This is a preprint of: "Poincaré–Pontryagin–Melnikov functions for a class of perturbed planar Hamiltonian equations", Salomón Rebollo-Perdomo, *Qual. Theory Dyn. Syst.*, 2016.
DOI: [10.1007/s12346-015-0185-5]

POINCARÉ–PONTRYAGIN–MELNIKOV FUNCTIONS FOR A CLASS OF PERTURBED PLANAR HAMILTONIAN EQUATIONS

Salomón Rebollo–Perdomo

Centre de Recerca Matemàtica, 08193 Bellaterra, Barcelona, Spain.
E-mail address: srebollo@crm.cat

Abstract. In this paper we extend a well-known algorithm for studying higher order Poincaré–Pontryagin–Melnikov functions of polynomial perturbed Hamiltonian equations. We consider a family of unperturbed equations whose associated Hamiltonian is not transversal to infinity, and its complexification is no a Morse polynomial. We prove that the first non-vanishing Poincaré–Pontryagin–Melnikov function of the displacement function, associated with the perturbed equation, is an Abelian integral, and we provide the algorithm to compute it. Our result generalizes the algorithm for the case when the Hamiltonian is transversal to infinity, and its complexification is a Morse polynomial. We apply our result to study the maximum number of zeros of the first non-vanishing Poincaré–Pontryagin–Melnikov function associated with some particular perturbed degenerated Hamiltonian equations.

1. Introduction

Consider the *perturbed Hamiltonian differential equation*

$$dF - \varepsilon\omega(\varepsilon) = 0, \tag{1_ε}$$

where $F := F(x, y)$ is a real polynomial, and $\omega(\varepsilon) = A(x, y, \varepsilon)dx + B(x, y, \varepsilon)dy$ is a 1-form on the real plane $\mathbb{R}^2$ such that $A(x, y, \varepsilon)$ and $B(x, y, \varepsilon)$ are real polynomials in x and y with coefficients that depend analytically on the real parameter $\varepsilon \in (\mathbb{R}, 0)$.

Suppose that the foliation defined by the *Hamiltonian differential equation* (1_0) has at least a continuous family of cycles (periodic orbits) $\gamma_c \subset F^{-1}(c)$. As is widely known [12, 17, 18], for ε small enough and non-zero, (1_ε) can have limit cycles bifurcating from the cycles γ_c of (1_0). A usual tool for studying these limit cycles is the displacement function associated with (1_ε) and the family $\{\gamma_c\}$:

$$L(\varepsilon, c) = \varepsilon L_1(c) + \varepsilon^2 L_2(c) + \varepsilon^3 L_3(c) + \cdots . \tag{2}$$

The coefficient $L_i(c)$ is the i-th order Poincaré–Pontryagin–Melnikov (PPM) function.

The limit cycles of (1_ε) that bifurcate from cycles of (1_0) are studied through the zeros of the first non-vanishing PPM function $L_k(c)$ with $k \geq 1$. Indeed, on one hand, the maximum number of isolated zeros, counting multiplicities, of $L_k(c)$ is an upper bound for the number of limit cycles of (1_ε) that bifurcate from (1_0); on the other hand, the number of distinct zeros of $L_k(c)$ can provide a lower bound for the number of these limit cycles. See for instance Proposition 26.1 and Remark 26.2 in [15]. Hence, we need to have a mechanism for knowing such a first non-vanishing function.

The classical Poincaré–Pontryagin formula says that the first order PPM function is given by an Abelian integral; however, if that function vanishes identically, then

2010 *Mathematics Subject Classification.* Primary: 34C07, 34C08, 37G15; Secondary: 34M45.
Key words and phrases. Limit cycle; Abelian integral; Perturbed equation; Hamiltonian equation.

the PPM functions of order $2, 3, \ldots$ have to be studied until either finding the first non-vanishing PPM function, or concluding that (1_ε) is integrable.

It is known that if F is transversal to infinity whose complexification is a Morse polynomial, then the first non-vanishing PPM function of the displacement function, associated with (1_ε), is an Abelian integral. See for instance Theorems 26.7 and 26.52 in [15]. The purpose of this paper is to show that this property remains for a larger family of polynomials F and to apply the delivered result for studying the maximum number of zeros of the first non-vanishing PPM function of some particular equations (1_ε).

In this paper we will assume that the polynomial $F(x, y)$ defining (1_0) is of the form $F = P(H)$, where $P = P(z)$ is a univariate polynomial of degree greater than one, and $H = H(x, y)$ is a bivariate polynomial. Under these assumptions the complexification of F is not a Morse polynomial, because F has degenerated singularities at $H^{-1}(Z(P'))$, where $P' = P'(z)$ denotes the derivative of $P(z)$ with respect to z, and $Z(P')$ denotes the finite set of solutions of $P'(z) = 0$.

Since the coefficients of $A(x, y, \varepsilon)$ and $B(x, y, \varepsilon)$ depend analytically on ε and $dF = P'(H)dH$, we can rewrite (1_ε) as

$$P'(H)dH - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \varepsilon^3\omega_3 - \cdots = 0, \tag{3_ε}$$

where $\omega_i = A_i(x, y)dx + B_i(x, y)dy$, for $i = 1, 2, \ldots$, is a polynomial 1-form.

We are assuming that $\deg(P) \geq 2$, then $P'(H)$ is not a constant; this implies that (3_0) is a *degenerated Hamiltonian differential equation*. Thus, we can say that (3_ε) is a *perturbed degenerated Hamiltonian differential equation*.

We note that $H^{-1}(Z(P'))$ is the set of points (x, y) where $P'(H)(x, y) = 0$. Hence, in $\mathbb{R}^2 \setminus H^{-1}(Z(P'))$ the perturbed equation (3_ε) is equivalent to

$$dH - \varepsilon\frac{\omega_1}{P'(H)} - \varepsilon^2\frac{\omega_2}{P'(H)} - \varepsilon^3\frac{\omega_3}{P'(H)} - \cdots = 0. \tag{4_ε}$$

If $\{\gamma_c\}$ is a family of cycles of (3_0), then it does not intersect $H^{-1}(Z(P'))$. Hence, as in the complement of $Z(P')$ the polynomial P is a local invertible map, we can reparametrize $\{\gamma_c\}$ by using $z = H|_{\{\gamma_c\}}$. We then have a family $\{\gamma_z\}$; moreover, γ_c and γ_z are the same cycle if $c = P(z)$. Therefore, we can write the displacement function associated with (3_ε) in terms of ε and z instead of ε and c.

The main result of this paper is the next theorem, which proves that under generic conditions on H the first non-vanishing PPM function $L_k(z)$ with $k \geq 2$ of the displacement function of (3_ε) is an Abelian integral, and we provide the algorithm to compute it.

Theorem 1. *Assume that H is transversal to infinity whose complexification is a Morse polynomial. If $L_k(z)$ with $k \geq 2$ is the first non-vanishing PPM function of the displacement function associated with (3_ε), then there are polynomials $q_1, \ldots, q_{k-1}$, $Q_1, \ldots, Q_{k-1}$, $\overline{q}$, and $\overline{Q}$ with $\overline{q} \equiv \overline{Q} \equiv 0$ if $k = 2$ such that*

$$L_k(z) = \int_{\gamma_z} \frac{\overline{\omega}_{k1} + \overline{\omega}_{k2} + \overline{\omega}_{k3}}{(P'(H))^{2k-3}},$$

where

$$\overline{\omega}_{k1} = (P'(H))^{2k-3}\omega_k,$$

$$\overline{\omega}_{k2} = \sum_{l=1}^{k-2}(P'(H))^{2(k-l)-3}\left[P'(H)q_l + (2l-1)P''(H)Q_l\right]\omega_{k-l},$$

and

$$\overline{\omega}_{k3} = \left(P'(H)\overline{q} + 2(k-2)P''(H)\overline{Q} + R_{k-1}(P'(H))^{2k-3}q_1\frac{(-Q_1)^{k-2}}{(k-2)!}\right)\omega_1$$

with $R_1 = 1/P'$ and $R_i = R_1R'_{i-1}$ for $i \geq 2$.

We observe that if the degree of P is one, then we can assume without loss of generality that $P' = 1$, which implies that $R_i = 0$ for $i \geq 2$. In this case Theorem 1 reduces to the formula given by Iliev [11, Lemma 2] for $F = (x^2 + y^2)/2$; therefore, Iliev's formula and Theorem 1 generalize the ideas of [5, 22].

As an application of Theorem 1 we study the maximum number of zeros of the first non-vanishing PPM function of (3_ε) when $H = (x^2 + y^2)/2$. Note that in such a case $dH = 0$ is the *linear center*, the simplest planar Hamiltonian equation having a family of cycles.

Theorem 2. *Let* $n = \sup\{\deg \omega_i \,|\, i \geq 1\}$, $p + 1 = \deg P$, *and* $H = (x^2 + y^2)/2$. *If* $L_k(c)$ *is the first non-vanishing PPM function of the displacement function associated with* (3_ε), *then an upper bound for the maximum number of isolated zeros in* $(0, \infty) \setminus Z(P')$, *counting multiplicities, of* $L_k(c)$ *is:*

$$Z_k(n,p) = \begin{cases} 0, & \text{if } n = 1; \\ \left[\frac{n-1}{2}\right], & \text{if } k = 1; \\ p(2k-3) + \left[\frac{n-1}{2}\right], & \text{if } k \geq 2 \text{ and } 2 \leq n \leq 2p+1; \\ p(k-2) + \left[\frac{k(n-1)}{2}\right], & \text{otherwise.} \end{cases}$$

In addition, the following statements hold.

(*a*) *For each* $k \in \{1, 2\}$, *each* $p \geq 1$, *and each* $n \geq 2$ *there exists a polynomial* $P(z)$ *of degree* $p+1$ *and a polynomial perturbation of degree* n *of* (3_0) *such that* $L_k(z)$ *has exactly* $Z_k(n,p)$ *simple zeros in* $(0, \infty) \setminus Z(P')$.

(*b*) *For* $k = 3$, $p \in \{1, 2\}$, $n \geq 2$, *there exists a polynomial* $P(z)$ *of degree* $p+1$ *and a polynomial perturbation of degree* n *of* (3_0) *such that* $L_3(z)$ *has exactly* $Z_3(n,p)$ *simple zeros in* $(0, \infty) \setminus Z(P')$.

Next tables give the numerical values of $Z_k(n,p)$ for $k = 1, 2, 3$ and $p = 1, 2$.

$k \backslash n$	1	2	3	4	5	$\cdots$	$2l$	$2l+1$	$\cdots$
1	0	0	1	1	2	$\cdots$	$l-1$	l	$\cdots$
2	0	1	2	3	4	$\cdots$	$2l-1$	$2l$	$\cdots$
3	0	3	4	5	7	$\cdots$	$3l-1$	$3l+1$	$\cdots$

TABLE 1. Number of zeros of $L_k(z)$ for the case $p = 1$

$k \backslash n$	1	2	3	4	5	6	7	$\cdots$	$2l$	$2l+1$	$\cdots$
1	0	0	1	1	2	2	3	$\cdots$	$l-1$	l	$\cdots$
2	0	2	3	3	4	5	6	$\cdots$	$2l-1$	$2l$	$\cdots$
3	0	6	7	7	8	9	11	$\cdots$	$3l$	$3l+2$	$\cdots$

TABLE 2. Number of zeros of $L_k(z)$ for the case $p = 2$

Theorem 2 improves the main results of [2] where the case $P(z) = z^{p+1}$ was considered; moreover, the theorem shows that when $P(z)$ is of degree greater than one $L_k(z)$ with $k \geq 2$ has more zeros than when the case $P(z)$ of degree one is considered, which was studied by Iliev in [11]. Thus, we believe that Theorem 1 can be applied to improve lower bounds for the number of limit cycles of polynomial differential equations by considering other polynomials H.

In this paper we will study (3_ε) in both real and complex planes, and it is organized as follows. The background and definitions in the real and complex planes are delivered in Section 2. In Section 3 we will study (3_ε) in the complex setting and we will derive the relationship between the PPM functions of the displacement functions associated with (3_ε) and (4_ε), respectively. The proof of Theorem 1 will be given in Section 4. In Section 5 we will study (3_ε) with $H = (x^2 + y^2)/2$ on the complex plane, and finally in Section 6 we will give the proof of Theorem 2.

2. Background and definitions

2.1. **Background in the real case.** The description of the displacement function associated with (1_ε) in the real case is as follows.

