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CENTER PROBLEM FOR SYSTEMS WITH TWO MONOMIAL NONLINEARITIES

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ABSTRACT. We study the center problem for planar systems with a linear center at the origin that in complex coordinates have a nonlinearity formed by the sum of two monomials. Our first result lists several centers inside this family. To the best of our knowledge this list includes a new class of Darboux centers that are also persistent centers. The rest of the paper is dedicated to try to prove that the given list is exhaustive. We get several partial results that seem to indicate that this is the case. In particular, we solve the question for several general families with arbitrary high degree and for cases of degree less or equal than 10. The CPU computing time used for obtaining all the significant Poincaré–Lyapunov constants for all these low degree cases has been of around 4 months. As a byproduct of our study we also obtain the highest known order for weak-foci of planar polynomial systems of some given degrees.

1. Introduction and statement of the main results

The center-focus problem consists in distinguishing whether a monodromic singular point is a center or a focus. For singular points with imaginary eigenvalues, usually called *nondegenerate singular points*, this problem was already solved by Poincaré and Lyapunov, see [19, 21, 22]. The solution consists in computing several quantities called commonly the *Poincaré–Lyapunov constants*, and study whether they are zero or not. There are different methods to compute them, for a brief survey of these methods see [4, 10, 12, 13, 20] and the references therein.

Despite the existence of many methods, the solution of the center-focus problem for simple families, like for instance the complete cubic systems or the quartic systems with homogeneous nonlinearities, has resisted all the attempts, see for instance [11, 18, 24]. For this reason, in this paper and following [17], we propose to push on this question in another direction. We study this problem for a natural family of differential systems with four real (two complex) free parameters but arbitrary degree. Before introducing this "simple" family of differential systems we recall the formulation of the center-focus problem for nondegenerate singular points in complex coordinates.

A real analytic planar differential system with a weak-focus can always be written as

$$\dot{x} = -y + \mathcal{P}(x,y) = -y + \sum_{j\geq 2} \mathcal{P}_j(x,y), \quad \dot{y} = x + \mathcal{Q}(x,y) = x + \sum_{j\geq 2} \mathcal{Q}_j(x,y),$$

where $\mathcal{P}_j$ and $\mathcal{Q}_j$ are real homogeneous polynomials of degree j. Equivalently, in complex notation, it writes as the equation

$$\dot{z} = iz + F(z,\bar{z}) = iz + \sum_{k+\ell\geq 2} f_{k,\ell} z^k \bar{z}^\ell, \tag{1}$$

where $z = x + iy$ and $f_{k,\ell}$ are complex numbers obtained from the coefficients of the polynomials $\mathcal{P}_j$ and $\mathcal{Q}_j$.

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The center-focus problem for equation (1) consists in finding necessary and sufficient conditions on the coefficients $f_{k,\ell}$ to distinguish if the origin is a center or a focus. Complex notation has often been used in several works, see for instance [9, 14, 18, 24, 25].

In this paper we study this problem for the family of differential equations

$$\dot{z} = iz + Az^k\bar{z}^\ell + Bz^m\bar{z}^n \tag{2}$$

where $k+\ell \le m+n$, $(k,\ell) \ne (m,n)$ and $A, B \in \mathbb{C}$. As we will see, the integer values

$$\alpha = k-\ell-1, \quad \beta = m-n-1,$$

will play a key role in our study. One of the reasons for this special role of both numbers is that when $\alpha = 0$ (resp. $\beta = 0$) the monomial $z^k\bar{z}^\ell$ (resp. $z^m\bar{z}^n$) appears as a *resonant* monomial in the Poincaré normal form of the complex differential equation.

Our first result lists the centers that we have found in family (2). To the best of our knowledge the centers presented in statement (d) when $k = m \ge 1$ are not known. They belong to a bigger new class of Darboux centers presented in Theorem 3. In fact, as we will see in Section 2, this family contains some new persistent and weakly persistent centers, see [4] for a first study of this class of centers.

