A FAMILY OF EXAMPLES OF
GENERALIZED PERFECT RINGS

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ABSTRACT. We construct a family of semiprimitive and non von Neumann regular rings satisfying that any right or left module is isomorphic to a quotient of its flat cover (in the sense of Enochs) by a small submodule. This answers in the negative a question posed by A. Amini, B. Amini, M. Ershad and H. Sharif (2007).

1. INTRODUCTION

Inspired by the fundamental work of Bass [3] on perfect rings and projective covers, A. Amini, B. Amini, M. Ershad and H. Sharif proposed in [1] to study a class of rings that they named *generalized perfect rings*.

Let R be a ring, and let F and M be right R-modules such that F_R is flat. Following [1], a module epimorphism $f \colon F \to M$ is said to be a *G-flat cover* of M if $\mathrm{Ker}\,(f)$ is a small submodule of F. Still following [1], a ring R is called *right generalized perfect* (right G-perfect, for short) if every right R-module has a flat cover. A ring R is called *G-perfect* if it is both left and right G-perfect.

It is clear from the definition that right perfect rings are right G-perfect rings, and also that von Neumann regular rings are G-perfect rings. Moreover, the class of right G-perfect rings is closed under finite products and quotients [1, Proposition 2.6].

A celebrated result by Bican, El Bashir and Enochs [4] shows that any module has a flat cover (see Definition 2.1). The relation between G-flat covers and flat covers (if any!) is quite unclear. In the case of perfect rings they coincide, and in the case of von Neumann regular rings flat covers are trivially G-flat covers but, in general, the converse is not true [1] (it happens that flat covers are unique up to isomorphism, while G-flat covers are not!).

Looking for a characterization of G-perfect rings, it was shown in [1] that if R is right G-perfect, then the Jacobson radical $J(R)$ is right T-nilpotent and, hence, idempotents lift modulo $J(R)$. Moreover, it was also proved that if R is right duo (i.e. all right ideals are two-sided ideals) and right G-perfect, then $R/J(R)$ is von Neumann regular. It was claimed that it was reasonable to conjecture that a right G-perfect ring is von Neumann regular modulo the Jacobson radical. In this work, we answer this conjecture in the negative by constructing semiprimitive G-perfect rings that are not von Neumann regular (see Examples 3.6).

Our examples are built using the following well-known pattern: let $S \hookrightarrow T$ be a ring inclusion, and consider the ring

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T, x \in S\} \subseteq T^{\mathbb{N}}.$$

Such a construction appears quite frequently in the literature. Our new input is the study of its category of modules. To do that it is very useful to consider a family of TTF-triples associated to such type of rings that relates the category of modules over R with the categories of T-modules and of S-modules, cf. Remark 3.2. In Theorem 3.5, we show that if S is right G-perfect and T is von Neumann regular, then R is also right G-perfect. We also show that if flat covers of S-modules are G-flat covers, then the same is true for R. Therefore, as far as we know, it could well be that for a G-perfect ring flat covers are always G-flat covers.

We have structured the paper in two sections. In the first one, we give some background material and we prove some preliminary results. In the second section, we specialize to the particular class of examples we are interested in. For the reader's sake, we have made the paper as self-contained as possible.

We are very grateful to the referee for his/her comments. Specially, for pointing out that the use of TTF-triples would give more light to our first and naive approach.

2. Background and preliminary results

All our rings are associative with 1, and ring homomorphisms preserve 1. The term module means unital module.

We recall the notion of cover. The definition is due to Auslander for the case of artin algebras and to Enochs for arbitrary rings.

Definition 2.1. Let R be a ring. Let $\mathcal{C}$ be a class of right R-modules, and let M_R be a right R-module. A module homomorphism $f \colon C \to M$ is a $\mathcal{C}$-precover of M if it satisfies that

(i) $C \in \mathcal{C}$;
(ii) $\mathrm{Hom}_R(C', f)$ is onto for any $C' \in \mathcal{C}$.

The homomorphism f is a $\mathcal{C}$-cover if, in addition, it is right minimal.

Recall that $f \colon C \to M$ is said to be *right minimal* if for any $g \in \mathrm{End}_R(C)$, $f = fg$ implies g bijective (cf. [2, §1.2]).

