This is a preprint of: "Periodic motion in perturbed elliptic oscillators revisited", Montserrat Corbera, Jaume Llibre, Clàudia Valls, *Astrophys. Space Sci.*, vol. 361:348, 1–8, 2016.
DOI: [10.1007/s10509-016-2927-5]

# PERIODIC MOTION IN PERTURBED ELLIPTIC OSCILLATORS REVISITED

MONTSERRAT CORBERA[1], JAUME LLIBRE[2] AND CLAUDIA VALLS[3]

Abstract. We analytically study the Hamiltonian system in $\mathbb{R}^4$ with Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2) - \varepsilon\, V_1(x, y)$$

being (a) $V_1(x, y) = -(xy^2 + ax^3)$ and (b) $V_1(x, y) = -(x^2 y + ax^3)$ with $a \in \mathbb{R}$, where $\varepsilon$ is a small parameter and $\omega_1$ and $\omega_2$ are the unperturbed frequencies of the oscillations along the $x$ and $y$ axis, respectively. For the potential (a) using averaging theory of first order we analytically find for each $a \in \mathbb{R}$ eight families of periodic solutions in every positive energy level of $H$ when the frequencies are not equal. For the potential (b) using averaging theory of first and second order we analytically find seven families of periodic solutions in every positive energy level of $H$ when the frequencies are not equal. Four of these seven families are defined for all $a \in \mathbb{R}$ whereas the other three are defined for all $a \neq 0$. Moreover, we provide the shape of all these families of periodic solutions. These Hamiltonians may represent the central parts of deformed galaxies and thus have been extensively used and studied numerically in order to describe local motion in galaxies near an equilibrium point.

## 1. Introduction and statement of the main results

After equilibrium points the periodic solutions are the most simple non–trivial solutions of a differential system. Their study is of special interest because the motion in their neighborhood can be determined by their kind of stability. The stable periodic orbits explain the dynamics of bounded regular motion, while the unstable ones helps to understand the possible chaotic motion of the system. So, periodic orbits play a very important role in understanding the orbital structure of a dynamical system.

Over the last half century dynamical systems perturbing a harmonic oscillator have been used extensively to describe the local motion, i.e. motion near an equilibrium point. The study of this motion have been made mainly using several numerical techniques, see for instance [2, 4, 5, 6, 8, 10, 14, 15, 16, 17, 23, 24] to cite just a few.

---

2010 *Mathematics Subject Classification.* Primary 34C05. Secondary 37C10.

*Key words and phrases.* galactic potential, periodic solution, averaging theory.

The general form of a potential for a two-dimensional dynamical system composed of two harmonic oscillators with cubic perturbing terms is

$$V = \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2) + \varepsilon V_1(x, y),$$

where $\omega_1$ and $\omega_2$ are the unperturbed frequencies of the oscillator along the $x$ and the $y$ axes, respectively, $\varepsilon$ is the small perturbation parameter and $V_1$ is the cubic function containing the perturbed terms. We will use the perturbation functions

$$(a) \qquad V_1(x, y) = -(xy^2 + ax^3),$$

and

$$(b) \qquad V_1(x, y) = -(x^2 y + ax^3),$$

with $a \in \mathbb{R}$. These perturbed oscillators are important because they describe the motion of a star under the gravity field of a galaxy, for more information see for instance the paper of Caranicolas [5] and the references quoted there.

The Hamiltonian associated to the potential $V$ is

$$H = H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y), \tag{1}$$

and the corresponding Hamiltonian system is

$$\begin{aligned} \dot{x} &= p_x, \\ \dot{y} &= p_y, \\ \dot{p_x} &= -\omega_1^2\, x - \varepsilon \frac{\partial V_1}{\partial x}, \\ \dot{p_y} &= -\omega_2^2\, y - \varepsilon \frac{\partial V_1}{\partial y}. \end{aligned} \tag{2}$$

As usual the dot denotes derivative with respect to the time $t \in \mathbb{R}$. Due to the physical meaning the frequencies $\omega_1$ and $\omega_2$ are both positive.

We note that system (2) for $\varepsilon = 0$ can be solved. It has the solutions on the energy level $H = h$ of the form

$$\begin{aligned} x(t) &= C_1 \cos(t\omega_1) + C_2 \sin(t\omega_1), \\ y(t) &= C_3 \cos(t\omega_2) + C_4 \sin(t\omega_2), \\ p_x(t) &= -C_1\omega_1 \sin(t\omega_1) + C_2\omega_1 \cos(t\omega_1), \\ p_y(t) &= -C_3\omega_2 \sin(t\omega_2) + C_4\omega_2 \cos(t\omega_2), \end{aligned} \tag{3}$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$ satisfy

$$h = \frac{1}{2}(\omega_1^2(C_1^2 + C_2^2) + \omega_2^2(C_3^2 + C_4^2)).$$

Note that the solutions of system (2) for $\varepsilon = 0$ given in (3) are periodic if and only if $\omega_2/\omega_1 = p/q$ with $p, q \in \mathbb{N}$ and $p, q$ coprime, where as usual $\mathbb{N}$

denotes the set of positive integers. The period of these periodic solutions is

$$T = \frac{2p\pi}{\omega_2} = \frac{2q\pi}{\omega_1}.$$

As far as we know there are no rigorous analytic studies of the existence of periodic solutions for the Hamiltonian system (2) when $V_1$ is as in cases (a) or (b) and $\omega_1 \neq \omega_2$. For the particular case $a = 0$ potentials (a) and (b) are the same interchanging the names of the variables $x$ and $y$. Periodic orbits of this particular case has been studied by several authors from both an analytical and numerical point of view by using different techniques, see for instance [11, 12, 13, 7] for $\omega_1 = \omega_2$, or [9] for a numerical study for some values $\omega_1 \neq \omega_2$. Miller in [20] studies the potential (a) with $\omega_1 = \omega_2 = 1$ and $a \neq 0$ by means of a Lissajous transformation, in particular he found six families of periodic orbits. The particular case $\omega_1 = \omega_2 = 1$ and $a = -1/3$ is the well know Hénon-Heiles potencial [16]. More global dynamics of the perturbed potential (a) with $\omega_1 = \omega_2$ and $a \neq 0$ where studied numerically in [5]. The perturbed potential (b) with with $\omega_2 = \omega_1$ has been studied analytically in [14], where the authors found six families of periodic orbits by using similar techniques than the ones in [20].

In this paper we will study the periodic orbits of the Hamiltonian system (2) with perturbed potentials (a) and (b) by using averaging theory. More precisely, for the perturbed potential (a) we will study the case $\omega_2 = \omega_1/2$ with first order averaging, and for the perturbed potential (b) we will study the cases $\omega_2 = 2\omega_1$ and $\omega_2 = 3\omega_1$ with first and second order averaging, respectively. These cases together with the case $\omega_2 = \omega_1$ are the unique cases that we are able to study with these averaging techniques. More precisely, for both systems (a) and (b), we will prove the existence of families of periodic solutions parameterized by the energy in every energy level $H = h > 0$, and these families will be given explicitly up to first order in the small parameter $\varepsilon$. The case $a = 0$ has been studied by several authors so it is not considered in this work.

Our first main result deals with the periodic solutions of the Hamiltonian system associated to the Hamiltonian system (2) with $V_1(x, y)$ given in (a).

**Theorem 1.** *Using averaging theory of first order for $|\varepsilon| \neq 0$ sufficiently small at every positive energy level $H = h$ of the Hamiltonian $H$ given in* (1) *with $V_1(x, y)$ given in (a) and with $\omega_2 = \omega_1/2 > 0$, we find for its associated Hamiltonian system* (2)*, eight periodic solutions (four linearly stable and four unstable) bifurcating from the periodic solutions of* (3) *with a period tending to $4\pi/\omega_1$ as $\varepsilon \to 0$. We denote $\tau = \omega_1 t$.*

(a) *The four linearly stable periodic solutions can be written in the form* $(\tilde{x}(t), \pm\tilde{y}(t), \tilde{p}_x(t), \pm\tilde{p}_y(t)) + O(\varepsilon)$ *where* $(\tilde{x}(t), \tilde{y}(t), \tilde{p}_x(t), \tilde{p}_y(t))$ *is, respectively*

$$\begin{aligned}
&\Big(\frac{\sqrt{2h}}{\sqrt{3}\omega_1}\cos\tau, \frac{4\sqrt{h}}{\sqrt{3}\omega_1}\cos\Big(\frac{\tau}{2}\Big), -\frac{\sqrt{2h}}{\sqrt{3}}\sin\tau, -\frac{2\sqrt{h}}{\sqrt{3}}\sin\Big(\frac{\tau}{2}\Big)\Big), \\
&\Big(\frac{\sqrt{2h}}{\sqrt{3}\omega_1}\cos\tau, \frac{4\sqrt{h}}{\sqrt{3}\omega_1}\sin\Big(\frac{\tau}{2}\Big), -\frac{\sqrt{2h}}{\sqrt{3}}\sin\tau, \frac{2\sqrt{h}}{\sqrt{3}}\cos\Big(\frac{\tau}{2}\Big)\Big).
\end{aligned} \tag{4}$$

(b) *The four unstable periodic orbits have a stable and an unstable manifold, and can be written in the form* $(\tilde{x}(t) + \varepsilon\tilde{x}_1(t), \pm\varepsilon\tilde{y}(t), \tilde{p}_x(t) + \varepsilon\tilde{p}_{x,1}(t), \pm\varepsilon\tilde{p}_y(t)) + O(\varepsilon^2)$ *where*

$$\begin{aligned}
&(\tilde{x}(t) + \varepsilon\tilde{x}_1(t), \varepsilon\tilde{y}(t), \tilde{p}_x(t) + \varepsilon\tilde{p}_{x,1}(t), \varepsilon\tilde{p}_y(t)) = \\
&\qquad (\tilde{x}_0(t), 0, \tilde{p}_{x,0}(t), 0) + \varepsilon(\tilde{x}_1(t), \tilde{y}_1(t), \tilde{p}_{x,1}(t), \tilde{p}_{y,1}(t))
\end{aligned}$$

*is, respectively*

$$\begin{aligned}
&\Big(\frac{\sqrt{2h}}{\omega_1}\cos(\tau), 0, -\sqrt{2h}\sin(\tau), 0\Big) + \varepsilon\Big(\frac{ah}{\omega_1^4}(3-\cos(2\tau)), \\
&\frac{\sqrt{55}ha}{\sqrt{2}\omega_1^2}(\sin(\tau/2)+\cos(\tau/2)), \frac{2ah}{\omega_1^3}\sin(2\tau), \frac{\sqrt{55}ha}{2\sqrt{2}\omega_1^3}(\cos(\tau/2)-\sin(\tau/2))\Big), \\
&\Big(\frac{\sqrt{2h}}{\omega_1}\cos(\tau), 0, -\sqrt{2h}\sin(\tau), 0\Big) + \varepsilon\Big(\frac{ah}{\omega_1^4}(3-\cos(2\tau)), \\
&\frac{\sqrt{55}ha}{\sqrt{2}\omega_1^2}(\sin(\tau/2)-\cos(\tau/2)), \frac{2ah}{\omega_1^3}\sin(2\tau), \frac{\sqrt{55}ha}{2\sqrt{2}\omega_1^3}(\cos(\tau/2)+\sin(\tau/2))\Big).
\end{aligned}$$

The proof of Theorem 1 is given in section 3.