Theorem 1. *The origin of equation* (2) *is a center when one of the following (nonexclusive) conditions hold:*

(*a*) $k = n = 2$ *and* $\ell = m = 0$ *(quadratic Darboux centers).*
(*b*) $\ell = n = 0$ *(holomorphic centers).*
(*c*) $A = -\bar{A}\, \mathrm{e}^{i\,\alpha\,\varphi}$ *and* $B = -\bar{B}\, \mathrm{e}^{i\,\beta\,\varphi}$ *for some* $\varphi \in \mathbb{R}$ *(reversible centers).*
(*d*) $k = m$ *and* $(\ell - n)\alpha \ne 0$ *(Hamiltonian or new Darboux centers).*

We remark that when in our work we say reversible center we refer to a center that has a line of symmetry in the sense already introduced by Poincaré. There is a more general notion of reversible center (with respect to general curves) which is not used in this paper. In this broader sense, it can be seen by using normal form theory, that every analytic nondegenerate center is reversible with respect to an analytic curve.

We also remark that in the above theorem, the names between brackets classifying the centers are only for orientation and they neither give an exclusive classification. For instance, in case (d) the Hamiltonian systems are the ones satisfying that $k = m = 0$ but in (c) there are reversible centers that are Hamiltonian as well, see Proposition 7(a).

The main goal of this paper is to investigate if the above list of centers is complete. We prove:

Theorem 2. *In the following cases the list of centers given in Theorem 1 is complete:*

(*a*) *When* $A\,B = 0$.
(*b*) *When* $\alpha\,\beta = 0$.
(*c*) *When* $(\alpha+\beta)(\alpha-\beta) = 0$.
(*d*) *When* k, ℓ, m *and* n *satisfy* $p\,\alpha + q\,\beta = (k+\ell-1)Q - (m+n-1)P = 0$, *for some* P, Q, p *and* q, *where* $P \le Q$ *and* $\mathcal{N}(P,Q)$ *are given in Table 1 and* $(p,q) \in \mathbb{N} \times \mathbb{Z}$ *are such that* $pP + |q|Q \le \mathcal{N}(P,Q)$.
(*e*) *When the nonlinearities are homogeneous* $(k+\ell = m+n = d)$ *and either* d *is even and* $d \le 34$ *or* d *is odd and* $d \le 57$.
(*f*) *When* $4 \le k+\ell+m+n \le 20$.

In cases (a) and (b) of Theorem 2 the result is already known, see [4, 17]. We provide short and different proofs that do not need the computation of Poincaré–Lyapunov constants, see Sections 3.2 and 3.3, respectively. Observe that all cases but (a) define varieties in the discrete space of the exponents, $(k,\ell,m,n) \in \mathbb{N}^4$. Specifically, the dimension of cases (b), (c), (d), (e) and (f) are 3, 3, 2, 0 and 0, respectively. In other words, while cases (e)

$P\backslash Q$	1	2	3	4	5	6
1	8	10	13	13	15	15
2	-	-	19	-	19	-
3	-	-	-	23	23	-

Table 1. Values of $\mathcal{N}(P,Q)$ for $P \le Q$ and coprime P and Q.

and (f) refer to fixed degree equations, all the other cases deal with unbounded degree families.

To the best of our knowledge the characterization of the centers for case (c) is new and it constitutes one of the main results of this paper, see Theorem 8. Also the results given in (d), (e) and (f) are new. A key point for our proof for cases (c) and (d), see Sections 3.4 and 3.5, respectively, has been a reparametrization of the degrees k,ℓ,m,n in terms of some P,Q that provides a compact expression of equation (2) in polar coordinates, see equation (9) below. We remark that our proof of (e) and (f) covers the low degree cases and includes all the particular ones solved in [17]. It uses the algorithm developed in [10], that we recall in Section 4 for completeness.

