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ON THE LOCUS OF SMOOTH PLANE CURVES WITH A FIXED AUTOMORPHISM GROUP

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ABSTRACT. Let M_g be the moduli space of smooth, genus g curves over an algebraically closed field K of zero characteristic. Denote by $M_g(G)$ the subset of M_g of curves δ such that G (as a finite non-trivial group) is isomorphic to a subgroup of $Aut(\delta)$, the full automorphism group of δ, and let $\widetilde{M_g(G)}$ be the subset of curves δ such that $G \cong Aut(\delta)$. Now, for an integer $d \geq 4$, let M_g^{Pl} be the subset of M_g representing smooth, genus g plane curves of degree d (in such case, $g = (d-1)(d-2)/2$) and consider the sets $M_g^{Pl}(G) := M_g^{Pl} \cap M_g(G)$ and $\widetilde{M_g^{Pl}(G)} := \widetilde{M_g(G)} \cap M_g^{Pl}$.

In this paper, we study some aspects of the irreducibility of $\widetilde{M_g^{Pl}(G)}$ and its interrelation with the existence of "normal forms", i.e. non-singular plane equations (depending on a set of parameters) such that a specialization of the parameters gives a certain non-singular plane model associated to the elements of $\widetilde{M_g^{Pl}(G)}$. In particular, we introduce the concept of being equation strongly irreducible (ES-Irreducible) for which the locus $\widetilde{M_g^{Pl}(G)}$ is represented by a single "normal form". Henn, in [11], and Komiya-Kuribayashi, in [13], observed that $\widetilde{M_3^{Pl}(G)}$ is ES-Irreducible. In this paper we prove that this phenomena does not occur for any odd $d > 4$. More precisely, let $\mathbb{Z}/m\mathbb{Z}$ be the cyclic group of order m, we prove that, for any odd integer $d \geq 5$, $\widetilde{M_g^{Pl}(\mathbb{Z}/(d-1)\mathbb{Z})}$ is not ES-Irreducible and the number of the irreducible components of such loci is at least two. Furthermore, we conclude the previous result when $d = 6$ for the locus $\widetilde{M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})}$.

Lastly, we prove the analogy of these statements when K is any algebraically closed field of positive characteristic p such that $p > (d-1)(d-2)+1$.

1. INTRODUCTION

Let K be an algebraically closed field of zero characteristic and fix an integer $d \geq 4$. We consider, up to K-isomorphism, a projective non-singular curve δ of genus $g = (d-1)(d-2)/2$ and assume that δ has a non-singular plane model, i.e. $\delta \in M_g^{Pl}$.

It is well known that any $\delta \in M_g^{Pl}(G)$ corresponds to a set $\{C_\delta\}$ of non-singular plane models in $\mathbb{P}^2(K)$ such that any two of them are K-isomorphic through a projective transformation $P \in PGL_3(K)$ (where $PGL_N(K)$ is the classical projective linear group of $N \times N$ invertible matrices over K), and their automorphism groups are conjugate. More concretely, fixing C, a non-singular plane model of δ, it is defined by a homogenous equation $F(X;Y;Z) = 0$ of degree d. Then, $Aut(C)$ is a finite subgroup of $PGL_3(K)$, and also we have $\rho(G) \preceq Aut(C)$ for some injective representation $\rho : G \hookrightarrow PGL_3(K)$. Moreover, $\rho(G) = Aut(C)$ whenever $\delta \in \widetilde{M_g^{Pl}(G)}$. For another non-singular plane model C' of δ, there exists $P \in PGL_3(K)$ where C' is defined by $F(P(X,Y,Z)) = 0$ and $P^{-1}\rho(G)P \preceq Aut(C')$ (respectively, $P^{-1}\rho(G)P = Aut(C')$ if $\delta \in \widetilde{M_g^{Pl}(G)}$) .

For an arbitrary, but a fixed degree d, classical and deep questions on the subject are: list the groups that appear as the exact automorphism groups of algebraic non-singular plane curves of degree d, and for each of such group, determine associated "normal forms", i.e. a finite set of homogenous equations $\{N_{1,G}, \ldots, N_{k,G}\}$ in X, Y, Z together with some parameters (under some restrictions) such that any specialization of a certain $N_{i,G}$ in K corresponds to a unique $\delta \in \widetilde{M_g^{Pl}(G)}$ (is the one that it is associated to the non-singular plane model given by the specialization of the normal form $N_{i,G}$), and given $\delta \in \widetilde{M_g^{Pl}(G)}$, exists a unique i_δ and a specialization of the parameters at K for $N_{i_\delta,G}$, such that one obtains a plane non-singular model associated to δ; in particular

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any specialization of the parameters of two distinct $N_{i,G}$ gives two non-singular plane models, which in turns relate to two non-isomorphic plane non-singular curves of $\widetilde{M_g^{Pl}(G)}$.

For $d=4$, Henn in [11] and Komiya-Kuribayashi in [13], answered the above natural questions. See also Lorenzo's thesis [14] § 2.1 and § 2.2, in order to fix some minor details. It appears, for $d=4$, the following phenomena: any element of $\widetilde{M_3^{Pl}(G)}$ has a non-singular plane model through some specialization of the parameters of a single normal form. If this phenomena appears for some g, we say that the locus $\widetilde{M_g^{Pl}(G)}$ is ES-Irreducible (see §2 for the precise definition). This is a weaker condition than the irreducibility of this locus inside of the moduli space M_g. In particular, it follows by Henn [11] and Komiya-Kuribayashi [13], the locus $\widetilde{M_3^{Pl}(G)}$ is always ES-Irreducible.

The motivation of this work is that we did not expect $\widetilde{M_g^{Pl}(G)}$ to be ES-Irreducible in general. In order to construct counter examples for which $\widetilde{M_g^{Pl}(G)}$ is not ES-Irreducible: we need first, a group G such that there exist at least two injective representations $\rho_i : G \hookrightarrow PGL_3(K)$ with $i=1,2$, which are not conjugate (i.e there is no transformation $P \in PGL_3(K)$ with $P^{-1}\rho_1(G)P = \rho_2(G)$, more details are included in §2), and for the zoo of groups that could appear for non-singular plane curves [10], we consider G, a cyclic group of order m. Secondly, one needs to prove the existence of two non-singular plane curves with automorphism groups are conjugate to $\rho_i(G)$ for each $i=1,2$.

The main results of the paper is that, for any odd degree $d(\geq 5)$, the locus $\widetilde{M_g^{Pl}(\mathbb{Z}/(d-1)\mathbb{Z})}$ is not ES-irreducible, and it has at least two irreducible components (recall that for $d=5$, by [2], we know that the only group G for which $\widetilde{M_6^{Pl}(G)}$ is not ES-Irreducible is for $\mathbb{Z}/4\mathbb{Z}$). For d even, in section § 5, we prove that $\widetilde{M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})}$ is not ES-irreducible. It is to be noted that, by our work in [1], we may conjecture that the locus $\widetilde{M_g^{Pl}(\mathbb{Z}/m\mathbb{Z})}$ could not be ES-Irreducible only if m divides d or $d-1$ (this is true at least until degree 9 by [1]). Concerning positive characteristic, in the last section (§ 6) of this paper we prove that the above examples of non-irreducible loci are also valid when K is an algebraically closed field of positive characteristic $p>0$, provided that the characteristic p is big enough, once we fix the degree d.

The irreducibility of the loci $\widetilde{M_g^{Pl}(\mathbb{Z}/m)}$ seems to be very deep problem. In §2, we give some insights that relate the above locus with subsets in classical loci of the moduli spaces. In particular, with the loci of curves of genus g with a prescribed cyclic Galois subcover. In this section, as an explicit example, we deal with the question for the locus $\widetilde{M_6^{Pl}(\mathbb{Z}/8)}$, which is ES-Irreducible, and is represented by a single normal form with only one paramater. In [1], we proved that $M_g^{Pl}(G)$ is irreducible when G has an element of order $(d-1)^2$, $d(d-1)$, $d(d-2)$ or d^2-3d+3, since this locus has only one element. In particular, we proved in [1] that $\widetilde{M_g^{Pl}(\mathbb{Z}/d(d-1))}$ and $\widetilde{M_g^{Pl}(\mathbb{Z}/(d-1)^2)}$ are irreducible.

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2. On the locus $M_g^{Pl}(G)$ and $\widetilde{M_g^{Pl}(G)}$.

Consider a projective non-singular curve δ of genus $g := \frac{(d-1)(d-2)}{2} \geq 2$ over K with G, a finite non-trivial group, inside $Aut(\delta)$. We always assume that δ admits a non-singular plane equation, and we consider δ up to K-isomorphism, as a point in $M_g^{Pl}(G)$.

Because linear systems g_d^2 are unique (up to multiplication by $P \in PGL_3(K)$ in $\mathbb{P}^2(K)$ [12, Lemma 11.28]), we always take C a plane non-singular model of δ, which is given by a projective plane equation $F(X;Y;Z)=0$ and $Aut(C)$ is a finite subgroup of $PGL_3(K)$ that fixes the equation F and is isomorphic to $Aut(\delta)$. Any other plane model of δ is given by $C_P : F(P(X;Y;Z))=0$ with $Aut(C_P)=P^{-1}Aut(C)P$ for some $P \in PGL_3(K)$ and C_P is K-equivalent or K-isomorphic to C. In particular, for $\delta \in M_g^{Pl}(G)$, exists $\rho : G \hookrightarrow PGL_3(K)$ where $\rho(G) \leq Aut(C)$ and $P^{-1}\rho(G)P \leq Aut(C_P)$.