We are interested in the case $\mathcal{C} = \mathcal{F}$ is the class of flat right modules, that is on flat precovers and on flat covers. We also consider the class

$$\mathcal{E} = \{B \in \mathrm{Mod}\text{-}R \mid \mathrm{Ext}^1_R(L, B) = 0 \text{ for any flat right } R\text{-module } L\}$$

which is called the class of *(Enochs) cotorsion modules*.

Let us recall some well known facts on flat covers that we will use in the sequel. Since any module is a homomorphic image of a projective module it easily follows that any flat precover is onto. As the class of flat modules is closed under extensions, Wakamatsu Lemma [7, Lemma 2.1.13] implies that the kernel of any flat cover is a cotorsion module. On the other hand, a result of Eklof and Trlifaj [6, Theorem 10] implies that any right R-module M fits into an exact sequence

$$0 \to B \to L \xrightarrow{g} M \to 0$$

where L is flat and B is a cotorsion module, cf. [4]. Since $\mathrm{Ext}^1_R(L', B) = 0$ for any flat module L', it follows that g is a flat precover.

If R is a right perfect ring then flat right R-modules are projective, and hence flat covers coincide with projective covers and with G-flat covers. In general, covers are unique up to isomorphism while G-flat covers are not [1, Example 3.1].

The following definition is quite well known.

Definition 2.2. *Let R be a ring. A pair $(\mathcal{X}, \mathcal{Y})$ of subclasses of* Mod-R *is said to be a torsion pair if*

(i) $\operatorname{Hom}_R(X, Y) = \{0\}$ *for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.*

(ii) *If X_R is a right R-module such that $\operatorname{Hom}_R(X, Y) = \{0\}$ for any $Y \in \mathcal{Y}$ then $X \in \mathcal{X}$.*

(iii) *If Y_R is a right R-module such that $\operatorname{Hom}_R(X, Y) = \{0\}$ for any $X \in \mathcal{X}$ then $Y \in \mathcal{Y}$.*

In this case, $\mathcal{X}$ is said to be a torsion class *and $\mathcal{Y}$ is a* torsion-free class. *The objects of $\mathcal{X}$ are called torsion modules and the objects in $\mathcal{Y}$ are called torsion-free modules.*

We refer to [8] for the basics on torsion pairs and their uses. We quickly recall the facts on torsion pairs that are more relevant to us.

Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair. If M_R is a right R-module, it makes sense to consider the largest submodule of M_R that is an object of $\mathcal{X}$, such submodule is called the torsion submodule of M and is denoted by $t(M)$. It is not difficult to see that t is indeed a functor and a radical. So that, there is an exact sequece

$$0 \to t(M) \to M \to M/t(M) \to 0$$

where $M/t(M) \in \mathcal{Y}$.

A class of modules $\mathcal{X}$ is torsion if and only if it is closed under isomorphisms, extensions, coproducts and quotients. Dually, a class of modules $\mathcal{Y}$ is a torsion-free class if it is closed under isomorphism, extensions, submodules and products.

Notice that if a class of modules $\mathcal{Y}$ is closed by products, coproducts, subobjects, quotients and extensions then $\mathcal{Y}$ is a torsion class and a torsion free class at the same time. Therefore, one has a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs. Such a triple is called a TTF-triple [8, Ch. VI §8].

Let $0 \longrightarrow M \xrightarrow{h} N \xrightarrow{f} K \longrightarrow 0$ be an exact sequence of right R-modules and let $L \xrightarrow{g} K \longrightarrow 0$ be an onto homomorphism. We consider the pullback of f and g to obtain a commutative diagram with exact rows and columns:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
X & = & X = Ker\,g \\
\downarrow{\scriptstyle\varepsilon_2} & & \downarrow \\
0 \longrightarrow M \xrightarrow{\varepsilon_1} & L' \xrightarrow{\pi_2} & L \longrightarrow 0 \\
\| & \downarrow{\scriptstyle\pi_1} & \downarrow{\scriptstyle g} \\
0 \longrightarrow M \xrightarrow{h} & N \xrightarrow{f} & K \longrightarrow 0 \\
& \downarrow & \downarrow \\
& 0 & 0
\end{array}
\tag{1}
$$

where $L' = \{(x, y) \in N \oplus L \mid f(x) = g(y)\}$. The maps $\pi_1 \colon L' \to N$ and $\pi_2 \colon L' \to L$ are restrictions of the canonical projections $\pi_1 \colon N \oplus L \to N$ and $\pi_2 \colon N \oplus L \to L$,

respectively. The homomorphism $\varepsilon_1 \colon M \to L'$ is defined by $\varepsilon_1(x) = (h(x), 0)$ for each $x \in M$, and $\varepsilon_2 \colon X \to L'$ is defined by $\varepsilon_2(y) = (0, y)$ for each $y \in X$.

