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ZERO-HOPF BIFURCATION IN THE GENERALIZED MICHELSON SYSTEM

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ABSTRACT. We provide sufficient conditions for the existence of two periodic solutions bifurcating from a zero–Hopf equilibrium for the differential system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = a + by + cz - x^2/2,$$

where a, b and c are real arbitrary parameters. The regular perturbation of this differential system provides the normal form of the so–called triple–zero bifurcation.

1. Introduction and statement of the main results

In the paper [1] it is proved that the normal form of the triple–zero bifurcation can be understood as a regular perturbation of the following generalized Michelson system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= a + by + cz - x^2/2, \end{aligned} \tag{1}$$

where a, b and c are real arbitrary parameters. The prime denotes derivative with respect to the independent variable t.

A *zero–Hopf equilibrium* is an equilibrium point of a 3–dimensional autonomous differential system, which has a zero eigenvalue and a pair of purely imaginary eigenvalues. Usually the *zero–Hopf bifurcation* is a two–parameter unfolding of a 3-dimensional autonomous differential system with a zero–Hopf equilibrium. The unfolding has an isolated equilibrium with a zero eigenvalue and a pair of purely imaginary eigenvalues if the two parameters take zero values, and the unfolding has different topological type of dynamics in the small neighborhood of this isolated equilibrium as the two parameters vary in a small neighborhood of the origin. For instance this zero–Hopf bifurcation has been studied in [6, 7, 8, 9, 11], and it has been shown that some complicated invariant sets of the unfolding could be bifurcated from the isolated zero–Hopf equilibrium under some conditions. Hence, in some cases zero–Hopf bifurcation could imply a local birth of “chaos” see for instance the articles of [2, 3, 4, 5, 11]).

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Our objective is to study analytically the periodic solutions of the zero–Hopf bifurcation for the generalized Michelson differential system (1). In the previous mentioned papers on the zero-Hopf bifurcation they do not use averaging theory for studying such kind of bifurcation. Our goal is to study analytically such a bifurcation using averaging theory which will allows to provide an explicit expression of the dominant terms of the periodic solution bifurcating from the zero-Hopf equilibrium. First, in the next proposition we characterize when the equilibrium point of the generalized Michelson system (1) is a zero-Hopf equilibrium point.

Proposition 1. *There is an one–parameter family of the generalized Michelson system* (1) *for which the origin of coordinates is a zero–Hopf equilibrium point. Namely* $a = 0, \quad b = -\omega^2, \quad c = 0.$

Proposition 1 is proved in section 2.

Theorem 2. *Assume that in the generalized Michelson system* (1) *we have*

$$a = \varepsilon^2(a_2 + \omega^4/2), \quad a_2 > 0, \quad b = -\omega^2 + \varepsilon b_1, \quad \omega > 0, \quad c = \varepsilon. \tag{2}$$

Then for $\varepsilon \neq 0$ *sufficiently small system* (1) *has two periodic solutions* $(x_i(t,\varepsilon), y_i(t,\varepsilon), z_i(t,\varepsilon))$ *bifurcating from the zero–Hopf equilibrium of Proposition 1, namely*

$$\begin{aligned}\Bigg(&\varepsilon\frac{V_i^* - \omega R^*\cos(\omega t) - \omega^2\sqrt{2a_2+\omega^4}}{\omega^2} + O(\varepsilon^2),\\ &\varepsilon R^*\sin(\omega t) + O(\varepsilon^2), \quad \varepsilon\omega R^*\cos(\omega t) + O(\varepsilon^2)\Bigg),\end{aligned} \tag{3}$$

where

$$R^* = 2\sqrt{a_2}\,\omega, \quad V_i^* = \omega^2((-1)^i\sqrt{2a_2+\omega^4} - \omega^2) \quad \text{for } i = 1, 2.$$

Moreover, these two periodic solutions are unstable.

Theorem 2 improves and extends the result of [10] where only one periodic solution was detected bifurcating from the zero–Hopf equilibrium for a subsystem of system (1).