We denote by $\rho(M_g^{Pl}(G))$ the loci given by $\delta \in M_g^{Pl}(G)$ such that G acts on a certain plane model associated to δ by $P^{-1}\rho(G)P$ for certain $P \in PGL_3(K)$, and similarly for $\rho(\widetilde{M_g^{Pl}(G)})$.

Denote by A_G the quotient set $\{\rho : G \hookrightarrow PGL_3(K)\}/\sim$ where $\rho_1 \sim \rho_2$ if and only if $\exists P \in PGL_3(K)$ such that $\rho_1(G) = P^{-1}\rho_2(G)P$, as usual $[\rho]$ denotes the class of ρ in A_G.

Clearly $M_g^{Pl}(G) = \cup_{[\rho]\in A_G}\rho(M_g^{Pl}(G))$.

Lemma 2.1. *The loci $\widetilde{M_g^{Pl}(G)}$ is the disjoint union of $\rho(\widetilde{M_g^{Pl}(G)})$ where $[\rho]$ runs the quotient set A_G.*

Proof. For $\delta \in \rho_1(\widetilde{M_g^{Pl}(G)}) \cap \rho_2(\widetilde{M_g^{Pl}(G)})$ means that it has a plane model C where $Aut(C) = P_1^{-1}\rho_1(G)P_1 = P_2^{-1}\rho_2(G)P_2$ for certain $P_1, P_2 \in PGL_3(K)$ therefore $\rho_1 \sim \rho_2$. □

Remark 2.2. *If $\delta \in \rho_1(M_g^{Pl}(G)) \cap \rho_2(M_g^{Pl}(G))$ with $[\rho_1] \neq [\rho_2] \in A_G$, and take C a plane model of δ, then $Aut(C) \leq PGL_3(K)$ should have two subgroups isomorphic to G which are not conjugate. A detailed study of the work of Blichfeldt [3] would give the list of G where the decomposition $M_g^{Pl}(G) = \cup_{[\rho]\in A_G}\rho(M_g^{Pl}(G))$ may not be disjoint, if any.*

Fix $[\rho] \in A_G$ then for $\delta \in \rho(M_g^{Pl}(G))$, we can associate infinitely many non-singular plane models which are K-isomorphic through a change of variables $P \in PGL_3(K)$, but we can consider only the models such that G is identified as automorphism group for the model exactly as $\rho(G) \leq PGL_3(K)$ for some fixed ρ in $[\rho] \in A_G$. Under this restriction, δ can be associated with a non-empty family of non-singular models of δ such that any two models are isomorphic, through a projective transformation P satisfying $P^{-1}\rho(G)P = \rho(G)$.

Recall that, it is a necessary condition for a projective plane curve of degree d to be non-singular that the defining equation of any model has degree $\geq d-1$ in each variable, and, once we fix a model, by a diagonal change of variables P, we can assume that we can chose a model such that the monomials with the maximal exponent have coefficients equal to 1, where for a non-zero monomial $cX^iY^jZ^k$ we define its exponent as $max\{i,j,k\}$. For a homogeneous polynomial F, the core of F is defined as the sum of all terms of F with the greatest exponent. Consequently, we reduce the case to the set of K-isomorphic non-singular plane models $F(X;Y;Z) = 0$ associated to δ with $\rho(G)$ fixes the equation (because are automorphism of such a model) and each term of the core of $F(X;Y;Z)$ is monic.

Lemma 2.3. *Let G be a non-trivial finite group and consider $\rho : G \hookrightarrow PGL_3(K)$ such that $\rho(M_g^{Pl}(G))$ is non-empty. There exists a single normal form, i.e. an homogenous polynomial $F_{\rho,G}(X;Y;Z) = 0$ of degree d in the variables X, Y and Z, endowed with certain parameters on the coefficients of the lower order terms (with some restrictions) representing the loci $\rho(M_g^{Pl}(G))$, more concretely, every specialization of the parameters at K (under the restriction on the parameters) of $F_{\rho,G}$ gives a plane non-singular model of an element of $\rho(M_g^{Pl}(G))$, and viceversa, for any element $\delta \in \rho(M_g^{Pl}(G))$ exists an specialization of the parameters at K for $F_{\rho,G}$ such that one obtains a plane non-singular model of δ in $\mathbb{P}^2(K)$. A similar statement holds for $\rho(\widetilde{M_g^{Pl}(G)})$ in such case we will name $F_{\rho,G,*}$ a single normal form. Moreover, such normal forms are unique up to a change of the variables X, Y, Z by $P \in PGL_3(K)$.*

Proof. Let $\sigma \in G$ be an automorphism of maximal order $m > 1$ and choose an element ρ in $[\rho] \in A$ such that, $\rho(\sigma)$ is diagonal of the form $diag(1, \xi_m^a, \xi_m^b)$ with $0 \leq a < b$ where ξ_m a primitive m-th root of unity in K. Following the same technique in [8] or [1] (for a general discussion), we can associate to the set $\rho(M_g^{Pl}(<\sigma>)$ a non-singular plane equation $F_{m,(a,b)}(X;Y;Z)$ with a certain set of parameters (which may have some restrictions in order to ensure the non-singularity), which is unique by construction which is a "normal form" for $\rho(M_g^{Pl}(<\sigma>)$. For example for $a \neq 0$ one argue following which of the reference points $\{(1:0:0), (0:1:0), (0:0:1)\}$ satisfy $F_{m,(a,b)}(X;Y;Z) = 0$. In particular when all reference points satisfy the normal form, we reduce that $F_{m,(a,b)}(X;Y;Z)$ is of the form $X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X + \sum_{j=2}^{\lfloor \frac{d}{2} \rfloor} \left(X^{d-j}L_{j,X} + Y^{d-j}L_{j,Y} + Z^{d-j}L_{j,Z}\right)$ where $L_{j,X}$ is an homogenous polynomial of degree j without the variable X and with parameters in the coefficients of the monomials. The first three factors implies that $a \equiv (d-1)a + b \equiv (d-1)b \,(mod\, m)$, obtaining that, $m|d^2 - 3d + 3$. The defining equation $F_{m,(a,b)}$ in such situation, follows immediately by checking monomials' invariance in each $L_{j,B}$. For example, rewrite $L_{j,X}$ as $\sum_{i=0}^{j}\beta_{j,i}Y^iZ^{j-i}$ (where $\beta_{j,i}$ are parameters) then $\beta_{j,i} = 0$

if $m \nmid ai+(j-i)b$, since $diag(1;\xi_m^a;\xi_m^b)\in Aut(C)$. Observe that in order to obtain such $F_{m,(a,b)}$ we chose a model for any $\delta\in\rho(M_g^{Pl}(G))$ satisfying $\rho(\sigma)=diag(1,\xi_m^a,\xi_m^b)$ and that the coefficients of the monomials of the core of the model are equal 1 when we restrict to $<\sigma>$, and in such assumptions we obtain a unique expression.

Now, in order to go from $\rho(<\sigma>)$ to $\rho(G)$, take generators u_G of G which does not belong to $<\sigma>$ and impose that $\rho(u_G)$ may retain invariant $F_{m,(a,b)}$ by imposing some specific algebraic relations between the parameters of $F_{m,(a,b)}$, this is done by comparing coefficients of monomials which may retain invariant. Then $F_{\rho,G}$ is obtained from $F_{m,(a,b)}$ imposing such algebraic relations between the coefficients of the monomials (i.e. certain parameters) of $F_{m,(a,b)}$.

We obtain $F_{\rho,G,*}$ from $F_{\rho,G}$. Recall that $\rho(G)\leq PGL_3(K)$ and for each finite group $\rho(G)\leq H\leq PGL_3(K)$ which exists a plane non-singular curve of genus g with automorphism group isomorphic to H, we need to impose that the generators of H which are not in $\rho(G)$ may not give invariant some monomial of $F_{\rho,G}$ in order to obtain $F_{\rho,G,*}$. In such case the relations that we need to impose are a complement of algebraic relations between the coefficients of monomials of $F_{\rho,G}$. □

Remark 2.4. *Observe that could happen that two different specializations of $F_{\rho,G}$ at K will give plane non-singular models of exactly the same curve $\delta\in\rho(M_g^{Pl}(G))$, this happens if exists P that goes from one model to another model which satisfies that $P^{-1}\rho(G)P=\rho(G)$ and $P^{-1}\rho(<\sigma>)P=\rho(<\sigma>)$. We could impose to $F_{\rho,G}$ that this phenomena will not occur by imposing more restrictions to the parameters, but we did not in our notion of "normal form". These further restrictions are recently explicit for $\rho(M_3^{Pl}(G))$ by Lorenzo [14], fixing missing details in the tables of Henn [11]. We also make explicit such restrictions and the ones that appears naturally during the proof of the above theorem for the particular case of $\rho(M_6^{Pl}(\mathbb{Z}/8))$ and $\rho(\widetilde{M_6^{Pl}(\mathbb{Z}/8)})$ at the end of this section.*

It is difficult to determine the groups G and $[\rho]\in A_G$ such that $\rho(M_g^{Pl}(G))$ is non-empty for some fixed g. Henn [11] obtained this determination for $g=3$, Badr-Bars [2] for $g=6$ and for a general implementation of any degree, we refer to [1] in which we formulate an algorithm to determine the ρ's when G is cyclic.