Our construction of G-flat covers will be done with such a pullback diagram. For further use we note the following lemma.

Lemma 2.3. *With the notation above, assume that X is a small submodule of L. If $Y \leq L'$ is such that $\varepsilon_1(M) \subseteq Y$, then $\varepsilon_2(X) + Y = L'$ implies $L' = Y$.*

Proof. Since $X \ll L$, $\pi_2(Y) = L$, because $\pi_2(L') = \pi_2(\varepsilon_2(X)) + \pi_2(Y) = X + \pi_2(Y) = L$.

Let $(x, y) \in L'$. Since $\pi_2(Y) = L$, there exists $(x', y) \in Y$ such that $g(y) = f(x') = f(x)$. Therefore, $x - x' \in Im(h)$, and so $(x - x', 0) \in \varepsilon_1(M) \subseteq Y$. Now, $(x, y) = (x', y) + (x - x', 0) \in Y$. This proves that $Y = L'$. $\square$

The next two lemmas will be useful to produce G-flat covers and flat covers by using diagram (1).

Lemma 2.4. *Let R be a ring. Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in $\mathrm{Mod}\text{-}R$ such that the associated torsion radical t is exact. Assume that in diagram (1), $M \in \mathcal{X}$ and K, $L \in \mathcal{Y}$.*

If X is small in L then $\varepsilon_2(X)$ is small in L'. In particular, if L_R and M_R are flat, then $\pi_1 \colon L' \to N$ is a G-flat cover of N.

Proof. Since $L \in \mathcal{Y}$ which is a torsion-free class, also $X \in \mathcal{Y}$. Let $Y \leq L'$ is such that $\varepsilon_2(X) + Y = L'$. Applying t to the exact sequence

$$0 \to \varepsilon_2(X) \cap Y \to \varepsilon_2(X) \oplus Y \to L' \to 0$$

yields that $t(L') = t(\varepsilon_2(X)) \oplus t(Y) = t(Y)$. Since $M \in \mathcal{X}$ and $L \in \mathcal{Y}$ it follows that $t(L') = \varepsilon_1(M)$. Therefore, $\varepsilon_1(M) \subseteq Y$. Now it follows from Lemma 2.3 that $\varepsilon_2(X)$ is small in L'.

The rest of the claim is clear. $\square$

Lemma 2.5. *Let R be a ring, let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in $\mathrm{Mod}\text{-}R$. Assume that in diagram (1), $M \in \mathcal{X}$ and K, $L \in \mathcal{Y}$. Then g is right minimal if and only if π_1 is right minimal.*

Proof. Throughout the proof we denote by t the torsion radical associated to the torsion pair. Since $M \in \mathcal{X}$ and $L \in \mathcal{Y}$, $t(L') \cong M$ and $L'/t(L') \cong L$. Also, because of diagram (1), $t(\pi_1) \colon t(L') \to t(N)$ is an isomorphism. Fix an isomorphism $\varphi \colon L \to L'/t(L')$

Assume g is right minimal, and let $h' \colon L' \to L'$ be such that $\pi_1 h' = \pi_1$. Then $t(\pi_1) t(h') = t(\pi_1)$, and since $t(\pi_1)$ is an isomorphism we deduce that $t(h')$ is an isomorphism. Now to conclude that h' is an isomorphism we only need to prove that the induced map $h \colon L'/t(L') \to L'/t(L')$ is an isomorphism. Indeed,

$$g\pi_2 = f\pi_1 = g\pi_2 h'.$$

Since $\pi_2 h' = h\varphi\pi_2$ and π_2 is an onto map, $g = gh\varphi$. Therefore $h\varphi$ and, hence, h are isomorphisms.

The converse follows using similar arguments. $\square$

The following lemma is a particular case of [5, Proposition 4.1.3].

Lemma 2.6. *Let $\varphi \colon R \to S$ be a ring morphism such that $_R S$ is flat. Let X_S and Y_S be right $S-$modules, then $\mathrm{Ext}_R^i(X, Y) \cong \mathrm{Ext}_S^i(X, Y)$ for any $i \geq 0$.*

The following proposition collects some well known facts on TTF-triples that will be useful in the sequel.