2. Proof of Proposition 1 and Theorem 2

Proof of Proposition 1. System (1) possesses the equilibrium points $(x, y, z) = (\pm\sqrt{2a}, 0, 0)$ if $a \geq 0$. The Jacobian matrix of system (1) at the equilibrium point $(\pm\sqrt{2a}, 0, 0)$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pm\sqrt{2a} & b & c \end{pmatrix}.$$

Its characteristic polynomial is $p(\lambda) = -\lambda^3 + c\lambda^2 + b\lambda \pm \sqrt{2a}$. In order to study the zero-Hopf bifurcation we force that $p(\lambda) = -(\lambda - \varepsilon)(\lambda^2 + \omega^2)$. $p(\lambda) + (\lambda - \varepsilon)(\lambda^2 + \omega^2) = 0$. This occurs if and only if the coefficients of this equation are $\pm\sqrt{2a} - \varepsilon\omega^2 = 0, b + \omega^2 = 0, c - \varepsilon = 0$. We obtain $a = \varepsilon^2 \dfrac{\omega^4}{2}, b = -\omega^2, c = \varepsilon$. This completes the proof of the proposition. $\square$

The differential system (1) satisfying (2) has two equilibria, namely

$$p_\pm = (\pm\varepsilon\sqrt{2a_2 + \omega^4}, 0, 0).$$

First we study the periodic solutions bifurcating from the zero–Hopf equilibrium near the equilibrium p_-.

For applying the averaging theory described in the appendix to system (1) satisfying (2) we translate the equilibrium point p_- to the origin by doing the change of variables

$$(x, y, z) = (x_1 - \varepsilon\sqrt{2a_2 + \omega^4}, y_1, z_1). \tag{4}$$

The differential system in the new variables (x_1, y_1, z_1) is

$$\begin{aligned} \dot{x_1} &= y_1, \\ \dot{y_1} &= z_1, \\ \dot{z_1} &= -\omega^2 y_1 - \frac{x_1^2}{2} + \varepsilon\left(b_1 y_1 + z_1 + x_1\sqrt{2a_2 + \omega^4}\right). \end{aligned} \tag{5}$$

We need to write the linear part of system (5) at the equilibrium point $(0, 0, 0)$ in its real Jordan normal form, i.e. into the form

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

in order to facilitate the application of the averaging theory, given by Theorem 3, for computing the zero–Hopf bifurcation. Then, doing the change of variables $(x_1, y_1, z_1) \to (X, Y, Z)$ given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{\omega} \\ 0 & 1 & 0 \\ \omega^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix},$$

the differential system (5) having its linear part in its real Jordan form is

$$\begin{aligned} \dot{X} = &\ -\frac{2Y\omega^6 + (Z - \omega X)^2}{2\omega^5} + \\ &\ \varepsilon\left(X + \frac{b_1 Y}{\omega} + \frac{1}{\omega^3}(Z - X\omega)\sqrt{2a_2 + \omega^4}\right), \\ \dot{Y} = &\ \omega X, \\ \dot{Z} = &\ -\frac{(Z - \omega X)^2}{2\omega^4} + \varepsilon\left(\omega X + b_1 Y + \frac{1}{\omega^2}(Z - \omega X)\sqrt{2a_2 + \omega^4}\right). \end{aligned} \tag{6}$$

Consider the cylindrical coordinates (r,θ,Z) defined by $X = r\cos\theta$, $Y = r\sin\theta$, $Z = Z$ then the differential system (6) becomes

(7)
$$\begin{aligned}
\dot{r} =& -\frac{\cos\theta(Z-\omega r\cos\theta)^2}{2\omega^5}+\\
& \varepsilon\left[\frac{r\cos\theta(\omega\cos\theta+b_1\sin\theta)}{\omega}-\frac{1}{\omega^3}\cos\theta(Z-\omega r\cos\theta)\sqrt{2a_2+\omega^4}\right],\\
\dot{\theta} =& \frac{2r\omega^6+(Z-r\omega\cos\theta)^2\sin\theta}{2r\omega^5}-\\
& \varepsilon\left[\frac{\sin\theta(\omega\cos\theta+b_1\sin\theta)}{\omega}-\frac{\sqrt{2a_2+\omega^4}}{\omega^3}\left[\frac{Z\sin\theta}{r}-\frac{\omega\sin(2\theta)}{2}\right]\right],\\
\dot{Z} =& -\frac{(Z-r\omega\cos\theta)^2}{2\omega^4}+\\
& \varepsilon\left[\omega r\cos\theta+b_1 r\sin\theta+\frac{1}{\omega^2}(Z-\omega r\cos\theta)\sqrt{2a_2+\omega^4}\right].
\end{aligned}$$