Our second main result deals with the periodic solutions of the Hamiltonian system associated to the Hamiltonian system (2) with $V_1(x, y)$ given in (b).

**Theorem 2.** *The following statements hold for the Hamiltonian system* (2) *with Hamiltonian* $H$ *given in* (1) *and* $V_1(x, y)$ *in (b).*

(a) *Using averaging theory of first order for* $|\varepsilon| \neq 0$ *sufficiently small at every positive energy level* $H = h$ *and with* $\omega_2 = 2\omega_1 > 0$, *we find for the Hamiltonian system* (2), *four periodic solutions (two linearly stable and two unstable) bifurcating from the periodic solutions of* (3) *with a period tending to* $2\pi/\omega_1$ *as* $\varepsilon \to 0$. *The two unstable periodic orbits have a stable and an unstable manifold, each one formed by two cylinders. All these periodic solutions can be written as* $(\tilde{x}(t), \tilde{y}(t), \tilde{p}_x(t), \tilde{p}_y(t)) + O(\varepsilon)$ *with* $(\tilde{x}(t), \tilde{y}(t), \tilde{p}_x(t), \tilde{p}_y(t))$ *being* *respectively,*

$$\Big(0, \pm\frac{\sqrt{h}}{\sqrt{2}\omega_1}\sin\tau, 0, \pm\sqrt{2h}\cos(\tau)\Big),$$
(5)
$$\Big(\frac{2\sqrt{h}}{\sqrt{3}\omega_1}\cos\Big(\frac{\tau}{2}\Big), \pm\frac{\sqrt{h}}{\sqrt{6}\omega_1}\cos\tau, -\frac{2\sqrt{h}}{\sqrt{3}}\sin\Big(\frac{\tau}{2}\Big), \mp\sqrt{\frac{2h}{3}}\sin\tau\Big),$$

*where* $\tau = 2\omega_1 t$.

(b) *Using averaging theory of second order for* $|\varepsilon| \neq 0$ *sufficiently small at every positive energy level* $H = h$ *and with* $\omega_2 = 3\omega_1 > 0$, *for each* $a \neq 0$ *we find for the Hamiltonian system* (2), *three periodic solutions (two linearly stable and one unstable) bifurcating from the periodic solutions of* (3) *with a period tending to* $2\pi/\omega_1$ *as* $\varepsilon \to 0$. *The unstable periodic orbit has a stable and an unstable manifold, each one formed by two cylinders. All these periodic solutions can be written as* $(\tilde{x}(t), \tilde{y}(t), \tilde{p}_x(t), \tilde{p}_y(t)) + O(\varepsilon)$ *with* $(\tilde{x}(t), \tilde{y}(t), \tilde{p}_x(t), \tilde{p}_y(t))$ *being respectively,*

$$\Big(\frac{r_i}{\omega_1}\cos\Big(\frac{\tau}{3}\Big), \mp\frac{\sqrt{2h - r_i^2}}{3\omega_1}\cos\tau, -r_i\sin\Big(\frac{\tau}{3}\Big), \pm\sqrt{2h - r_i^2}\sin\tau\Big) \tag{6}$$

*for* $i = 1, 2, 3$, *where* $\tau = 3\omega_1 t$ *and* $r_1, r_2, r_3$ *as well as the stability of the solutions for each* $r_i$ *are given in the proof of the theorem.*

The proof of Theorem 2 is given in section 4.

In section 2 we present a summary of the results on the averaging theory that we shall need for proving our results.

## 2. The averaging theory of first and second order

In this section we summarize the averaging theory of second order, it provides sufficient conditions for the existence of periodic solutions for a periodic differential system depending on a small parameter. See [3] for additional details and for the proofs of the results stated in this section.

**Theorem 3.** *Consider the differential system*

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon), \tag{7}$$

*where* $F_1$, $F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous and T-periodic functions in the first variable, and* $D$ *is an open subset of* $\mathbb{R}^n$. *Assume that the following hypotheses hold.*

(i) $F_1(t, \cdot) \in C^1(D)$ *for all* $t \in \mathbb{R}$, $F_1$, $F_2$, $R$ *and* $D_x F_1$ *are locally Lipschitz with respect to* $x$, *and* $R$ *is differentiable with respect to* $\varepsilon$.

*We define* $f_1,\ f_2 : D \to \mathbb{R}^n$ *as*

$$f_1(z) = \int_0^T F_1(s,z)ds, \tag{8}$$

$$f_2(z) = \int_0^T [D_z F_1(s,z) \int_0^s F_1(t,z)dt + F_2(s,z)]ds. \tag{9}$$

(ii) *For* $V \subset D$ *an open and bounded set and for each* $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\backslash\{0\}$, *there exist* $a \in V$ *such that*

(ii.1) *if* $f_1(z) \not\equiv 0$, *then* $f_1(a) = 0$ *and* $d_B(f_1, a) \neq 0$, *where* $d_B(f_1, a)$ *denotes the Brouwer degree of the function* $f_1 : V \to \mathbb{R}^n$ *at the fixed point* $a$*; and*

(ii.1) *if* $f_1(z) \equiv 0$ *and* $f_2(z) \not\equiv 0$, *then* $f_2(a) = 0$ *and* $d_B(f_2, a) \neq 0$.

*Then for* $|\varepsilon| > 0$ *sufficiently small, there exists a* $T-$*periodic solution* $\varphi(t,\varepsilon)$ *of the system such that* $\varphi(0,\varepsilon) \to a$ *when* $\varepsilon \to 0$. *The kind of stability or instability of the limit cycle* $\varphi(t,\varepsilon)$ *is given by the eigenvalues of the Jacobian matrix* $D_z(f_1(z) + \varepsilon f_2(z))|_{z=a}$.

Note that a sufficient condition for showing that the Brouwer degree of a function $f$ at a fixed point $a$ is non–zero, is that the Jacobian of the function $f$ at $a$ (when it is defined) is non–zero, see [19].

Under the assumption (ii.1) Theorem 3 provides the *averaging theory of first order*, and it provides the *averaging theory of second order* when assumption (ii.2) holds.

## 3. Proof of Theorem 1

For proving Theorem 1 we shall use Theorem 3, so the first step is to write system (2) in such a way that conditions of Theorem 3 be satisfied.

We observe that system (2) with $V_1(x,y)$ given in (a) is invariant by the symmetry $(x,y,p_x,p_y) \mapsto (x,-y,p_x,-p_y)$. This implies that if $(x(t),y(t), p_x(t),p_y(t))$ is a solution so is $(x(t),-y(t),p_x(t),-p_y(t))$.

First we write system (2) and the Hamiltonian (1) in polar coordinates

$$\begin{aligned} x &= \frac{r\,\cos\theta}{\omega_1}, & p_x &= r\,\sin\theta, \\ y &= \frac{\rho\,\cos(\alpha+\omega_2\theta/\omega_1)}{\omega_2}, & p_y &= \rho\,\sin(\alpha+\omega_2\theta/\omega_1), \end{aligned} \tag{10}$$

and we get the system of equations

$$
\begin{aligned}
\dot r &= \varepsilon\left(\frac{3ar^2\cos^2\theta}{\omega_1^2}+\frac{\rho^2}{\omega_2^2}\cos^2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right)\sin\theta,\\
\dot\theta &= -\omega_1+\varepsilon\left(\frac{3a}{\omega_1^2}r\cos^3\theta+\frac{\rho^2}{r\omega_2^2}\cos\theta\cos^2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right),\\
\dot\rho &= \frac{r\varepsilon\rho}{\omega_1\omega_2}\cos\theta\sin\left(2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right),\\
\dot\alpha &= -\varepsilon\left(\frac{3a\omega_2}{\omega_1^3}r\cos^3\theta+\frac{\rho^2-2r^2}{r\omega_1\omega_2}\cos\theta\cos^2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right),
\end{aligned}
\tag{11}
$$

and the Hamiltonian

$$
H=\frac{1}{2}\left(r^2+\rho^2\right)-\frac{\varepsilon}{\omega_1}r\cos\theta\left(\frac{a}{\omega_1^2}r^2\cos^2\theta+\frac{\rho^2}{\omega_2^2}\cos^2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right).
$$

We note that system (11) is periodic in the variable $\theta$ if and only if $\omega_2 = p\,\omega_1/(2q)$ for some $p, q \in \mathbb{N}$ coprime. Moreover its period is $2q\pi$.