Proposition 2.7. *Let R and S be rings such that there is an exact sequence*

$$0 \to I \to R \xrightarrow{\varphi} S \to 0$$

where φ is a ring morphism such that $_RS$ becomes a flat module. Consider the following classes of modules

$$\mathcal{X} = \{X \in \mathrm{Mod}\text{-}R \mid XI = X\}$$

$$\mathcal{Y} = \{Y \in \mathrm{Mod}\text{-}R \mid YI = \{0\}\}$$

$$\mathcal{Z} = \{Z \in \mathrm{Mod}\text{-}R \mid \mathrm{ann}_Z(I) = \{0\}\}$$

then $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF-triple such that the torsion pair $(\mathcal{X}, \mathcal{Y})$ is hereditary and $\mathrm{Ext}_R^i(X, Y) = 0$ for any $i \geq 0$, $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Moreover, the torsion radical associated to the torsion class $\mathcal{X}$ is naturally equivalent to the exact functor $- \otimes_R I$, and the torsion radical associated to the class $\mathcal{Y}$ is naturally equivalent to the functor $\mathrm{Hom}_R(S, -)$.

Proof. Notice that the class $\mathcal{Y}$ is equivalent to the category of right S-modules, so that $\mathcal{Y}$ has all the closure properties that ensure that it is a torsion class and a torsion-free class. Then it is easy to check that the classes $\mathcal{X}$ and $\mathcal{Z}$ are the ones that complete the TTF-triple.

Since $_RS$ is flat, $_RI$ is a pure submodule of R and, hence, it is also flat and an idempotent ideal. Therefore, if t denotes the radical associated to the torsion class $\mathcal{X}$, it follows that $t(M) = MI \cong M \otimes_R I$ for any right R-module M. From this it easily follows that t is naturally equivalent to the exact funcor $- \otimes_R I$.

Since t is exact, $\mathcal{X}$ is closed under submodule and then $\mathcal{Y}$ is closed by injective hulls [8, Proposition VI.3.1]. Therefore, if $Y \in \mathcal{Y}$, then its injective hull as an R-module must be an S-module, so its injective hull as an R-module must coincide with its injective hull as an S-module. This implies that, for $Y \in \mathcal{Y}$, we can consider an injective R-coresolution $I^\bullet$ of Y consisting of injective modules in $\mathcal{Y}$. Now, if $X \in \mathcal{X}$, then the cocomplex $\mathrm{Hom}_R(X, I^\bullet)$ is zero, so that it has zero cohomology. This shows that $\mathrm{Ext}_R^i(X, Y) = 0$ for any $i \geq 0$. $\qquad \square$

Corollary 2.8. *Let R and S be rings such that there is an exact sequence*

$$0 \to I \to R \xrightarrow{\varphi} S \to 0$$

where φ is a ring morphism such that S becomes a flat R-module on the right and on the left. Then:

    (i) *M_R is flat if and only if $M \otimes_R S$ is a flat right S-module and MI is a flat right R-module.*

    (ii) *Let M be a right S-module, then M is cotorsion as a right R-module if and only if it is cotorsion as an S-module.*

Proof. Notice that since S is flat on both sides as R-module then also I is a flat R-module on both sides.

(i). By Proposition 2.7 and following the notation in that proposition, for any right R-module M there is an exact sequence

$$0 \to MI \to M \to M/MI \to 0 \qquad (*)$$

with $MI \in \mathcal{X}$ and $M/MI \in \mathcal{Y}$.

Assume M_R is flat. The functors $MI \otimes_R -$ and $M/MI \otimes_R -$ are naturally equivalent to $M \otimes_R I \otimes_R -$ and $M \otimes_R S \otimes_R -$, respectively. These functors are exact because I_R and S_R are flat. Therefore, MI and M/MI are flat right R-modules. In particular, $M/MI \cong M \otimes_R S$ is flat as a right S-module.

Assume now that MI is a flat right R-module and that M/MI is flat as a right S-module. The functor $M/MI \otimes_R -$ is exact because it is naturally equivalent to $M/MI \otimes_S S \otimes_R -$ and S_R is flat. Therefore, M/MI is flat as a right R-module. Using the exact sequence $(*)$ we deduce that M_R is flat.