Let $\{\gamma_c\}$ be a family of cycles of (1_0) with c varying over an open interval $D = (a, b) \subset \mathbb{R}$. Let Σ be a transversal section to $\{\gamma_c\}$. If Σ' is compact subset of Σ, then there is $\varepsilon_0 = \varepsilon_0(\Sigma') > 0$ such that for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the subset Σ' remains transversal to the foliation defined by (1_ε). That implies that the orbit through a point $\sigma' \in \Sigma'$, after making one round, intersects Σ at a point σ. Let $\gamma_{(\varepsilon,\sigma')}$ be the piece of the orbit joining (ε, σ') and (ε, σ). The transversal Σ can be parametrized by $c = F|_\Sigma$; in other words we identify Σ with D, and Σ' has an identification with a subset D' of D. Hence we can take $c = F(\sigma')$ and $\mathcal{P}_\varepsilon(c) = F(\sigma)$; furthermore, we can write $\gamma_{(\varepsilon,c)}$ instead $\gamma_{(\varepsilon,\sigma')}$. Therefore, the *first return map* is defined by $(\varepsilon, c) \mapsto \mathcal{P}_\varepsilon(c)$, and the *displacement function* associated with (1_ε) and $\{\gamma_c\}$ is

$$L : (-\varepsilon_0, \varepsilon_0) \times D' \to \mathbb{R}, \quad (\varepsilon, c) \mapsto \int_{\gamma_{(\varepsilon,c)}} dF = \mathcal{P}_\varepsilon(c) - c,$$

which is analytic and can be expressed as the power series in ε given in (2).

We know [15, Theorem 26.3] that the first order PPM function in (2) is given by the Abelian integral

$$L_1(c) = \int_{\gamma_c} \omega_1, \tag{5}$$

which can be treated and understood in a better way if we consider it on the complex plane $\mathbb{C}^2$ since its analytic continuation depends on the singular values and on the monodromy of the complexification of F: by considering x and y as complex variables.

On the other hand, although the study of limit cycles of planar differential equations was originally stated on $\mathbb{R}^2$, one can consider it on $\mathbb{C}^2$ [7–10, 12, 14, 16, 21] because a real (limit) cycle of (1_ε) is a complex (limit) cycle of its complexification [13, p. 340]. Moreover, the first return map and the displacement function have a natural complexification as we will show in next subsection.

2.2. **Background in the complex case.** From now on we will study (1_ε) on $\mathbb{C}^2$, that is, we will consider F as a polynomial in the ring $\mathbb{C}[x, y]$ of complex polynomials in two complex variables with coefficients in $\mathbb{C}$, ε as a small complex parameter, and ω as a complex 1-form, i.e. we think $A(x, y, \varepsilon)$ and $B(x, y, \varepsilon)$ as complex polynomials in x and y with coefficients that depend analytically on $\varepsilon \in (\mathbb{C}, 0)$.

For studying the (complex) limit cycles of (1_ε) on the complex plane that bifurcate from (1_0) we have to recall some definitions and some properties of the elements that define the equation (1_ε). For that, we will divide this subsection into three parts. In the first one, we will provide the notion of cycle and limit cycle in the complex setting, as well as the definition of the holonomy map, which is the complexification of the first return map. The second part is devoted to recalling some properties of complex polynomials in order to know the structure of the foliation defined by the Hamiltonian equation (1_0), and in the third one we will give some definitions related to complex polynomial 1-forms, which link their analytical or algebraic properties with respect to the Hamiltonian equation (1_0).

2.2.1. *Complex cycle, complex limit cycle, and the holonomy map.* We know that for each $\varepsilon \in (\mathbb{C}, 0)$ the differential equation (1_ε) defines a 1-dimensional complex foliation with singularities $\mathcal{F}_\varepsilon$ on $\mathbb{C}^2$: $\mathcal{F}_\varepsilon$ is a foliation by Riemann surfaces with singularities on $\mathbb{C}^2$.

Definition 1. *Consider a leaf Γ of $\mathcal{F}_\varepsilon$. Let $\gamma \subset \Gamma$ be a real curve homeomorphic to the unit circle $\mathbb{S}^1$, and let $[\gamma]$ be its free homotopy class on Γ. If γ is not homotopic to a point on Γ, then $[\gamma]$ is called a* ***complex cycle*** *of (1_ε).*

To give the definition of complex limit cycle, we need to recall the construction of the holonomy map, which is as follows. We consider a complex cycle $[\gamma]$ of (1_{ε_0}). Let U be an annular neighborhood of γ in Γ, and let V be a tubular neighborhood of U in $(\mathbb{C}, \varepsilon_0) \times \mathbb{C}^2$: $V \subset (\mathbb{C}, \varepsilon_0) \times \mathbb{C}^2$ is an open set containing U, and there is a retraction $\pi : V \to U$ and a biholomorphism $\psi : V \to U \times \mathbb{D}^2$, where $\mathbb{D} \subset \mathbb{C}$ is the unit disc, such that the diagram

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\pi"'] & U \times \mathbb{D}^2 \arrow[dl, "P_1"] \\ U & \end{tikzcd}$$

commutes, where $P_1 : U \times \mathbb{D}^2 \to U$ is the projection on the first factor.

Let p_0 be a point of γ and consider a parametrization $\gamma : [0, 1] \to \Gamma$ of γ such that $p_0 = \gamma(0) = \gamma(1)$ (we are identifying $\gamma(t)$ with its image γ). Because of the commutativity of the previous diagram, the set $L := \pi^{-1}(p_0) = \psi^{-1}\left(\{p_0\} \times \mathbb{D}^2\right)$ is a 2-dimensional transversal section to $\{\mathcal{F}_\varepsilon\}$, for $\varepsilon \in (\mathbb{C}, \varepsilon_0)$, at p_0. Thus, for each point $(\varepsilon, p) \in L$ close to (ε_0, p_0), the curve γ may be lifted to a unique curve $\gamma_{(\varepsilon,p)} : [0, 1] \to \mathbb{C}^2$ that lies on the leaf $\Gamma_{(\varepsilon,p)}$ passing through (ε, p) and that covers γ under the retraction π. In other words, $\pi(\gamma_{(\varepsilon,p)}(t)) = \gamma(t)$ for all $t \in [0, 1]$, which implies that $\gamma_{(\varepsilon,p)}(1) \in L$. See [3, Ch. IV] for details. Hence, there is an open subset L' of L such that the map $f_\gamma : L' \to L$, $(\varepsilon, p) \mapsto \gamma_{(\varepsilon,p)}(1)$ is well-defined, and it is analytic because of the analytic dependence of the leaves of $\{\mathcal{F}_\varepsilon\}$ on initial conditions.

We can identify the transversal L with $(\mathbb{C}, \varepsilon_0) \times \mathbb{D}$ by using the parametrization

$$G : (\mathbb{C}, \varepsilon_0) \times \mathbb{D} \to L, \quad (\varepsilon, c) \mapsto \psi^{-1}(p_0, \varepsilon, c),$$

with $G(\varepsilon_0, 0) = (\varepsilon_0, p_0)$. Thus, the subset L' is identified with $(\mathbb{C}, \varepsilon_0) \times D'$, where D' is an open subset of $\mathbb{D}$. Now, since $\gamma_{(\varepsilon,p)}(1) = (\varepsilon, \hat{p}) \in L$, we can identify (ε, p) and $(\varepsilon, \hat{p})$ with (ε, c) and $(\varepsilon, \Delta_\gamma(\varepsilon, c))$ for some $c \in D'$ and $\Delta_\gamma(\varepsilon, c) \in \mathbb{D}$, respectively. The *holonomy map associated to* γ is then defined by the analytic map

$$\Delta_\gamma : (\mathbb{C}, \varepsilon_0) \times D' \to D, \quad (\varepsilon, c) \mapsto \Delta_\gamma(\varepsilon, c).$$

Remark 1. *The lifted curve $\gamma_{(\varepsilon,c)}$ contained in $\Gamma_{(\varepsilon,c)}$ joins the initial point (ε, c) with the end point $(\varepsilon, \Delta_\gamma(\varepsilon, c))$, and from the continuity of the solutions of (1_ε) on initial conditions, it follows that $\lim_{\varepsilon \to \varepsilon_0} \gamma_{(\varepsilon,c)} = \gamma_{(\varepsilon_0,c)}$.*

Remark 2. *The map f_γ does not depend, up to analytic conjugation, on the representative of $[\gamma]$, the parametrization of the representative, the point p_0, or the transversal section L (i.e. the retraction π and the biholomorphism ψ). This property implies that if γ' is another representative of $[\gamma]$ and $\Delta_{\gamma'}(\delta, c)$ is the corresponding holonomy map, then $\Delta_\gamma(\delta, c)$ and $\Delta_{\gamma'}(\delta, c)$ are analytically conjugated.*

Definition 2. *A complex cycle* $[\gamma]$ *of* (1_{ε_0}) *is a* ***complex limit cycle*** *if a holonomy map* $\Delta_\gamma(\varepsilon, c)$*, restricted to* $\varepsilon = \varepsilon_0$*, has an isolated fixed point at* $c = 0$*.*

Remark 3. *Any real (limit) cycle* γ *of the real differential equation* (1_{ε_0}) *defines a complex (limit) cycle* $[\gamma]$ *of the complexification of* (1_{ε_0})*, and the holonomy map* $\Delta_\gamma(\varepsilon, c)$ *is the complexification of the first return map associated with* γ*.*

2.2.2. *Properties of complex polynomials.* It is well-known that for each $F \in \mathbb{C}[x, y]$ there is a finite set $\Sigma_F \subset \mathbb{C}$ such that

$$F : \mathbb{C}^2 \setminus F^{-1}(\Sigma_F) \to \mathbb{C} \setminus \Sigma_F \tag{6}$$

is a locally trivial smooth fibration. See for instance [1]. This finite set Σ_F is the *set of singular values of* F, which is composed of the values in $\mathbb{C}$ coming from the finite singular points of F and of the "singular points at infinity" of F [4]. Every value $c \in \mathbb{C} \setminus \Sigma_F$ is a *generic value* of F and the corresponding fiber

$$F^{-1}(c) := \{(x, y) \in \mathbb{C}^2 \mid F(x, y) - c = 0\} \subset \mathbb{C}^2,$$

which is an affine nonsingular algebraic curve, is a *generic fiber* of F.

A complex polynomial is *primitive* if its generic fiber is irreducible; in such a case, the generic fiber is diffeomorphic to a compact Riemann surface of genus $g \geq 0$ punctured at $h \geq 1$ different points. These are the *points at infinity* of the fiber, and the polynomial is called *type* (g, h) [20].

We know that for each $F \in \mathbb{C}[x, y]$ there are a primitive bivariate polynomial $H \in \mathbb{C}[x, y]$ and a univariate polynomial $P \in \mathbb{C}[z]$, such that $F = P(H)$. See for example [6]. Note that if $F = P(H)$ and P is of degree one, then F is primitive. Hence, if F is not primitive, then its generic fiber is diffeomorphic to the finite disjoint union of at least two punctured compact Riemann surfaces.

A polynomial $F \in \mathbb{C}[x, y]$ of degree $n + 1 \geq 2$ is *transversal to infinity*, if its homogeneous part of degree $n + 1$ factors out as the product of $n + 1$ pairwise different linear forms; furthermore, F is a *Morse polynomial*, if it has n^2 singular points and n^2 singular values.

The *Hamiltonian foliation* $\mathcal{F}_0 = \{dF = 0\}$ defined by (1_0) is given by the fibers of F, thus, $\mathcal{F}_0$ defines a foliation by punctured compact Riemann surfaces of finite genus with singularities on $\mathbb{C}^2$.

2.2.3. *Properties of complex polynomial 1-foms.* Here we will recall two properties that a complex polynomial 1-form ω can have with respect to the foliation $\mathcal{F}_0$. Such properties will be useful in the study of the higher order PPM functions of the displacement function associated with (1_ε).

Definition 3. *A 1-form* ω *is* ***analytically relatively exact*** *with respect to the Hamiltonian foliation* $\mathcal{F}_0$ *if for each value* $c \in \mathbb{C}$ *and each homological cycle* $\delta \in H_1(F^{-1}(c), \mathbb{Z})$ *the integral of* ω *along* δ *is zero:*

$$\int_\delta \omega = 0.$$

Definition 4. *A 1-form* ω *is* ***algebraically relatively exact*** *with respect to the Hamiltonian foliation* $\mathcal{F}_0$*, if there are polynomials* Q*,* $q \in \mathbb{C}[x, y]$ *such that*

$$\omega = dQ + qdF.$$

Clearly each polynomial 1-form that is algebraically relatively exact is also analytically relatively exact with respect to $\mathcal{F}_0$. The inverse connection between these two properties is given by the following result.

Theorem 3. (Ilyashenko [12], Gavrilov [7].) *Suppose that $F \in \mathbb{C}[x,y]$ has isolated critical points in $\mathbb{C}^2$ and that $F^{-1}(c)$ is connected for each $c \in \mathbb{C}$. A complex polynomial 1-form ω is then algebraically relatively exact with respect to $\mathcal{F}_0$ if and only if it is analytically relatively exact with respect to it.*

3. Perturbed degenerated Hamiltonian equations

We suppose that the polynomial $F(x,y)$ defining the Hamiltonian system (1_0) is non-primitive; hence $F = P(H)$ for a univariate polynomial $P(z)$ of degree $r \geq 2$ and a bivariate primitive polynomial $H(x,y)$. Thus, we have the perturbed degenerated Hamiltonian differential equation (3_ε).

