2+d^4)t^4+\dots \\ &= L(f)t + \frac{1}{2}L(f^2)t^2 + \frac{1}{3}L(f^3)t^3 + \frac{1}{4}L(f^4)t^4 + \dots\end{aligned}$$

Since the system $L(f) = L(f^2) = L(f^4) = 0$ has no solutions in the variables a, b, c, d the statement (d.1) is proved.

A solution a, b, c, d of the system $L(f) = L(f^3) = 0$ must satisfies the equation $a + d = 0$. So, statement (d.2) follows.

The solutions a, b, c, d of the system $L(f) = L(f^2) = 0$ are the ones satisfying the two equations $bc+d^2 = -1$ and $a+d = 0$. Hence, statement (d.3) is proved. Hence Theorem 2 is proved. □

Proof of Theorem 3. Let f be a continuous self–map of $\mathbb{S}^n \times \mathbb{S}^m$ with $n \neq m$. It is known that the induced linear maps are $f_{*0} = (1)$, $f_{*n} = (a)$, $f_{*m} = (b)$ with $a, b \in \mathbb{Z}$, $f_{*n+m} = (D)$, where $D \in \mathbb{Z}$ is the degree of the map f and $f_{*i} = (0)$ for $i \in \{0, ..., n+m\}$, $i \neq 0, n, m, n+m$ (see for more details [2]).

By Poincaré duality, or again by a direct consideration with the cup-product, we have $ab = \deg(f)$. For homomorphisms we have the additional restriction $a, b \in \{\pm 1\}$.

From (3) the Lefschetz zeta function of f is of the form

$$\mathcal{Z}_f(t) = \frac{(1-at)^{(-1)^{n+1}}(1-bt)^{(-1)^{m+1}}(1-Dt)^{(-1)^{n+m+1}}}{1-t}. \tag{5}$$

Let f be an orientation preserving homeomorphism, n and m even. Therefore the degree D of f is 1. By (2) and (5) we have that

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m &= \log\left(\frac{1}{(1-at)(1-bt)(1-t)^2}\right) \\ &= (2+a+b)t + \frac{1}{2}(2+a^2+b^2)t^2 \\ &\quad + \frac{1}{3}(2+a^3+b^3)t^3 + \frac{1}{4}(2+a^4+b^4)t^4 + \dots \\ &= L(f)t + \frac{1}{2}L(f^2)t^2 + \frac{1}{3}L(f^3)t^3 + \frac{1}{4}L(f^4)t^4 + \dots\end{aligned}$$

Since the system $L(f) = L(f^2) = 0$ has no solutions in the variables a, b the statement (a.1) is proved.

Since the unique solution of the system $L(f) = L(f^3) = 0$ is $(a, b) = (-1, -1)$, the statement (a.2) follows.

Let f be an orientation reversing homeomorphism, n and m even. Therefore the degree D of f is -1. By (2) and (5) we have that

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m &= \log\left(\frac{1}{(1-at)(1-bt)(1-t^2)}\right) \\ &= (a+b)t + \frac{1}{2}(2+a^2+b^2)t^2 \\ &\quad + \frac{1}{3}(a^3+b^3)t^3 + \frac{1}{4}(2+a^4+b^4)t^4 + \dots \\ &= L(f)t + \frac{1}{2}L(f^2)t^2 + \frac{1}{3}L(f^3)t^3 + \frac{1}{4}L(f^4)t^4 + \dots\end{aligned}$$

Since the systems $L(f) = L(f^2) = L(f^3) = 0$ and $L(f) = L(f^2) = L(f^4) = 0$ have no solutions in the variables a, b the statement (b) is proved.

Let f be an orientation reversing homeomorphism, n is even and m is odd. Therefore the degree D of f is -1. By (2) and (5) we have that

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m &= \log\left(\frac{(1-bt)(1+t)}{(1-at)(1-t)}\right) \\ &= (2+a-b)t + \frac{1}{2}(a^2-b^2)t^2 \\ &\quad + \frac{1}{3}(2+a^3-b^3)t^3 + \frac{1}{4}(a^4-b^4)t^4 + \dots \\ &= L(f)t + \frac{1}{2}L(f^2)t^2 + \frac{1}{3}L(f^3)t^3 + \frac{1}{4}L(f^4)t^4 + \dots\end{aligned}$$

Since the unique solution of the systems $L(f) = L(f^2) = 0$ and $L(f) = L(f^3) = 0$ is $(a, b) = (-1, 1)$, the statement (c) follows.