Note that in system (11), the equations of $\dot r, \dot\theta$ and $\dot\alpha$ depend in $\rho^2$ instead of $\rho$. We thus introduce the new variable $\Gamma = \rho^2$. In this new variable system (11) becomes

$$
\begin{aligned}
\dot r &= \varepsilon\left(\frac{3ar^2\cos^2\theta}{\omega_1^2}+\frac{\Gamma}{\omega_2^2}\cos^2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right)\sin\theta,\\
\dot\theta &= -\omega_1+\varepsilon\left(\frac{3a}{\omega_1^2}r\cos^3\theta+\frac{\Gamma}{r\omega_2^2}\cos\theta\cos^2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right),\\
\dot\Gamma &= \frac{2r\varepsilon\Gamma}{\omega_1\omega_2}\cos\theta\sin\left(2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right),\\
\dot\alpha &= -\varepsilon\left(\frac{3a\omega_2}{\omega_1^3}r\cos^3\theta+\frac{\Gamma-2r^2}{r\omega_1\omega_2}\cos\theta\cos^2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right),
\end{aligned}
\tag{12}
$$

and the Hamiltonian becomes

$$
H=\frac{1}{2}\left(r^2+\Gamma\right)-\frac{\varepsilon}{\omega_1}r\cos\theta\left(\frac{a}{\omega_1^2}r^2\cos^2\theta+\frac{\Gamma}{\omega_2^2}\cos^2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right).
\tag{13}
$$

Now in system (12) we take as independent variable the angular variable $\theta$ and it becomes

$$
r' = \frac{\dot r}{\dot\theta},\qquad \Gamma' = \frac{\dot\Gamma}{\dot\theta},\qquad \alpha' = \frac{\dot\alpha}{\dot\theta},
\tag{14}
$$

where the prime denotes derivative with respect to $\theta$. We compute $\Gamma$ by solving equation $H = h$, and we get

$$
\Gamma=\frac{\omega_2^2\left(2ar^3\varepsilon\cos^3\theta+\omega_1^3\left(2h-r^2\right)\right)}{\omega_1^2\left(\omega_1\omega_2^2-2r\varepsilon\cos\theta\cos^2\left(\alpha+\frac{\theta\omega_2}{\omega_1}\right)\right)}=\Gamma_0+\Gamma_1\varepsilon+O(\varepsilon^2),
\tag{15}
$$

where

$$\Gamma_0 = 2h - r^2,$$
$$\Gamma_1 = \frac{2r\cos\theta}{\omega_1^3\omega_2^2}\Big(ar^2\omega_2^2\cos^2\theta + (2h-r^2)\omega_1^2\cos^2\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big)\Big).$$

We substitute the expression of $\Gamma$ into (12) and we develop the right-hand side in power series of $\varepsilon$ up to second order. Therefore, at each energy level $H = h$ the equations of motion can be written as

$$r' = \varepsilon F_{11} + \varepsilon^2 F_{21} + O(\varepsilon^3), \quad \alpha' = \varepsilon F_{12} + \varepsilon^2 F_{22} + O(\varepsilon^3), \tag{16}$$

where

$$F_{11} = \frac{-3a}{\omega_1^3}r^2\cos^2\theta\sin\theta + \frac{1}{\omega_1\omega_2^2}(r^2-2h)\cos^2\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big)\sin\theta,$$
$$F_{12} = \frac{3a\omega_2}{\omega_1^4}r\cos^3\theta + \frac{2h-3r^2}{r\omega_1^2\omega_2}\cos\theta\cos^2\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big),$$
$$F_{21} = \frac{\sin(2\theta)}{2r\omega_1^6\omega_2^4}\Big(-9a^2r^4\omega_2^4\cos^4\theta + 4ar^2\omega_1^2\omega_2^2(r^2-3h)\cos^2\theta$$
$$\cos^2\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big) + \omega_1^4(r^4-4h^2)\cos^4\Big(\alpha+\frac{\theta\omega_2}{\omega_1})\Big)\Big),$$
$$F_{22} = \frac{\cos^2\theta}{r^2\omega_1^7\omega_2^3}\Big(9a^2r^4\omega_2^4\cos^4\theta + 2ar^2\omega_1^2\omega_2^2(6h-5r^2)\cos^2\theta$$
$$\cos^2\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big) + \omega_1^4(r^2-2h)^2\cos^4\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big)\Big).$$

In order that the differential system (16) be in the normal form (7) for applying the averaging theory, this system must be periodic in the variable $\theta$. System (14) is periodic in the variable $\theta$ when $\omega_2 = p\omega_1/(2q)$ for $p, q \in \mathbb{N}$ and its period is $2q\pi$. Then system (16) is in the normal form (7) for applying the averaging theory with $T = 2q\pi$, $\mathbf{x} = (r, \alpha)$, $t = \theta$, $F_1(\theta, \mathbf{x}) = (F_{11}, F_{12})$, $F_2(\theta, \mathbf{x}) = (F_{21}, F_{22})$ and $\varepsilon^2 R(\theta, \mathbf{x}, \varepsilon)$ is $O(\varepsilon^3)$. We also observe that $F$ and $R$ are $\mathcal{C}^2$ in $\mathbf{x}$ and $2q\pi$–periodic in $\theta$. After some computations, from (8) we get

$$f_1(\mathbf{x}) = \int_0^{2q\pi} F_1(\theta, \mathbf{x})d\theta = (f_{11}(\mathbf{x}), f_{12}(\mathbf{x})),$$

with

$$f_{11}(\mathbf{x}) = \int_0^{2q\pi} F_{11}\,d\theta = \begin{cases} 0 & p \neq q, \\ \dfrac{2\pi q}{\omega_1^3}(2h-r^2)\sin(2\alpha) & p = q, \end{cases}$$

and

$$f_{12}(\mathbf{x}) = \int_0^{2q\pi} F_{12}\,d\theta = \begin{cases} 0 & p \neq q, \\ \dfrac{\pi q}{r\omega_1^3}(2h-3r^2)\cos(2\alpha) & p = q. \end{cases}$$

We shall always consider the case $p \neq 2q$ (i.e. $\omega_1 \neq \omega_2$), because the case $\omega_1 = \omega_2$) was studied in [20].

*Case* 1: $p = q$. Since $p$ and $q$ are coprime, we take $q = 1$; that is, $\omega_2 = \omega_1/2$. The solutions of system $f_1(\mathbf{x}) = (f_{11}(\mathbf{x}), f_{12}(\mathbf{x})) = 0$ are $(r_{(1,j)}, \alpha_{(1,j)}) = (\sqrt{2h}, \pi(1+2j)/4)$ and $(r_{(2,j)}, \alpha_{(2,j)}) = (\sqrt{2h/3}, \pi j/2)$ with $j = 0, 1, 2, 3$.

Now we compute the Jacobian matrix of $f_1$ and we get

$$\begin{aligned} \mathcal{J} = \quad D_{\mathbf{x}} f_1 &= \begin{pmatrix} \dfrac{\partial f_{11}}{\partial r} & \dfrac{\partial f_{11}}{\partial \alpha} \\ \dfrac{\partial f_{12}}{\partial r} & \dfrac{\partial f_{12}}{\partial \alpha} \end{pmatrix} \\ &= \begin{pmatrix} -\dfrac{4\pi r \sin(2\alpha)}{\omega_1^3} & \dfrac{4\pi(2h - r^2)\cos(2\alpha)}{\omega_1^3} \\ \dfrac{-\pi(2h + 3r^2)\cos(2\alpha)}{r^2 \omega_1^3} & \dfrac{2\pi(3r^2 - 2h)\sin(2\alpha)}{r\omega_1^3} \end{pmatrix}. \end{aligned} \tag{17}$$

By evaluating the determinant of $\mathcal{J}$ on the solutions $(r_{(i,j)}, \alpha_{(i,j)})$ for $i = 1, 2$ and $j = 0, 1, 2, 3$ we obtain

$$\det(\mathcal{J})_{(r,\alpha)=(r_{(1,j)},\alpha_{(1,j)})} = -\frac{32\pi^2 h}{\omega_1^6} \neq 0,$$
$$\det(\mathcal{J})_{(r,\alpha)=(r_{(2,j)},\alpha_{(2,j)})} = \frac{32\pi^2 h}{\omega_1^6} \neq 0.$$

If follows from Theorem 3 that for any given $h > 0$ and for $|\varepsilon|$ sufficiently small, system (16) has eight $2\pi$–periodic solutions. They are $(r^{(i,j)}(\theta, \varepsilon)$, $\alpha^{(i,j)}(\theta, \varepsilon))$ for $i = 1, 2$ and $j = 0, 1, 2, 3$, and $(r^{(i,j)}(\theta, \varepsilon), \alpha^{(i,j)}(\theta, \varepsilon))$ tends to $(r_{(i,j)}, \alpha_{(i,j)})$ when $\varepsilon \to 0$.

The eigenvalues of the matrix $\mathcal{J}$ evaluated at $(r, \alpha) = (r_{(1,j)}, \alpha_{(1,j)})$ for $j = 0, 1, 2, 3$ are

$$\lambda_{1,2} = \pm \frac{4\pi\sqrt{2h}}{\omega_1^3},$$

and the eigenvalues of the matrix $\mathcal{J}$ evaluated at $(r, \alpha) = (r_{(2,j)}, \alpha_{(2,j)})$ for $j = 0, 1, 2, 3$ are

$$\lambda_{1,2} = \pm i \frac{4\pi\sqrt{2h}}{\omega_1^3},$$

where $i = \sqrt{-1}$. So we have that the periodic solutions $(r^{(1,j)}(\theta, \varepsilon), \alpha^{(1,j)}(\theta, \varepsilon))$ for $j = 0, 1, 2, 3$ are linearly unstable, and the periodic solutions $(r^{(2,j)}(\theta, \varepsilon)$, $\alpha^{(2,j)}(\theta, \varepsilon))$ for $j = 0, 1, 2, 3$ are linearly stable. Since the eigenvalues of the matrix (17) evaluated at $(r^{(2,j)}(\theta, \varepsilon)$, $\alpha^{(2,j)}(\theta, \varepsilon))$ provide the stability of the fixed point corresponding to the Poincaré map defined in a neighborhood of the periodic solution associated to $(r^{(2,j)}(\theta, \varepsilon)$, $\alpha^{(2,j)}(\theta, \varepsilon))$ (see for instance the proof of Theorem 11.6 of [22]), and this fixed point is locally a saddle, we obtain that the four unstable orbits have a stable and an unstable manifolds formed by two cylinders.

Now we shall go back through the changes of variables in order to see how the $2\pi$–periodic solutions $(r^{(i,j)}(\theta,\varepsilon),\alpha^{(i,j)}(\theta,\varepsilon))$, with $i=1,2$ and $j=0,1,2,3$ of the differential system (16) are written in the original variables $(x,y,p_x,p_y)$. By substituting $(r^{(i,j)}(\theta,\varepsilon),\alpha^{(i,j)}(\theta,\varepsilon))$ into equation $H=h$ with $H$ given in (13) we get $\Gamma^{(i,j)}(\theta,\varepsilon)$. Then

$$(r^{(i,j)}(\theta,\varepsilon),\Gamma^{(i,j)}(\theta,\varepsilon),\alpha^{(i,j)}(\theta,\varepsilon)),$$

is a $2\pi$–periodic solution for the differential system (14). This solution provides the $2\pi/\omega_1$–periodic solution for the differential system (12)

$$\begin{aligned}&(r^{(i,j)}(\theta^{(i,j)}(t,\varepsilon),\varepsilon),\theta^{(i,j)}(t,\varepsilon),\Gamma^{(i,j)}(\theta^{(i,j)}(t,\varepsilon),\varepsilon),\alpha^{(i,j)}(\theta^{(i,j)}(t,\varepsilon),\varepsilon))=\\ &\Big(r_{(i,j)}+O(\varepsilon),-\omega_1 t+O(\varepsilon),2h-r_{(i,j)}^2+O(\varepsilon),\alpha_{(i,j)}+O(\varepsilon)\Big).\end{aligned}$$

Now, we introduce the variable $\rho^{(i,j)}(\theta^{(i,j)}(t,\varepsilon),\varepsilon)=\sqrt{\Gamma^{(i,j)}(\theta^{(i,j)}(t,\varepsilon),\varepsilon)}$. Going back to the change of variables (10) we get the $2\pi/\omega_1$ periodic solutions of system (2) with $V_1=-(xy^2+ax^3)$.