3.1. The complex displacement function.

Let c_0 be a generic value of F, and consider a fixed complex cycle $[\gamma_{c_0}]$ of (1_0). We take $\gamma := \gamma_{c_0}$ as a representative loop of $[\gamma_{c_0}]$. We can transport this loop continuously into the neighboring fibers according to the fibration (6). Let $\{\gamma_c\}$ be the family of resulting loops. The transportation depends on the representative loop, but the free homotopy classes $\{[\gamma_c]\}$ of the obtained loops are well-defined. The parameter c varies in a small enough neighborhood $D(c_0)$ of c_0 contained in the set of generic values of F. Following the same idea as in Subsection 2.1, and abusing of the notation, we denote by Σ a transversal section to $\{\gamma_c\}$ which can be identified with $D(c_0)$ by parametrizing Σ with $c = F|_\Sigma$. If $D'(c_0)$ is a compact subset of $D(c_0)$, then there exists $\rho_0 = \rho_0(D'(c_0)) > 0$ such that for each $|\varepsilon| < \rho_0$ the corresponding subset Σ' in Σ remains transversal to the foliation defined by (1_ε). Thus, from Subsection 2.2 we get the well-defined analytic holonomy map $\Delta_\gamma(\varepsilon, c)$ which is defined in $\{(\varepsilon, c) \,|\, |\varepsilon| < \rho_0, c \in D'(c_0)\}$.

The *complex displacement function* associated with (1_ε) and γ is defined as

$$L_{F,\gamma}(\varepsilon, c) = \int_{\gamma_{(\varepsilon,c)}} dF = \Delta_\gamma(\varepsilon, c) - c.$$

Since $F = P(H)$, there are $r \geq 2$ different generic values $z_1, \ldots, z_r$ of H such that the generic fiber $F^{-1}(c_0)$ of F is the disjoint union of the r generic fibers $\mathcal{L}_{z_1}, \ldots, \mathcal{L}_{z_r}$ of H. $\mathcal{L}_{z_i}$, for each $i = 1, \ldots, r$, is diffeomorphic to a finite punctured compact Riemann surface of finite genus. Hence, the loop γ defining the fixed cycle $[\gamma_{c_0}]$ is contained in one of the fibers $\mathcal{L}_{z_1}, \ldots, \mathcal{L}_{z_r}$ of H. Without loss of generality we can suppose that $\gamma \subset \mathcal{L}_{z_1}$; thus, every transported loop γ_c is contained in a generic fiber $\mathcal{L}_z$ of H, where z varies in the neighborhood $D(z_1) := P^{-1}(D(c_0))$ of z_1 which is contained in the set of generic values of H. We can then reparametrize the family $\{\gamma_c\}$ by using z, as a result we have the family $\{\gamma_z\}$ with $z \in D(z_1)$. Therefore, analogous to the previous construction, the complex displacement function associated with (4_ε) and γ is defined as

$$L_{H,\gamma}(\varepsilon, z) = \int_{\gamma_{(\varepsilon,z)}} dH = \Delta_\gamma(\varepsilon, z) - z.$$

3.2. Poincaré–Pontryagin–Melnikov functions.

By construction, the displacement function $L_{F,\gamma}(\varepsilon, c)$ is analytic, and it admits a power series

$$L_{F,\gamma}(\varepsilon, c) = \varepsilon L_1(c) + \varepsilon^2 L_2(c) + \cdots . \tag{7}$$

In addition, since we have the parametrization $P : D(z_1) \to D(c_0)$, we can write $L_{F,\gamma}(\varepsilon, c)$ in terms of (ε, z); thus we have

$$L_{F,\gamma}(\varepsilon, z) := L_{F,\gamma}(\varepsilon, P(z)) = \varepsilon L_1(z) + \varepsilon^2 L_2(z) + \cdots . \tag{8}$$

Analogously,

$$L_{H,\gamma}(\varepsilon, z) = \varepsilon l_1(z) + \varepsilon^2 l_2(z) + \cdots . \tag{9}$$

Of course, if $k \geq 1$ and $L_k(z)$ vanishes identically on $D(z_1)$, then $L_k(c)$ vanishes identically on $D(c_0)$. In addition, if $L_k(z)$ is the first non-vanishing coefficient in (8), then its maximum number of zeros in $D(z_1)$ coincides with the maximum number of zeros in $D(c_0)$ of the first non-vanishing coefficient $L_k(c)$ in (7). Furthermore, we know that the first non-vanishing PPM function of $L_{F,\gamma}(\varepsilon, z)$ depends only on the free homotopy class $[\gamma_z]$ of γ_z. See [10].

Now we will derive the relationship between the PPM functions $L_k(z)$ and $l_k(z)$ of $L_{F,\gamma}(\varepsilon, z)$ and $L_{H,\gamma}(\varepsilon, z)$, respectively.

From the definition of $L_{F,\gamma}(\varepsilon, c)$ it follows that

$$\begin{aligned} L_{F,\gamma}(\varepsilon, z) &= F(\gamma_{(\varepsilon,P(z))}(1)) - F(\gamma_{(\varepsilon,P(z))}(0)) \\ &= P(H(\gamma_{(\varepsilon,P(z))}(1))) - P(H(\gamma_{(\varepsilon,P(z))}(0))) \\ &= P'(\psi(\varepsilon, z))(H(\gamma_{(\varepsilon,z)}(1)) - H(\gamma_{(\varepsilon,z)}(0))) \\ &= P'(\psi(\varepsilon, z))(L_{H,\gamma}(\varepsilon, z)) \end{aligned}$$

for an analytic function $\psi(\varepsilon, z)$ which has the form

$$\psi(\varepsilon, z) = z + \varepsilon\psi_1(z) + \varepsilon^2\psi_2(z) + \cdots .$$

Hence $P'(\psi(\varepsilon, z))$ can be written as

$$P'(\psi(\varepsilon, z)) = P'(z) + \varepsilon\widetilde{\psi}_1(z) + \varepsilon^2\widetilde{\psi}_2(z) + \cdots ,$$

whereby we obtain

$$L_{F,\gamma}(\varepsilon, z) = \varepsilon P'(z)l_1(z) + \varepsilon^2 \left(P'(z)l_2(z) + l_1(z)\widetilde{\psi}_1(z)\right) + \cdots .$$

Thus, if $L_k(z)$ is the first non-vanishing PPM function of $L_{F,\gamma}(\varepsilon, z)$ then

$$L_k(z) = P'(z)l_k(z), \tag{10}$$

with $l_k(z)$ being the first PPM function of $L_{H,\gamma}(\varepsilon, z)$ that does not vanish identically. Hence, for determining the form of $L_k(z)$ in terms of the perturbation of (3_0) we will obtain a recursive formula for $l_k(z)$.

The Poincaré–Pontryagin formula says that the first coefficient in (9) is given by the integral of $\omega_1/P'(H)$ along γ_z, and as $P'(H)$ is constant on γ_z it follows that

$$l_1(z) = \frac{1}{P'(z)} \int_{[\gamma_z]} \omega_1. \tag{11}$$

To compute the first coefficient $l_k(z)$ in (9) that does not vanish identically with $k \geq 2$, we construct inductively a sequence of polynomial 1-forms by assuming that

- the polynomial H has only isolated critical points in $\mathbb{C}^2$ and that the fiber $\mathcal{L}_z$ of H is connected for each $z \in \mathbb{C}$.

The construction of the sequence is as follows.

- Let $\widetilde{\omega}_1 = \omega_1$.
- If the 1-forms $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_{k-1}$ are already constructed and are analytically relatively exact with respect to the Hamiltonian foliation $\{dH = 0\}$, then from the Ilyashenko–Gavrilov theorem (Theorem 3) they are algebraically relatively exact with respect to $\{dH = 0\}$: for $i = 1, 2, \ldots, k-1$ there exist polynomials $Q_i, q_i \in \mathbb{C}[x, y]$ such that $\widetilde{\omega}_i = dQ_i + q_i dH$.
- We then define

$$\widetilde{\omega}_k := (P')^{2k-2}\omega_k + \sum_{i=1}^{k-1}(P')^{2(k-i-1)} \left[P'q_i + (2i-1)P''Q_i\right] \omega_{k-i}, \tag{12}$$

  where $P' = P'(H)$ and $P'' = P''(H)$.

The following two propositions will allow us to obtain the formula for $l_k(z)$, and as a result the expression of $L_k(z)$.

Proposition 1. *If $\widetilde{\omega}_k$ is the first non analytically relatively exact 1-form in the sequence $\widetilde{\omega}_1,\dots,\widetilde{\omega}_{k-1},\widetilde{\omega}_k$ constructed inductively by* (12)*, then $l_1(z),\dots,l_{k-1}(z)$ vanish identically in $D(z_1)$, and*

$$l_k(z)=\int_{[\gamma_z]}\frac{\widetilde{\omega}_k}{(P')^{2k-1}}.$$

Proposition 2. *Suppose $k\geq 2$. If $\widetilde{\omega}_k$ is the first non analytically relatively exact 1-form in the sequence $\widetilde{\omega}_1,\dots,\widetilde{\omega}_{k-1},\widetilde{\omega}_k$ constructed inductively by* (12)*, then there are polynomials $\overline{q}$ and $\overline{Q}$ with $\overline{q}=\overline{Q}\equiv 0$ if $k=2$ such that*

$$\frac{\widetilde{\omega}_k}{(P')^{2k-1}}=\frac{\overline{\omega}_{k1}+\overline{\omega}_{k2}+\overline{\omega}_{k3}}{(P')^{2k-2}}+\frac{R_k(-Q_1)^{k-1}}{(k-1)!}\,\omega_1$$

where

$$\overline{\omega}_{k1}=(P')^{2k-3}\omega_k,\quad \overline{\omega}_{k2}=\sum_{l=1}^{k-2}(P')^{2(k-l)-3}\left[P'q_l+(2l-1)P''Q_l\right]\omega_{k-l},\tag{13}$$

and

$$\overline{\omega}_{k3}=\left(P'\overline{q}+2(k-2)P''\overline{Q}+\frac{R_{k-1}(P')^{2k-3}q_1(-Q_1)^{k-2}}{(k-2)!}\right)\omega_1\tag{14}$$

with $R_1=1/P'$ and $R_i=R_1R_{i-1}'$ for $i\geq 2$.

The proof of the previous propositions and the next lemma will be given at the end of the next section.

Lemma 1. *Let $P(z)$ be a polynomial of degree $p+1$. If we define $R_1(z)=1/P'(z)$ and $R_i(z)=R_1(z)R_{i-1}'(z)$ for $i\geq 2$, then $(P'(z))^{2i-1}R_i(z)$ is a polynomial of degree $(i-1)(p-1)$.*

4. Proof of Theorem 1

We will prove Theorem 1 by assuming Propositions 1 and 2.

Proof of Theorem 1. By assumption H is a real polynomial transversal to infinity and its complexification is a Morse polynomial, then from [15, Theorem 26.52] it follows that we can consider the sequence $\widetilde{\omega}_1,\dots,\widetilde{\omega}_k$ according to (12). Moreover, the polynomials $q_1,\dots,q_{k-1}$ and $Q_1,\dots,Q_{k-1}$ involved in the construction of the sequence are real polynomials because H and ω_i with $i\geq 1$ are real objects. Analogously, the polynomials $\overline{q}$ and $\overline{Q}$ in the definition of $\overline{\omega}_{k3}$ (as in (14)) are real.

From (10) we know that $L_k(z)=P'(z)l_k(z)$, and from Proposition 1 we have

$$l_k(z)=\int_{\gamma_z}\frac{\widetilde{\omega}_k}{(P')^{2k-1}}.$$

By assuming Proposition 2 it follows that

$$l_k(z)=\int_{\gamma_z}\frac{\overline{\omega}_{k1}+\overline{\omega}_{k2}+\overline{\omega}_{k3}}{(P')^{2k-2}}+\int_{\gamma_z}R_k\frac{(-Q_1)^{k-1}}{(k-1)!}\omega_1,$$

and a simple computation shows that

$$R_k\frac{(-Q_1)^{k-1}}{(k-1)!}\omega_1=d\widehat{Q}_k+(\widehat{q}_{k1}+\widehat{q}_{k2})\,dH,\tag{15}$$

where

$$\widehat{Q}_k=-R_k\frac{(-Q_1)^k}{(k)!},\quad \widehat{q}_{k1}=R_kq_1\frac{(-Q_1)^{k-1}}{(k-1)!},\quad\text{and}\quad \widehat{q}_{k2}=-R_k'\frac{(-Q_1)^k}{(k)!}.\tag{16}$$

As a result,

$$\int_{\gamma_z}R_k\frac{(-Q_1)^{k-1}}{(k-1)!}\omega_1\equiv 0.\tag{17}$$

Therefore,

$$L_k(z) = P'(z)\int_{\gamma_z} \frac{\overline{\omega}_{k1}+\overline{\omega}_{k2}+\overline{\omega}_{k3}}{(P')^{2k-2}} = \int_{\gamma_z} \frac{\overline{\omega}_{k1}+\overline{\omega}_{k2}+\overline{\omega}_{k3}}{(P')^{2k-3}},$$

where $\overline{\omega}_{k1}$, $\overline{\omega}_{k2}$, and $\overline{\omega}_{k3}$ are as in (13) and (14), which are precisely the expressions given in the statement of the theorem. $\square$

Proof of Proposition 1. The assertion is true for $k = 1$ because of the Poincaré–Pontryagin formula (11). To complete the proof we will proceed by induction on k. We assume that the theorem is true for $k-1$, and we will prove it for k.