Statements (e) and (f) of Theorem 2 are proved with a case by case study, computing all the necessary Poincaré–Lyapunov constants to solve the center-focus problem. In Sections 3.6 and 3.7, we detail some of these results, including the total CPU time for each of them. For instance, the results for cases (d), (e), and (f) needed around 15, 39, and 58 days of CPU time, respectively. We have used MAPLE 18 in a Xeon computer (CPU E5-450, 3.0 GHz, RAM 32 Gb) with GNU Linux for all the computations except one special case that has needed more RAM memory. In Section 3.7 we include some remarks about the computational difficulties.

In general, to predict the exact number of Poincaré–Lyapunov constants needed to solve the center-focus problem or, equivalently, to know which is the highest order weak-focus in a given family is a difficult and intriguing question. Our study for proving statement (e), see Section 3.6, suggests an answer in terms of k,ℓ,m and n for the homogeneous nonlinearities case. As a consequence of our results we also give the highest known order for weak-foci of polynomial systems of odd degree d when $d \le 87$, see Proposition 10. In Section 3.7 we also detail the values that we have found for the case (f) and we explain why such a prediction is complicated for the nonhomogeneous family. In particular, the highest order weak-focus found for this last case has been 88, it corresponds to the Poincaré–Lyapunov constant V_{177}, and it happens when $(k,\ell,m,n) = (8,0,1,11)$.

Notice that from statements (a) and (b) of Theorem 2 the center-focus problem for equation (2) is totally solved when $\alpha\beta = 0$ or $AB = 0$. Therefore, as we will see in Section 3.4, instead of dealing with equation (2) we can reduce our study by considering

$$\dot{z} = iz + z^k \bar{z}^\ell + C z^m \bar{z}^n, \tag{3}$$

with $k+\ell \le m+n$, $(k,\ell) \ne (m,n)$, $\alpha\beta \ne 0$ and $0 \ne C \in \mathbb{C}$. We also remark that for equation (3) the characterization of the reversible centers given in Theorem 1(c) reduces to

$$C^{|q|} + (-1)^{p+|q|+1}\bar{C}^{|q|} = 0, \tag{4}$$

where $(p,q) \in \mathbb{N}\times\mathbb{Z}$ are the coprime values such that $p\alpha + q\beta = 0$.

The results of this paper suggest the following challenging question: Is the list of centers of equation (3) presented in Theorem 1 exhaustive? All our attempts seem to indicate that the answer is yes.

In the particular case of homogeneous nonlinearities the above question reduces to: Is it true that when $k+\ell = m+n \ge 3$ all the centers of equation (3) are reversible?

2. Sufficient center conditions

It is said that the origin of (1) is a *persistent center* if it is center for $\dot z = iz+\lambda F(z,\bar z)$ for all $\lambda \in \mathbb{C}$ and the origin is a *weakly persistent center* if it is a center for $\dot z = iz+\mu F(z,\bar z)$ for all $\mu \in \mathbb{R}$, see [4]. The following theorem gives a new and large family of planar differential systems with persistent or weakly persistent centers at the origin.

Theorem 3. *Consider the differential equation* (1) *where* $F(z,\bar z) = z^k f(\bar z) = z^k \sum_{\ell\geq 0} f_\ell \bar z^\ell$ *for* $k \geq 0$*, and* F *starts at the origin at least with second degree terms. Then the origin is a center if and only if either* $k \in \{0,1\}$ *or* $k > 1$ *and* $\operatorname{Re}(f_{k-1}) = 0$*. Indeed, in all cases the origin is a weakly persistent center, Hamiltonian when* $k = 0$ *and of Darboux type when* $k \geq 1$*. Moreover, it is a persistent center when* $k \in \{0,1\}$ *or* $k \geq 1$ *and* $f_{k-1} = 0$*.*

Proof. Firstly observe that for $k > 0$, $U(z,\bar z) = z\bar z = 0$ is an invariant algebraic curve for $\dot z = iz + z^k f(\bar z)$ because

$$\dot U(z,\bar z) = \dot z\bar z + z\dot{\bar z} = 2\operatorname{Re}\big(z^{k-1}F(\bar z)\big)U(z,\bar z).$$

We start proving that the function $U^{-k}(z,\bar z) = (z\bar z)^{-k}$, constructed from this invariant algebraic curve, is an integrating factor of the differential equation. This is precisely the definition of a Darboux integrable equation, see for instance the survey [15].