For the case $i=2$, we get the four $2\pi/\omega_1$ periodic solutions of system (2) with $V_1=-(xy^2+ax^3)$ given in (4). We note that these four solutions are different because changing the independent variable $\tau=\omega_1 t$ we cannot pass from one of them to the others. This completes the proof of statement (a) of Theorem 1.

For the case $i=1$, note that $r_{(1,j)}=2h$ and so $\rho^{(1,j)}(\theta^{(1,j)}(t,0),0)=\sqrt{\Gamma^{(1,j)}(\theta^{(1,j)}(t,0),0)}=0$ (see (15)). Going back to the change of variables (10) for the case $i=1$, the terms of order 0 of the solutions of system (2) with $V_1=-(xy^2+ax^3)$ are the same for the four solutions $(r^{(1,j)}(\theta,\varepsilon),\alpha^{(1,j)}(\theta,\varepsilon))$ with $j=0,1,2,3$. They are

$$(\widetilde{x}(t),\widetilde{y}(t),\widetilde{p}_x(t),\widetilde{p}_y(t))=\left(\frac{\sqrt{2h}\cos(t\omega_1)}{\omega_1},0,-\sqrt{2h}\sin(t\omega_1),0\right).$$

So the four solutions $(r^{(1,j)}(\theta,\varepsilon),\alpha^{(1,j)}(\theta,\varepsilon))$ with $j=0,1,2,3$ could give the same periodic orbit of the initial system (2). Next we will see that this is not the case by computing the first tems in power series of $\varepsilon$ of the solutions. In particular we will see that each solution $(r^{(1,j)}(\theta,\varepsilon),\alpha^{(1,j)}(\theta,\varepsilon))$ provides a different periodic solution of system (2).

By abuse of notation let

$$
\begin{aligned}
r^{(1,j)}(t,\varepsilon) &= r^{(1,j)}(\theta^{(1,j)}(t,\varepsilon),\varepsilon) = r_0 + \sum_{k=1}^{\infty} r_k(t)\varepsilon^k, \\
\theta^{(1,j)}(t,\varepsilon) &= -\omega_1 t + \sum_{k=1}^{\infty} \theta_k(t)\varepsilon^k, \\
\rho^{(1,j)}(t,\varepsilon) &= \sqrt{\Gamma^{(1,j)}(\theta^{(1,j)}(t,\varepsilon),\varepsilon)}, \\
\alpha^{(1,j)}(t,\varepsilon) &= \alpha^{(1,j)}(\theta^{(1,j)}(t,\varepsilon),\varepsilon) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k(t)\varepsilon^k.
\end{aligned} \tag{18}
$$

Going back to the change of variables (10) we get

$$
\begin{aligned}
x^{(1,j)}(t,\varepsilon) &= \frac{r^{(1,j)}(t,\varepsilon)\cos(\theta^{(1,j)}(t,\varepsilon))}{\omega_1}, \\
y^{(1,j)}(t,\varepsilon) &= r^{(1,j)}(t,\varepsilon)\sin(\theta^{(1,j)}(t,\varepsilon)), \\
p_x^{(1,j)}(t,\varepsilon) &= \frac{\rho^{(1,j)}(t,\varepsilon)}{\omega_2}\cos\left(\alpha^{(1,j)}(t,\varepsilon) + \frac{\omega_2\theta^{(1,j)}(t,\varepsilon)}{\omega_1}\right), \\
p_y^{(1,j)}(t,\varepsilon) &= \rho^{(1,j)}(t,\varepsilon)\sin\left(\alpha^{(1,j)}(t,\varepsilon) + \frac{\omega_2\theta^{(1,j)}(t,\varepsilon)}{\omega_1}\right).
\end{aligned} \tag{19}
$$

We substitute the power series (18) into the solution (19), and then the solution (19) into system (2) with $V_1 = -(xy^2 + ax^3)$. Let

$$
\begin{aligned}
G_1 &= \dot{x} - p_x = \sum_{k=1}^{\infty} G_{1k}\varepsilon^k, \\
G_2 &= \dot{y} - p_y, \\
G_3 &= \dot{p_x} + \omega_1^2\, x - \varepsilon\left(3ax^2 + y^2\right) = \sum_{k=1}^{\infty} G_{3k}\varepsilon^k, \\
G_4 &= \dot{p_y} + \omega_2^2\, y - 2xy\varepsilon.
\end{aligned}
$$

We develop both sides of equations $G_i = 0$ with $i = 1, \dots, 4$ in power series of $\varepsilon$ and we compute the first terms in $\varepsilon$ of (18). In particular, we compute the terms up to order four for $(r,\theta)$ and the terms up to order two for $\alpha$. Notice that if $r = 2h$ then $\Gamma^{(1,j)} = O(\varepsilon)$, and consequently $\rho^{(1,j)} = O(\sqrt{\varepsilon})$. Thus in the expansion of $G_2$ and $G_4$ for each solution $(r^{(1,j)}(\theta,\varepsilon), \alpha^{(1,j)}(\theta,\varepsilon))$ with $j = 0,1,2,3$ it appears terms in $\sqrt{\varepsilon}$ which do not disappear until we substitute the solution $(r,\theta)$ up to order three in $\varepsilon$. On the other hand, as we will see later on, the first term in power series in $\varepsilon$ which is different on both solutions $(x^{(1,1)}(t), p_x^{(1,1)}(t))$ and $(x^{(1,2)}(t), p_x^{(1,2)}(t))$ is the term of order 3. To compute this term we need terms in $\varepsilon$ up to order four of $(r,\theta)$ and up to order two of $\alpha$ in (18). These terms are computed following the next procedure.

The terms $G_{11}$ and $G_{31}$ only depend on $\dot r_1(t)$ and $\dot\theta_1(t)$, then by solving the system of equations $G_{11}=0$ and $G_{31}=0$ we get $\dot r_1(t)$ and $\dot\theta_1(t)$, and integrating we get $r_1(t)$ and $\theta_1(t)$. Now we substitute $r_1$, $\dot r_1$, $\theta_1$ and $\dot\theta_1$ into $G_{12}$ and $G_{32}$. As above, $G_{12}$ and $G_{32}$ only depend on $\dot r_2(t)$ and $\dot\theta_2(t)$. Solving the system of equations $G_{12}=0$, $G_{32}=0$ we get $\dot r_2(t)$ and $\dot\theta_2(t)$, and integrating we get $r_2(t)$ and $\theta_2(t)$. We substitute these solutions into $G_{13}=0$ and $G_{33}=0$ proceeding as above and we get $\dot r_3(t)$, $\dot\theta_3(t)$, $r_3(t)$ and $\theta_3(t)$. Now we substitute $r_k$, $\dot r_k$, $\theta_k$ and $\dot\theta_k$ for $k=1,2,3$ into $G_2$ and we develop in power series of $\varepsilon$, we see that the terms of order 0 and 1 of $G_2$ are equal to zero and the term of order 2 only depends on $\dot\alpha_1(t)$. Equating to zero the second ordre term of the development of $G_2$ we get $\dot\alpha_1(t)$, and integrating we get $\alpha_1(t)$. By substituting $r_k$, $\dot r_k$, $\theta_k$ and $\dot\theta_k$ for $k=1,2,3$, $\dot\alpha_1(t)$ and $\alpha_1(t)$ into $G_{14}=0$ and $G_{34}=0$ we obtain $\dot r_4(t)$, $\dot\theta_4(t)$, $r_4(t)$ and $\theta_4(t)$. Then substituting $r_k$, $\dot r_k$, $\theta_k$ and $\dot\theta_k$ for $k=1,2,3,4$, $\dot\alpha_1(t)$ and $\alpha_1(t)$ into $G_2$ we obtain $\dot\alpha_2(t)$ and $\alpha_2(t)$. We could compute higher order terms by following this pattern.

Once we have the coefficients of the power series (18) we substitute them into (19), and then we develop again the solution (19) into power series of $\varepsilon$. Due to the invariance by the symmetry $(x,y,p_x,p_y)\mapsto(x,-y,p_x,-p_y)$ we have

$$
\begin{aligned}
x^{(1,j)}(t,\varepsilon) &= x^{(1,j-2)}(t,\varepsilon), \quad y^{(1,j)}(t,\varepsilon) = -y^{(1,j-2)}(t,\varepsilon),\\
p_x^{(1,j)}(t,\varepsilon) &= p_x^{(1,j-2)}(t,\varepsilon), \quad p_y^{(1,j)}(t,\varepsilon) = -p_y^{(1,j-2)}(t,\varepsilon),
\end{aligned}
$$

for $j=3,4$. Moreover, these are the results that we have obtained

$$
\begin{aligned}
x^{(1,1)}(t,\varepsilon) &= x_0^{(1)}(t) + x_1^{(1)}(t)\varepsilon + x_2^{(1)}(t)\varepsilon^2 + x_3^{(1)}(t)\varepsilon^3 + O(\varepsilon^4),\\
y^{(1,1)}(t,\varepsilon) &= y_1^{(1)}(t)\varepsilon + y_2^{(1)}(t)\varepsilon^2 + y_3^{(1)}(t)\varepsilon^3 + O(\varepsilon^4),\\
p_x^{(1,1)}(t,\varepsilon) &= {p_x}_0^{(1)}(t) + {p_x}_1^{(1)}(t)\varepsilon + {p_x}_2^{(1)}(t)\varepsilon^2 + {p_x}_3^{(1)}(t)\varepsilon^3 + O(\varepsilon^4),\\
p_y^{(1,1)}(t,\varepsilon) &= {p_y}_1^{(1)}(t)\varepsilon + {p_y}_2^{(1)}(t)\varepsilon^2 + {p_y}_3^{(1)}(t)\varepsilon^3 + O(\varepsilon^4),\\
x^{(1,2)}(t,\varepsilon) &= x_0^{(1)}(t) + x_1^{(1)}(t)\varepsilon + x_2^{(1)}(t)\varepsilon^2 + x_3^{(2)}(t)\varepsilon^3 + O(\varepsilon^4),\\
y^{(1,2)}(t,\varepsilon) &= y_1^{(2)}(t)\varepsilon + y_2^{(2)}(t)\varepsilon^2 + y_3^{(2)}(t)\varepsilon^3 + O(\varepsilon^4),\\
p_x^{(1,2)}(t,\varepsilon) &= {p_x}_0^{(1)}(t) + {p_x}_1^{(1)}(t)\varepsilon + {p_x}_2^{(1)}(t)\varepsilon^2 + {p_x}_3^{(2)}(t)\varepsilon^3 + O(\varepsilon^4),\\
p_y^{(1,2)}(t,\varepsilon) &= {p_y}_1^{(2)}(t)\varepsilon + {p_y}_2^{(2)}(t)\varepsilon^2 + {p_y}_3^{(2)}(t)\varepsilon^3 + O(\varepsilon^4),
\end{aligned}
$$