By hypothesis, $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_{k-1}$ are analytically relatively exact 1-forms with respect to the foliation $\{dH = 0\}$. Hence by the Ilyashenko–Gavrilov theorem there are polynomials $Q_i, q_i \in \mathbb{C}[x, y]$ such that $\widetilde{\omega}_i = dQ_i + q_i dH$ for $i = 1, 2, \ldots, k-1$.

A straightforward computation shows that

$$\frac{\widetilde{\omega}_i}{(P')^{2i-1}} = d\widetilde{Q}_i + \widetilde{q}_i dH, \quad \widetilde{Q}_i = \frac{Q_i}{(P')^{2i-1}}, \quad \widetilde{q}_i = \frac{q_i}{(P')^{2i-1}} + \frac{(2i-1)P''Q_i}{(P')^{2i}}. \tag{18}$$

Following [5], we multiply (4_ε) by $\big(1 + \varepsilon\widetilde{q}_1 + \cdots + \varepsilon^{k-1}\widetilde{q}_{k-1}\big)$. So, we have

$$0 = \big(1 + \varepsilon\widetilde{q}_1 + \cdots + \varepsilon^{k-1}\widetilde{q}_{k-1}\big)\left(dH - \varepsilon\frac{\omega_1}{P'} - \varepsilon^2\frac{\omega_2}{P'} - \cdots\right). \tag{19}$$

By expanding and reordering terms we get

$$\begin{aligned} 0 = dH &- \varepsilon\left(\frac{\omega_1}{P'} - \widetilde{q}_1 dH\right) - \varepsilon^2\left(\frac{\omega_2}{P'} + \frac{\widetilde{q}_1\omega_1}{P'} - \widetilde{q}_2 dH\right) \\ &- \cdots - \varepsilon^{k-1}\left(\frac{\omega_{k-1}}{P'} + \sum_{l=1}^{k-2}\frac{\widetilde{q}_l\omega_{k-1-l}}{P'} - \widetilde{q}_{k-1}dH\right) - \varepsilon^k\left(\frac{\omega_k}{P'} + \sum_{l=1}^{k-1}\frac{\widetilde{q}_l\omega_{k-l}}{P'}\right) - \cdots. \end{aligned}$$

Now by using the expression of $\widetilde{q}_i$ given in (18) we obtain

$$\begin{aligned} \frac{\omega_i}{P'} + \sum_{l=1}^{i-1}\frac{\widetilde{q}_l\omega_{i-l}}{P'} &= \frac{\omega_i}{P'} + \sum_{l=1}^{i-1}\left(\frac{q_l}{(P')^{2l-1}} + \frac{(2l-1)P''Q_l}{(P')^{2l}}\right)\frac{\omega_{i-l}}{P'} \\ &= \frac{\omega_i}{P'} + \sum_{l=1}^{i-1}\left[P'q_l + (2l-1)P''Q_l\right]\frac{\omega_{i-l}}{(P')^{2l+1}} \\ &= \frac{\omega_i}{P'} + \frac{\sum_{l=1}^{i-1}(P')^{2(i-l-1)}\left[P'q_l + (2l-1)P''Q_l\right]\omega_{i-l}}{(P')^{2i-1}} \\ &= \frac{(P')^{2i-2}\omega_i + \sum_{l=1}^{i-1}(P')^{2(i-l-1)}\left[P'q_l + (2l-1)P''Q_l\right]\omega_{i-l}}{(P')^{2i-1}}. \end{aligned}$$

We note that the numerator of the right-hand side of the last expression is precisely the definition of $\widetilde{\omega}_i$. Clearly (19) can be written then as

$$0 = dH - \varepsilon\left(\frac{\widetilde{\omega}_1}{P'} - \widetilde{q}_1 dH\right) - \cdots - \varepsilon^{k-1}\left(\frac{\widetilde{\omega}_{k-1}}{(P')^{2k-3}} - \widetilde{q}_{k-1}dH\right) - \varepsilon^k\left(\frac{\widetilde{\omega}_k}{(P')^{2k-1}}\right) - \cdots$$

and as $d\widetilde{Q}_i = \widetilde{\omega}_i/(P')^{2i-1} - \widetilde{q}_i dH$ because of (18) we see that

$$0 = dH - \varepsilon\, d\widetilde{Q}_1 - \cdots - \varepsilon^{k-1}\, d\widetilde{Q}_{k-1} - \varepsilon^k\left(\frac{\widetilde{\omega}_k}{(P')^{2k-1}}\right) - \cdots.$$

By integrating along the curve solution $\gamma_{(\varepsilon,z)}$ of (4_ε) we get

$$\begin{aligned} 0 &= \int_{\gamma_{(\varepsilon,z)}} dH - \varepsilon^k \int_{\gamma_{(\varepsilon,z)}} \frac{\widetilde{\omega}_k}{(P')^{2k-1}} - \sum_{i=1}^{k-1} \varepsilon^i \int_{\gamma_{(\varepsilon,z)}} d\widetilde{Q}_i + O\left(\varepsilon^{k+1}\right) \\ &= L_{H,\gamma}(\varepsilon, z) - \varepsilon^k \int_{\gamma_{(\varepsilon,z)}} \frac{\widetilde{\omega}_k}{(P')^{2k-1}} - \sum_{i=1}^{k-1} \varepsilon^i \int_{\gamma_{(\varepsilon,z)}} d\widetilde{Q}_i - O\left(\varepsilon^{k+1}\right). \end{aligned}$$

In addition, $\int_{\gamma_{(\varepsilon,z)}} d\widetilde{Q}_i = \varepsilon^k \widehat{Q}_i(\varepsilon, z)$. Hence

$$\begin{aligned} 0 &= L_{H,\gamma}(\varepsilon, z) - \varepsilon^k \int_{\gamma_{(\varepsilon,z)}} \frac{\widetilde{\omega}_k}{(P')^{2k-1}} - O\left(\varepsilon^{k+1}\right) \\ &= \varepsilon^k \left(l_k(z) - \int_{\gamma_{(\varepsilon,z)}} \frac{\widetilde{\omega}_k}{(P')^{2k-1}} \right) - O\left(\varepsilon^{k+1}\right). \end{aligned}$$

Finally, by multiplying by $1/\varepsilon^k$, taking $\lim_{\varepsilon\to 0}$, and using Remark 1, the result follows because the integral depends only on the free homotopy class $[\gamma_z]$ of $\gamma_z = \gamma(0, z)$. □

Proof of Proposition 2. We proceed by induction on k. The base case is $k = 2$; in such a case, from (13) and (14) it follows that

$$\overline{\omega}_{21} = P'\omega_2, \quad \overline{\omega}_{22} = 0, \quad \text{and} \quad \overline{\omega}_{23} = R_1 P' q_1 \omega_1 = q_1 \omega_1$$

because $R_1 P' = (1/P')P' = 1$.

On the other hand, $\widetilde{\omega}_1 = dQ_1 + q_1 dH$ for some polynomials Q_1 and q_1. Then

$$\widetilde{\omega}_2 = (P')^2 \omega_2 + \left[P' q_1 + P'' Q_1\right] \omega_1 = P' \left(P'\omega_2 + q_1\omega_1\right) + P'' Q_1 \omega_1.$$

Thus, as $R_2 = R_1\, R_1' = (1/P')(1/P')' = -P''/(P')^3$ we obtain

$$\frac{\widetilde{\omega}_2}{(P')^3} = \frac{P'\omega_2 + q_1\omega_1}{(P')^2} + \frac{P'' Q_1 \omega_1}{(P')^3} = \frac{\overline{\omega}_{21} + \overline{\omega}_{22} + \overline{\omega}_{23}}{(P')^2} - R_2 Q_1 \omega_1.$$

Hence, the proposition is true for $k = 2$.

Now, we assume that the result is true for $k - 1$, and we will prove it for k.

From (12) we know that

$$\widetilde{\omega}_k = (P')^{2k-2} \omega_k + \sum_{l=1}^{k-1} (P')^{2(k-l-1)} \left[P' q_l + (2l-1) P'' Q_l\right] \omega_{k-l}.$$

By using (13) it is easy to see that we can rewrite $\widetilde{\omega}_k$ as

$$\widetilde{\omega}_k = P'\overline{\omega}_{k1} + P'\overline{\omega}_{k2} + (P' q_{k-1} + (2k-3) P'' Q_{k-1})\omega_1.$$

If we prove that

$$\left(P' q_{k-1} + (2k-3) P'' Q_{k-1}\right) \omega_1 = P' \overline{\omega}_{k3} + (P')^{2k-2} R_{k-1}' \frac{(-Q_1)^{k-1}}{(k-1)!} \omega_1, \tag{20}$$

then

$$\frac{\widetilde{\omega}_{k-1}}{(P')^{2k-1}} = \frac{\overline{\omega}_{k1} + \overline{\omega}_{k2} + \overline{\omega}_{k3}}{(P')^{2k-2}} + \frac{R_{k-1}'}{P'} \frac{(-Q_1)^{k-1}}{(k-1)!} \omega_1.$$

We then obtain

$$\frac{\widetilde{\omega}_{k-1}}{(P')^{2k-1}} = \frac{\overline{\omega}_{k1} + \overline{\omega}_{k2} + \overline{\omega}_{k3}}{(P')^{2k-2}} + R_k \frac{(-Q_1)^{k-1}}{(k-1)!} \omega_1,$$

which proves that the result is true for k. Therefore, to complete the proof we must show that (20) holds. Next we will prove it.

Since $\widetilde{\omega}_k$ is the first non analytically relatively exact 1-form with respect to the foliation $\{dH = 0\}$,

$$\int_{[\delta_z]} \widetilde{\omega}_{k-1} = 0 \qquad \text{for each cycle } [\delta_z] \in H_1(H^{-1}(z), \mathbb{Z}).$$

Moreover, from the induction hypothesis we have

$$\frac{\widetilde{\omega}_{k-1}}{(P')^{2k-3}} = \frac{\overline{\omega}_{(k-1)1} + \overline{\omega}_{(k-1)2} + \overline{\omega}_{(k-1)3}}{(P')^{2k-4}} + R_{k-1}\frac{(-Q_1)^{k-2}}{(k-2)!}\omega_1;$$

it then follows that

$$\int_{[\delta_z]} \widetilde{\omega}_{k-1} = P'(z) \int_{[\gamma_z]} \overline{\omega}_{(k-1)1} + \overline{\omega}_{(k-1)2} + \overline{\omega}_{(k-1)3}$$
$$+ (P'(z))^{2k-3} \int_{[\delta_z]} R_{k-1}\frac{(-Q_1)^{k-2}}{(k-2)!}\omega_1 = 0.$$

In (17) the second integral in the right-hand side of the previous equation was proven to vanish identically. Therefore,

$$\int_{[\delta_z]} \overline{\omega}_{(k-1)1} + \overline{\omega}_{(k-1)2} + \overline{\omega}_{(k-1)3} = 0 \quad \text{for each cycle } [\delta_z] \in H_1(H^{-1}(z), \mathbb{Z}),$$

in other words, $\overline{\omega}_{(k-1)1} + \overline{\omega}_{(k-1)2} + \overline{\omega}_{(k-1)3}$ is analytically relatively exact with respect to the foliation $\{dH = 0\}$. The Ilyashenko-Gavrilov Theorem then implies that

$$\overline{\omega}_{(k-1)1} + \overline{\omega}_{(k-1)2} + \overline{\omega}_{(k-1)3} = d\overline{Q} + \overline{q}dH,$$

whence

$$\frac{\overline{\omega}_{(k-1)1} + \overline{\omega}_{(k-1)2} + \overline{\omega}_{(k-1)3}}{(P')^{2k-4}} = d\left(\frac{\overline{Q}}{(P')^{2k-4}}\right) + \left(\frac{\overline{q}}{(P')^{2k-4}} + \frac{(2k-4)\overline{Q}P''}{(P')^{2k-3}}\right) dH,$$

and by using (15) and (16) we get

$$\frac{\widetilde{\omega}_{k-1}}{(P')^{2k-3}} = d\left(\frac{\overline{Q}}{(P')^{2k-4}} + \widehat{Q}_{k-1}\right) + \left(\frac{\overline{q}}{(P')^{2k-4}} + \frac{(2k-4)\overline{Q}P''}{(P')^{2k-3}} + \widehat{q}_{(k-1)1} + \widehat{q}_{(k-1)2}\right) dH.$$

On the other hand, from (18) we know that

$$\frac{\widetilde{\omega}_{k-1}}{(P')^{2k-3}} = d\left(\frac{Q_{k-1}}{(P')^{2k-3}}\right) + \left(\frac{q_{k-1}}{(P')^{2k-3}} + \frac{(2k-3)P''Q_{k-1}}{(P')^{2k-2}}\right) dH.$$

By comparing the right-hand sides of the two previous equations, we then obtain after a straightforward computation that

$$Q_{k-1} = P'\overline{Q} - R_{k-1}(P')^{2k-3}\frac{(-Q_1)^{k-1}}{(k-1)!} \tag{21}$$

and

$$q_{k-1} = P'\overline{q} - P''\overline{Q} + (P')^{2k-3}R_{k-1}q_1\frac{(-Q_1)^{k-2}}{(k-2)!}$$
$$+ (P')^{2k-3}\left(R'_{k-1} + (2k-3)R_{k-1}\frac{P''}{P'}\right)\frac{(-Q_1)^{k-1}}{(k-1)!}.$$