(ii). By (i) and Proposition 2.7, a right R-module M is cotorsion if and only if $\operatorname{Ext}^i_R(F, M) = 0$ for any flat module F that is either in $\mathcal{X}$ or in $\mathcal{Y}$. If, in addition, M is a right S module, then it is in $\mathcal{Y}$ so we only need to check that for any flat R-module F in $\mathcal{Y}$, that is, for any flat right S-module, $\operatorname{Ext}^1_R(F, M) = \operatorname{Ext}^1_S(F, M) = 0$ (cf. Lemma 2.6). This yields that a right S-module is cotorsion as an R-module if and only if it is cotorsion as an S-module. $\qquad\square$

3. The examples

Now we start to study the particular construction we are interested in.

Proposition 3.1. *Let $S \subseteq T$ be an extension of rings. Let*

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T, x \in S\}.$$

Then, the following statements hold.

(i) *The map $\varphi \colon R \to S$ defined by $\varphi(x_1, x_2, \ldots, x_n, x, x, \ldots) = x$ is a ring homomorphism with kernel*

$$I = \bigoplus_{\mathbb{N}} T = \bigoplus_{i \in \mathbb{N}} e_i R,$$

where $e_i = (0, \ldots, 0, 1^{(i)}, 0, 0, \ldots)$ for any $i \in \mathbb{N}$.

(ii) *I is a two-sided, countably generated idempotent ideal of R which is pure and projective on both sides. Therefore, S is flat as a right and as a left R-module.*

(iii) *For any $i \in \mathbb{N}$, the canonical projection into the i-th component $\pi_i \colon R \to T$ has kernel $(1-e_i)R$ so that T is projective as a right and as a left R-module via the R-module structure induced by π_i.*

Proof. Statement (i) is clear. To prove (ii) note that since $\{e_i\}_{i \in \mathbb{N}}$ is a family of orthogonal central idempotents of R, $I^2 = I$, I is projective on both sides and, moreover, I is a pure ideal of R on both sides. Now the existence of the pure exact sequence

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\varphi} S \longrightarrow 0$$

yields that S is flat as a right and as a left R-module.

Statement (iii) is clear. $\qquad\square$

Remark 3.2. *Let R be a ring as in Proposition 3.1. In view of Proposition 2.7, there is a TTF-triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ associated to the pure exact sequence*

$$0 \to I \to R \xrightarrow{\varphi} S \to 0$$

where $\mathcal{X} = \{X \in \operatorname{Mod} - R \mid X = \oplus_{i \in \mathbb{N}} X e_i\}$, $\mathcal{Y} = \{Y \in \operatorname{Mod} - R \mid YI = \{0\}\}$ $\mathcal{Z} = \{Z \in \operatorname{Mod} - R \mid \operatorname{ann}_Z(I) = \{0\}\}$. Also, for any $i \in \mathbb{N}$, the split sequence

$$0 \to R(1 - e_i) \to R \xrightarrow{\pi_i} T \to 0$$

yields a corresponding (split) TTF*-triple* $(\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i)$.

Proposition 3.3. *Let R be a ring as in the statement of Proposition 3.1. Then,*

- (i) $J(R)$ *contains* $J = \bigoplus_{\mathbb{N}} J(T)$. *Moreover, J is essential on both sides into* $J(R)$. *In particular, $J(R) = 0$ if and only if $J(T) = 0$.*
- (ii) R *is von Neumann regular if and only if S and T are von Neumann regular.*

Proof. (i). Let $x \in J(T)$. Then, for any $i \in \mathbb{N}$, $x(i) = (0, \ldots, 0, x^{(i)}, 0, 0, \ldots) \in J$ is an element of $J(R)$. Since $J(R)$ is an ideal, it follows that it contains J. Now we show that $J_R \leq_e J(R)_R$, the statement on the left follows by a symmetric argument.

Let $0 \neq a = (x_1, x_2, \ldots, x_n, x, x, \ldots) \in J(R)$. There exists $i \in \mathbb{N}$ such that $0 \neq ae_i = (0, \ldots, 0, y^{(i)}, 0, 0, \ldots) \in J(R)$ for some $y \in T$. Then, for any $t \in T$, $(1, \ldots, 1, \ldots) - ae_i(0, \ldots, 0, t^{(i)}, 0, 0, \ldots)$ is an invertible element of R and this is equivalent to say that $1 - yt$ is an invertible element of T. Hence $y \in J(T)$, so that ae_i is a non-zero element of $J \cap aR$.