Definition 2.5. Write $M_g^{Pl}(G)$ as $\cup_{[\rho]\in A_G}\rho(M_g^{Pl}(G))$, we define the number of the equation components of $M_g^{Pl}(G)$ to be the number of elements $[\rho]\in A_G$ such that $\rho(M_g^{Pl}(G))$ is not empty. We say that $M_g^{Pl}(G)$ is equation irreducible if $M_g^{Pl}(G)=\rho(M_g^{Pl}(G))$ for a certain $[\rho]\in A_G$. For $\widetilde{M_g^{Pl}(G)}=\cup_{[\rho]\in A_G}\rho(\widetilde{M_g^{Pl}(G)})$, we define the number of the strongly equation irreducible components of $\widetilde{M_g^{Pl}(G)}$ to be the number of the elements $[\rho]\in A_G$ such that $\rho(\widetilde{M_g^{Pl}(G)})$ is not empty.

We say that $\widetilde{M_g^{Pl}(G)}$ is equation strongly irreducible (or simply, ES-irreducible) if it is not empty and $\widetilde{M_g^{Pl}(G)}=\rho(\widetilde{M_g^{Pl}(G)})$ for some $[\rho]\in A_G$.

Of course, if $\widetilde{M_g^{Pl}(G)}$ is not ES-irreducible then it is not irreducible and the number of the strongly irreducible equation components of $\widetilde{M_g^{Pl}(G)}$ is a lower bound for the number of irreducible components.

In this language, we can formulate the main result in [11] as follows

Theorem 2.6 (Henn, Komiya-Kuribayashi)**.** *If G is a non-trivial group that appears as the full automorphism group of a non-singular plane curve of degree 4, then $\widetilde{M_3^{P}(G)}$ is ES-Irreducible.*

Remark 2.7. *Henn in [11], observed that $M_3^{Pl}(\mathbb{Z}/3)$ already has two irreducible equation components, but one of such components has a bigger automorphism group namely, S_3 the symmetry group of of order 3.*

To finish this section, we state some natural questions concerning the locus $\rho(M_g^{Pl}(G))$ (and similar questions can be state for $\rho(\widetilde{M_g^{Pl}(G)})$) with different loci on moduli spaces of genus g curves:

Question 2.8. *Is it true that all the elements of $\rho(M_g^{Pl}(G))$ the corresponding Galois covers $\delta\to\delta/G$ have fixed ramification data?*

We believe that the answer to this question for $K = \mathbb{C}$ (i.e. Riemann surfaces) should be always true from the work of Breuer [4]. See Remark 4.4 for the explicit Galois subcover and the ramification data for the locus $\rho(M_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$ and §2.1 for the loci $\rho(M_6^{Pl}(\mathbb{Z}/8\mathbb{Z}))$.

Question 2.9. *Is $\rho(M_g^{Pl}(G))$ an irreducible set when G is a cyclic group?*

It is to be noted that when $K = \mathbb{C}$, Cornalba [7], with G cyclic of prime order, and Catanese [5], for general order, obtained that the locus of smooth projective curves of genus g with a cyclic Galois subcover of group isomorphic to G with a prescribed ramification is irreducible.

Concerning the irreducibility question, we prove in [1] that if G has an element of large order $(d-1)^2$, $d(d-1)$, $d(d-2)$ or d^2-3d+3 then $\rho(M_g^{Pl}(G))$ has at most one element therefore, is irreducible. At §2.1, we deal on irreducibility for the ES-Irreducibility loci $M_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$ where the single "normal form" has only one parameter.

Moreover, Catanese, Lönne and Perroni in [6, §2] defines a topological invariant for the loci $M_g(G)$ which is trivial if it is irreducible.

Question 2.10. *Consider G, a non-trivial group, where the set A_G is given by one element (see next section on groups G with A_G given by a single element). Is it true that the topological invariant in [6, §2] is trivial for $M_g(G)$ in order to be irreducible? Is it true that $M_g^{Pl}(G)$ are irreducible?*

2.1. The loci $M_6^{Pl}(\mathbb{Z}/8)$ and $\widetilde{M_6^{Pl}(\mathbb{Z}/8\mathbb{Z})}$.

Consider in M_6 an element δ which has a smooth non-singular plane model with an effective action of the cyclic group of order 8 in particular, $\delta \in M_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$. Following [1], [8] or the table §4 in this note, $M_6^{Pl}(\mathbb{Z}/8) = \rho(M_6^{Pl}(\mathbb{Z}/8))$ with $\rho(\mathbb{Z}/8\mathbb{Z}) =< diag(1, \xi_8, \xi_8^4) >$ where ξ_8 is a 8-th primitive root of unity in K an such loci has a "normal form" $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$ with β a parameter taking values at K such that always $\beta \neq \pm 2$ because is non-singular. Therefore, we can associate to δ a fix plane non-singular model of the form $X^5 + Y^4Z + XZ^4 + \beta_\delta X^3Z^2 = 0$ for certain $\beta_\delta \in K$ (but may be β_δ not unique in K).

Now, let us compute all non-singular plane models of the form $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$ that can be associated to the fixed curve δ. This corresponds to models obtained by a change of variables through a transformation $P \in PGL_3(K)$ such that $P^{-1} < (diag(1, \xi_8, \xi_8^4) > P =< diag(1, \xi_8, \xi_8^4) >$ and the new model has a similar form $X^5 + Y^4Z + XZ^4 + \beta' X^3Z^2 = 0$.

Without any loss of generality, we can suppose that $P^{-1}diag(1, \xi_8, \xi_8^4)P = diag(1, \xi_8, \xi_8^4)$ hence in order to have the same eigenvalues which are pairwise distinct, we may assume that P is a diagonal matrix, say $P = diag(1, \lambda_2, \lambda_3)$. Therefore, we get an equation of the form: $X^5 + \lambda_2^4\lambda_3 Y^4Z + \lambda_3^4 XZ^4 + \beta_\delta \lambda_3^2 X^3Z^2 = 0$. From which we must have $\lambda_2^4\lambda_3 = \lambda_3^4 = 1$, thus λ_3^2 is 1 or -1. Hence, we obtain a bijection map

$$\varphi : M_6^{Pl}(\mathbb{Z}/8\mathbb{Z}) \to \mathbb{A}^1(K) \setminus \{-2, 2\}/\sim$$
$$\delta \mapsto [\beta_\delta] = \{\beta_\delta, -\beta_\delta\}$$

where $a \sim b \Leftrightarrow b = a \text{ or } a = -b$. Moreover, by the work that we did in [2], we know that $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$ has a bigger automorphism group than $\mathbb{Z}/8\mathbb{Z}$ if and only if $\beta = 0$, therefore, we have a bijection map

$$\tilde{\varphi} : \widetilde{M_6^{Pl}(\mathbb{Z}/8\mathbb{Z})} \to \mathbb{A}^1(K) \setminus \{-2, 0, 2\}/\sim$$
$$\delta \mapsto [\beta_\delta] = \{\beta_\delta, -\beta_\delta\}$$

and observe that $0 \in \mathbb{A}^1(K)$ is the only point which had no identification by the relation rule $\sim$. The above sets, when K is the complex field, are irreducible.

Moreover, if we consider the Galois cyclic cover of degree 8 given by the action of the automorphism of order 8 on $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$, we obtain that it ramifies at the points $(0 : 1 : 0), (0 : 0 : 1)$ with ramification index 8 as well as the four points $(1 : 0 : h)$ where $1 + h^4 + \beta h^2 = 0$ with ramification index 2 if $\beta \neq \pm 2$. That is, $M_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$ is inside the locus of curves in M_6 which have a cyclic Galois subcover of degree 8 to a genus zero curve and which ramifies at 6 points, 2 points with ramification index 8 and the other 4 points with ramification index 4.

3. Preliminaries on automorphism on plane curves

Given $\delta \in M_g^{Pl}$, with $Aut(\delta)$ non-trivial, we fix C a plane non-singular model of degree d. By an abuse of notation, once and for all, we also denote C by a non-singular projective plane curve. Then $Aut(C)$ is a finite subgroup of $PGL_3(K)$ and it satisfies one of the following situations (for more details, see Mitchell [15]):

(1) fixes a point $\mathcal{P}$ and a line L with $P \notin L$ in $\mathbb{P}^2(K)$,
(2) fixes a triangle, i.e. exists 3 points $S := \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ of $\mathbb{P}^2(K)$, such that is fixed as a set,
(3) $Aut(C)$ is conjugate of a representation inside $PGL_3(K)$ of one of the finite primitive group namely, the Klein group $PSL(2,7)$, the icosahedral group A_5, the alternating group A_6, the Hessian groups $Hess_{216}$, $Hess_{72}$ or $Hess_{36}$.

We recall that for a non-zero monomial $cX^iY^jZ^k$ we define its exponent as $max\{i,j,k\}$. For a homogeneous polynomial F, the core of F is defined as the sum of all terms of F with the greatest exponent. Let C_0 be a smooth plane curve, a pair $(C, \underline{G})$ with $\underline{G} \leq Aut(C)$ is said to be a descendant of C_0 if C is defined by a homogeneous polynomial whose core is a defining polynomial of C_0 and $\underline{G}$ acts on C_0 under a suitable coordinate system.