Let f be an orientation reversing homeomorphism, n is odd and m is even. Therefore the degree D of f is -1. By (2) and (5) we have that

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m &= \log\left(\frac{(1-at)(1+t)}{(1-bt)(1-t)}\right) \\ &= (2+b-a)t + \frac{1}{2}(b^2-a^2)t^2 \\ &\quad + \frac{1}{3}(2+b^3-a^3)t^3 + \frac{1}{4}(b^4-a^4)t^4 + \dots \\ &= L(f)t + \frac{1}{2}L(f^2)t^2 + \frac{1}{3}L(f^3)t^3 + \frac{1}{4}L(f^4)t^4 + \dots\end{aligned}$$

Since the unique solution of the systems $L(f) = L(f^2) = 0$ and $L(f) = L(f^3) = 0$ is $(a, b) = (1, -1)$, the statement (d) follows.

Let f be an orientation preserving homeomorphism, n and m are odd. Therefore the degree D of f is 1. By (2) and (5) we have that

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{L(f^m)}{m}t^m &= \log\left(\frac{(1-at)(1-bt)}{(1-t)^2}\right)\\ &= (2-a-b)t+\frac{1}{2}(2-a^2-b^2)t^2\\ &\quad +\frac{1}{3}(2-a^3-b^3)t^3+\frac{1}{4}(2-a^4-b^4)t^4+\dots\\ &= L(f)t+\frac{1}{2}L(f^2)t^2+\frac{1}{3}L(f^3)t^3+\frac{1}{4}L(f^4)t^4+\dots\end{aligned}$$

Since the unique solution of the systems $L(f)=L(f^2)=0$ and $L(f)=L(f^3)=0$ is $(a,b)=(1,1)$, the statement (e) is proved. This completes the proof of the theorem. □

Proof of Theorem 4. Let f be a continuous self–map of $\mathbb{C}P^n$ with $n\geq 1$. We know that the induced linear maps are $f_{*q}=(a^{\frac{q}{2}})$ for $q\in\{0,2,4,...,2n\}$ with $a\in\mathbb{Z}$, and $f_{*q}=(0)$ otherwise (see for more details [12, Corollary 5.28]).

From (3) the Lefschetz zeta function of f has the form

$$\mathcal{Z}_f(t)=\left(\prod_q(1-a^{q/2}t)\right)^{-1}, \tag{6}$$

where q runs over $\{0,2,4,...,2n\}$.

Let f be a continuous self–map of $\mathbb{H}P^n$ with $n\geq 1$. We know that the induced linear maps are $f_{*q}=(a^{\frac{q}{4}})$ for $q\in\{0,4,8,...,4n\}$ with $a\in\mathbb{Z}$, and $f_{*q}=(0)$ otherwise (see for more details [12, Corollary 5.33]).

From (3) the Lefschetz zeta function of f has the form

$$\mathcal{Z}_f(t)=\left(\prod_q(1-a^{q/4}t)\right)^{-1}, \tag{7}$$

where q runs over $\{0,4,8,...,4n\}$.

Suppose now that f is a homeomorphism of either $\mathbb{C}P^n$ or $\mathbb{H}P^n$, then $a=\pm1$. By (2), (6) and (7) we have that

$$\sum_{m=1}^{\infty}\frac{L(f^m)}{m}t^m=\begin{cases}\log\dfrac{1}{(1-t)^n} & \text{if } a=1,\\[2ex] \log\dfrac{1}{(1-t^2)^{(n-1)/2}(1-t)} & \text{if } a=-1 \text{ and } n \text{ is odd},\\[2ex] \log\dfrac{1}{(1-t^2)^{n/2}} & \text{if } a=-1 \text{ and } n \text{ is even}.\end{cases}$$

Since

$$\log \frac{1}{(1-t)^n} = nt + \frac{1}{2}nt^2 + \frac{1}{3}nt^3 + \frac{1}{4}nt^4 + \frac{1}{5}nt^5 + \frac{1}{6}nt^6 + \ldots,$$

$$\log \frac{1}{(1-t^2)^{(n-1)/2}(1-t)} = t + \frac{1}{2}nt^2 + \frac{1}{3}t^3 + \frac{1}{4}nt^4 + \frac{1}{5}t^5 + \frac{1}{6}nt^6 + \ldots,$$

$$\log \frac{1}{(1-t^2)^{n/2}} = \frac{1}{2}nt^2 + \frac{1}{4}nt^4 + \frac{1}{6}nt^6 + \ldots,$$

We have that $L(f) \neq 0$ if $a = 1$, or $a = -1$ and n is odd, and that $L(f) = 0$ and $L(f^2) \neq 0$ if $a = -1$ and n is even. Then theorem follows. □

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