It is easy to see that if X is the vector field associated to a differential equation $\dot z = G(z,\bar z)$ then

$$\operatorname{div}(X) = 2\operatorname{Re}\Big(\frac{\partial}{\partial z}\big(G(z,\bar z)\big)\Big).$$

Therefore, if X is the vector field associated to $\dot z = \big(iz + z^k f(\bar z)\big)(z\bar z)^{-k}$ it holds that

$$\operatorname{div}(X) = 2\operatorname{Re}\Big(\frac{\partial}{\partial z}\big(\big(iz + z^k f(\bar z)\big)(z\bar z)^{-k}\big)\Big) = 2(1-k)\operatorname{Re}\big(i\,(z\bar z)^{-k}\big) \equiv 0,$$

as we wanted to show. Let us compute a real first integral of our differential equation. From the equation

$$\frac{\partial H_1(z,\bar z)}{\partial \bar z} = \big(iz + z^k f(\bar z)\big)(z\bar z)^{-k},$$

we obtain that

$$H_1(z,\bar z) = \begin{cases} \frac{(z\bar z)^{1-k}}{1-k} i + g(\bar z) - \overline{g(\bar z)}, & \text{if} \quad k \neq 1,\\ \log(z\bar z) i + g(\bar z) - \overline{g(\bar z)}, & \text{if} \quad k = 1,\end{cases}$$

where g satisfies $g'(u) = f(u)u^{-k}$. Therefore the real function

$$H_2(z,\bar z) = \begin{cases} \frac{(z\bar z)^{1-k}}{2(1-k)} + \operatorname{Im}\big(g(\bar z)\big), & \text{if} \quad k \neq 1,\\ \frac{\log(z\bar z)}{2} + \operatorname{Im}\big(g(\bar z)\big), & \text{if} \quad k = 1,\end{cases}$$

is a candidate to be a first integral of the equation in $\mathbb{C}\setminus\{0\}$. Notice that, since the origin is a monodromic critical point, to prove that it is a center it suffices to construct a smooth first integral of $\dot z = iz + z^k f(\bar z)$ that is continuous at the origin.

When $k = 0$ we have that

$$H(z,\bar z) = H_1(z,\bar z) = \frac{z\bar z}{2} + \operatorname{Im}(g(\bar z)),$$

where $g(u) = \displaystyle\int_0^u f(s)\,ds$ is a smooth first integral at the origin. Therefore, in this case we are done.

When $k = 1$, consider the first integral

$$H(z,\bar z) = \mathrm{e}^{2H_1(z,\bar z)} = z\bar z\,\mathrm{e}^{2\operatorname{Im}(g(\bar z))},$$

with $g(u)=\displaystyle\int_0^u f(s)/s\,ds$. It is smooth at the origin because by hypotheses $f(0)=0$. Hence, for $k=1$ we have also proved that the origin is a center.

Finally consider $k>1$. In this case we define

$$H(z,\bar z)=\frac{1}{H_1(z,\bar z)}=\frac{(z\bar z)^{k-1}}{\frac{1}{2(1-k)}+(z\bar z)^{k-1}\operatorname{Im}(g(\bar z))},$$

where

$$g(u)=\int_{u_0}^u \frac{f(s)}{s^k}\,ds=\int_{u_0}^u\sum_{\ell\ge 0} f_\ell s^{\ell-k}\,ds=\sum_{\ell\ge 0,\,\ell\ne k-1}\frac{f_\ell}{\ell-k+1}u^{\ell-k+1}+f_{k-1}\log u+g_0,$$

for some $g_0\in\mathbb{C}$.