Moreover, as

$$(P')^{2k-3}\left(R'_{k-1} + (2k-3)R_{k-1}\frac{P''}{P'}\right) = \left((P')^{2k-3}R_{k-1}\right)',$$

then

$$q_{k-1} = P'\overline{q} - P''\overline{Q} + \frac{(P')^{2k-3}R_{k-1}q_1(-Q_1)^{k-2}}{(k-2)!} + \frac{\left((P')^{2k-3}R_{k-1}\right)'(-Q_1)^{k-1}}{(k-1)!}. \tag{22}$$

Therefore,

$$
\begin{aligned}
P'q_{k-1} + (2k-3)P''Q_{k-1} = P' &\left(P'\overline{q}_{k-1} - \overline{Q}_{k-1}P'' + \frac{(P')^{2k-3}R_{k-1}q_1(-Q_1)^{k-2}}{(k-2)!} \right) \\
&+ (P')^{2k-2}\left(R'_{k-1} + (2k-3)R_{k-1}\frac{P''}{P'} \right)\frac{(-Q_1)^{k-1}}{(k-1)!} \\
&+ (2k-3)P''\left(P'\overline{Q}_{k-1} - R_{k-1}(P')^{2k-3}\frac{(-Q_1)^{k-1}}{(k-1)!} \right),
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
P'q_{k-1} + (2k-3)P''Q_{k-1} = P' \Bigg(&P'\overline{q}_{k-1} + (2k-4)\overline{Q}_{k-1}P'' \\
&+ \frac{(P')^{2k-3}R_{k-1}q_1(-Q_1)^{k-2}}{(k-2)!} \Bigg) \\
&+ (P')^{2k-2}R'_{k-1}\frac{(-Q_1)^{k-1}}{(k-1)!}.
\end{aligned}
$$

Thus, by using (14) we get (20). □

Proof of Lemma 1. Let $P' = P'(z)$ and $R_i = R_i(z)$. Since $P'R_1 = P'(1/P') = 1$, the result holds for $k = 1$.

Now, we assume that the result is true for $k-1$, and we will prove it for k. By the induction hypothesis, $(P')^{2(k-1)-1}R_{k-1} = (P')^{2k-3}R_{k-1}$ is a polynomial of degree $(k-2)(p-1)$, then its derivative $\left((P')^{2k-3}R_{k-1}\right)'$ is of degree $(k-2)(p-1)-1$. Hence, the degree of $P'\left((P')^{2k-3}R_{k-1}\right)'$ is $(k-2)(p-1)-1+p = (k-1)(p-1)$.

As

$$
\begin{aligned}
P'\left((P')^{2k-3}R_{k-1}\right)' &= P'\left((P')^{2k-3}R'_{k-1} + (2k-3)R_{k-1}(P')^{2k-4}P''\right) \\
&= (P')^{2k-1}R_k + (2k-3)R_{k-1}(P')^{2k-3}P''
\end{aligned}
$$

and $R_{k-1}(P')^{2k-3}P''$ is a polynomial of degree $(k-2)(p-1)+(p-1) = (k-1)(p-1)$, it follows that $(P')^{2k-1}R_k$ is a polynomial of degree $(k-1)(p-1)$. □

5. PPM functions of (3_ε) with $H = (x^2+y^2)/2$

In this section H will denote the polynomial $H(x,y) = (x^2+y^2)/2 : \mathbb{C}^2 \to \mathbb{C}$. This polynomial has the following properties: the origin of $\mathbb{C}^2$ is the unique singular point of H, the fiber $\mathcal{L}_0 = H^{-1}(0)$ is the connected union of the two complex lines $\{x - \sqrt{-1}y = 0\}$ and $\{x + \sqrt{-1}y = 0\}$, and for $z \neq 0$ the map

$$
\varphi_z : \mathbb{C} \setminus \{0\} \to \mathcal{L}_z, \qquad t \mapsto \left(\frac{t^2+z}{\sqrt{2}t}, \frac{\sqrt{-1}(z-t^2)}{\sqrt{2}t} \right) \tag{23}
$$

is a parametrization of $\mathcal{L}_z$. Thus, H satisfies the hypothesis of the Ilyashenko-Gavrilov theorem (Theorem 3). Hence, we can consider the sequence of polynomial 1-forms $\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots$, constructed inductively in (12) with $H = (x^2+y^2)/2$. Moreover, if we assume that $\widetilde{\omega}_k$ with $k \geq 2$ is the first non analytically relatively exact 1-form in the sequence $\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_k$, then from Proposition 2 it follows that

$$
\widetilde{\omega}_k = P'\left(\overline{\omega}_{k1} + \overline{\omega}_{k2} + \overline{\omega}_{k3}\right) + (P')^{2k-1}R_k\frac{(-Q_1)^{k-1}}{(k-1)!}\omega_1,
$$

where

$$
\overline{\omega}_{k1} = (P')^{2k-3}\omega_k, \qquad \overline{\omega}_{k2} = \sum_{l=1}^{k-2}(P')^{2(k-l)-3}\left[P'q_l + (2l-1)P''Q_l\right]\omega_{k-l},
$$

and

$$\overline{\omega}_{k3} = \left(P'\overline{q} + 2(k-2)P''\overline{Q} + R_{k-1}(P')^{2k-3} q_1 \frac{(-Q_1)^{k-2}}{(k-2)!} \right) \omega_1$$

with $R_1 = 1/P'$, $R_i = R_1 R'_{i-1}$ for $i \geq 2$, and $\overline{q} \equiv \overline{Q} \equiv 0$ if $k = 2$. Moreover, the polynomials $q_1, \ldots, q_{k-1}$ and $Q_1, \ldots, Q_{k-1}$ satisfy

$$\widetilde{\omega}_1 = dQ_1 + q_1 dH,\ \widetilde{\omega}_2 = dQ_2 + q_2 dH, \ldots,\ \widetilde{\omega}_{k-1} = dQ_{k-1} + q_{k-1} dH.$$

Next, we will state a result about the degree of $\widetilde{\omega}_i$ for $i = 1, 2, \ldots, k$ and another about the degree of $\overline{\omega}_{k1}$, $\overline{\omega}_{k2}$, and $\overline{\omega}_{k3}$, which will be useful in the proof of Theorem 2. We will prove them later on.

Proposition 3. *If $\deg \omega_i \leq n$ for $i \geq 1$, $\deg P = p+1$, and $\widetilde{\omega}_k$ with $k \geq 1$ is the first non analytically relatively exact 1-form in the sequence $\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_k$, then for $i = 1, 2, \ldots, k$ we have*

$$\deg \widetilde{\omega}_i \leq \begin{cases} 4p(i-1) + n, & \text{if } n \leq 2p+1; \\ 2p(i-1) + i(n-1) + 1, & \text{otherwise.} \end{cases}$$

Proposition 4. *If $\deg \omega_i \leq n$ for $i \geq 1$, $\deg P = p+1$, and $\widetilde{\omega}_k$ with $k \geq 2$ is the first non analytically relatively exact 1-form in the sequence $\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_k$, then*

(a) $\deg \overline{\omega}_{k1} = 2p(2k-3) + n$

(b) $\deg \overline{\omega}_{k2} \leq \begin{cases} 4pk - 8p + 2n - 1, & \text{if } n \leq 2p+1; \\ (k-1)(2p+n-1) + 1, & \text{if } n > 2p+1. \end{cases}$

(c) $\deg \overline{\omega}_{k3} \leq \begin{cases} 4p(k-2) + 2n - 1, & \text{if } n \leq 2p+1; \\ (k-2)(2p+n-1) + 2n - 1, & \text{if } n > 2p+1. \end{cases}$

In the study of PPM functions related to (3_ε), as well as in the proof of the previous two propositions, is essential the following result, which give us the structure of a polynomial 1-form with respect to the Hamiltonian H.

Proposition 5. [11] *Each polynomial 1-form ω of degree n can be written as*

$$\omega = dQ + qdH + \tilde{q}(H) y dx,$$

where Q and q are polynomials of degree at most $n+1$ and $n-1$, respectively, and $\tilde{q}$ is a polynomial of degree at most $\left[\frac{n-1}{2}\right]$.

We will provide a proof of Proposition 5 at the end of this section, which is different from the one given in [11].

On the other hand, since H is transversal to infinity and it is a Morse polynomial, the first non vanishing PPM function $L_k(z)$ of the displacement function associated with (3_ε) is an Abelian integral because of Theorem 1. This property implies that $L_k(z)$ depends only on the homological cycles of $\mathcal{L}_z$. By abusing of the notation we will denote by $[\gamma_z]$ the homological class of a loop γ_z in $\mathcal{L}_z$.

As $\mathcal{L}_z$ is homeomorphic to $\mathbb{C} \setminus \{0\}$, then it has only one nontrivial homological cycle. The homological cycle $[\gamma_z]$ with $\gamma_z = \varphi_z(\alpha)$, where α is the unit circle in $\mathbb{C} \setminus \{0\}$ is the generator of $H_1(\mathcal{L}_z, \mathbb{Z})$.

By assuming Propositions (3), (4), and (5) we will now prove the next theorem, which gives the maximum number of isolated zeros of the first non-vanishing PPM function associated with (3_ε).

Theorem 4. *Let $n = \sup\{\deg(\omega_i) \,|\, i \geq 1\}$ and $p+1 = \deg P(z)$. If $L_k(c)$ is the first non-vanishing PPM function of the displacement function associated with (3_ε),* *then an upper bound for the maximum number of isolated zeros in* $(\mathbb{C}\setminus\{0\})\setminus Z(P')$*, counting multiplicities, of* $L_k(c)$ *is*

$$Z_k(n,p)=\begin{cases}0, & \text{if } n=1;\\ \left[\frac{n-1}{2}\right], & \text{if } k=1;\\ p(2k-3)+\left[\frac{n-1}{2}\right], & \text{if } k\geq 2 \text{ and } 2\leq n\leq 2p+1;\\ p(k-2)+\left[\frac{k(n-1)}{2}\right], & \text{otherwise.}\end{cases}$$

Proof. Firstly we assume that $k=1$. From (10) and (11) we then obtain the Poincaré–Pontryagin formula:

$$L_1(z)=\int_{[\gamma_z]}\omega_1.$$

From Proposition 5 it follows that $\omega_1=dQ_1+q_1dH+\widetilde{q}_1(H)ydx$ with $\widetilde{q}_1$ a univariate polynomial of degree at most $[(\deg\omega_1-1)/2]$, where $\deg\omega_1$ is the degree of ω_1. Thus,

$$L_1(z)=\int_{[\gamma_z]}dQ_1+q_1dH+\widetilde{q}_1(H)ydx=\widetilde{q}_1(z)\int_{[\gamma_z]}ydx.$$

By using the parametrization $\varphi_z(t)$ given by (23) we have

$$\int_{[\gamma_z]}ydx=\int_{\gamma_z}ydx=\int_\alpha\varphi*(ydx)=\int_\alpha\frac{\sqrt{-1}(z-t^2)}{\sqrt{2}t}\frac{\sqrt{2}(t^2-z)}{2t^2}dt=-2\pi z.$$

This implies that the degree of $\tilde{q}_1$ is an upper bound for the maximum number of isolated zeros, counting multiplicities, of $L_1(z)$ in $\mathbb{C}\setminus\{0\}$.

By assumption, $\deg\omega_1\leq n$. Thus, $L_1(z)$ has at most $[(n-1)/2]$ isolated zeros, counting multiplicities, in $\mathbb{C}\setminus\{0\}$. Therefore the result is true for $k=1$.

We now assume $k\geq 2$. From Theorem 1 we know that

$$L_k(z)=\int_{\gamma_z}\frac{\overline{\omega}_{k1}+\overline{\omega}_{k2}+\overline{\omega}_{k3}}{(P'(H))^{2k-3}}=\frac{1}{(P'(z))^{2k-3}}\int_{\gamma_z}\overline{\omega}_{k1}+\overline{\omega}_{k2}+\overline{\omega}_{k3},$$

where $\overline{\omega}_{k1}$, $\overline{\omega}_{k2}$, and $\overline{\omega}_{k3}$ are as in (13) and (14). Furthermore, as

$$\overline{\omega}_k:=\overline{\omega}_{k1}+\overline{\omega}_{k2}+\overline{\omega}_{k3}=dQ_k+q_kdH+\widetilde{q}_k(H)ydx$$

because of Proposition 5, we then have

$$L_k(z)=\frac{2\pi z}{(P'(z))^{2k-3}}\widetilde{q}_k(z).$$

Proposition 5 says that $\widetilde{q}_k$ is a polynomial of degree at most $[(\deg\overline{\omega}_k-1)/2]$. Therefore, to complete the proof we must prove that

$$\deg\overline{\omega}_k\leq\begin{cases}2p(2k-3)+n, & \text{if } n\leq 2p+1;\\ 2p(k-2)+k(n-1)+1, & \text{if } n>2p+1.\end{cases}$$

Next we will prove this assertion.