where

$$\begin{aligned}
x_0^{(1)}(t) =& \frac{\sqrt{2h}}{\omega_1}\cos(t\omega_1),\\
x_1^{(1)}(t) =& \frac{ah}{\omega_1^4}(3-\cos(2t\omega_1)),\\
y_1^{(k)}(t) =& \sqrt{\frac{55a^2}{2}}\frac{h}{\omega_1^4}\Big(\sin\Big(\frac{t\omega_1}{2}\Big) \pm \cos\Big(\frac{t\omega 1}{2}\Big)\Big),\\
p_{x0}^{(1)}(t) =& -\sqrt{2h}\sin(t\omega_1),\\
p_{x1}^{(1)}(t) =& \frac{2ah}{\omega_1^3}\sin(2t\omega_1),\\
p_{y1}^{(k)}(t) =& \sqrt{\frac{55a^2}{2}}\frac{h}{2\omega_1^3}\Big(\cos\Big(\frac{t\omega_1}{2}\Big) \mp \sin\Big(\frac{t\omega_1}{2}\Big)\Big),
\end{aligned}$$

and $x_2^{(1)}, x_3^{(1)}, y_2^{(k)}(t), y_3^{(1)}(t), p_{x2}^{(1)}(t), p_{x3}^{(k)}(t), p_{y2}^{(k)}(t), p_{y3}^{(k)}(t)$ are given in the appendix. In the expressions $y_1^{(k)}(t)$ and $p_{y1}^{(k)}(t)$ the upper sign corresponds to $k=1$ and the lower sign to $k=2$.

In short, we get four $2\pi/\omega_1$ periodic solutions of system (2) with $V_1 = -(xy^2+ax^3)$ which completes the proof of statement (b) of Theorem 1.

*Case* 2: $p \neq q$ (we recall that we are also under the assumptions that $p \neq 2q$). Since $f_{11}(\mathbf{x}) = 0$ and $f_{12}(\mathbf{x}) = 0$, we need to consider averaging of second order. We compute for our system the integral $\int_0^s F_1(t,z)dt$ of (9), and after tedious computations we compute (9), and we get $f_2(\mathbf{x}) = (f_{21}(r,\alpha), f_{22}(r,\alpha))$ where $f_{21}(r,\alpha) = 0$ and

$$\begin{aligned}
f_{22}(r,\alpha) =& \frac{\pi}{4p^3\omega_1^6(p^2-q^2)}\big(r^2(15a^2p^6 - 15a^2p^4q^2 - 48ap^4q^2 + 48ap^2q^4\\
&- 48q^6)48ahp^4q^2 - 48ahp^2q^4 - 32hp^2q^4 + 96hq^6\big).
\end{aligned}$$

Note that there are no values of $a \in \mathbb{R}$ and $p,q \in \mathbb{N}$ such that $f_{22}(r,\alpha)$ is identically zero, so we cannot go to higher order in the averaging theory for trying to obtain information about the periodic solutions. Moreover, since the function $f_{21}$ is identically 0, the averaging theory of second order cannot be applied and we do not get any information on the periodic solutions of (16) in this case. This completes the proof of Theorem 1 because we have shown that there are no other periodic solutions which can be found with the averaging theory.

## 4. Proof of Theorem 2

For proving Theorem 2 we shall use Theorem 3. The first step is to write system (2) with $V_1 = -(x^2y + ax^3)$ in such a way that conditions of Theorem 3 be satisfied. As in Section 3 we write system (2) and the

Hamiltonian (1) in polar coordinates (10) and we get the system of equations

$$
\begin{aligned}
\dot{r} &= \frac{\varepsilon r \sin(2\theta)}{2\omega_1^2\omega_2}\left(3ar\omega_2\cos\theta + 2\rho\omega_1\cos\left(\alpha + \frac{\theta\omega_2}{\omega_1}\right)\right),\\
\dot{\theta} &= -\omega_1 + \frac{\varepsilon\cos^2\theta}{\omega_1^2\omega_2}\left(3ar\omega_2\cos\theta + 2\rho\omega_1\cos\left(\alpha + \frac{\theta\omega_2}{\omega_1}\right)\right),\\
\dot{\rho} &= \frac{\varepsilon r^2\cos^2\theta}{\omega_1^2}\sin\left(\alpha + \frac{\theta\omega_2}{\omega_1}\right),\\
\dot{\alpha} &= \frac{\varepsilon\cos^2\theta}{\rho\omega_1^3}\left(-3a\rho r\omega_2\cos\theta + \omega_1\left(r^2 - 2\rho^2\right)\cos\left(\alpha + \frac{\theta\omega_2}{\omega_1}\right)\right),
\end{aligned} \tag{20}
$$

and the Hamiltonian

$$
H = \frac{1}{2}\left(r^2 + \rho^2\right) - \frac{\varepsilon}{\omega_1^2} r^2\cos^2\theta\left(\frac{ar\cos\theta}{\omega_1} + \frac{\rho}{\omega_2}\cos\left(\alpha + \frac{\theta\omega_2}{\omega_1}\right)\right). \tag{21}
$$

We note that system (20) is periodic in the variable $\theta$ if and only if $\omega_2 = p\omega_1/q$ for some $p, q \in \mathbb{N}$ coprime. Moreover its period is $2q\pi$. We write system (20) by taking as the new independent variable the angular variable $\theta$ and we obtain

$$
r' = \frac{\dot{r}}{\dot{\theta}}, \qquad \rho' = \frac{\dot{\rho}}{\dot{\theta}}, \qquad \alpha' = \frac{\dot{\alpha}}{\dot{\theta}}, \tag{22}
$$

where the prime denotes derivative with respect to $\theta$. We compute $\rho = \rho_0 + \rho_1\varepsilon + O(\varepsilon^2)$ by solving equation $H = h$ and we get

$$
\begin{aligned}
\rho_0 &= \sqrt{2h - r^2},\\
\rho_1 &= \frac{r^2\cos^2\theta}{\omega_1^3}\left(\frac{ar\cos\theta}{\sqrt{2h - r^2}} + \frac{\omega_1}{\omega_2}\cos\left(\alpha + \frac{\theta\omega_2}{\omega_1}\right)\right).
\end{aligned} \tag{23}
$$

We substitute the expression of $\rho$ into (22) and we develop the resulting equations in power series of $\varepsilon$ up to second order. In the energy level $H = h > 0$ the equations of motion become

$$
r' = \varepsilon F_{11} + \varepsilon^2 F_{21} + O(\varepsilon^3), \quad \alpha' = \varepsilon F_{12} + \varepsilon^2 F_{22} + O(\varepsilon^3), \tag{24}
$$

with

$$
\begin{aligned}
F_{11} =& -\frac{r\sin(2\theta)}{2\omega_1^3\omega_2}\Big(3ar\omega_2\cos\theta + 2\omega_1\sqrt{2h-r^2}\cos\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big)\Big),\\
F_{12} =& \frac{\cos^2\theta}{\omega_1^4\sqrt{2h-r^2}}\Big(3ar\omega_2\sqrt{2h-r^2}\cos\theta + \omega_1(4h-3r^2)\cos\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big)\Big),\\
F_{21} =& \frac{r\sin\theta\cos^3\theta}{\omega_1^6\omega_2^2\sqrt{2h-r^2}}\Big(-9a^2r^2\omega_2^2\sqrt{2h-r^2}\cos^2\theta + 2ar\omega_1\omega_2(5r^2-12h)\\
&\cos\theta\cos\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big) + 2\omega_1^2\sqrt{2h-r^2}(r^2-4h)\cos^2\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big)\Big),\\
F_{22} =& \frac{\cos^4\theta}{\omega_1^7\omega_2(2h-r^2)^{3/2}}\Big(9a^2r^2\omega_2^2(2h-r^2)^{3/2}\cos^2\theta\\
&+ 2ar\omega_1\omega_2(24h^2-25hr^2+7r^4)\cos\theta\cos\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big)\\
&+ \omega_1^2\sqrt{2h-r^2}(16h^2-16hr^2+5r^4)\cos^2\Big(\alpha+\frac{\theta\omega_2}{\omega_1}\Big)\Big).
\end{aligned}
$$

In order that the differential system (24) be in the normal form (7) for applying the averaging theory we need that this system be periodic in the variable $\theta$. This implies that $\omega_2/\omega_1$ be rational. So from now on we assume that $\omega_2 = p\omega_1/q$ with $p, q \in \mathbb{N}$. We note that system (22) is $2q\pi$–periodic in the variable $\theta$. Then system (24) is in the normal form (7) for applying the averaging theory with $T = 2q\pi$, $\mathbf{x} = (r, \alpha)$, $t = \theta$, $F_1(\theta, \mathbf{x}) = (F_{11}, F_{12})$, $F_2(\theta, \mathbf{x}) = (F_{21}, F_{22})$ and $\varepsilon^2 R(\theta, \mathbf{x}, \varepsilon)$ is $O(\varepsilon^3)$. We also observe that $F$ and $R$ are $\mathcal{C}^2$ in $\mathbf{x}$ and $2q\pi$–periodic in $\theta$ in an open set not containing $r = \sqrt{2h}$. After some computations, from (8) we get

$$
f_1(\mathbf{x}) = \int_0^{2q\pi} F_1(\theta, \mathbf{x})d\theta = (f_{11}(\mathbf{x}), f_{12}(\mathbf{x})),
$$

with

$$
f_{11}(\mathbf{x}) = \int_0^{2q\pi} F_{11}\, d\theta = \begin{cases} 0 & p \neq 2q,\\ \dfrac{\pi q r\sqrt{2h-r^2}\sin\alpha}{2\omega_1^3} & p = 2q,\end{cases}
$$

$$
f_{12}(\mathbf{x}) = \int_0^{2q\pi} F_{12}\, d\theta = \begin{cases} 0 & p \neq 2q,\\ \dfrac{\pi q\left(4h-3r^2\right)\cos\alpha}{2\omega_1^3\sqrt{2h-r^2}} & p = 2q.\end{cases}
$$