To ensure that the above function is not multivaluated in $\mathbb{C}$ we need to impose that $\operatorname{Im}(f_{k-1}\log\bar z)$ is not multivaluated. This forces that $\operatorname{Re}(f_{k-1})=0$. Under this hypothesis the function $H(z,\bar z)$ is well defined in $\mathbb{C}$ and moreover it is continuous at the origin, because

$$\lim_{z\to 0}(z\bar z)^{k-1}\operatorname{Im}(g(\bar z))=0.$$

Hence, when $k>1$ and $\operatorname{Re}(f_{k-1})=0$ the origin of our differential equation has a center, as we wanted to prove. When, $\operatorname{Re}(f_{k-1})\ne 0$ the same expression of H implies that the origin is not a center. Therefore, the characterization of the centers follows.

It is clear that when $\operatorname{Re}(f_{k-1})=0$ the centers are weakly persistent and when $f_{k-1}=0$ they are persistent. Hence the theorem is proved. □

We will also use the following well-known proposition to characterize the reversible and the holomorphic centers, see [3, Prop. 7] and [8]. For completeness we also include its proof.

Proposition 4. *Equation* (1) *has a center at the origin when one of the following conditions holds:*

(*a*) *There exists* $\varphi\in\mathbb{R}$ *such that* $f_{k,\ell}=-\bar f_{k,\ell}\,\mathrm{e}^{i(k-\ell-1)\varphi}$ *for all* k,ℓ *(reversible center).*

(*b*) $F(z,\bar z)\equiv F(z)$ *(holomorphic center).*

Proof. (a) Assume that $f_{k,\ell}=-\bar f_{k,\ell}\,\mathrm{e}^{i(k-\ell-1)\varphi}$ holds for all k and ℓ. Let us prove that if $z=Z(t)$ is a solution of the differential equation (1), then $z=\mathrm{e}^{-i\varphi}\,\bar Z(-t)$ is a solution as well. To do this, consider the transformation $w=\mathrm{e}^{-i\varphi}\,\bar z$, $s=-t$. Then

$$\begin{aligned}\frac{dw}{ds}&=-\,\mathrm{e}^{-i\varphi}\,\dot{\bar z}=-\,\mathrm{e}^{-i\varphi}\big(-i\bar z+\sum_{k+\ell\ge 2}\bar f_{k,\ell}\,\bar z^k z^\ell\big)\\&=iw-\sum_{k+\ell\ge 2}\bar f_{k,\ell}\,\mathrm{e}^{i(k-\ell-1)\varphi}\,w^k\bar w^\ell=iw+\sum_{k+\ell\ge 2}f_{k,\ell}\,w^k\bar w^\ell=iw+F(w,\bar w),\end{aligned}$$

as we wanted to prove. Fix a small enough neighborhood of the origin. In this neighborhood, let $\Gamma^+=\{Z(t)\,:\,0\le t\le t_1>0\}$ be a piece of the solution of (1) between two consecutive cuts of $Z(t)$ with the straight line with slope $\theta=-\varphi/2$. These cuts exist because the origin is a monodromic weak-focus. Therefore,

$$Z(0)=r_0\,\mathrm{e}^{-i\varphi/2},\quad Z(t_1)=r_1\,\mathrm{e}^{-i\varphi/2+i\pi},$$

for some $r_0>0,r_1>0$, small enough. Then $\Gamma^-=\{W(t)=\mathrm{e}^{-i\varphi}\,\bar Z(-t)\,:\,-t_1\le t\le 0\}$ is another piece of the solution of (1), and moreover