We have $\deg\overline{\omega}_k=\max\{\deg\overline{\omega}_{k1},\deg\overline{\omega}_{k2},\deg\overline{\omega}_{k3}\}$. Thus, from Proposition 4 it follows that if $n\leq 2p+1$, then

$$\deg\overline{\omega}_k=\max\{2p(2k-3)+n,4pk-8p+2n-1,4p(k-2)+2n-1\},$$

and if $n>2p+1$, then

$$\deg\overline{\omega}_k=\max\{2p(2k-3)+n,(k-1)(2p+n-1)+1,(k-2)(2p+n-1)+2n-1\}.$$

On the other hand, easy computations show that if $n\leq 2p+1$, then

$$2p(2k-3)+n\geq 4pk-8p+2n-1$$

and

$$2p(2k-3)+n\geq 4pk-8p+2n-1;$$

and, if $n > 2p+1$, then

$$2p(2k-3)+n \leq (k-2)(2p+n-1)+2n-1$$

and

$$(k-1)(2p+n-1)+1 \leq (k-2)(2p+n-1)+2n-1.$$

Therefore,

$$\deg \overline{\omega}_k \leq \begin{cases} 2p(2k-3)+n, & \text{if } n \leq 2p+1; \\ (k-2)(2p+n-1)+2n-1, & \text{if } n > 2p+1. \end{cases}$$

Finally, since $(k-2)(2p+n-1)+2n-1 = 2p(k-2)+k(n-1)+1$, we obtain the desired result. □

Proof of Proposition 3. The result holds for $k=1$ because $\deg \widetilde{\omega}_1 = \deg \omega_1 \leq n$.

We now assume that the result is true for $i \leq k-1$, and we will prove it for k. We will split the proof in two parts: $n \leq 2p+1$ and $n > 2p+1$.

Suppose $n \leq 2p+1$. Since $\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_{k-1}$ are analytically relatively exact polynomial 1-forms with respect to $\{dH=0\}$, there are polynomials $q_1, \ldots, q_{k-1}$ and $Q_1, \ldots, Q_{k-1}$ such that

$$\widetilde{\omega}_1 = dQ_1 + q_1 dH, \quad \widetilde{\omega}_2 = dQ_2 + q_2 dH, \ldots, \quad \widetilde{\omega}_{k-1} = dQ_{k-1} + q_{k-1} dH.$$

By induction hypothesis $\deg \widetilde{\omega}_i \leq 4p(i-1)+n$ for $i = 1, 2, \ldots, k-1$, then from Proposition 5 it follows that

$$\deg q_i \leq 4p(i-1)+n-1 \quad \text{and} \quad \deg Q_i \leq 4p(i-1)+n+1.$$

On the other hand, by definition

$$\widetilde{\omega}_k = (P')^{2k-2}\omega_k + \sum_{l=1}^{k-1} (P')^{2(k-l-1)} \left[P' q_l + (2l-1) P'' Q_l\right] \omega_{k-l}.$$

As $\deg P = p+1$, $P' = P'(H)$, and $P'' = P''(H)$, then

$$\deg (P')^{2k-2}\omega_k \leq 2p(2k-2)+n = 4pk-4p+n,$$

$$\deg P' q_l \leq 2p + 4p(l-1)+n-1 = 4pl-2p+n-1,$$

and

$$\deg P'' Q_l \leq 2(p-1)+4p(l-1)+n+1 = 4pl-2p+n-1,$$

whence a simple computation shows that

$$\deg\,(P')^{2(k-l-1)} \left[P' q_l + (2l-1)P'' Q_l\right] \omega_{k-l} \leq 4pk-10p+2n-1.$$

Hence

$$\deg \widetilde{\omega}_k \leq \max\{4p(k-2)+n, 4pk-10p+2n-1\}.$$

Since $4p(k-2)+n-(4pk-10p+2n-1) = 2p-n+1$ and $2p-n+1 \geq 0$ by assumption, $4p(k-2)+n \geq 4pk-10p+2n-1$; therefore,

$$\deg \widetilde{\omega}_k \leq 4p(k-2)+n.$$

The proof in the case $n > 2p+1$ is analogous. □

Proof of Proposition 4. The statement (a) follows easily.

For proving statements (b) and (c) we will proceed by induction on k. If $k=2$, then $\overline{\omega}_{22} = 0$ and $\overline{\omega}_{23} = R_1 P' q_1 \omega_1 = q_1 \omega_1$. From Proposition 3 we get $\deg \widetilde{\omega}_1 \leq n$. Thus, $\deg q_1 \leq n-1$ because of Proposition 5, whence

$$\deg \overline{\omega}_{22} = 0 \qquad \text{and} \qquad \deg \overline{\omega}_{23} = 2n-1$$

since $\deg \omega_1 \leq n$. Hence we have proved the assertion for $k=2$.

Suppose now that statements (b) and (c) are true for $k-1$, and we will prove them for k.

From Proposition 3 it follows that for $i = 1, \ldots, k-1$ we have

$$\deg \widetilde{\omega}_i \leq \begin{cases} 4p(i-1)+n, & \text{if } n \leq 2p+1; \\ 2p(i-1)+i(n-1)+1, & \text{if } n > 2p+1. \end{cases}$$

Thus, by Proposition 5 we conclude that for $i = 1, \ldots, k-1$ either

$$\deg q_i \leq 4p(i-1)+n-1 \quad \text{and} \quad \deg Q_i \leq 4p(i-1)+n+1, \quad \text{if } n \leq 2p+1,$$

or

$$\deg q_i \leq 2p(i-1)+i(n-1) \quad \text{and} \quad \deg Q_i \leq 2p(i-1)+i(n-1)+2, \quad \text{if } n > 2p+1.$$

Hence for $l = 1, \ldots, k-2$ we have either

$$\deg P'q_l \leq 4pl-2p+n-1 \quad \text{and} \quad \deg P''Q_l \leq 4pl-2p+n-1, \quad \text{if } n \leq 2p+1,$$

or

$$\deg P'q_l \leq 2pl+l(n-1) \quad \text{and} \quad \deg P''Q_l \leq 2pl+l(n-1), \quad \text{if } n > 2p+1.$$

We the obtain

$$\deg\, [P'q_l + (2l-1)P''Q_l]\, \omega_{i-l} \leq \begin{cases} (4pl-2p+n-1)+n, & \text{if } n \leq 2p+1; \\ 2pl+l(n-1)+n, & \text{if } n > 2p+1. \end{cases}$$

Therefore, the degree of $(P')^{2(k-l)-3}\, [P'q_l + (2l-1)P''Q_l]\, \omega_{k-l}$ is at most either $4pk-8p+2n-1$, if $n \leq 2p+1$, or $4pk-6p+l(n-2p-1)+n$, if $n > 2p+1$. In addition, by using that $1 \leq l \leq k-2$ we get

$$\begin{aligned} 4pk-6p+l(n-2p-1)+n &\leq 4pk-6p+(k-2)(n-2p-1)+n \\ &= (k-1)(2p+n-1)+1. \end{aligned}$$

This implies that

$$\deg\, (P')^{2(k-l)-3}\, [P'q_l + (2l-1)P''Q_l]\, \omega_{k-l}$$

is at most $4pk-8p+2n-1$, if $n \leq 2p+1$, or $(k-1)(2p+n-1)+$, if $n > 2p+1$.

Hence, the statement (b) follows because

$$\deg\, \overline{\omega}_{k2} = \max_{1 \leq l \leq k-2} \left\{ \deg\, (P')^{2(k-l)-3}\, [P'q_l + (2l-1)P''Q_l]\, \omega_{k-l} \right\}.$$

Now, we will give the proof of statement (c). First, we recall that

$$\overline{\omega}_{k3} = \left(P'\overline{q} + 2(k-2)P''\overline{Q} + R_{k-1}(P')^{2k-3} q_1 \frac{(-Q_1)^{k-2}}{(k-2)!} \right) \omega_1.$$

Thus,

$$\deg\, \overline{\omega}_{k3} \leq \max \left\{ \deg P'\overline{q}, \deg P''\overline{Q}, \deg R_{k-1}(P')^{2k-3} q_1 (-Q_1)^{k-2} \right\} + n.$$

On the other hand,

$$\deg R_{k-1}(P')^{2k-3} q_1 (-Q_1)^{k-2} \leq \deg R_{k-1}(P')^{2k-3} + (n-1) + (k-2)(n+1)$$

because $\deg q_1 \leq n-1$ and $\deg Q_1 \leq n+1$. Proving the following inequalities is then sufficient for finishing the proof:

$$\deg R_{k-1}(P')^{2k-3} \leq \begin{cases} (4p-n-1)(k-2), & \text{if } n \leq 2p+1; \\ 2(k-2)(p-1), & \text{if } n > 2p+1; \end{cases} \tag{24}$$

$$\deg P''\overline{Q} \leq \begin{cases} 4p(k-2)+n-1, & \text{if } n \leq 2p+1; \\ (k-2)(2p+n-1)+n-1, & \text{if } n > 2p+1; \end{cases} \tag{25}$$

$$\deg P'\overline{q} \leq \begin{cases} 4p(k-2)+n-1, & \text{if } n \leq 2p+1; \\ (k-2)(2p+n-1)+n-1, & \text{if } n > 2p+1. \end{cases} \tag{26}$$

The rest of the proof is devoted to proving (24), (25), and (26).

By Lemma 1 we have

$$\deg\, R_{k-1}(P')^{2k-3} \leq 2(k-2)(p-1); \tag{27}$$

moreover, for $n \leq 2p+1$ we have $2(p-1) \leq 4p-n-1$, which implies (24).

We know that $\deg P'' = 2p-2$, then for proving (25) we will prove that

$$\deg \overline{Q} \leq \begin{cases} 4p(k-2)+n+1-2p, & \text{if } n \leq 2p+1; \\ (k-2)(2p+n-1)+n+1-2p, & \text{if } n > 2p+1. \end{cases} \tag{28}$$

The polynomial $\overline{Q}$ satisfies (21), or equivalently,

$$P'\overline{Q} = Q_{k-1} + \frac{R_{k-1}(P')^{2k-3}(-Q_1)^{k-1}}{(k-1)!}.$$

From (27) and a simple calculation we obtain

$$\deg R_{k-1}(P')^{2k-3}(-Q_1)^{k-1} \leq (k-2)(2p+n-1)+n+1;$$

in addition, we know that

$$\deg Q_{k-1} \leq \begin{cases} 4p(k-2)+n+1, & \text{if } n \leq 2p+1; \\ (k-2)(2p+n-1)+n+1, & \text{if } n > 2p+1. \end{cases}$$

Moreover, if $n \leq 2p+1$, then

$$4p(k-2)+n+1 \geq (k-2)(2p+n-1)+n+1.$$

As a result,

$$\deg P'\overline{Q} \leq \begin{cases} 4p(k-2)+n+1, & \text{if } n \leq 2p+1; \\ (k-2)(2p+n-1)+n+1, & \text{if } n > 2p+1, \end{cases}$$

whence we can deduce (28).

For proving (26) we can proceed analogously to the previous case.

The polynomial $\overline{q}$ satisfies (22), or equivalently,

$$P'\overline{q} = q_{k-1} + P''\overline{Q} - \frac{R_{k-1}(P')^{2k-3}q_1(-Q_1)^{k-2}}{(k-2)!} - \frac{\left[(P')^{2k-3}R_{k-1}\right]'(-Q_1)^{k-1}}{(k-1)!}.$$

On the other hand, we know that

$$\deg q_{k-1} \leq \begin{cases} 4p(k-2)+n-1, & \text{if } n \leq 2p+1; \\ (k-2)(2p+n-1)+n-1, & \text{if } n > 2p+1. \end{cases}$$

In addition, we have (25) and by using (27) it is easy to see that

$$\deg R_{k-1}(P')^{2k-3}q_1(-Q_1)^{k-2} \leq (k-2)(2p+n-1)+n-1$$

and

$$\deg \left[(P')^{2k-3}R_{k-1}\right]'(-Q_1)^{k-1} \leq (k-2)(2p+n-1)+n-1.$$

Therefore, we it is clear that $\deg P'\overline{q} \leq (k-2)(2p+n-1)+n-1$, if $n > 2p+1$; furthermore, if $n \leq 2p+1$, then

$$\deg P'\overline{q} \leq \max\{4p(k-2)+n-1, (k-2)(2p+n-1)+n-1\}.$$

Finally, (26) holds because for $n \leq 2p+1$ we have

$$4p(k-2)+n-1 \geq (k-2)(2p+n-1)+n-1.$$

□

Proof of Proposition 5. Let P_n be the quotient space of Ω_n modulo $E_n + \{qdH\}$, where Ω_n the vector space of complex polynomial 1-forms of degree at most n, E_n is the subspace of exact polynomial 1-forms of degree at most n, and $\{qdH\} = \{qdH \in \Omega_n \,|\, q \in \mathbb{C}[x,y]\}$ denotes the subspace in Ω_n generated by dH.