*Case* 1: $p = 2q$. Since $p$ and $q$ are coprime, we take $q = 1$. The solutions of system $f_1(\mathbf{x}) = (f_{11}(\mathbf{x}), f_{12}(\mathbf{x})) = 0$ are $(r_{(1,j)}, \alpha_{(1,j)}) = (0, \pi/2 + j\pi)$ and

$(r_{(2,j)}, \alpha_{(2,j)}) = (2\sqrt{h/3}, j\pi)$ with $j = 0, 1$.

$$
\begin{aligned}
\mathcal{J} = \quad D_{\mathbf{x}} f_1 = & \begin{pmatrix} \dfrac{\partial f_{11}}{\partial r} & \dfrac{\partial f_{11}}{\partial \alpha} \\ \dfrac{\partial f_{12}}{\partial r} & \dfrac{\partial f_{12}}{\partial \alpha} \end{pmatrix} \\
= & \left( \begin{array}{cc} \dfrac{\pi\left(h - r^2\right)\sin\alpha}{\sqrt{2h - r^2}\omega_1^3} & \dfrac{\pi r\sqrt{2h - r^2}\cos\alpha}{2\omega_1^3} \\ \dfrac{\pi r\left(3r^2 - 8h\right)\cos\alpha}{2\left(2h - r^2\right)^{3/2}\omega_1^3} & -\dfrac{\pi\left(4h - 3r^2\right)\sin\alpha}{2\sqrt{2h - r^2}\omega_1^3} \end{array} \right).
\end{aligned}
$$

By evaluating the determinant of $\mathcal{J}$ on the solutions $(r_{(i,j)}, \alpha_{(i,j)})$ for $i = 1, 2$ and $j = 0, 1$ we get

$$
\begin{aligned}
\det(\mathcal{J})_{(r,\alpha)=(r_{(1,j)},\alpha_{(1,j)})} &= -\frac{\pi^2 h}{\omega_1^6} \neq 0, \\
\det(\mathcal{J})_{(r,\alpha)=(r_{(2,j)},\alpha_{(2,j)})} &= \frac{2\pi^2 h}{\omega_1^6} \neq 0.
\end{aligned}
$$

If follows from Theorem 3 that for any given $h > 0$ and for $|\varepsilon|$ sufficiently small, system (24) has the four $2\pi$–periodic solutions that are $(r^{(i,j)}(\theta, \varepsilon), \alpha^{(i,j)}(\theta, \varepsilon))$ for $i = 1, 2$ and $j = 0, 1$ such that the solution $(r^{(i,j)}(\theta, \varepsilon), \alpha^{(i,j)}(\theta, \varepsilon))$ tends to $(r_{(i,j)}, \alpha_{(i,j)})$ when $\varepsilon \to 0$.

The eigenvalues of the matrix $\mathcal{J}$ evaluated at $(r, \alpha) = (r_{(1,j)}, \alpha_{(1,j)})$ for $j = 0, 1$ are

$$
\lambda_1 = -(-1)^j \frac{\sqrt{2h}\pi}{\omega_1^3}, \qquad \lambda_2 = (-1)^j \frac{\pi\sqrt{h}}{\sqrt{2}\,\omega_1^3},
$$

and the eigenvalues of the matrix $\mathcal{J}$ evaluated at $(r, \alpha) = (r_{(2,j)}, \alpha_{(2,j)})$ for $j = 0, 1$ are

$$
\lambda_{1,2} = \pm \frac{i\sqrt{2h}\pi}{\omega_1^3}.
$$

So we have that the periodic solutions $(r^{(1,j)}(\theta, \varepsilon), \alpha^{(1,j)}(\theta, \varepsilon))$ for $j = 0, 1$ are linearly unstable and the periodic solutions $(r^{(2,j)}(\theta, \varepsilon), \alpha^{(2,j)}(\theta, \varepsilon))$ for $j = 0, 1$ are linearly stable. Note that the unstable orbits have a stable and an unstable manifolds formed by two cylinders.

Now we shall go back through the changes of variables in order to see how the $2\pi$–periodic solutions $(r^{(i,j)}(\theta, \varepsilon), \alpha^{(i,j)}(\theta, \varepsilon))$, with $i = 1, 2$ and $j = 0, 1$ of the differential system (24) looks in the initial Hamiltonian system (2) with $V_1 = -(x^2 y + ax^3)$. By substituting $(r^{(i,j)}(\theta, \varepsilon), \alpha^{(i,j)}(\theta, \varepsilon))$ into equation $H = h$ with $H$ given in (21) we get $\rho^{(i,j)}(\theta, \varepsilon)$. Then

$$
(r^{(i,j)}(\theta, \varepsilon), \rho^{(i,j)}(\theta, \varepsilon), \alpha^{(i,j)}(\theta, \varepsilon)),
$$

is a $2\pi$–periodic solution for the differential system (22). This solution provides the $2\pi/\omega_1$–periodic solution for the differential system (20)

$$(r^{(i,j)}(\theta^{(i,j)}(t,\varepsilon),\varepsilon), \theta^{(i,j)}(t,\varepsilon), \rho^{(i,j)}(\theta^{(i,j)}(t,\varepsilon),\varepsilon), \alpha^{(i,j)}(\theta^{(i,j)}(t,\varepsilon),\varepsilon)) = \Big(r_{(i,j)} + O(\varepsilon), -\omega_1 t + O(\varepsilon), \sqrt{2h - r_{(i,j)}^2} + O(\varepsilon), \alpha_{(i,j)} + O(\varepsilon)\Big).$$

Going back to the change of variables (10) we get the four $2\pi/\omega_1$ periodic solutions of system (2) with $V_1 = -(x^2y + ax^3)$ given in (5), which clearly are different solutions. This completes the proof of statement (a) of Theorem 3.

*Case* 2: $p \neq 2q$. Since $f_{11}(\mathbf{x}) = 0$ and $f_{12}(\mathbf{x}) = 0$, we need to consider averaging of second order. After tedious computations, from (9), we get $f_2(\mathbf{x}) = (f_{21}(r,\alpha), f_{22}(r,\alpha))$ where

$$f_{21}(r,\alpha) = \begin{cases} 0 & p \neq q,\ p \neq 3q,\ p \neq 5q, \\ 0 & p = 5q, \\ g_0(r,\alpha) & p = q, \\ -\dfrac{\pi a q r^2 \sqrt{2h - r^2}\sin\alpha}{2\omega_1^6} & p = 3q, \end{cases}$$

and

$$f_{22}(r,\alpha) = \begin{cases} g_1(r,\alpha) & p \neq q,\ p \neq 3q,\ p \neq 5q, \\ g_2(r,\alpha) & p = 5q, \\ g_3(r,\alpha) & p = q, \\ g_4(r,\alpha) & p = 3q, \end{cases}$$

where

$$\begin{aligned} g_0(r,\alpha) &= \frac{\pi r q \sin\alpha}{2\omega_1^6}\big(5ar\sqrt{2h - r^2} + \cos\alpha(8h - 4r^2)\big), \\ g_1(r,\alpha) &= \frac{\pi\left(3p^2r^2\left(5a^2p^2 - 20a^2q^2 + q^2\right) - 8hq^4\right)}{2p\omega_1^6\left(p^2 - 4q^2\right)}, \\ g_2(r,\alpha) &= \frac{\pi q}{210\omega_1^6}(75(105a^2 + 1)r^2 - 8h), \\ g_3(r,\alpha) &= \frac{\pi q}{\omega_1^6}\Big(\frac{3(15a^2 - 1)r^2 + 12\cos(2\alpha)(h - r^2) + 8h}{6} \\ &\quad + \frac{5ar\cos\alpha(3h - 2r^2)}{\sqrt{2h - r^2}}\Big), \\ g_4(r,\alpha) &= -\frac{\pi q}{30\omega_1^6}\left(8h - 27(25a^2 + 1)r^2 + \frac{30ar\left(3h - 2r^2\right)\cos\alpha}{\sqrt{2h - r^2}}\right). \end{aligned}$$

*Case* 2.1: $p \neq q$, $p \neq 3q$ and $p \neq 5q$. There are no $a \in \mathbb{R}$ and $p, q \in \mathbb{N}$ such that $f_{22}(r,\alpha)$ be identically 0. Since the function $f_{21}$ is identically 0 and $f_{22}$ is not identically 0, the averaging theory does not provide any information on the periodic solutions of (24) in this case.

*Case* 2.2: $p = 5q$. As in the previous case the averaging theory does not provide any information on the periodic solutions of (24).

*Case* 2.2: $p = q$. This case corresponds to the case $\omega_1 = \omega_2$ studied in [14]. So it is not considered in this work.

*Case* 2.3: $p = 3q$. Since $p$ and $q$ are coprime, we take $q = 1$. We seek solutions of system $f_2(\mathbf{x}) = 0$ for which the Jacobian of $f_2$ evaluated at the solution be non–zero. The solutions of equation $f_{21}(r, \alpha) = 0$ are $r = 0$, $r = \sqrt{2h}$ and $\alpha = j\pi$ for $j = 0, 1$. The solution $r = 0$ does not provide solutions of $f_2(\mathbf{x}) = 0$, because $f_{22}(0, \alpha) = -(4h\pi)/(15\omega_1^6) \neq 0$. The solution $r = \sqrt{2h}$ is not valid because in this case system $f_2(\mathbf{x}) = 0$ is not defined. Now we analyze the solutions $\alpha = j\pi$. For this, we need an auxiliary result.

**Lemma 4.** *Assume $h > 0$ and let $t_1(a, h)$, $t_2(a, h)$, $t_3(a, h)$ be the three real solutions of the polynomial*

$$
\begin{aligned}
&(455625a^4 + 40050a^2 + 729)t^3 - (911250a^4h + 94500a^2h + 1890h)t^2 \\
&+ (29700a^2h^2 + 928h^2)t - 128h^3 = 0,
\end{aligned}
$$

*ordered from big to small. Then the following statements hold for the equation $f_{22}(r, j\pi) = 0$.*

(a) *For $a > 0$ it has a unique positive real solution $r = \sqrt{t_2(a, h)}$ when $j = 0$, and two positive real solutions $r = \sqrt{t_1(a, h)}$ and $r = \sqrt{t_3(a, h)}$ when $j = 1$.*

(b) *For $a < 0$ it has two positive real solutions $r = \sqrt{t_1(a, h)}$ and $r = \sqrt{t_3(a, h)}$ when $j = 0$, and one positive real solution $r = \sqrt{t_2(a, h)}$ when $j = 1$.*

(c) *The Jacobian matrix of $f_2$ evaluated at $(r, \alpha) = (r(a, h), j\pi)$, where $r(a, h)$ is any solution of $f_{22}(r, j\pi) = 0$ with $a \neq 0$ and $j = 0, 1$, is different from zero.*

*Proof.* If $r \neq \sqrt{2h}$, equation $f_{22}(r, j\pi) = 0$ is equivalent to equation

$$
(8h - 27(25a^2 + 1)r^2)\sqrt{2h - r^2} = -30ar(3h - 2r^2)(-1)^j. \tag{25}
$$

Squaring both sides of (25) we get the polynomial equation (independent of $j$)

$$
\begin{aligned}
&(455625a^4 + 40050a^2 + 729)r^6 - (911250a^4h + 94500a^2h + 1890h)r^4 \\
&+ (29700a^2h^2 + 928h^2)r^2 - 128h^3 = 0.
\end{aligned} \tag{26}
$$

Equation (26) has the solutions of (25) and probably new ones.