Theorem 3.1 (Harui)**.** *(see [10] §2) Let $\underline{G}$ be a subgroup of $Aut(C)$. Then $\underline{G}$ satisfies one of the following statements:*

(1) *$\underline{G}$ fixes a point on C and then it is cyclic.*
(2) *$\underline{G}$ fixes a point not lying on C and it satisfies a short exact sequence of the form*
$$1 \to N \to G \to G' \to 1,$$
with N a cyclic group of order dividing d and G' is isomorph to a cyclic group C_m of order m, a Dihedral group D_{2m}, A_4 , A_5 or S_4, where m is an integer $\leq d-1$. Moreover, if $G' \cong D_{2m}$, then $m|(d-2)$ or N is trivial.
(3) *$\underline{G}$ is conjugate (by certain $P \in PGL_3(K)$) to a subgroup of $Aut(F_d)$ where F_d is the Fermat curve $X^d + Y^d + Z^d$ and $(\underline{G}, C)$ is a descendant of F_d. In particular, $|G| \,|\, 6d^2$.*
(4) *$\underline{G}$ is conjugate to a subgroup of $Aut(K_d)$ where K_d is the Klein curve curve $X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X$ and $(\underline{G}, C)$ is a descendant of K_d. Therefore $|G| \,|\, 3(d^2 - 3d + 3)$.*
(5) *G is conjugate to a finite primitive subgroup $PGL_3(K)$ namely, the Klein group $PSL(2,7)$, the icosahedral group A_5, the alternating group A_6, or the Hessian groups $Hess_{216}$, $Hess_{72}$, $Hess_{36}$.*

The Hessian group: A representation of the Hessian group of order 216 inside $PGL_3(K)$ is given by $Hess_{216} =< S, T, U, V >$ with,

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}; \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad V = \frac{1}{\omega - \omega^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}; \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};$$

always ω means a primitive 3rd root of unity. Also, we consider the primitive subgroups of order 36 $Hess_{36}$, one of them is $< S, T, V >$ and the primitive subgroup of order 72, $Hess_{72} =< S, T, V, UVU^{-1} >$. Recall that there are exactly 3 primitive subgroups of order 36 for the above fixed representation, see [9]. It should be noted that, representations of $Hess_{216}$ inside $PGL_3(K)$ forms a unique set up to conjugation (see Mitchell [15] page 217). Grove in [9, §23, p.25], proved that the any representation of a Hessian group of order 36 or 72 is given by matrices fixing certain fix triangle and another matrices with a particular movement in the triangle permuting the vertices. One can extend these groups in a bigger one of 216 matrices, corresponding to a representation of $Hess_{216}$ in $PGL_3(K)$, by allowing more movements permuting the vertices of the fix triangle. Moreover two of the Hessian groups of order 36, in a fixed representation of $Hess_{216}$ in $PGL_3(K)$, are related by a change of variables involving certain permutation of the vertices of the triangle, therefore any of the exactly 3 primitive subgroups of order 36 of a fix representation of the $Hess_{216}$ in $PGL_3(K)$ are always conjugate. Therefore, given an injective representation of $Hess_{72}$ or $Hess_{36}$ inside $PGL_3(K)$, it extends to an injective representation of $Hess_{216}$ in $PGL_3(K)$, (and because the three $Hess_{36}$ are conjugate), their representations inside $PGL_3(K)$ are unique up to conjugation.

Remark 3.2. *In particular, for the Hessian groups $Hess_{216}$, $Hess_{72}$ and $Hess_{36}$, the locus $\widetilde{M_g^{Pl}(Hess_*)}$ is ES-Irreducible as long as is not empty (where $* \in \{36, 72, 216\}$) because the set A_{Hess_*} is trivial (with the notation of §2).*

With the interest to answer when $\widetilde{M_g^{Pl}(G)}$, is ES-irreducible or not, and the classical result of Klein on the uniqueness up to conjugation on finite subgroups inside $PGL_2(K)$, one could ask the following question in group theory,

Question 3.3. *Let $\underline{G}$ be a non-trivial and non-cyclic finite subgroup of $PGL_3(K)$. Is it true that exists $\underline{G}$ such that the set $A_{\underline{G}}$ has at least two elements?*

4. Cyclic groups in smooth plane curves of degree 5 and $\widetilde{M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})}$.

Note that, we study non-singular plane curves $C : F(X;Y;Z) = 0$ of degree $d \geq 4$ such that $Aut(C)$ is non-trivial, up to K-isomorphism (that is, two of them are K-isomorphic if one transforms to the other by a change of variables $P \in PGL_3(K)$) and we denote by C_P the plane curve $F(P(X;Y;Z)) = 0$.

By a change of variables, we can suppose that the cyclic group of exact order m acting on a smooth plane curve of degree 5 is given in $PGL_3(K)$ by a diagonal matrix $diag(1;\xi_m^a;\xi_m^b)$, where ξ_m is an m-th primitive root of unity, and $0 \leq a < b < m$ are positive integers. We call this element by Type $m,(a,b)$. Following the same proof of [8, §6.5] (or see [1], for a general treatment with an algorithm of computation for any degree d), we obtain a "normal form" associated to type $m,(a,b)$ corresponding to the loci $\rho(M_6^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ with $\rho(\mathbb{Z}/m) =< diag(1;\xi_m^a;\xi_m^b) >$:

Type: $m, (a,\, b)$	$F_{m,(a,b)}(X;Y;Z)$
$20,(4,5)$	$X^5+Y^5+XZ^4$
$16,(1,12)$	$X^5+Y^4Z+XZ^4$
$15,(1,11)$	$X^5+Y^4Z+YZ^4$
$13,(1,10)$	$X^4Y+Y^4Z+Z^4X$
$10,(2,5)$	$X^5+Y^5+XZ^4+\beta_{2,0}X^3Z^2$
$8,(1,4)$	$X^5+Y^4Z+XZ^4+\beta_{2,0}X^3Z^2$
$5,(1,2)$	$X^5+Y^5+Z^5+\beta_{3,1}X^2YZ^2+\beta_{4,3}XY^3Z$
$5,(0,1)$	$Z^5+L_{5,Z}$
$4,(1,3)$	$X^5+X\big(Z^4+Y^4+\beta_{4,2}Y^2Z^2\big)+\beta_{2,1}X^3YZ$
$4,(1,2)$	$X^5+X\big(Z^4+Y^4\big)+\beta_{2,0}X^3Z^2+\beta_{3,2}X^2Y^2Z+\beta_{5,2}Y^2Z^3$
$4,(0,1)$	$Z^4L_{1,Z}+L_{5,Z}$
$3,(1,2)$	$X^5+Y^4Z+YZ^4+\beta_{2,1}X^3YZ+X^2\big(\beta_{3,0}Z^3+\beta_{3,3}Y^3\big)+\beta_{4,2}XY^2Z^2$
$2,(0,1)$	$Z^4L_{1,Z}+Z^2L_{3,Z}+L_{5,Z}$

where $L_{i,U}$ means a homogeneous polynomial of degree i that does not contain the variable U with parameters in the coefficients in the monomials, and $\beta_{i,j}$ are parameters taking values in K. (It remains to introduce the algebraic restrictions that should be imposed on the parameters $\beta_{i,j}$ so that the defining equation $F_{m,(a,b)}(X;Y;Z) = 0$ is non-singular, which will be omitted).

By the above table, we find that the locus $M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})$ is not empty, only for the values m which are included in the previous list. Moreover, for $m \neq 4,5$, we have $\widetilde{M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})} = \rho(M_6^{Pl}(\mathbb{Z}/m\mathbb{Z}))$, where ρ is obtained such that $\rho(\mathbb{Z}/m\mathbb{Z}) =< diag(1,\xi_m^a,\xi_m^b) >$. Thus, the corresponding loci $\widetilde{M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})}$, where $m \neq 4,5$, are ES-Irreducible provided that they are non-empty.

Now, we consider the remaining cases of the loci $\widetilde{M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})}$ with $m = 4$ or 5:

Obviously, the plane model of type $5,(1,2)$ have always a bigger automorphism group by permuting X and Z. Therefore, there is at most one "normal form" that defines curves of degree 5 whose full automorphism group is isomorphic to $\mathbb{Z}/5\mathbb{Z}$, (observe that the number of the conjugacy classes of representations of $\mathbb{Z}/5\mathbb{Z}$ in $PGL_3(K)$ is three). In particular, $\widetilde{M_6^{Pl}(\mathbb{Z}/5\mathbb{Z})}$ is ES-Irreducible if it is non-empty. More precisely, $\widetilde{M_6^{Pl}(\mathbb{Z}/5\mathbb{Z})} = \rho(M_6^{Pl}(\mathbb{Z}/5\mathbb{Z}))$, where $\rho(\mathbb{Z}/5\mathbb{Z}) =< diag(1,1,\xi_5) >$ in this case.

On the other hand, for the cyclic groups of order 4, we have: Type $4,(1,3)$ is not irreducible, since it is of the form $X\cdot G(X;Y;Z)$. Hence, it is singular, and will be out of the scope of this note. Therefore, we have $M_6^{Pl}(\mathbb{Z}/4\mathbb{Z})=\rho_1(M_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))\cup\rho_2(M_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$, where ρ_1 corresponds to Type $4,(0,1)$ and ρ_2 to Type $4,(1,2)$.

4.1. **On type $4,(0,1)$.** Consider the non-singular plane curve which is defined by the equation

$$\tilde{C}:\ X^5+Y^5+Z^4X+\beta X^3Y^2,$$

where $\beta\neq 0$. This curve admits an automorphism of order 4 namely, $\sigma:=[X;Y;\xi_4Z]$ that fixes pointwise the line $Z=0$ (its axis) and the point $[0:0:1]$ off this line (its center). We call the elements of $PGL_3(K)$ that fix similar geometric constructions, homologies (for the element $diag(1;\xi_m^a;\xi_m^b)\in PGL_3(K)$ with $0\leq a<b<m$, is an homology when $a=0$). It follows, by Mitchell [15] §5, that $Aut(\tilde{C})$ should fix a point, a line or a triangle.

If $Aut(\tilde{C})$ fixes a triangle and neither a line nor a point is leaved invariant then, $\tilde{C}$ is a descendant of the Fermat curve F_5 or the Klein curve K_5 (Harui [10], §5). But this is impossible, because $4\nmid|Aut(F_5)|(=150)$, and $4\nmid|Aut(K_5)|(=39)$. Therefore, $Aut(\tilde{C})$ should fix a line and a point off that line.