$$\begin{aligned}W(0)&=\mathrm{e}^{-i\varphi}\,\bar Z(0)=\mathrm{e}^{-i\varphi}\,r_0\,\mathrm{e}^{i\varphi/2}=r_0\,\mathrm{e}^{-i\varphi/2}=Z(0),\\W(-t_1)&=\mathrm{e}^{-i\varphi}\,\bar Z(t_1)=\mathrm{e}^{-i\varphi}\,r_1\,\mathrm{e}^{i\varphi/2-i\pi}=r_1\,\mathrm{e}^{-i\varphi/2+i\pi}=Z(t_1).\end{aligned}$$

By the uniqueness of solutions, joining both pieces Γ^+ and Γ^- we obtain that the solution passing through $Z(0)$ is periodic, with period $2t_1$. As a consequence, the origin of equation (1) is a center, as we wanted to show.

(b) When $F(z,\bar z)\equiv F(z)$ we can write equation (1) as

$$\dot z = iz + F(z) = z\,G(z),$$

with G a holomorphic function such that $G(0)=i$. Following [8] we consider the holomorphic map

$$\Phi(z) = z\exp\Big(\int_0^z \frac{i-G(s)}{s\,G(s)}ds\Big).$$

Notice that it is well defined and invertible in a neighborhood of $z=0$ because $\Phi(0)=0$ and $\Phi'(0)\neq 0$. In fact, let us see that the local change of variables $w=\Phi(z)$ is a local holomorphic conjugacy between $\dot z = F(z)$ and $z = iz$, or in other words, a holomorphic linearization of the differential equation. In fact, if $w=\Phi(z)$ we have

$$\begin{aligned}\dot w = \Phi'(z)\,\dot z =&\exp\Big(\int_0^z \frac{i-G(s)}{s\,G(s)}ds\Big)\Big(1+z\,\frac{i-G(z)}{z\,G(z)}\Big)z\,G(z)\\ =&i\,z\exp\Big(\int_0^z \frac{i-G(s)}{s\,G(s)}ds\Big) = i\,\Phi(z) = i\,w.\end{aligned}$$

Therefore the origin of equation (1) is an (isochronous) center, as we wanted to prove. □

Proof of Theorem 1. Equation (2) under condition (a) is a well-known class of Darboux quadratic centers, see [26]. The result under conditions (b) and (c) is a corollary of Proposition 4. The proof for family (d) is a straightforward consequence of Theorem 3. □

3. Proof of Theorem 2

This section is divided into seven subsections. In the first one we recall the definition of the Poincaré–Lyapunov constants. Each one of the remainder subsections corresponds to the proof of one of the statements of Theorem 2. The results dealing with cases (a), (b), and (c) are proved analytically. The remainder ones are obtained by doing almost all the algebraic computations with MAPLE 18. For case (d), the significant center conditions are obtained not working directly with the differential equation in polar coordinates but using a preliminary simplification that, as we will see, will also constitute the key point for studying case (c). The Poincaré–Lyapunov constants for cases (e) and (f) are obtained by using the method developed in [10]. We briefly summarize it in an appendix, see Section 4.

3.1. **Poincaré–Lyapunov constants.** Differential equation (1) can be written in polar coordinates, $z = r\,\mathrm{e}^{i\theta}$, as

$$\dot r = \sum_{j\geq 2}\mathrm{Re}(S_j(\theta))r^j, \qquad \dot\theta = 1+\sum_{j\geq 2}\mathrm{Im}(S_j(\theta))r^{j-1},$$

where $S_j(\theta) = \bar z F_j(z,\bar z)|_{z=\mathrm{e}^{i\theta}}$. Then, in a neighborhood of $r=0$, it can be studied through the differential equation

$$\frac{dr}{d\theta} = \frac{\sum_{j\geq 2}\mathrm{Re}(S_j(\theta))\,r^j}{1+\sum_{j\geq 2}\mathrm{Im}(S_j(\theta))\,r^{j-1}}. \tag{5}$$