By doing the change of variables $t = r^2$ in (26) we obtain a new cubic polynomial equation $\overline{g}(t) = 0$ with positive discriminant

$$
a^2 \left(405a^2 + 13\right)^2 \left(5843390625a^6 + 604158750a^4 + 19231425a^2 + 194672\right) h^6,
$$

unless a positive real constant. So the polynomial $\overline{g}(t) = 0$ has three real positive roots, for more details about the discriminant of a cubic polynomial see [1]. Using the Descartes rule on the signs of a polynomial we can see that these roots cannot be negative, and of course they are non-zero, so they are positive and we denote them as $t_1(a,h) > t_2(a,h) > t_3(a,h)$ for $a \neq 0$ and $h > 0$.

Now we study which of these solutions provide solutions of equation $f_{22}(r, j\pi) = 0$. It is not difficult to check that the factors $K_1 = (3h - 2r^2)$ and $K_2 = (8h - 27(25a^2+1)r^2)$ in $f_{22}(r, j\pi)$ do not change their sing on the solutions $r = \pm\sqrt{t_i(a,h)}$ for all $i = 1, 2, 3$. In particular,

$$\begin{cases} K_1, K_2 < 0 & \text{on } r = \pm\sqrt{t_1(a,h)}, \\ K_1 > 0, K_2 < 0 & \text{on } r = \pm\sqrt{t_2(a,h)}, \\ K_1, K_2 > 0 & \text{on } r = \pm\sqrt{t_3(a,h)}. \end{cases}$$

Analyzing the signs of the two summands of $f_{22}(r, j\pi)$ we conclude that $f_{22}(r, j\pi) = 0$ has the following solutions: $r = -\sqrt{t_1(a,h)}$, $r = \sqrt{t_2(a,h)}$ and $r = -\sqrt{t_3(a,h)}$ when either $j = 0$ and $a > 0$ or $j = \pi$ and $a < 0$; and $r = \sqrt{t_1(a,h)}$, $r = -\sqrt{t_2(a,h)}$ and $r = \sqrt{t_3(a,h)}$ when either $j = 1$ and $a > 0$ or $j = 0$ and $a < 0$. This proves statements (a) and (b).

To prove statement (c) we seek for the solutions $r = r(a,h)$ of the equation $f_{22}(r, j\pi) = 0$ such that the Jacobian of $f_2$ evaluated at $(r, \alpha) = (r(a,h), j\pi)$ is equal to zero. The Jacobian of $f_2$ evaluated at $\alpha = j\pi$ is

$$\det(\mathcal{J}) = \frac{\pi^2 a^2 r^2 \left(3h^2 - 6hr^2 + 2r^4\right)}{\omega_1^{12}\left(r^2 - 2h\right)} + \frac{9\pi^2 a \left(25a^2+1\right)(-1)^j r^3 \sqrt{2h - r^2}}{10\omega_1^{12}}.$$

We transform equation $\det(\mathcal{J}) = 0$ into a polynomial equation in the variable $t = r^2$ as we have done with equation $f_{22}(r, j\pi) = 0$ and we get

$$\begin{aligned}\overline{g}_1(t) = {} & \pi^4 a^2 t^2(-900a^2h^4 + 72(5625a^4 + 500a^2 + 9)h^3 t \\ & - 12(50625a^4 + 4450a^2 + 81)h^2t^2 + 6(50625a^4 + 4450a^2 + 81)ht^3 \\ & + (-50625a^4 - 4450a^2 - 81)t^4).\end{aligned}$$

We compute the resultant between the polynomial $\overline{g}(t)$ and $\overline{g}_1(t)$ with respect to the variable $t$ and we obtain a polynomial $P(a,h)$, in the variables $a$ and $h$, with the property that if the polynomials $\overline{g}(t)$ and $\overline{g}_1(t)$ have a commun root, this occurs for values of $(a,h)$ such that $P(a,h) = 0$, for more information about the resultant of two polynomials see for instance [18, 21]. Since $P(a,h)$ is zero if and only if $a = 0$ (recall that $h > 0$). Therefore there are no solutions of system $\overline{g}(t) = 0$, $\overline{g}_1(t) = 0$ with $a \neq 0$, and consequently there are no solutions of system $f_2(\mathbf{x}) = 0$ with $a \neq 0$ having Jacobian equal to zero. On the other hand it is easy to check that the solution $f_{22}(r, j\pi) = 0$ for $a = 0$, $r = \frac{2}{3}\sqrt{\frac{2}{3}}\sqrt{h}$ has Jacobian equal to zero. This completes the proof of the lemma. □

Let $r_1 = \sqrt{t_1(a,h)}$, $r_2 = \sqrt{t_2(a,h)}$ and $r_3 = \sqrt{t_3(a,h)}$; and let $\mathbf{x}_1 = (r_2, 0)$, $\mathbf{x}_2 = (r_1, \pi)$, $\mathbf{x}_3 = (r_3, \pi)$, $\mathbf{x}_4 = (r_1, 0)$, $\mathbf{x}_5 = (r_3, 0)$, and $\mathbf{x}_6 = (r_2, \pi)$.

From Lemma 4 together with Theorem 3, for any given $h > 0$ and for $|\varepsilon|$ sufficiently small, system (24) when $a > 0$ has three $2\pi$–periodic solutions $(r^i(\theta,\varepsilon),\alpha^i(\theta,\varepsilon))$ with $i = 1, 2, 3$, such that $(r^i(\theta,\varepsilon),\alpha^i(\theta,\varepsilon))$ tends to $\mathbf{x}_i$ when $\varepsilon \to 0$. When $a < 0$ system (24) has three $2\pi$–periodic solutions $(r^i(\theta,\varepsilon),\alpha^i(\theta,\varepsilon))$ with $i = 4, 5, 6$, such that $(r^i(\theta,\varepsilon),\alpha^i(\theta,\varepsilon))$ tends to $\mathbf{x}_i$ when $\varepsilon \to 0$.

Now we analyze the stability of these periodic solutions from the eigenvalues of the Jacobian matrix $\mathcal{J}$ of $f_2(\mathbf{x})$ evaluated at $\mathbf{x}_i$ for $i = 1, \ldots, 6$.

The eigenvalues of $\mathcal{J}$ evaluated at $(r, j\pi)$ are

$$\lambda_{1,2} = \pm\pi/(\sqrt{10}\omega_1^6)\sqrt{m(r,a,h)}$$

where

$$\begin{aligned} m(r,a,h) = &-\frac{10a^2r^2\left(3h^2-6hr^2+2r^4\right)}{r^2-2h} \\ &-9a\left(25a^2+1\right)(-1)^j r^3\sqrt{2h-r^2}. \end{aligned}$$

We are interested in the sign of $m(r,a,h)$ on the solutions of $f_{22}(r,j\pi)$. By proceeding as in Lemma 4 we see that there is no $(a,h)$ with $a \neq 0$ and $h > 0$ such that $m(r,a,h)$ evaluated on the solutions of $f_{22}(r,j\pi)$ be 0. Moreover we can see that if $a > 0$ then $m(r,a,h)|_{\mathbf{x}_1} < 0$, $m(r,a,h)|_{\mathbf{x}_2} < 0$ and $m(r,a,h)|_{\mathbf{x}_3} > 0$, and if $a < 0$ then $m(r,a,h)|_{\mathbf{x}_4} < 0$, $m(r,a,h)|_{\mathbf{x}_5} > 0$ and $m(r,a,h)|_{\mathbf{x}_6} < 0$. Therefore the periodic solutions $(r^i(\theta,\varepsilon),\alpha^i(\theta,\varepsilon))$ with $i = 3, 5$ are linearly unstable and the ones with $i = 1, 2, 4, 6$ are linearly stable. Clearly, the unstable orbits have a stable and an unstable manifolds formed by two cylinders.

Now we shall go back through the changes of variables in order to see how the $2\pi$–periodic solutions $(r^i(\theta,\varepsilon),\alpha^i(\theta,\varepsilon))$, with $i = 1, \ldots, 6$ of the differential system (24) looks in the initial Hamiltonian system (2) with $V_1 = -(x^2y + ax^3)$. By substituting $(r^i(\theta,\varepsilon),\alpha^i(\theta,\varepsilon))$ into equation $H = h$ with $H$ given in (21) we get $\rho^i(\theta,\varepsilon)$. Then

$$(r^i(\theta,\varepsilon),\rho^i(\theta,\varepsilon),\alpha^i(\theta,\varepsilon)),$$

is a $2\pi$–periodic solution for the differential system (22). This solution provides the $2\pi/\omega_1$–periodic solution for the differential system (20)

$$\begin{aligned} &(r^i(\theta^i(t,\varepsilon),\varepsilon),\theta^i(t,\varepsilon),\rho^i(\theta^i(t,\varepsilon),\varepsilon),\alpha^i(\theta^i(t,\varepsilon),\varepsilon)) = \\ &\quad (r+O(\varepsilon), -\omega_1 t+O(\varepsilon), \sqrt{2h-r^2}+O(\varepsilon), \alpha+O(\varepsilon))|_{(r,\alpha)=\mathbf{x}^i}. \end{aligned}$$