Now, the point $(0:0:1)$ is an inner Galois point of $\tilde{C}$, by Lemma 3.7 in [10]. Also, it is unique, by Yoshihara [17], §2, Theorem 4. Therefore, this point must be fixed by $Aut(\tilde{C})$. Moreover, the axis $Z=0$ is also leaved invariant by Mitchell [15], §4. In particular, $Aut(\tilde{C})$ is cyclic by Lemma 11.44 in [12], and automorphisms of $\tilde{C}$ are all diagonal of the form $[X;vY;tZ]$. This in turns implies that $v^5=v^2=t^4=1$. Hence, $v=1$ and t is a 4-th root of unity. This shows that $Aut(\tilde{C})$ is cyclic of order 4.

Therefore, with the above argument we conclude the following result.

Proposition 4.1. *The locus set $\rho_1(\widetilde{M_6^{Pl}(\mathbb{Z}/4\mathbb{Z})})$ is non-empty.*

4.2. **On type $4,(1,2)$.** Consider the non-singular plane curve defined by the equation

$$\tilde{\tilde{C}}:\ X^5+X(Z^4+Y^4)+\beta Y^2Z^3,$$

where $\beta\neq 0$. This curve admits a cyclic subgroup of automorphisms generated by $\tau:=[X;\xi_4Y;\xi_4^2Z]$. For the same reason as above (i.e $4\nmid|Aut(K_5)|,|Aut(F_5)|$), $\tilde{\tilde{C}}$ is not a descendant of the Fermat curve F_5 or the Klein curve K_5. Moreover, $Aut(\tilde{\tilde{C}})$ is not conjugate to an icosahedral group A_5 (no elements of order 4), the Klein group $PSL(2,7)$, the Hessian group $Hess_{216}$ or the alternating group A_6 (since by [10], Theorem 2.3, $|Aut(\tilde{\tilde{C}})|\leq 150$).

Now, we claim to prove that $Aut(\tilde{\tilde{C}})$ is also not conjugate to any of the Hessian subgroups namely, $Hess_{36}$ or $Hess_{72}$, and therefore it should fix a line and a point off that line: Let C be a non-singular plane curve of degree 5 such that $Aut(C)$ is conjugate, through $P\in PGL_3(K)$, to $Hess_*$ with $*\in\{36,72,216\}$. Then $Aut(C_P)$ is given by the usual presentation inside $PGL_3(K)$ of the above Hessian groups. In particular, $Aut(C_P)$ always has the following five elements: $[Z;Y;X]$, $[X;Z;Y]$, $[Y;X;Z]$, $[Y;Z;X]$ and $[X;\omega Y;\omega^2Z]$, where ω is a primitive 3-rd root of unity. Because C_P is invariant by $[Z;Y;X]$, $[X;Z;Y]$, $[Y;X;Z]$ and $[Y;Z;X]$, then C_P must be of the form: $u(X^5+Y^5+Z^5)+a(X^4Z+X^4Y+Y^4X+Y^4Z+Z^4X+Z^4Y)+G(X;Y;Z)$, where $u,a\in K$, and $G(X;Y;Z)$ is a homogenous polynomial of degree at most three in each variable. Now, imposing that $[X;\omega Y;\omega^2Z]\in Aut(C_P)$, we obtain that $u=0$ and $a=0$, a contradiction to non-singularity. Therefore, there is no non-singular, degree 5 plane curve whose automorphism group is conjugate to one of the Hessian groups. This proves our claim.

It follows, by the previous discussion, that $Aut(\tilde{\tilde{C}})$ should fix a line and a point off that line. Moreover, $\tau\in Aut(\tilde{\tilde{C}})$ is of the form $diag(1;a;b)$ such that $1,a,b$ (resp. $1,a^3,b^3$) are pairwise distinct then, automorphisms of $\tilde{\tilde{C}}$ are of the forms $\tau_1:=[X;vY+wZ;sY+tZ]$, $\tau_2:=[vX+wZ;Y;sX+tZ]$ or $\tau_3:=[vX+wY;sX+tY;Z]$ (because the fixed point is one of the reference points $[1:0:0]$, $[0:1:0]$ or $[0:0:1]$, and the fixed line is one of the reference lines $X=0$, $Y=0$ or $Z=0$).

If $\tau_1\in Aut(\tilde{\tilde{C}})$ then $s=0=w$ (Coefficient of Y^5 and Z^5), and we have the same conclusion, if τ_2 (resp. τ_3) $\in Aut(\tilde{\tilde{C}})$ from the coefficients of X^3Y^2 and Y^4Z (resp. Z^3X and YZ^4). Hence, automorphisms of $\tilde{\tilde{C}}$ are all diagonal of the form $[X;vY;sZ]$. Moreover, $v^4=s^4=v^2s^3=1$, hence $v=\xi_4^r$, $s=\xi_4^{r'}$ with $(r,r')\in\{(0,0),\,(2,0),\,(1,2),\,(3,2)\}$. That is, $Aut(\tilde{\tilde{C}})$ is cyclic of order 4.

Consequently, the following results follow.

Proposition 4.2. *The locus set* $\rho_2(\widetilde{M_6^{Pl}(\mathbb{Z}/4\mathbb{Z})})$ *is non-empty.*

Corollary 4.3. *The locus set* $\widetilde{M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})}$ *is ES-Irreducible if and only if* $m \neq 4$*. If* $m = 4$ *then* $\widetilde{M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})}$ *has exactly two irreducible equation components, and hence the number of its irreducible components is at least two.*

Remark 4.4. *Observe that for any element of* $\rho_1(M_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$*, the Galois cover of degree 4 corresponding to* $\rho_1(M_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$*:*

$$C_1 := Z^4L_{1,Z} + L_{5,Z} = 0 \to C_1/ < [X;Y;\xi_4Z] >$$

is ramified exactly at six points with ramification index 4. Indeed, the fixed points of σ^i *for* $i = 1, 2, 3, 4$ *in* $\mathbb{P}^2(K)$ *are all the same set where* $\sigma = diag(1, 1, \xi_4)$*, therefore, we only need to consider the ramification points of* σ*, in particular, the ramification index is always 4. Now, by the Hurwitz formula we get* $10 = 4(2g_0 - 2) + 3k$ *where* g_0 *is the genus of* $C_1/ < [X, Y, \xi_4Z] >$ *hence we are forced to* $g_0 = 0$ *and* $k = 6$*. On the other hand, for any element of* $\rho_2(M_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$*, the Galois cover*

$$C_2 := X^5 + X(Z^4 + Y^4) + \beta_{2,0}X^3Z^2 + \beta_{3,2}X^2Y^2Z + \beta_{5,2}Y^2Z^3 = 0 \to C_2/ < [X;\xi_4Y;\xi_4^2Z]$$

is ramified at the points $(0 : 1 : 0), (0 : 0 : 1)$ *with ramification index 4 and at the 4 points namely,* $(1 : 0 : h)$ *where* $1 + h^4 + \beta_{2,0}h^2 = 0$ *with ramification index 2 provided that* $\beta_{2,0} \neq \pm 2$*. The situation with* $\beta_{2,0} = \pm 2$ *is that the equation is singular or non-geometrically irreducible, which is not of our concern in this work.*

Remark 4.5. *Given* G*, a non-trivial finite group, such that* $\widetilde{M_6^{Pl}(G)}$ *is non-empty. By a tedious work, one can show that* $\widetilde{M_6^{Pl}(G)}$ *is ES-Irreducible, except for the case* $G \cong \mathbb{Z}/4\mathbb{Z}$ *(for more details, we refer to [2]).*

Theorem 4.6. *Let* $d \geq 5$ *be an odd integer, and consider* $g = (d-1)(d-2)/2$ *as usual. Then* $M_g^{Pl}(\widetilde{\mathbb{Z}/(d-1)\mathbb{Z}})$ *is not ES-Irreducible, and it has at least two irreducible components.*

Proof. The above argument for concrete curves of Type $4, (0, 1)$ and Type $4, (1, 2)$ is valid for any odd degree $d \geq 5$ and the proof is quite similar. In other words, let $\tilde{C}$ and $\tilde{\tilde{C}}$ be the non-singular plane curves of types $d - 1, (0, 1)$ and $d - 1, (1, 2)$ defined by the equations $X^d + Y^d + Z^{d-1}X + \beta X^{d-2}Y^2 = 0$, and $X^d + X(Z^{d-1} + Y^{d-1}) + \beta Y^2Z^{d-2} = 0$, where $\beta \neq 0$. Then, $Aut(\tilde{C})$ and $Aut(\tilde{\tilde{C}})$ are non-conjugate cyclic groups of order $d - 1$, and are generated by $[X;Y;\xi_{d-1}Z]$ and $[X;\xi_{d-1}Y;\xi_{d-1}^2Z]$ respectively. Therefore, they belong to two different $[\rho]'s$.

On type $d - 1, (0, 1)$**:** With a homology of order $d - 1 \geq 4$ inside $Aut(\tilde{C})$, we conclude that $Aut(\tilde{C})$ fixes a point, a line or a triangle (See [15], §5). Furthermore, the center $(0 : 0 : 1)$ of this homology is an inner Galois point, by Lemma 3.7 in [10]. Also, it is unique, by Theorem 4 in [17]. Therefore, it should be fixed by $Aut(\tilde{C})$, and also the axis $Z = 0$ is leaved invariant, by Theorem 4 in [15]. Hence, $Aut(\tilde{C})$ is cyclic, by Lemma 11.44 in [12], and automorphisms of $\tilde{C}$ are of the form $diag(1; v; t)$ such that $v^d = t^{d-1} = v = 1$. That is, $|Aut(\tilde{C})| = d - 1$.

