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LENGTH PRODUCT OF HOMOLOGICALLY INDEPENDENT LOOPS FOR TORI

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ABSTRACT. We prove that any Riemannian torus of dimension m with unit volume admits m homologically independent closed geodesics whose length product is bounded from above by m^m.

The goal of this note is to prove the following analog of Minkowski's second theorem.

Theorem 1. *Let $(\mathbb{T}^m, g)$ be a Riemannian torus of dimension $m \geq 2$. There exist m homologically independent closed geodesics $(\gamma_1, \ldots, \gamma_m)$ whose length product satisfies*

$$\prod_{i=1}^{m} \ell_g(\gamma_i) \leq m^m \cdot \mathrm{vol}(\mathbb{T}^m, g).$$

Given a closed manifold M of dimension m and a class $\zeta \neq 0 \in H^1(M; \mathbb{Z}_2)$ set

$$L(\zeta) := \inf\{\ell_g(\gamma) \mid \gamma \text{ is a closed curve with } \langle \zeta, [\gamma] \rangle \neq 0\}$$

where $[\gamma]$ denotes the homology class in $H_1(M; \mathbb{Z}_2)$ corresponding to the curve γ and $\langle \cdot, \cdot \rangle$ the pairing between $\mathbb{Z}_2$-cohomology and homology. The central result of this note is the following statement from which Theorem 1 can be deduced.

Theorem 2. *Let (M, g) be a closed Riemannian manifold of dimension $m \geq 1$ and suppose that there exist (not necessarily distinct) cohomology classes $\zeta_1, \ldots, \zeta_m$ in $H^1(M; \mathbb{Z}_2)$ whose cup product $\zeta_1 \cup \ldots \cup \zeta_m \neq 0$ in $H^m(M; \mathbb{Z}_2)$. Then*

$$\prod_{i=1}^{m} L(\zeta_i) \leq m^m \cdot \mathrm{vol}\,(M, g).$$

In order to show this result we follow the approach by L. Guth [Gut10] involving nearly minimal hypersurfaces in his alternative proof of Gromov isosystolic inequality [Gro83] in the special case of manifolds whose $\mathbb{Z}_2$-cohomology has maximal cup-length.

Theorem 1 can be deduced from this result as follows. Choose a sequence of $\mathbb{Z}_2$-homologically independent closed geodesics $(\gamma_1, \ldots, \gamma_m)$ in $(\mathbb{T}^m, g)$ corresponding to the m first successive minima, that is

$$\ell_g(\gamma_k) = \min\{\lambda \mid \text{there exist } k \ \mathbb{Z}_2\text{-homologically independent closed curves of length at most } \lambda\}.$$

The dual basis $(\zeta_1, \ldots, \zeta_m)$ to the basis $([\gamma_1], \ldots, [\gamma_m])$ satisfies the condition $\zeta_1 \cup \ldots \cup \zeta_m \neq 0$ and that $L(\zeta_k) = \ell_g(\gamma_k)$ which prove Theorem 1.

The rest of this note is devoted to the proof of Theorem 2. Instead of considering balls like in Guth's argument we inductively construct a set which is longer in the appropriate direction and whose structure is described as follows. By a strictly increasing sequence of closed submanifolds of M we mean a sequence $Z_0 \subset Z_1 \subset \ldots \subset Z_{m-1} \subset Z_m = M$ of closed manifolds Z_i of dimension i for $i = 0, \ldots, m$. In particular Z_0 is a finite collection of points of M and Z_{m-1} an hypersurface. Given such a sequence and m positive numbers $R_1, \ldots, R_m$ we define another sequence of subsets $D_1 \subset \ldots \subset D_m$ by induction as follows:

$$D_1 := \{z \in Z_1 \mid d_g(z, Z_0) \leq R_1\} \subset Z_1,$$

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and for $k = 2, \ldots, m$

$$D_k := \{z \in Z_k \mid d_g(z, D_{k-1}) \leq R_k\} \subset Z_k.$$

Fix $\delta > 0$. We will prove by induction that there exists a strictly increasing sequence $Z_0 = \{z_0\} \subset Z_1 \subset \ldots \subset Z_{m-1} \subset Z_m = M$ of closed submanifolds of M such that

- the homology class $[Z_{i-1}] \in H_{i-1}(Z_i; \mathbb{Z}_2)$ is the Poincaré dual of the restriction $\zeta'_i := \zeta_{i|Z_i}$;
- for any sequence $\{R_i\}_{i=1}^m$ of positive numbers if $2\sum_{k=1}^i R_k < L(\zeta'_i)$ for $i = 1, \ldots, m$ then

$$\operatorname{vol} D_m \geq 2^m \prod_{i=1}^m R_i - O(\delta).$$

As $L(\zeta_i) \leq L(\zeta'_i)$ (with equality at least for $i = m$), ordering $\zeta_1, \ldots, \zeta_m$ such that $L(\zeta_1) \leq \ldots \leq L(\zeta_m)$, taking $R_i \to \frac{L(\zeta_i)}{2m}^-$ and then letting $\delta \to 0$ in the above inequality implies Theorem 2.