Going back to the change of variables (10) we get $2\pi/\omega_1$ periodic solutions of system (2) with $V_1 = -(x^2y+ax^3)$. Denoting $\mathcal{R}_{r_i} = \sqrt{2h-r_i^2}$, their first order in $\varepsilon$ is: For $a > 0$,

$$
\begin{aligned}
&\Big(\frac{r_2\cos(t\omega_1)}{\omega_1}, \frac{\mathcal{R}_{r_2}\cos(3t\omega_1)}{3\omega_1}, -r_2\sin(t\omega_1), -\mathcal{R}_{r_2}\sin(3t\omega_1)\Big),\\
&\Big(\frac{r_1\cos(t\omega_1)}{\omega_1}, -\frac{\mathcal{R}_{r_1}\cos(3t\omega_1)}{3\omega_1}, -r_1\sin(t\omega_1), \mathcal{R}_{r_1}\sin(3t\omega_1)\Big),\\
&\Big(\frac{r_3\cos(t\omega_1)}{\omega_1}, -\frac{\mathcal{R}_{r_3}\cos(3t\omega_1)}{3\omega_1}, -r_3\sin(t\omega_1), \mathcal{R}_{r_3}\sin(3t\omega_1)\Big),
\end{aligned}
\tag{27}
$$

and for $a < 0$

$$
\begin{aligned}
&\Big(\frac{r_1\cos(t\omega_1)}{\omega_1}, \frac{\mathcal{R}_{r_1}\cos(3t\omega_1)}{3\omega_1}, -r_1\sin(t\omega_1), -\mathcal{R}_{r_1}\sin(3t\omega_1)\Big),\\
&\Big(\frac{r_3\cos(t\omega_1)}{\omega_1}, \frac{\mathcal{R}_{r_3}\cos(3t\omega_1)}{3\omega_1}, -r_3\sin(t\omega_1), -\mathcal{R}_{r_3}\sin(3t\omega_1)\Big),\\
&\Big(\frac{r_2\cos(t\omega_1)}{\omega_1}, -\frac{\mathcal{R}_{r_2}\cos(3t\omega_1)}{3\omega_1}, -r_2\sin(t\omega_1), \mathcal{R}_{r_2}\sin(3t\omega_1)\Big).
\end{aligned}
\tag{28}
$$

The first two periodic solutions in (27) are stable and the third one is unstable, whereas the first and the third periodic solutions in (28) are stable and the second one is unstable. Clearly these three solutions are different, so this completes the proof of statement (b) Theorem 2.

## Appendix

$$
\begin{aligned}
x_2^{(1)}(t) =& \frac{a^2h^{3/2}}{8\sqrt{2}\omega_1^7}\Big(120t\omega_1\sin(t\omega_1) + 19\cos(t\omega_1) + 6\cos(3t\omega_1)\Big),\\
x_3^{(k)}(t) =& \frac{a^2h^2}{8\omega_1^{10}}\Big(220 + 285a \pm 55\sin(t\omega_1) \mp 110t\omega_1\cos(t\omega_1)\\
&- 120at\omega_1\sin(2t\omega_1) - 137a\cos(2t\omega_1) - 2a\cos(4t\omega_1)\Big),\\
y_2^{(k)}(t) =& \frac{\sqrt{55}h^{3/2}\sqrt{a^2}}{2\omega_1^7}\Big( - 2(1 \mp t\omega_1)\sin\Big(\frac{t\omega_1}{2}\Big) - \sin\Big(\frac{3t\omega_1}{2}\Big)\\
&\pm 2(1 \pm t\omega_1)\cos\Big(\frac{t\omega_1}{2}\Big) \mp \cos\Big(\frac{3t\omega_1}{2}\Big)\Big),\\
y_3^{(1)}(t) =& \sqrt{\frac{11}{10}}\frac{\sqrt{a^2}h^2}{768\omega_1^{10}}\Big(3\sin\Big(\frac{t\omega_1}{2}\Big)(10243a^2 \mp 1280t\omega_1 \pm 7680at\omega_1\\
&+ 21120a + 1280t^2\omega_1^2 - 1600) \pm 3\cos\Big(\frac{t\omega_1}{2}\Big)(10243a^2\\
&\pm 1280t\omega_1 \mp 7680at\omega_1 + 21120a + 1280t^2\omega_1^2 - 1600)
\end{aligned}
$$

$$- 1920(a \pm 2t\omega_1 - 5)\sin\left(\frac{3t\omega_1}{2}\right) + 640(a+1)\sin\left(\frac{5t\omega_1}{2}\right)$$
$$\pm 1920(a \mp 2t\omega_1 - 5)\cos\left(\frac{3t\omega_1}{2}\right) \pm 640(a+1)\cos\left(\frac{5t\omega_1}{2}\right)\Big),$$

$$p_{x_2}^{(1)}(t) = \frac{a^2h^{3/2}}{8\sqrt{2}\omega_1^6}\big(101\sin(t\omega_1) - 18\sin(3t\omega_1) + 120t\omega_1\cos(t\omega_1)\big),$$

$$p_{x_3}^{(k)}(t) = \frac{a^2h^2}{8\omega_1^9}\big(\pm 110t\omega_1\sin(t\omega_1) \mp 55\cos(t\omega_1)$$
$$+ 154a\sin(2t\omega_1) - 240at\omega_1\cos(2t\omega_1) + 8a\sin(4t\omega_1)\big),$$

$$p_{y_2}^{(k)}(t) = \frac{\sqrt{55}\sqrt{a^2}h^{3/2}}{4\omega_1^6}\Big(2(1\pm t\omega_1)\cos\left(\frac{t\omega_1}{2}\right) - 3\cos\left(\frac{3t\omega_1}{2}\right)$$
$$\pm \sin\left(\frac{t\omega_1}{2}\right)\Big(5 \mp 2t\omega_1 + 6\cos(t\omega_1)\Big)\Big),$$

$$p_{y_3}^{(k)}(t) = \sqrt{\frac{11}{10}}\frac{\sqrt{a^2}h^2}{1536\omega_1^9}\Big(\mp 3\sin\left(\frac{t\omega_1}{2}\right)\Big(10243a^2 \mp 3840t\omega_1 \mp 7680at\omega_1$$
$$+ 5760a + 1280t^2\omega_1^2 + 960\Big) \pm 1920\sin\left(\frac{3t\omega_1}{2}\right)\Big(11 - 3a \pm 6t\omega_1\Big)$$
$$+ 3\cos\left(\frac{t\omega_1}{2}\right)(10243a^2 \pm 3840t\omega_1 \pm 7680at\omega_1 + 5760a$$
$$+ 1280t^2\omega_1^2 + 960) \mp 3200(a+1)\sin\left(\frac{5t\omega_1}{2}\right)$$
$$+ 3200(a+1)\cos\left(\frac{5t\omega_1}{2}\right) + 1920\cos\left(\frac{3t\omega_1}{2}\right)(11 - 3a \mp 6t\omega_1)\Big).$$

In the expressions the upper sign corresponds to $k = 1$ and the lower sign to $k = 2$.

## 5. Acknowledgements

The first two authors are partially supported by MINECO/FEDER grants number MTM2013-40998-P. The second author is also supported by an AGAUR grant number 2014SGR-568, and the grants FP7-PEOPLE-2012-IRSES 318999 and 316338. The third author is partially supported by FCT/Portugal through UID/MAT/04459/2013.

## References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9-th printing, New York: Dover, p. 17, 1972.

[2] B. Barbanis, Escape regions of a quartic potential, Celest. Mech. Dyn. Astron., 48, 57–77 (1990)

[3] A. Buică and J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, Bull. Sci. Math. 128 (2004), 7–22.

[4] N.D. Caranicolas, A map for a group of resonant cases in quartic galactic hamiltonian, J. Astrophys. Astron., 22, 309–319 (2001)

[5] N.D. Caranicolas, Orbis in global and local galactic potentials, Astron. Astrophys. Trans., 23, 241–252 (2004)

[6] N.D. Caranicolas and G.I. Karanis, Motion in a potential creating a weak bar structure, Astron. Astrophys., 342, 389–394 (1999)
[7] N.D. Caranicolas and K.A. Innanen, Periodic motion in perturbed elliptic oscillators, Astron. J., 103, 1308–1312 (1992)
[8] N.D. Caranicolas and N. D. Zotos, Investigating the nature of motion in 3D perturbed elliptic oscillators displaying exact periodic orbits, Nonlinear Dyn., 69, 1795-1805 (2012)
[9] G. Contopoulos, Orbits in highly perturbed dynamical systems. I. Periodic orbits, Astron. J., 75, 96–107 (1970)
[10] G. Contopoulos, Orbits in highly perturbed dynamical systems. II. Stability of periodic orbits, Astron. J., 75, 108–130 (1970)
[11] G. Contopoulos and M. Moutsoulas, Resonance cases and small divisors in a thrird integral of motion II, Astron. J. 70, 817-835 (1965)
[12] G. Contopoulos and M. Zikides, Periodic orbits and ergodic components of a resonant dynamical system, Astron. Astrophys. 90, 198–203 (1980)
[13] E. Davoust, Periodic orbits in a two–dimensional galactic potential, Cel. Mech. 31, 303–315 (1983)
[14] A. Elipe, B. Miller and M. Vallejo, Bifurcations in a non–symmetric cubic potential, Astron. Atrophys. 300, 722–725 (1995).
[15] A. Giorgilli and L. Galgani, From integrals from an autonomous Hamiltonian system near an equilibrium ponit, Celest. Mech., 17, 267–280 (1978)
[16] M. Hénon and C. Heiles, The applicability of the third integral of motion: some numerical experiments, Astron. J., 69, 73–79 (1964)
[17] G.I. Karanis and L.Ch. Vozikis, Fast detection of chaotic behavior in galactic potentials, Astron. Nachr., 329, 403–412 (2007)
[18] S. Lang, Algebra, 3rd. Edition, Addison–Wesley (1993).
[19] N.G. Lloyd, Degree Theory, Cambridge Trends in Mathematics 73, Cambridge Univesity Press, Cambridge, New York-Melbourne, 1978.
[20] B.R. Miller, The Lissajous transformation III. Parametric bifurcations, Celest. Mech. & Dynam. Astron. 51, 251–270 (1991)
[21] P. Olver, Classical invariant theory, London Math. Soc. Student Texts, Vol. 44, Cambridge Univ. Press, New York (1999).
[22] F. Verhulst, Nonlinear dierential equations and dynamical systems, Universitext, Springer, 1991.
[23] E.E. Zotos, Application of new dynamical spectra of orbits in Hamiltonian systems, Nonlinear Dyn., 69, 2041–2063 (2012)
[24] E.E. Zotos, The fast norm vector indicator (FNVI) method: a new dynamical parameter for detecting order and chaos in Hamiltonian systems, Nonlinear Dyn., 70, 951–978 (2012)

[1] Departament de Tecnologies Digitals i de la Informació, Escola Politècnica Superior, Universitat de Vic - Universitat Central de Catalunya (UVic-UCC), C. de la Laura, 13, 08500 Vic, Catalonia, Spain
*E-mail address*: montserrat.corbera@uvic.cat

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
*E-mail address*: jllibre@mat.uab.cat

[3] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049-001, Lisboa, Portugal
*E-mail address*: cvalls@math.ist.utl.pt