On type $d - 1, (1, 2)$**:** First, we prove that $Aut(\tilde{\tilde{C}})$ fixes a line and a point off this line. We consider the case $d \geq 7$ (For $d = 5$, we refer to the previous results). The alternating group A_6 has no elements of order $d - 1 \geq 6$. The Klein group $PSL(2, 7)$, which is the only simple group of order 168, has no elements of order ≥ 8, and also there are no elements of order 6 inside (for more details, we refer to [16]). Therefore, the primitive groups A_5, A_6, and $PSL(2, 7)$ do not appear as the full automorphism group. Moreover, elements inside the Hessian group $Hess_{216} \cong SmallGroup(216, 153)$ have orders $1, 2, 3, 4$ and 6. Then $Hess_*$ with $* \in \{36, 72, 216\}$ do not appear as the full automorphism group, except possibly for $d = 7$. On the other hand, $d-1 \nmid 3(d^2-3d+3)$ hence $\tilde{\tilde{C}}$ is not a descendant of the Klein curve K_d. Furthermore, $\tilde{\tilde{C}}$ is not a descendant of the Fermat curve F_d, because $d - 1 \nmid 6d^2$ (except for $d = 7$).

Finally, it remains to deal with the case $d = 7$ for the Hessian groups or for being a Fermat's descendant. By the same line of argument as for the claim of Type $4,(1,2)$, we can show that non of the Hessian groups could appear for a non-singular, degree 7, plane curve. Also, the automorphisms of the Fermat curve F_7 are of the forms $[X;\xi_7^aY;\xi_7^bZ]$, $[\xi_7^bZ;\xi_7^aY;X]$, $[X;\xi_7^bZ;\xi_7^aY]$, $[\xi_7^aY;X;\xi_7^bZ]$, $[\xi_7^aY;\xi_7^bZ;X]$, $[\xi_7^bZ;X;\xi_7^aY]$. One can easily verify that non of them has order 6. Consequently, we exclude the possibility of being a Fermat's descendant.

Now, the full automorphism group should fix a line and a point off this line. Thus automorphisms of $\tilde{\tilde{C}}$ have the forms $[X;vY+wZ;sY+tZ]$, $[vX+wZ;Y;sX+tZ]$ or $[vX+wY;sX+tY;Z]$, since $[X;\xi_{d-1}Y;\xi_{d-1}^2Z]\in Aut(\tilde{\tilde{C}})$.

If $[X;vY+wZ;sY+tZ]\in Aut(\tilde{\tilde{C}})$ then $s=0=w$ (Coefficient of Y^d and Z^d), and the same conclusion follows if $[vX+wZ;Y;sX+tZ]$ (resp. $[vX+wY;sX+tY;Z]$) $\in Aut(\tilde{\tilde{C}})$ from the coefficients of $X^{d-2}Y^2$ and $Y^{d-1}Z$ (resp. $Z^{d-2}X^2$ and YZ^{d-1}). Hence, automorphisms of $\tilde{\tilde{C}}$ are all diagonal of the form $diag(1;v;s)$. Moreover, $v^{d-1}=s^{d-1}=v^2s^{d-2}=1$ that is, $v=\xi_{d-1}^r$ and $s=\xi_{d-1}^{r'}$ such that $d-1|2r-r'$. Therefore, automorphisms of $\tilde{\tilde{C}}$ are $[X;\xi_{d-1}^rY;\xi_{d-1}^{2r}Z]$ with $r\in 0,1,...,d-2$. Hence, $Aut(\tilde{\tilde{C}})$ is cyclic of order $d-1$, which was to be shown.

□

5. On the locus $\widetilde{M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})}$.

By a similar argument as the degree 5 case, we obtain the following "normal forms" for $\rho(M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z}))$, (see the full table on degree 6 in [1]):

Type: $m,(a,b)$	$F_{m,(a,b)}(X;Y;Z)$
$3,(0,1)$	$Z^6+Z^3L_{3,Z}+L_{6,Z}$
$3,(1,2)$	$X^5Y+Y^5Z+Z^5X+\mu_1Z^2X^4+\mu_2X^2Y^4+\mu_3Y^2Z^4+\alpha_1X^3Y^2Z+\alpha_2XY^3Z^2+\alpha_3X^2YZ^3$

where μ_i,α_i denote parameters taking values in K in order to give non-singular models for the respective loci $\rho(M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z}))$.

5.1. On type $3,(1,2)$.

Proposition 5.1. *Let $\delta\in M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$ such that δ admits a non-singular plane model $\tilde{C}$ of the form*

$$X^5Y+Y^5Z+Z^5X+\mu_1Z^2X^4+\mu_2X^2Y^4+\mu_3Y^2Z^4+\alpha_1X^3Y^2Z+\alpha_2XY^3Z^2+\alpha_3X^2YZ^3=0.$$

Then, $Aut(\tilde{C})$ either fixes a line and a point off that line or it fixes a triangle.

Proof. It suffices to show that $Aut(\tilde{C})$ is not conjugate to any of the finite primitive groups inside $PGL_3(K)$ namely, the Klein group $PSL(2,7)$, the icosahedral group A_5, the alternating group A_6, the Hessian group $Hess_{216}$ or to any of its subgroups $Hess_{72}$ or $Hess_{36}$, and the result follows by Mitchell in [15].

Let $\tau\in Aut(\tilde{C})$ be an element of order 2 such that $\tau\sigma\tau=\sigma^{-1}$, where $\sigma:=[X;\omega Y;\omega^2Z]$ then τ has one of the forms $[X;\beta Z;\beta^{-1}Y]$, $[\beta Y;\beta^{-1}X;Z]$ or $[\beta Z;Y;\beta^{-1}X]$. But non of these transformations retains $\tilde{C}$, hence $Aut(\tilde{C})$ does not contain an S_3 as a subgroup. Consequently, $Aut(\tilde{C})$ is not conjugate to A_5 or A_6. Moreover, it is well known that $PSL(2,7)$ contains an octahedral group of order 24 (but not an isocahedral group of order 60), and since all elements of order 3 in $PSL(2,7)$ are conjugate (for more details, we refer to [16]). Then, by the same argument as before, we conclude that $Aut(\tilde{C})$ is not conjugate to $PSL(2,7)$. Lastly, assume that $Aut(\tilde{C})$ is conjugate, through a transformation P, to one of the Hessian groups say, $Hess_*$. Then, we can consider $P^{-1}SP=\lambda S$, because we did not fix the plane model for a curve whose automorphism group is $Hess_*$. In particular, P should be of the form $[Y;\gamma Z;\beta X]$, $[Z;\gamma X;\beta Y]$ or $[X;\gamma Y;\beta Z]$, but non of them transform $\tilde{C}$ to $\tilde{C}_P$ with $\{[X;Z;Y],[Y;X;Z],[Z;Y;X]\}\subseteq Aut(\tilde{C}_P)$. Therefore, $Aut(\tilde{C})$ is not conjugate to any of the Hessian groups, and we have done. □

Now, we state and prove the main result for this section:

Theorem 5.2. *Consider an element* $\delta \in M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$ *that has a non-singular plane model* $\tilde{C}$ *of the form* $\tilde{C}: X^5Y + Y^5Z + Z^5X + \alpha_3 X^2YZ^3$ *with* $\alpha_3 \neq 0$*. The full automorphism group of such* δ *is cyclic of order* 3*, and is generated by the transformation* $\sigma: (x; y; z) \mapsto (x; \omega y; \omega^2 z)$*.*

Proof. It follows, by Proposition 5.1, that $Aut(\tilde{C})$ either fixes a line and a point off that line or it fixes a triangle. We treat each of these two cases.

(1) If $Aut(\tilde{C})$ fixes a line L and a point P off this line, then L must be one of the reference lines $B = 0$, where $B \in \{X, Y, Z\}$, and P is one of the reference points namely, $[1:0:0]$, $[0:1:0]$, or $[0:0:1]$ (being $\sigma \in Aut(\tilde{C})$). Consequently, $Aut(\tilde{C})$ is cyclic, since all the reference points lie on $\tilde{C}$. Also, automorphisms of $\tilde{C}$ are of the forms

$$\tau_1 := [X; vY + wZ; sY + tZ],\ \tau_2 := [vX + wZ; Y; sX + tZ] \ \ or\ \ \tau_3 := [vX + wY; sX + tY; Z]$$

For τ_1 to be in $Aut(\tilde{C})$, we must have $w = 0 = s$ (coefficients of X^5Z and XY^5), and similarly, for τ_2 (resp. τ_3) through the coefficients of Y^5X and Z^6 (resp. YZ^5 and X^5Z). That is, elements of $Aut(\tilde{C})$ are all diagonal of the form $diag(1; v; t)$ such that $tv^4 = 1 = t^3$ and $t^5 = v$. Thus, $t = \xi_3^a$ and $v = \xi_3^{2a}$, where ξ_3 is a primitive 3-rd root of unity, and hence, $|Aut(C)| = 3$.

(2) If $Aut(\tilde{C})$ fixes a triangle and there exist neither a line nor a point leaved invariant, then by Harui [10], $\tilde{C}$ is a descendant of the Fermat curve $F_6: X^6+Y^6+Z^6$ or the Klein curve $K_6: X^5Y+Y^5Z+Z^5X$. Hence, $Aut(\tilde{C})$ is conjugate to a subgroup of $Aut(F_6) =< [\xi_6 X; Y; Z], [X; \xi_6 Y; Z], [Y; Z; X], [X; Z; Y] >$ or to a subgroup of $Aut(K_6) =< [Z; X; Y],\ [X; \xi_{21} Y; \xi_{21}^{-4} Z] >$.