(ii). Assume that R is von Neumann regular. Since S is a homomorphic image of R, it is also von Neumann regular.

The canonical projection $\pi_1 \colon R \to T$ defined by $\pi_1(x_1, x_2, \ldots, x_n, x, x, \ldots) = x_1$ is also an onto ring homomorphism, hence T is von Neumann regular.

It is clear that if S and T are von Neumann regular, then so is R. $\qquad\square$

Lemma 3.4. *Let R be a ring as in the statement of Proposition 3.1. Let M_R be a right R-module. Then M_R is flat if and only if $M \otimes_R S$ is a flat right S-module and, for any $i \in \mathbb{N}$, Me_i is a flat right T-module.*

Proof. Applying Corollary 2.8 with the flat epimorphism φ, it follows that M_R is flat if and only if $M \otimes_R S$ is a flat right S-module and Me_i is a flat right R-module for any $i \in \mathbb{N}$.

Applying Corollary 2.8 with the flat epimorphism π_i, it follows that Me_i is a flat right R-module if and only if it is a flat T-module. $\qquad\square$

Now we are ready to prove our main result:

Theorem 3.5. *Let $S \subseteq T$ be an extension of rings. Assume T is von Neumann regular and that S is right G-perfect. Then*

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T, x \in S\}$$

is a right G-perfect ring such that $J(R) = 0$.

Moreover, if S is a ring such that flat covers are G-flat covers, then also R satisfies this property.

Proof. By Proposition 3.3, it readily follows that $J(R) = 0$.

Let N be any right R-module. By Proposition 3.1, there is a pure exact sequence

$$0 \longrightarrow NI \cong \bigoplus_{i \in \mathbb{N}} Ne_i \longrightarrow N \xrightarrow{f} N/NI \longrightarrow 0.$$

Since T is von Neumann regular, for any $i \in \mathbb{N}$, Ne_i is a flat T-module. Hence NI is flat as a right R-module by Lemma 3.4.

Let $0 \to X \to L \xrightarrow{g} N/NI \to 0$ be a G-flat cover of the right S-module N/NI. Considering the pullback of h and f yields the following diagram with exact rows and columns

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
X & = & X = Ker\,h \\
\downarrow & & \downarrow \\
0 \longrightarrow NI \longrightarrow L' & \xrightarrow{\pi_2} & L \longrightarrow 0 \\
\quad \| \quad\quad \downarrow^{\pi_1} & & \downarrow^{g} \\
0 \longrightarrow NI \longrightarrow N & \xrightarrow{f} & N/NI \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Since, by Proposition 3.1, the radical associated to the torsion pair $(\mathcal{X}, \mathcal{Y})$ is exact and $L \in \mathcal{Y}$, it follows from Lemma 2.4 that π_1 is a G-flat cover of N.

Now assume, in addition, that $0 \to X \to L \xrightarrow{h} N/NI \to 0$ is a flat cover of the right S-module $N \otimes_R S$. In particular, X_S is cotorsion. By Corollary 2.8, X_R is also a cotorsion module, hence $0 \to X \to L' \xrightarrow{\pi_1} N \to 0$ is a flat precover of N. By Lemma 2.5 it follows that π_1 is also a flat cover. $\square$

Finally, we show how to produce the examples claimed in the Introduction from Theorem 3.5.

Examples 3.6. (a) *Let F be a field, and let S be any finite dimensional F-algebra such that $J(S) \neq 0$. Since S is artinian, it is G-perfect. If $\dim_F(S) = n$, then $S \hookrightarrow T = \mathbb{M}_n(F)$ which is von Neumann regular. Therefore,*

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T, x \in S\}$$

is G-perfect by Theorem 3.5. By Proposition 3.3, $J(R) = 0$ and R is not von Neumann regular.

For a particular realization of such a ring R consider, for example, $S = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. In this case, T can be taken to be $M_2(F)$.

(b) *Let R be as in (a). Then, $R \subseteq \prod \mathbb{M}_n(F) = T'$ which is a von Neumann regular ring and $R' = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T', x \in R\}$ is also a G-perfect ring.*

Note that, by Theorem 3.5, in the above examples flat covers are G-flat covers.

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