The set $\left\{x^i y^j dx, x^i y^j dy \,|\, i+j = 0,1,...,n\right\}$ is a basis of Ω_n. E_n is isomorphic to the set $\{F \in \mathbb{C}[x,y] \,|\, 1 \le \deg F \le n+1\}$ and $\{qdH\}$ is isomorphic to the set $\{q \in \mathbb{C}[x,y] \,|\, 0 \le \deg q \le n-1\}$. Thus, we have

$$\dim \Omega_n = (n+1)(n+2),\ \dim E_n = \frac{(n+2)(n+3)}{2} - 1, \text{ and } \dim\{qdH\} = \frac{n(n+1)}{2}.$$

Hence, the dimension of the quotient space P_n is

$$(n+1)(n+2) - \frac{(n+2)(n+3)}{2} + 1 - \frac{n(n+1)}{2} + \dim E_n \cap \{qdH\},$$

which after a simple computation reduces to $\dim E_n \cap \{qdH\}$. In addition, from Proposition 6 below it follows that $\dim E_n \cap \{qdH\} = [(n-1)/2] + 1$. Therefore, $\dim \mathrm{P}_n = [(n-1)/2] + 1$.

To end the proof we will show that the set $\mathcal{B}_n := \left\{H^s ydx \,\middle|\, s = 0,1,...,[(n-1)/2]\right\}$ is a basis for the quotient space P_n. For that, seeing that the elements of $\mathcal{B}_n$ are non-zero linearly independent elements in P_n is sufficient.

Firstly, that $\mathcal{B}_n \subset \Omega_n$ is clear, and since H is constant along γ_z,

$$\int_{[\gamma_z]} H^s ydx = z^s \left(\int_{[\gamma_z]} ydx \right) = 2\pi\, z^{s+1}.$$

Hence, $H^s ydx$ for $s = 0,1,...,[(n-1)/2]$ is non-zero in P_n, and all them are linearly independent. □

Proposition 6. *If $f, g \in \mathbb{C}[x,y]$ are two polynomials, then $gdf = dG$ for a polynomial G if and only if $g = \tilde{q} \circ f$ for a univariate polynomial $\tilde{q}$.*

Proof. This is consequence of the Stein factorization theorem [19]. □

6. Proof of Theorem 2

In this section we will consider (3_ε) with $H(x,y) = (x^2+y^2)/2 : \mathbb{R}^2 \to \mathbb{R}$. The results of previous section can then be restricted to this real case. Moreover, now $\gamma_z = H^{-1}(z)$ for $z \in (0,\infty)$ is a circle, centering at the origin of $\mathbb{R}^2$ of radius $\sqrt{2z}$.

Proof of Theorem 2. The first part of the theorem is a corollary of Theorem 4. Hence, proving the second part remains: statements (a) and (b).

Proof of (a). The assertion follows easily for $k = 1$. Indeed, considering in (3_ε) the polynomial $P'(z) = z^p$ and the 1-form $\omega_1 = \alpha(H)ydx$ is sufficient, where $\alpha(H) = \prod_{i=1}^{Z_1(n,p)}(H - z_i)$ with $z_1, z_2, \ldots, z_{Z_1(n,p)}$ a sequence of points in $(0,\infty)$.

For $k = 2$ we will split the proof in two cases: $n \le 2p+1$ and $n > 2p+1$.

Case 1. Assume $n \le 2p+1$. In (3_ε) we take

$$P'(z) = \prod_{i=1}^{p}(z-i),\ \omega_1 = d(ax) + q_1 dH \text{ with } q_1 = y, \text{ and } \omega_2 = \prod_{i=1}^{\left[\frac{n-1}{2}\right]} \left(H - (p+i)\right) ydx.$$

Thus, $L_1(z) \equiv 0$, and from Theorem 1 we have

$$L_2(z) = \frac{1}{P'(z)} \int_{\gamma_z} \overline{\omega}_{21} + \overline{\omega}_{22} + \overline{\omega}_{23} = \frac{1}{P'(z)} \int_{\gamma_z} P'(H)\omega_2 + q_1\omega_1.$$

By using the expressions of P', ω_1, and ω_2 we obtain

$$L_2(z) = \frac{1}{P'(z)}\left(\prod_{i=1}^{p+\left[\frac{n-1}{2}\right]}(z-i)+a\right)\int_{\gamma_z} ydx = \frac{-2\pi z}{P'(z)}\left(\prod_{i=1}^{p+\left[\frac{n-1}{2}\right]}(z-i)+a\right).$$

Hence, for a small enough, $L_2(z)$ has $p+\left[\frac{n-1}{2}\right]$ simple zeros in $(0,\infty)\setminus Z(P')$.

Case 2. Assume $n > 2p+1$. In (3_ε) we take the polynomial $P'(z) = (z+1)^p$, the 1-form $\omega_1 = dQ_1 + q_1 dH$ with $q_1 = yx^{n-2}$, and

$$Q_1 = \begin{cases} \displaystyle\sum_{i=1}^{m} \frac{a_{2i+1}}{2i+1}x^{2i+1}, & \text{if } n = 2m; \\ \displaystyle\sum_{i=1}^{m+1} \frac{a_{2i}}{2i}x^{2i}, & \text{if } n = 2m+1. \end{cases} \tag{24}$$

Moreover, the 1-form $\omega_2 = \alpha(H)\, ydx$ with $\alpha(H) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} d_i H^i$. Thus, $L_1(z) \equiv 0$ and

$$q_1\omega_1 = q_1 dQ_1 + q_1^2 dH = \begin{cases} \displaystyle\sum_{i=1}^{m} a_{2i+1}x^{2(m+i-1)}\, ydx + q_1^2 dH, & \text{if } n = 2m; \\ \displaystyle\sum_{i=1}^{m+1} a_{2i}x^{2(m+i-1)}\, ydx + q_1^2 dH, & \text{if } n = 2m+1. \end{cases}$$

A simple computation gives

$$\int_{\gamma_z} x^{2j} ydx = -2\pi A_j z^{j+1} \qquad \text{with} \quad A_j = \frac{1}{2^{j+1}(2j+1)}\binom{2(j+1)}{j+1}. \tag{25}$$

Hence, we get that for $n = 2m$

$$L_2(z) = -\frac{2\pi z}{P'(z)}\left((z+1)^p\sum_{i=0}^{m-1} d_i z^i + \sum_{i=1}^{m} a_{2i+1}A_{m+i-1}z^{m+i-1}\right),$$

and for $n = 2m+1$

$$L_2(z) = -\frac{2\pi z}{P'(z)}\left((z+1)^p\sum_{i=0}^{m} d_i z^i + \sum_{i=1}^{m+1} a_{2i}A_{m+i-1}z^{m+i-1}\right).$$

By using that

$$(z+1)^p\sum_{i=0}^{r} d_i z^i = \sum_{i=0}^{r}\left(\sum_{\mu=0}^{\min\{i,p\}}\binom{p}{\mu}d_{i-\mu}\right)z^i + \sum_{i=r+1}^{r+p}\beta_i z^i,$$

it follows that in both cases $n = 2m$ and $n = 2m+1$ the second function $L_2(z)$ takes the form

$$L_2(z) = -\frac{2\pi z}{P'(z)}G_{n-1}(z),$$

where G_{n-1} is a polynomial of degree $Z_2(n,p) = n-1$ with n independent coefficients, which implies that $L_2(z)$ can have $Z_2(n,p)$ simple zeros in $(0,\infty)\setminus Z(P')$.

Proof of (b). Again, we will consider two cases: $n \le 2p+1$ and $n > 2p+1$.

Case 1. Assume $n \le 2p+1$. For each $p \in \{1,2\}$ and each $2 \le n \le 2p+1$ we consider in (3_ε) the polynomial $P'(z) = P_p'(z) = \prod_{i=p+1}^{2p}(z-i)$ and the polynomial 1-forms $\omega_1 = dQ_{1p} + q_1 dH$, where $q_1 = y$ and $Q_{1p} = y\left(ax + P_p''(y^2)\right)$, $\omega_2 = d(bx)$, and $\omega_3 = (yx^{n-1} + \alpha(H))\, ydx$ with $\alpha(H) = \prod_{i=1}^{m}(H-i)$ with $m = \left[\frac{n-1}{2}\right]$.

By construction $n = \max_{1\le i\le 3}\{\deg\omega_i\}$ and $L_1(z) \equiv 0$. Moreover,

$$P'(H)\omega_2 + q_1\omega_1 = d\overline{Q} + \overline{q}dH,$$

where $\overline{Q} = P'(H)bx + q_1 Q_{1p} + \int_0^x as^2\,ds - \int_0^y sP_p''(s^2)\,ds$ and $\overline{q} = q_1^2 - ax - P''(H)bx$, which implies that $L_2(z) \equiv 0$. From Theorem 1 we then have

$$L_3(z) = \frac{1}{(P'(z))^3}\int_{\gamma_z} \overline{\omega}_{31} + \overline{\omega}_{32} + \overline{\omega}_{33}$$

where $\overline{\omega}_{31} = (P'(H))^3\,\omega_3$, $\overline{\omega}_{32} = P'(H)\,[P'(H)q_1 + P''(H)Q_{1p}]\,\omega_2$, and $\overline{\omega}_{33} = T\omega_1$ with $T = \left[P'(H)\overline{q} + 2P''(H)\overline{Q} + P''(H)q_1Q_{1p}\right]$.

A long but easy calculation shows that

$$\begin{aligned}\overline{\omega}_{32} + \overline{\omega}_{33} = {} & b\,(P'(H))^2\,ydx + P'(H)P''(H)d(bxQ_{1p}) + P'(H)(q_1^2 - ax)dQ_{1p}\\ & + P''(H)\left(3q_1Q_{1p} + 2\int_0^x as^2\,ds - 2\int_0^y sP_p''(s^2)\,ds\right)dQ_{1p} + Tq_1\,dH.\end{aligned}$$

In addition, a simple computation gives

$$\int_{\gamma_z}(q_1^2 - ax)dQ_{1p} = -2\pi za(pz + P_p''(z))$$

and

$$\int_{\gamma_z}\left(3q_1Q_{1p} + 2\int_0^x as^2\,ds - 2\int_0^y sP_p''(s^2)\,ds\right)dQ_{1p} = 2\pi z\left(\frac{az}{2}\right)P_p''(z),$$

whence

$$\int_{\gamma_z}\overline{\omega}_{32} + \overline{\omega}_{33} = 2\pi z\left(b\left(P_p'(z)\right)^2 - aP_p'(z)\left(pz + P_p''(z)\right) + \left(P_p''(z)\right)^2\left(\frac{az}{2}\right)\right).$$

Therefore,

$$L_3(z) = \frac{2\pi z}{\left(P_p'(z)\right)^3}\left[\left(P_p'(z)\right)^3\alpha(z) + b\left(P_p'(z)\right)^2 - \frac{a}{2}\,G(z)\right],$$

where $G(z) = \left(2P_p'(z)\left(pz + P_p''(z)\right) - z\left(P_p''(z)\right)^2\right)$. Moreover,

$$G(z) = \begin{cases} 2z^2 - 3z - 4, & \text{if } p = 1;\\ 4z^3 - 42z^2 + 145z - 168, & \text{if } p = 2.\end{cases}$$

For b small enough, $(P'(z))^2\,(P'(z)\alpha(c) + b)$ has $p+m$ simple zeros $z_1, \ldots, z_{p+m}$ different from $1, \ldots, p+m$ and p zeros of multiplicity 2 at the zeros of $P'(z)$. It is easy to see that $G(z) < 0$ in $[0, 2p]$. Thus, for $|a| \ll |b|$ small enough, $L_3(z)$ has $p+m$ simple zeros close to $z_1, \ldots, z_{p+m}$ and two simple zeros close to each zero of $P'(z)$. Therefore, $L_3(z)$ has $3p + m = 3p + \left[\frac{n-1}{2}\right]$ simple zeros in $(\mathbb{R}\setminus\{0\})\setminus Z(P')$.

Case $n > 2p+1$. Following [11], for each $p \in \{1, 2\}$ and each $n > 2p+1$ we consider in (3_ε) the following objects.

- The polynomial $P'(z) = z - 1$ if $p = 1$ and $P'(z) = (z-1)(z-2)$ if $p = 2$.
- $\omega_1 = d(yf(x)) + q_1\,dH$ where $q_1 = yx^{n-2}$ and with

$$f(x) = \begin{cases} \displaystyle\sum_{i=0}^{m-1}\frac{b_{2i+1}}{2i+1}x^{2i+1} + \frac{x^{2m}}{2m}, & \text{if } n = 2m;\\ \displaystyle\sum_{i=0}^{m}\frac{b_{2i+1}}{2i+1}x^{2i+1}, & \text{if } n = 2m+1;\end{cases} \tag{26}$$

- $\omega_2 = dQ_1$, where Q_1 is given by (24).
- $\omega_3 = \alpha(H)\,ydx$, where $\alpha(H) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} d_iH^i$.

Thus, by construction $n = \max_{1\le i\le 3}\{\deg \omega_i\}$ and $L_1(z) \equiv 0$. Moreover,

$$P'(H)\omega_2 + q_1\omega_1 = d\overline{Q} + \overline{q}dH,$$

where

$$\overline{Q} = P'(H)Q_1 + yq_1 f(x) + (n-1)\int_0^x s^{n-1} f(s)\,ds - 2(n-2)H\int_0^x s^{n-3} f(s)\,ds$$

and

$$\overline{q} = q_1^2 + 2(n-2)\int_0^x s^{n-3} f(s)\,ds - x^{n-2} f(x) - P''(H)Q_1,$$

whence $L_2(z) \equiv 0$.