Denote by $r(\theta;\eta)$ the solution of (5) such that $r=\eta\geq 0$ when $\theta=0$. For r small enough, we can write

$$r(\theta;\eta) = \eta + \sum_{j=2}^{\infty} v_j(\theta)\,\eta^j,$$

with $v_j(0) = 0$ for $j \geq 2$. The Poincaré return map is defined as

$$\Pi(\eta) = r(2\pi, \eta) = \eta + \sum_{k=2}^{\infty} v_k(2\pi)\, \eta^k.$$

Furthermore, when $v_2(2\pi) = \cdots = v_{N-1}(2\pi) = 0$, the value of $V_N = v_N(2\pi)$ is the *N-th Poincaré–Lyapunov constant* of equation (1). It is well known that the first N such that $V_N \neq 0$ is always odd, see [1, p. 243]. The number N_0, where $N = 2N_0 + 1$, is called the *order* of the weak-focus.

3.2. **The case $AB = 0$.** As we have already said, the result in this case is well known, see [4, 17]. We provide a proof for the sake of completeness. It uses the next general result that gives necessary conditions for a system to have a center.

Lemma 5. *Let $\dot z = G(z, \bar z)$ and $\dot z = H(z, \bar z)$ be two smooth differential equations with a critical point at the origin. If one of the equations has a center at the origin and*

$$\mathrm{Im}(G(z, \bar z)\overline{H(z, \bar z)}) = \gamma(z\bar z)^j + O(2j + 1), \quad \gamma \in \mathbb{R},$$

then a necessary condition for the other equation to have a center at the origin is $\gamma = 0$.

Proof. Notice that $\mathrm{Im}(G(z, \bar z)\overline{H(z, \bar z)})$ gives the scalar product between the vector field associated to $\dot z = G(z, \bar z)$ and the orthogonal of the vector field associated to $\dot z = H(z, \bar z)$. Therefore, if $\gamma \neq 0$, the above equality implies that, in a neighborhood of the origin, the level curves of the solutions of the equation having a center are without contact for the flow associated to the other equation, giving the impossibility of having a center for the second equation. Hence $\gamma = 0$ is a necessary condition to have a center for the other differential equation. □

Lemma 6. *The differential equation $\dot z = iz + Az^k \bar z^\ell$ has a center at the origin if and only if either $\alpha = 0$ and $\mathrm{Re}(A) = 0$ or $\alpha \neq 0$.*

Proof. When $\alpha \neq 0$ all the differential equations have a reversible center by Theorem 1(c) because the equation $A = -\bar A\, \mathrm{e}^{i\alpha\varphi}$ has always solution. When $\alpha = 0$, consider the equation $\dot z = G(z, \bar z) = iz$ with a center at the origin and $\dot z = H(z, \bar z) = iz + Az^{\ell+1}\bar z^\ell$. Then $\mathrm{Im}(G(z, \bar z)\overline{H(z, \bar z)}) = \mathrm{Re}(A)\,(z\bar z)^{\ell+1}$ and by Lemma 5 the proof finishes. □

3.3. **The case $\alpha\beta = 0$.** The proof of Theorem 2(b) follows from the next result.

Proposition 7. *Equation (2) with $\alpha\,\beta = 0$ has a center at the origin if and only if one of the following three conditions holds.*

(*a*) $\alpha = \beta = 0$ *and* $\mathrm{Re}(A) = \mathrm{Re}(B) = 0$ *(reversible Hamiltonian center).*
(*b*) $\alpha = 0, \beta \neq 0$ *and* $\mathrm{Re}(A) = 0$ *(reversible center).*
(*c*) $\alpha \neq 0, \beta = 0$ *and* $\mathrm{Re}(B) = 0$ *(reversible center).*

Proof. The three conditions are included in Theorem 1(c), therefore their sufficiency is proved. To show that they are necessary we will apply again Lemma 5.