The case $m = 1$ is trivial, so suppose that $m > 1$ and that the statement is proved for dimensions at most $m - 1$. Let $\mathbf{Z_{m-1}} \in H_{m-1}(M; \mathbb{Z}_2)$ be the Poincaré dual to ζ_m. We fix a smooth embedded and closed hypersurface Z_{m-1} which is *δ-minimizing* in the homology class $\mathbf{Z_{m-1}}$: any other smooth hypersurface Z' representing $\mathbf{Z_{m-1}}$ satisfies $\operatorname{vol}_{m-1} Z' \geq \operatorname{vol}_{m-1} Z_{m-1} - \delta$.

The restriction to Z_{m-1} of the cohomologic classes $\zeta_1, \ldots, \zeta_{m-1}$ gives a family of cohomologic classes $\zeta''_1, \ldots, \zeta''_{m-1}$ in $H^1(Z_{m-1}; \mathbb{Z}_2)$ such that $\zeta''_1 \cup \ldots \cup \zeta''_{m-1} \neq 0$ in $H^{m-1}(Z_{m-1}; \mathbb{Z}_2)$. By the induction hypothesis there exists a strictly increasing sequence $Z_0 = \{z_0\} \subset Z_1 \subset \ldots \subset Z_{m-2}$ of submanifolds of Z_{m-1} such that

- for $i \leq m - 1$ the homology class of Z_{i-1} is the Poincaré dual of the restriction of ζ''_i to Z_i, which coincides with ζ'_i defined as the restriction of ζ_i to Z_i ;
- for any sequence $\{R_i\}_{i=1}^{m-1}$ of positive numbers if $2\sum_{k=1}^i R_k < L(\zeta'_i)$ for $i = 1, \ldots, m-1$ then

$$\operatorname{vol}_{m-1} D_{m-1} \geq 2^{m-1} \prod_{k=1}^{m-1} R_i - O(\delta).$$

Now fix a sequence $\{R_i\}_{i=1}^m$ of positive numbers such that $2\sum_{k=1}^i R_k < L(\zeta'_i)$ for $i = 1, \ldots, m$. We will need the following Lemma:

Lemma 0.1. *Let c be a 1-cycle in D_m. Then there exist loops $\gamma_1, \ldots, \gamma_k \subset D_m$ with $l_g(\gamma_i) < L(\zeta'_m)$ for $i = 1, \ldots, k$ and such that c is homologous to the 1-cycle $\gamma_1 + \ldots + \gamma_k$.*

Proof of the Lemma. We proceed as in the Curve Factoring Lemma (see [Gut10]). Just observe that any point of D_m can be connected to z_0 through a path in D_m of length at most $\sum_{k=1}^m R_k$. □

Using this Lemma, we can prove the following analog of the version of Stability Lemma due to Nakamura [Nak13].

Lemma 0.2. *For any $r \leq R_m$*

$$\operatorname{vol}_{m-1}\{x \in M \mid d_g(x, D_{m-1}) = r\} \geq 2\,(\operatorname{vol}_{m-1}\{x \in Z_{m-1} \mid d_g(x, D_{m-1}) \leq r\} - \delta).$$

Proof of the Lemma. First note we only have to prove the inequality for $r = R_m$ which writes as follows:

$$\operatorname{vol}_{m-1}\{x \in M \mid d_g(x, D_{m-1}) = R_m\} \geq 2\,(\operatorname{vol}_{m-1}(Z_{m-1} \cap D_m) - \delta).$$

We argue as in the proof of the Stability Lemma in [Nak13]. First note that

$$[Z_{m-1} \cap D_m] = 0 \in H_{m-1}(D_m, \partial D_m; \mathbb{Z}_2).$$

For this recall the following argumentation due to [Gut10]. If this is not the case, by the Poincaré-Lefschetz duality the cycle $Z_{m-1} \cap D_m$ has a non-zero algebraic intersection number with an absolute cycle c in D_m. Using Lemma 0.1 we find a finite number of loops $\gamma_1, \ldots, \gamma_k \subset D_m$ with $l_g(\gamma_i) < L(\zeta_m)$ for $i = 1, \ldots, k$ and such that c is homologous to the 1-cycle $\gamma_1 + \ldots + \gamma_k$. But this implies that for some $i = 1, \ldots, k$ the intersection with Z_{m-1} is not zero which gives a contradiction with the definition of $L(\zeta_m)$. Now the proof proceeds *mutatis mutandis* as in [Nak13]. □

Then by the coarea formula

$$\begin{aligned}\operatorname{vol}_m D_m &= \int_0^{R_m} \operatorname{vol}_{m-1}\{x \in M \mid d_g(x, D_{m-1}) = r\} dr \\ &\geq \int_0^{R_m} 2\left(\operatorname{vol}_{m-1}\{x \in Z_{m-1} \mid d_g(x, D_{m-1}) \leq r\} - \delta\right) dr \\ &\geq \int_0^{R_m} 2\left(\operatorname{vol}_{m-1} D_{m-1} - \delta\right) dr \\ &\geq 2R_m\left(\operatorname{vol}_{m-1} D_{m-1} - \delta\right)\end{aligned}$$

which proves the assertion.

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