As in the previous case, from Theorem 1 we have

$$L_3(z) = \frac{1}{(P'(z))^3}\int_{\gamma_z} \varpi_{31} + \varpi_{32} + \varpi_{33}$$

where now $\varpi_{31} = (P'(H))^3\omega_3$, $\varpi_{32} = P'(H)\left[P'(H)q_1 + P''(H)yf(x)\right]\omega_2$, and $\varpi_{33} = T\omega_1$ with $T = \left[P'(H)\overline{q} + 2P''(H)\overline{Q} + P''(H)q_1(yf(x))\right]$.

Following the ideas of previous case, we get

$$\begin{aligned}\varpi_{32} + \varpi_{33} =& (P'(H))^2 q_1 dQ_1 + P'(H)P''(H)d(yf(x)Q_1) \\ &+ P'(H)S_1(x,y)d(yf(x)) + P''(H)S_2(x,y)d(yf(x)) + Tq_1\,dH,\end{aligned}$$

where

$$S_1(x,y) = q_1^2 + 2(n-2)\int_0^x s^{n-3} f(s)\,ds - x^{n-2} f(x)$$

and

$$S_2(x,y) = 3yq_1 f(x) + 2(n-1)\int_0^x s^{n-1} f(s)\,ds - 4(n-2)H\int_0^x s^{n-3} f(s)\,ds.$$

In addition, straightforward computations yield

$$S_1(x,y)d(yf(x)) = dQ_{11} + q_{11}\,dH + \tilde{q}_{11}\,ydx$$

and

$$S_2(x,y)d(yf(x)) = dQ_{22} + q_{22}\,dH + \tilde{q}_{22}\,ydx$$

for some polynomials Q_{11}, q_{11}, Q_{22}, q_{22}, as well as with

$$\tilde{q}_{11} = x^{n-3} f(x)\left(2(n-1)x^n - 4(n-2)Hx^{n-2} + xf'(x) - (n-2)f(x)\right),$$

and

$$\tilde{q}_{22} = \frac{1}{2}x^{n-3}(f(x))^2\left(2(n-2)H - (n-1)x^2\right).$$

Hence

$$\int_{\gamma_z} \varpi_{32} + \varpi_{33} = (P'(z))^2\int_{\gamma_z} q_1 dQ_1 + P'(z)\int_{\gamma_z} \tilde{q}_{11}\,ydx + P''(z)\int_{\gamma_z} \tilde{q}_{22}\,ydx.$$

Now if $n = 2m$, then from (26) we obtain $f(x) = f_1 + x^{2m}/(2m)$, where

$$f_1 = f_1(x) = \sum_{i=1}^{m-1}\frac{b_{2i+1}}{2i+1}x^{2i+1}.$$

Thus,

$$\begin{aligned}\tilde{q}_{11} =& \frac{x^{4m-3}}{2m}\left(xf_1' + (8m^2 - 6m + 4)f_1\right) - 8(m-1)x^{4m-5} f_1 H \\ &+ x^{2m-3} f_1\left(xf_1' - 2(m-1)f_1\right) + \frac{4m^2 - 2m + 1}{m^2}x^{6m-3} - 8(m-1)x^{6m-5}H\end{aligned}$$

and

$$\tilde{q}_{22} = \left(\frac{x^{2m-3}f_1^2}{2} + \frac{f_1 x^{4m-3}}{m} + \frac{x^{6m-3}}{4m}\right)\left(4(m-1)H - (2m-1)x^2\right).$$

Moreover, since f_1 is an odd polynomial, $x^{2m-3}f_1\,(xf_1' - 2(m-1)f_1)$, x^{6m-3}, x^{6m-5}, and $x^{2m-3}f_1^2$ are odd polynomials.

On the other hand, from the symmetry of γ_z with respect to the y-axis it follows that if $g(x)$ is an odd polynomial, then $\int_{\gamma_z} g(x)\,ydx = 0$ for all $z \in (0,\infty)$.

Therefore, from this property and by using the expression of f_1 we have

$$\int_{\gamma_z} \tilde{q}_{11}\,ydx = \sum_{i=0}^{m-1} \frac{b_{2i+1}}{2i+1} \int_{\gamma_z} \left(\frac{8m^2-6m+2i+5}{2m}x^2 - 8(m-1)H\right) x^{2(2m+i-2)}$$

and

$$\int_{\gamma_z} \tilde{q}_{22}\,ydx = \sum_{i=0}^{m-1} \frac{b_{2i+1}}{m(2i+1)} \int_{\gamma_z} \left(4(m-1)H - (2m-1)x^2\right) x^{2(2m+i-1)}.$$

Moreover, by applying (25) we get

$$\int_{\gamma_z} \tilde{q}_{11}\,ydx = -2\pi z \sum_{i=0}^{m-1} \frac{b_{2i+1}}{2i+1} \left(C_{m,i}A_{2m+i-1} - (8m-8)A_{2m+i-2}\right) z^{2m+i-1},$$

where $C_{m,i} = (8m^2-6m+2i+5)/(2m)$, and

$$\int_{\gamma_z} \tilde{q}_{22}\,ydx = -2\pi z \sum_{i=0}^{m-1} \frac{b_{2i+1}}{m(2i+1)} \left((4m-4)A_{2m+i-1} - (2m-1)A_{2m+i}\right) z^{2m+i}.$$

Additionally, from (24) and (25) we get

$$\int_{\gamma_z} q_1 dQ_1 = \begin{cases} -2\pi z \displaystyle\sum_{i=1}^{m} a_{2i+1}\,A_{m+i-1}\,z^{m+i-1}, & \text{if } n = 2m; \\ -2\pi z \displaystyle\sum_{i=1}^{m+1} a_{2i}\,A_{m+i-1}\,z^{m+i-1}, & \text{if } n = 2m+1. \end{cases}$$

Therefore, for $n = 2m$ we have obtained that

$$\begin{aligned} L_3(z) = \frac{-2\pi z}{(P'(z))^3}\Bigg(&(P'(z))^3 \sum_{i=0}^{m-1} d_i z^i + (P'(z))^2 \sum_{i=1}^{m} \tilde{a}_{m,i}\,z^{m+i-1} \\ &+ P'(z) \sum_{i=0}^{m-1} \tilde{b}_{m,i}\,z^{2m+i-1} + P''(z) \sum_{i=0}^{m-1} \hat{b}_{m,i}\,z^{2m+i}\Bigg), \end{aligned}$$

where $\tilde{a}_{m,i} = a_{2i+1}\,A_{m+i-1}$, $\tilde{b}_{m,i} = \frac{b_{2i+1}}{2i+1}\left(C_{m,i}A_{2m+i-1} - (8m-8)A_{2m+i-2}\right)$, and $\hat{b}_{m,i} = \frac{b_{2i+1}}{m(2i+1)}\left((4m-4)A_{2m+i-1} - (2m-1)A_{2m+i}\right)$.

By reordering the terms in previous equation, we get

$$L_3(z) = \begin{cases} \frac{-2\pi z}{(P'(z))^3}\left(\beta_0 + \beta_1 z + \cdots + \beta_{3m-1}z^{3m-1}\right), & \text{if } p = 1; \\ \frac{-2\pi z}{(P'(z))^3}\left(\beta_0 + \beta_1 z + \cdots + \beta_{3m}z^{3m}\right), & \text{if } p = 2, \end{cases}$$

where the coefficients $\beta_0, \ldots, \beta_{3m-2+p}$ are independent linear functions depending on the $3m$ parameters $d_0, \ldots, d_{m-1}$, $a_3, \ldots, a_{2m+1}$, and $b_1, \ldots, b_{2m-1}$. Therefore, for suitable $3m-2+p$ different points in $(0,\infty) \setminus Z(P)$ there are values of $d_0, \ldots, d_{m-1}$, $a_3, \ldots, a_{2m+1}$, and $b_1, \ldots, b_{2m-1}$ such that $L_3(z)$ has a simple zero at each one of the selected points. Thus, $L_3(z)$ has exactly $Z_3(p, 2m) = 3m-2+p$ simple zeros.

Finally, similar ideas can be used for the case $n = 2m+1$. $\square$

References

[1] S. A. Broughton, *On the topology of polynomial hypersurfaces*, Singularities, Part 1 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 167–178. MR 713056 (85d:14033)

[2] A. Buică, J. Giné, and J. Llibre, *Bifurcation of limit cycles from a polynomial degenerate center*, Adv. Nonlinear Stud. **10** (2010), no. 3, 597–609. MR 2676635 (2011k:34063)

[3] C. Camacho and A. Lins Neto, *Geometric theory of foliations*, Birkhäuser Boston Inc., Boston, MA, 1985, Translated from the Portuguese by Sue E. Goodman. MR 824240 (87a:57029)

[4] A. H. Durfee, *Five definitions of critical point at infinity*, Singularities (Oberwolfach, 1996), Progr. Math., vol. 162, Birkhäuser, Basel, 1998, pp. 345–360. MR 1652481 (2000a:32054)

[5] J. P. Françoise, *Successive derivatives of a first return map, application to the study of quadratic vector fields*, Ergodic Theory Dynam. Systems **16** (1996), no. 1, 87–96. MR 1375128 (97a:58131)

[6] M. Furushima, *Finite groups of polynomial automorphisms in the complex affine plane. I*, Mem. Fac. Sci. Kyushu Univ. Ser. A **36** (1982), no. 1, 85–105. MR 659047 (84m:14016)

[7] L. Gavrilov, *Petrov modules and zeros of Abelian integrals*, Bull. Sci. Math. **122** (1998), no. 8, 571–584. MR 1668534 (99m:32043)

[8] ______, *The infinitesimal 16th Hilbert problem in the quadratic case*, Invent. Math. **143** (2001), no. 3, 449–497. MR 1817642 (2002g:34069)

[9] ______, *Higher order Poincaré-Pontryagin functions and iterated path integrals*, Ann. Fac. Sci. Toulouse Math. (6) **14** (2005), no. 4, 663–682. MR 2188587 (2006i:34074)

[10] L. Gavrilov and I. D. Iliev, *The displacement map associated to polynomial unfoldings of planar Hamiltonian vector fields*, Amer. J. Math. **127** (2005), no. 6, 1153–1190. MR 2183522 (2006h:34065)

[11] I. D. Iliev, *The number of limit cycles due to polynomial perturbations of the harmonic oscillator*, Math. Proc. Cambridge Philos. Soc. **127** (1999), no. 2, 317–322. MR 1705462 (2000d:34054)

[12] Y. Ilyashenko, *The appearance of limit cycles under a perturbation of the equation $dw/dz = -R_z/R_w$, where $R(z, w)$ is a polynomial*, Mat. Sb. (N.S.) **78 (120)** (1969), 360–373, Math. USSR, Sb. 78 (120) (1969), no. 3, 353–364.

[13] ______, *Centennial history of Hilbert's 16th problem*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 3, 301–354. MR 1898209 (2003c:34001)

[14] ______, *Selected topics in differential equations with real and complex time*, Normal forms, bifurcations and finiteness problems in differential equations, NATO Sci. Ser. II Math. Phys. Chem., vol. 137, Kluwer Acad. Publ., Dordrecht, 2004, pp. 317–354. MR 2083252 (2006i:37002)

[15] Y. Ilyashenko and S. Yakovenko, *Lectures on analytic differential equations*, Graduate Studies in Mathematics, vol. 86, American Mathematical Society, Providence, RI, 2008. MR 2363178 (2009b:34001)

[16] P. Mardešić, *An explicit bound for the multiplicity of zeros of generic Abelian integrals*, Nonlinearity **4** (1991), no. 3, 845–852. MR 1124336 (92h:58163)

[17] H. Poincaré, *Sur le problème des trois corps et les équations de la dynamique.*, Acta Math. **13** (1890), 1–270.

[18] L. S. Pontryagin, *Über autoschwingungssysteme, die den hamiltonschen nahe liegen.*, Phys. Z. Sowjetunion **6** (1934), 25–28 (German), Zh. Eksp. Teor. Fiz. 4 (1934), 234–238. (in Russian.).

[19] K. Stein, *Analytische Projektion komplexer Mannigfaltigkeiten*, Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, Georges Thone, Liège, 1953, pp. 97–107. MR 0060602 (15,695i)

[20] M. Suzuki, *Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace* $\mathbf{C}^2$, J. Math. Soc. Japan **26** (1974), no. 2, 241–257. MR 0338423 (49 #3188)

[21] S. Yakovenko, *Complete abelian integrals as rational envelopes*, Nonlinearity **7** (1994), no. 4, 1237–1250. MR 1284690 (95d:34049)

[22] ______, *A geometric proof of the Bautin theorem*, Concerning the Hilbert 16th problem, Amer. Math. Soc. Transl. Ser. 2, vol. 165, Amer. Math. Soc., Providence, RI, 1995, pp. 203–219. MR 1334344 (96j:34056)