- Suppose first that $Aut(\tilde{C})$ is conjugate (through P) to a subgroup of $Aut(F_6)$. Then, it suffices to assume that $P^{-1}SP \in \{S, [Y; Z; X], [Y; \xi_6 Z; X], [Y; \xi_6^2 Z; X]\}$, since any element of order 3 in $Aut(F_6)$, which is not a homology, is conjugate to one of those inside $Aut(F_6)$. Now, if $P^{-1}SP = S$ then $P \in PGL_3(K)$ is of the form $[Y; \gamma Z; \beta X]$, $[Z; \gamma X; \beta Y]$ or $[X; \gamma Y; \beta Z]$, but non of them transforms $\tilde{C}$ to $\tilde{C}_P$ with core $X^6 + Y^6 + Z^6$, a contradiction. Furthermore, if $P^{-1}SP = [Y; Z; X]$ (resp. $= [Y; \xi_6 Z; X]$ or $= [Y; \xi_6^2 Z; X]$), then P has the form $\begin{pmatrix} \lambda & 1 & \lambda^2 \\ \omega\lambda\beta_2 & \beta_2 & \omega^2\lambda^2\beta_2 \\ \omega^2\lambda\beta_3 & \beta_3 & \lambda^2\omega\beta_3 \end{pmatrix}$, where $\lambda^3 = 1$ (resp. $\lambda^3 = \xi_6$ or $\lambda^3 = \xi_6^2$). We thus get $\tilde{C}_P$ of the form $v_1X^6 + v_2Y^6 + v_3Z^6 +$ *lower terms* such that the system $v_1 = v_3 = v_3 = 1$ has no solutions in K^{*2}, a contradiction. Consequently, $\tilde{C}$ is not a descendant of the Fermat curve F_6.
- Secondly, suppose that $\tilde{C}$ is a descendant of the Klein curve K_6. This should happen through a change of the variables $P \in PGL_3(K)$ such that $\tilde{C}_P: X^5Y + Y^5Z + Z^5X + lower\ terms$. We claim to show that $P^{-1}SP = \lambda S$ for some $\lambda \in K^*$. Indeed, elements of order 3 inside $Aut(K_6)$, which are not homologies, are $S, S^{-1}, [\xi_{21}^a Y; \xi_{21}^{-4a} Z; X]$ and $[\xi_{21}^{-4a} Z; X; \xi_{21}^a Y]$, and it is enough to consider the situation $P^{-1}SP \in \{S, S^{-1}, [\xi_{21}^a Y; \xi_{21}^{-4a} Z; X], [\xi_{21}^{-4a} Z; X; \xi_{21}^a Y]\}$ with $a = 0, 1, 2$, because any other value is conjugate inside $Aut(K_6)$ to one of these transformations.
  If $P^{-1}SP = \lambda S^{-1}$ then P fixes one of the variables and permutes the others. Hence, the resulting core is different from $X^5Y + Y^5Z + Z^5X$, a contradiction.
  If $P^{-1}SP = \lambda[\xi_{21}^a Y; \xi_{21}^{-4a} Z; X]$ (resp. $[\xi_{21}^{-4a} Z; X; \xi_{21}^a Y]$) then P has the form

$$\begin{pmatrix} \lambda\xi_{21}^{-a} & 1 & \lambda^2\xi_{21}^{-a} \\ \lambda\xi_{21}^{-a}\omega\beta_2 & \beta_2 & \lambda^2\xi_{21}^{-a}\omega^2\beta_2 \\ \lambda\xi_{21}^{-a}\omega^2\beta_3 & \beta_3 & \lambda^2\xi_{21}^{-a}\omega\beta_3 \end{pmatrix} \ (resp.\ \begin{pmatrix} \lambda^2\xi_{21}^{-18a} & 1 & \lambda\xi_{21}^{-a} \\ \lambda^2\xi_{21}^{-18a}\omega^2\beta_2 & \beta_2 & \lambda\xi_{21}^{-a}\omega\beta_2 \\ \lambda^2\xi_{21}^{-18a}\omega\beta_3 & \beta_3 & \lambda\xi_{21}^{-a}\omega^2\beta_3 \end{pmatrix})$$

  where $\lambda^3 = \xi_{21}^{-3a}$. For both transformations, we must have $\beta_3\beta_2^5 + \left(\delta_3\beta_3^3 + 1\right)\beta_2 + \beta_3^5 = 0$ so that X^6, Y^6, Z^6 do not appear. Therefore, by imposing the condition X^5Z, XY^5 and YZ^5 do not appear as well, we get $\delta_3 = 0$, which is already excluded. Consequently, $P^{-1}SP = \lambda S$, and we proved the claim. Now, P has one of the forms $[Y; \gamma Z; \beta X]$, $[Z; \gamma X; \beta Y]$ or $[X; \gamma Y; \beta Z]$. Therefore, $\tilde{C}_P$ is defined by an equation of the form $\lambda_0(X^5Y + Y^5Z + Z^5X) + \lambda_1 G(X; Y; Z)$, where $G(X; Y; Z) \in \{X^2YZ^3, Y^2ZX^3, Z^2XY^3\}$. In particular, $[\mu_1 Z; X; \mu_2 Y] \notin Aut(\tilde{C}_P)$, and

$Aut(\tilde{C}_P) \preceq < \tau := [X; \xi_{21}Y; \xi_{21}^{-4}Z] >$. Moreover, $\tau^r \in Aut(\tilde{C}_P)$ if and only if $7|r$. Hence, $Aut(\tilde{C})$ is cyclic of order 3.

This completes the proof. □

5.2. **On type** $3, (0, 1)$.

Proposition 5.3. *If $\delta \in M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$ has a non-singular plane model $\tilde{\tilde{C}}$ of the form $Z^6 + Z^3 L_{3,Z} + L_{6,Z}$, then $Aut(\tilde{\tilde{C}})$ is either conjugate to the Hessian group $Hess_{216}$ or it leaves invariant a point, a line or a triangle.*

Proof. The result is an immediate consequence, since $Aut(\tilde{\tilde{C}})$ contains a homology (i.e. leaves invariant a line pointwise and a point off this line) of period 3 namely, $\sigma' := [X; Y; \omega Z]$, and $Hess_{216}$ is the only multiplicative group that contains such homologies and does not leave invariant a point, a line or a triangle (See Theorem 9, [15]). □

Now, we can prove our main result for this section.

Theorem 5.4. *The automorphisms group of an element $\delta \in M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$ with a non-singular plane model $\tilde{\tilde{C}}$ of the form $Z^6 + X^5Y + XY^5 + \alpha_3 Z^3 X^3 = 0$ such that $\alpha_3 \neq 0$ is cyclic of order 3, and is generated by the automorphism $\sigma' : (x; y; z) \mapsto (x; y; \omega z)$.*

Proof. Suppose that $Aut(\tilde{\tilde{C}})$ is conjugate, through a transformation P, to the Hessian group $Hess_{216}$. Then, we can assume, without loss of generality, that $P^{-1}\sigma' P = \lambda \sigma'$ for some $\lambda \in K^*$. Hence, $P = [\alpha_1 X + \alpha_2 Y; \beta_1 X + \beta_2 Y; Z]$ and clearly, $\{[Z; Y; X], [X; Z; Y]\} \nsubseteq Aut(\tilde{\tilde{C}}_P)$, a contradiction. Therefore, by Proposition 5.3, we deduce that $Aut(\tilde{\tilde{C}})$ should fix a point, a line or a triangle.

In what follows, we treat each case.

(1) If $Aut(\tilde{\tilde{C}})$ fixes a line and a point off that line, and if $\tilde{\tilde{C}}$ admits a bigger non cyclic automorphism group, then $Aut(\tilde{\tilde{C}})$ satisfies a short exact sequence of the form $1 \to C_3 \to Aut(\tilde{\tilde{C}}) \to G' \to 1$, where G' is conjugate to C_m ($m = 2, 3$ or 4), D_{2m} ($m = 2$ or 4), A_4, S_4 or A_5.

If G' is conjugate to C_3, A_4, S_4 or A_5, then there exists, by Sylow's theorem, a subgroup H of automorphisms of $\tilde{\tilde{C}}$ of order 9. In particular, H is conjugate to C_9 or $C_3 \times C_3$, but both cases do not occur. Indeed, if H is conjugate to C_9 then $Aut(\tilde{\tilde{C}})$ has an element of order 9, which is not possible because $9 \nmid d-1, d, (d-1)^2, d(d-2), d(d-1), d^2 - 3d + 3$ with $d = 6$ (for more details, we refer to [1]). Moreover, if H is conjugate to $C_3 \times C_3$ then there exists $\tau \in Aut(\tilde{\tilde{C}})$ of order 3 such that $\tau\sigma' = \sigma'\tau$. Hence, $\tau = [vX + wY; sX + tY; Z]$, and comparing the coefficients of Z^3Y^3 and X^6 in $\tilde{\tilde{C}}_\tau$, we get $w = 0 = s$ and $v^5 t = vt^5 = v^3 = 1$. Thus $\tau \in < \sigma' >$, a contradiction.

By a similar argument, we exclude the cases C_4 and D_{2m}, because for each $SmallGroup(6m, ID)$, there must be an element τ of order 2 or 4 which commutes with σ'.

Finally, if G' is conjugate to C_2 then there exists an element τ of order 2 such that $\tau\sigma'\tau = \sigma'^{-1}$ and one can easily verify that such an element does not exists.

We conclude that $Aut(\tilde{\tilde{C}})$ should be cyclic (in particular, is commutative). Hence, it can not be of order > 3 (otherwise; there must be an element $\tau \in Aut(\tilde{\tilde{C}})$ of order > 3 which commutes with σ', and by a previous argument such elements do not exist).