Doing the rescaling $(r,Z)=(\varepsilon R,\varepsilon V)$ we obtain

(8)
$$\begin{aligned}
\dot{R} =& -\varepsilon\frac{1}{2\omega^5}\cos\theta\big(V\left(V-2\omega^2\sqrt{\omega^4+2a_2}\right)+\\
& R\omega\left(\omega R\cos^2\theta-2\omega^3 b_1\sin\theta-2\left(\omega^4-\sqrt{\omega^4+2a_2}\omega^2+V\right)\cos\theta\right)\big),\\
\dot{\theta} =& \ \omega+\varepsilon\frac{1}{2R\omega^5}\sin\theta\big(V\left(V-2\omega^2\sqrt{\omega^4+2a_2}\right)+\\
& R\omega\left(\omega R\cos^2\theta-2\omega^3 b_1\sin\theta-2\left(\omega^4-\sqrt{\omega^4+2a_2}\omega^2+V\right)\cos\theta\right)\big),\\
\dot{V} =& \ \varepsilon\frac{1}{2\omega^4}\big(R\omega\left(2\omega^3 b_1\sin\theta-\omega R\cos^2\theta+2\left(\omega^4-\sqrt{\omega^4+2a_2}\omega^2+V\right)\cos\theta\right)\\
& -V\left(V-2\omega^2\sqrt{\omega^4+2a_2}\right)\big).
\end{aligned}$$

Therefore taking θ as the new independent variable of the differential system (8), its solutions in the region $\dot{\theta}>0$ can be studied analyzing the solution of the differential system

$$\begin{aligned}
\frac{dR}{d\theta} =& -\varepsilon\frac{1}{2\omega^6}\cos\theta\Big[V\left(V-2\omega^2\sqrt{2a_2+\omega^4}\right)-2\omega R(V+\omega^4-\\
& \omega^2\sqrt{2a_2+\omega^4})\cos\theta+\omega^2R^2\cos^2\theta-2b_1\omega^4R\sin\theta\Big]+O(\varepsilon^2)\\
=& \ \varepsilon F_1(\theta,R,V)+O(\varepsilon^2),
\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{dV}{d\theta} = & -\varepsilon\frac{1}{2\omega^5}\Big[V\left(V-2\omega^2\sqrt{2a_2+\omega^4}\right)-2\omega R(V+\omega^4- \\ & \omega^2\sqrt{2a_2+\omega^4}\cos\theta)+\omega^2R^2\cos\theta^2-2b_1\omega^4R\sin\theta\Big]+O(\varepsilon^2) \\ = & \ \varepsilon F_2(\theta,R,V)+O(\varepsilon^2).\end{aligned}$$

We shall apply the averaging theory described in the appendix to the differential system (9). Using the notation of the appendix we have $t=\theta$, $T=2\pi$, $\mathbf{x}=(R,V)^T$ and

$$F(\theta,R,V)=\begin{pmatrix}F_1(\theta,R,V)\\ F_2(\theta,R,V)\end{pmatrix},\quad\text{and}\quad f(R,V)=\begin{pmatrix}f_1(R,V)\\ f_2(R,V)\end{pmatrix}.$$

It is immediate to check that system (9) satisfies all the assumptions of Theorem (3) of the appendix. So we will apply it to system (9).

Now we compute the integrals (12), i.e.

$$\begin{aligned}f_1(R,V)= & \ \frac{1}{2\pi}\int_0^{2\pi}F_1(\theta,R,V)d\theta=\frac{R(V+\omega^4-\omega^2\sqrt{2a_2+\omega^4}}{2\omega^5}, \\ f_2(R,V)= & \ \frac{1}{2\pi}\int_0^{2\pi}F_2(\theta,R,V)d\theta=-\frac{2V^2+R^2\omega^2-4V\omega^2\sqrt{2a_2+\omega^4}}{4\omega^5}.\end{aligned}$$

The system $f_1(R,V)=f_2(R,V)=0$ has a unique solution (R^*,V^*) with $R^*>0$, namely

$$\begin{aligned}R^* = & \ 2\sqrt{a_2}\omega, \\ V^* = & \ \omega^2(\sqrt{2a_2+\omega^4}-\omega^2),\end{aligned}$$

because by assumptions $a_2>0$.

The Jacobian (13) at (R^*,V^*) takes the value $\dfrac{a_2}{\omega^6}>0$.