(2) If $Aut(\tilde{\tilde{C}})$ fixed a triangle and neither a point nor a line is fixed, then it follows, by Harui [10], that $\tilde{\tilde{C}}$ is a descendant of the Fermat curve F_6 or the Klein curve K_6. The last case does not happen, because $Aut(K_6)$ does not have elements of order 3 whose Jordan form is the the same as σ' (i.e a homology). Now, suppose that $\tilde{\tilde{C}}$ is a descendant of F_6 that is, $\tilde{\tilde{C}}$ can be transformed (through P) into a curve $\tilde{\tilde{C}}_P$ whose core is $X^6 + Y^6 + Z^6$. Then, $P = [\alpha_1 X + \alpha_2 Y;\ \beta_1 X + \beta_2 Y;\ Z]$, since there are only two sets of homologies in $Aut(F_6)$ of order 3 namely, $\{[\omega X; Y; Z],\ [X; \omega Y; Z],\ [X; Y; \omega Z]\}$ and $\{[\omega^2 X; Y; Z],\ [X; \omega^2 Y; Z],\ [X; Y; \omega^2 Z]\}$ (recall that the two sets are not conjugate in $PGL_3(K)$. Also, elements of the first set are all conjugate inside $Aut(F_6)$ to $[X; Y; \omega Z]$. So it suffices to consider the

situation $P^{-1}\sigma P=\lambda\sigma$). Now, $\tilde{\tilde{C}}_P$ has the form

$$\mu_0X^6+\mu_1Y^6+Z^6+\alpha_3(\alpha_1X+\alpha_2Y)^3Z^3+\mu_2X^5Y+\mu_3X^4Y^2+\mu_4X^3Y^3+\mu_5X^2Y^4+\mu_6XY^5,$$

where $\mu_0:=\alpha_1\beta_1\left(\alpha_1^4+\beta_1^4\right)(=1)$ and $\mu_1:=\alpha_2\beta_2\left(\alpha_2^4+\beta_2^4\right)(=1)$. In particular, $(\alpha_1\beta_1)(\alpha_2\beta_2)\neq 0$ therefore, $[X;vZ;wY]$, $[vZ;wY;X]$, $[wY;vZ;X]$, and $[vZ;X;wY]\notin Aut(\tilde{\tilde{C}}_P)$, because of the monomial XY^2Z^3. Moreover, $[wY;X;vZ]\in Aut(\tilde{\tilde{C}}_P)$ only if $\alpha_1=\alpha_2$ and $w=v^3=1$. Hence

$$\tilde{\tilde{C}}_P:\ Z^6+\alpha_3\alpha_1^3(X+Y)^3Z^3+\alpha_1(X+Y)\left(\beta_1X+\beta_2Y\right)\left(\alpha_1^4(X+Y)^4+(\beta_1X+\beta_2Y)^4\right).$$

Consequently, $\beta_1=\beta_2$ (because we are assuming $[Y;X;vZ]\in Aut(\tilde{\tilde{C}}_P)$), a contradiction to invertibility of P.

Finally, if $[X,\xi_6^rY,\xi_6^{r'}Z]\in Aut(\tilde{\tilde{C}}_P)$ then $r=0$ and $2|r'$, since $\alpha_1\alpha_2\neq 0$. That is, $|Aut(\tilde{\tilde{C}}_P)|=3$, which was to be shown.

□

As a conclusion of the results that are introduced in this section, we get the following result.

Corollary 5.5. *The locus $\widetilde{M_{10}^{Pl}}(\mathbb{Z}/3\mathbb{Z})$ is not ES-Irreducible, and it has at least two irreducible components.*

6. Positive characteristic

Now, suppose that $\mathbb{K}$ is an algebraically closed field of positive characteristic $p>0$. Consider a non-singular plane curve C in $\mathbb{P}^2(\mathbb{K})$ of degree d and assume that the order of $Aut(C)$ is coprime with p, $p\nmid d(d-1)$, $p\geq 7$ and the order of $Aut(F_d)$ and $Aut(K_d)$ are coprime with p where $F_d: X^d+Y^d+Z^d=0$ is the Fermat curve and $K_d: X^{d-1}Y+Y^{d-1}Z+Z^{d-1}X=0$ is the Klein curve. Then, all the techniques that appeared in Harui [10], can be applied: Hurwitz bound, Arakawa and Oiakawa inequalities and so on. In particular, the arguments of all the previous sections hold.

Consider the p-torsion of the degree 0 Picard group of C, which is a finitely generated $\mathbb{Z}/(p)$-module of dimension γ(always $\gamma\leq g$ where g is the genus of C), we call γ the p-rank of C.

For a point $\mathcal{P}$ of C denote by $Aut(C)_{\mathcal{P}}$ the subgroup of $Aut(C)$ that fixes the place $\mathcal{P}$.

Lemma 6.1. *Assume that $Aut(C)_{\mathcal{P}}$ is prime to p for any point $\mathcal{P}$ of C and the p-rank of C is trivial. Then $Aut(C)$ is prime to p.*

Proof. Consider $\sigma\in Aut(C)$ of order p, then the extension $\mathbb{K}(C)/\mathbb{K}(C)^{\sigma}$ is a finite extension of degree p and is unramified everywhere (because if it ramifies at a place P then σ will be an element of $Aut(C)_{\mathcal{P}}$ giving a contradiction). But, if $\gamma=0$ (i.e. the p-rank is trivial for C) then, from Deuring-Shafarevich formula [12, Theorem11.62], we obtain that $\frac{\gamma-1}{\gamma'-1}=p$ where γ' is the p-rank for $C/<\sigma>$ which is impossible. Therefore, such extensions do not exist. □

Lemma 6.2. *Consider C a plane non-singular curve of degree $d\geq 4$. If $p>(d-1)(d-2)+1$, then $Aut(C)_{\mathcal{P}}$ is coprime with p for any point $\mathcal{P}$ of the curve C.*

Proof. By [12, Theorem 11.78] the maximal order of the p-subgroup of $Aut(C)_{\mathcal{P}}$ is at most $\frac{4p}{(p-1)^2}g^2$. Hence, with $g=\frac{(d-1)(d-2)}{2}$ and assuming that $p>\frac{4p}{(p-1)^2}g^2$, we obtain the result. □

Lemma 6.3. *Let C be a non-singular curve of genus $g\geq 2$ defined over an algebraic closed field $\mathbb{K}$ of characteristic $p>0$. Suppose that C has an unramified subcover of degree p, i.e. $\Phi: C\to C'$ of degree p. Then C' has genus ≥ 2, $g\equiv 1(mod\ p)$ and $\gamma\equiv 1(mod\ p)$. In particular, for the existence of such subcover, one needs to assume that $p<g$.*

Proof. The Hurwitz formula for Φ gives the equality $(2g-2)=p(2g'-2)$ where g' is the genus of C'. We have $g'\neq 0$ or 1 because $g\geq 2$, therefore $g'\geq 2$ and $g-1\equiv 0(mod\ p)$. Now, consider Deuring-Shafaravich formula, which in such unramified extension could be read as $\gamma-1=p(\gamma'-1)$ where γ' the p-rank of C'. If $\gamma=1$ then there is nothing to prove and if $\gamma>1$ then the congruence is clear. Finally, if $\gamma=0$ then this situation is not possible. □

Corollary 6.4. *Let C be a non-singular plane curve of degree d and genus $g \geq 2$ defined over an algebraic closed field $\mathbb{K}$ of characteristic $p > 0$. Suppose that $p > (d-1)(d-2)+1 > g$. Then the order of $Aut(C)$ is coprime with p.*

Proof. Suppose $\sigma \in Aut(C)$ of order p, then $\mathbb{K}(C)/\mathbb{K}(C)^{\sigma}$ is a separable degree p extension, and by Lemma 6.2, it is unramified everywhere. By Lemma 6.3 we find that such extensions do not exist. □

And as a direct consequence of the above lemmas and because all techniques in the previous sections, from [10], are applicable when $Aut(C)$ is coprime with p, then we obtain:

Corollary 6.5. *Assume $p > 13$. The automorphism groups of the curves $\tilde{C} : X^5 + Y^5 + Z^4X + \beta X^3Y^2$ and $\tilde{\tilde{C}} : \ X^5 + X(Z^4 + Y^4) + \beta Y^2Z^3$ such that $\beta \neq 0$, are cyclic of order 4. Moreover, $\tilde{C}$ is not isomorphic to $\tilde{\tilde{C}}$ for any choice of the parameters.*

Proof. Only we need to mention that the linear g_2-systems for the immersion of the curve inside $\mathbb{P}^2$ are unique up to conjugation in $PGL_3(\mathbb{K})$ see [12, Lemma 11.28] (also the curves $\tilde{C}$ and $\tilde{\tilde{C}}$ have cyclic covers of degree 4 with different type of the cover, from Hurwitz equation, therefore they belong to different irreducible components in the moduli space of genus 6 curves). □

Corollary 6.6. *For $p > 13$ we have that the locus $\widetilde{M_6^{Pl}(\mathbb{Z}/4\mathbb{Z})}$ of the moduli space of positive characteristic, has at least two irreducible components.*

Similarly we obtain the following result from results in §4,

Corollary 6.7. *For $p > (d-1)(d-2)+1$ where $d \geq 5$ is an odd integer, the locus $\widetilde{M_g^{Pl}(\mathbb{Z}/(d-1)\mathbb{Z})}$ of the moduli space over positive characteristic p is not ES-Irreducible and it has at least two strongly equation components. In particular, it has at least two irreducible components.*

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