Theorem 3 garantes for $\varepsilon\neq$ sufficiently small the existence of a periodic solution $(R(\theta,\varepsilon),V(\theta,\varepsilon))$ of system (9) such that $(R(0,\varepsilon),V(0,\varepsilon))\to(R^*,V^*)$ when $\varepsilon\to0$. That is, system (9) has the periodic solution

$$(R(\theta,\varepsilon),V(\theta,\varepsilon))=\Big(R^*+O(\varepsilon),V^*+O(\varepsilon)\Big).$$

This periodic solution writes in system (8) as

$$(R(t,\varepsilon),\theta(t,\varepsilon),V(t,\varepsilon))=\Big(R^*+O(\varepsilon),\omega t+O(\varepsilon),V^*+O(\varepsilon)\Big).$$

In system (7) it becomes

$$(r(t,\varepsilon),\theta(t,\varepsilon),Z(t,\varepsilon))=\Big(\varepsilon R^*+O(\varepsilon^2),\omega t+O(\varepsilon),\varepsilon V^*+O(\varepsilon^2)\Big).$$

Passing to periodic solution to system (6) we get

$$(X(t,\varepsilon),Y(t,\varepsilon),Z(t,\varepsilon)) = \Big(\varepsilon R^*\cos(\omega t)+O(\varepsilon^2), \varepsilon R^*\sin(\omega t)+O(\varepsilon^2), \varepsilon V^*+O(\varepsilon^2)\Big).$$

In system (5) the periodic solution writes

$$(x_1(t,\varepsilon),y_1(t,\varepsilon),z_1(t,\varepsilon)) = \Big(\varepsilon\frac{V^*-\omega R^*\cos(\omega t)}{\omega^2}+O(\varepsilon^2), \varepsilon R^*\sin(\omega t)+O(\varepsilon^2), \varepsilon\omega R^*\cos(\omega t)+O(\varepsilon^2)\Big).$$

Finally for system (1) the periodic solution becomes the solution (3) for $i=2$ of the statement of Theorem 2.

Computing the eigenvalues of the Jacobian matrix

$$\left.\frac{\partial(f_1,f_2)}{\partial(R,V)}\right|_{(R,V)=(R^*,V^*)}$$

we obtain

$$\frac{\omega^5 \pm \sqrt{\omega^{10}-4a_2\omega^6}}{2\omega^6}.$$

So, by statement (b) of Theorem 3 the periodic solution associated to the zero (R^*,V^*) is unstable, because the real part of the two eigenvalues always is positive.

If instead of translating the equilibrium point p_- at the origin doing the change of variables (4), we translate at the origin the equilibrium point p_+ and repeat all the previous computations we shall obtain for system (1) the periodic solution (3) for $i=1$ of the statement of Theorem 2. This completes the proof of Theorem 2.

Appendix: averaging theory of first order

Now we shall present the basic results from the averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [12].

Consider the differential equation

$$\dot{\mathbf{x}} = \varepsilon F(t,\mathbf{x}) + \varepsilon^2 G(t,\mathbf{x},\varepsilon), \quad \mathbf{x}(0)=\mathbf{x}_0 \tag{10}$$

with $\mathbf{x}\in D$, where D is an open subset of $\mathbb{R}^n$, $t\geq 0$. Moreover we assume that both $F(t,\mathbf{x})$ and $G(t,\mathbf{x},\varepsilon)$ are $T-$periodic in t. We also consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f(\mathbf{y}), \quad \mathbf{y}(0)=\mathbf{x}_0, \tag{11}$$

where

$$f(\mathbf{y}) = \frac{1}{T}\int_0^T F(t,\mathbf{y})dt. \tag{12}$$

Under certain conditions, equilibrium solutions of the averaged equation (11) provide $T-$periodic solutions of the differential equation (10).

Theorem 3. *Consider the two initial value problems (10) and (11). Suppose:*

(i) *F, its Jacobian $\partial F/\partial x$, its Hessian $\partial^2 F/\partial x^2$, G and its Jacobian $\partial G/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty)\times D$ and $\varepsilon \in (0,\varepsilon_0]$.*

(ii) *F and G are $T-$periodic in t (T independent of ε).*

Then the following statements hold.

(a) *If p is an equilibrium point of the averaged equation* (11) *and*

$$\det\left(\frac{\partial f}{\partial \mathbf{y}}\right)\Bigg|_{\mathbf{y}=p} \neq 0, \tag{13}$$

then there exists a $T-$periodic solution $\varphi(t,\varepsilon)$ of equation (10) *such that $\varphi(0,\varepsilon)\to p$ as $\varepsilon \to 0$.*

(b) *The stability or instability of the periodic solution $\varphi(t,\varepsilon)$ is given by the stability or instability of the equilibrium point p of the averaged system* (11)*. In fact the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t,\varepsilon)$.*

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