(a)-(b) We take $G(z, \bar z) = iz$ and $H(z, \bar z) = iz + Az^{\ell+1}\bar z^\ell + Bz^m \bar z^n$. A simple computation gives that

$$\mathrm{Im}(G(z, \bar z)\overline{H(z, \bar z)}) = \mathrm{Re}(A)(z\bar z)^{\ell+1} + \mathrm{Im}(i\,\bar B)z^{n+1}\bar z^m.$$

Therefore, $\mathrm{Re}(A) = 0$ is a first necessary center condition. In case (b) we are done. In case (a), then $m = n + 1$ and therefore $\mathrm{Im}(i\,\bar B)z^{n+1}\bar z^m = \mathrm{Re}(B)(z\bar z)^{n+1}$ and $\mathrm{Re}(A) = \mathrm{Re}(B) = 0$ are the center conditions, as we wanted to prove.

(c) Here we take $G(z, \bar z) = iz + Az^k\bar z^\ell + \mathrm{Im}(B)i\,z^{n+1}\bar z^n$ and $H(z, \bar z) = iz + Az^k\bar z^\ell + Bz^{n+1}\bar z^n$. By Theorem 1(c) the origin of $\dot z = G(z, \bar z)$ is a center. Direct computations lead to

$$\mathrm{Im}(G(z, \bar z)\overline{H(z, \bar z)}) = \mathrm{Re}(B)(z\bar z)^{n+1} + O(2n + 3).$$

Hence, $\mathrm{Re}(B) = 0$ is the necessary condition to have a center. □

3.4. **The case** $(\alpha+\beta)(\alpha-\beta)=0$**.** This section proves Theorem 2(c). Here we will assume that $\alpha\beta\neq 0$ and $AB\neq 0$ because the cases $\alpha\beta=0$ or $AB=0$ are already studied in the previous sections. Under these conditions there exists $\lambda\in\mathbb{C}$ such that the change of variables $w=\lambda z$ writes equation (2) as

$$\dot z = iz + z^k\bar z^\ell + Cz^m\bar z^n, \tag{6}$$

for some $C=c_1+ic_2\in\mathbb{C}\setminus\{0\}$. Consider the following reparametrization

$$k+\ell-1=PM, \quad m+n-1=QM,$$

with $P=Q=1$ when (6) has homogeneous nonlinearities or $P<Q$ and coprime otherwise. Then equation (6) writes

$$\dot z = iz + (z\bar z)^{\frac{PM}{2}} z^{\frac{\alpha+2}{2}}\bar z^{\frac{-\alpha}{2}} + C(z\bar z)^{\frac{QM}{2}} z^{\frac{\beta+2}{2}}\bar z^{\frac{-\beta}{2}}. \tag{7}$$

This equation in polar coordinates, $z=r\,\mathrm{e}^{i\theta}$, is

$$\frac{dr}{d\theta} = \frac{a_P(\theta)r^{PM+1}+a_Q(\theta)r^{QM+1}}{1+b_P(\theta)r^{PM}+b_Q(\theta)r^{QM}}, \tag{8}$$

where

$$\begin{aligned}
a_P(\theta) &= \frac{1}{2}(\mathrm{e}^{i\alpha\theta}+\mathrm{e}^{-i\alpha\theta}),\\
a_Q(\theta) &= \frac{1}{2}((c_1+ic_2)\,\mathrm{e}^{i\beta\theta}+(c_1-ic_2)\,\mathrm{e}^{-i\beta\theta}),\\
b_P(\theta) &= \frac{1}{2i}(\mathrm{e}^{i\alpha\theta}-\mathrm{e}^{-i\alpha\theta}),\\
b_Q(\theta) &= \frac{1}{2i}((c_1+ic_2)\,\mathrm{e}^{i\beta\theta}-(c_1-ic_2)\,\mathrm{e}^{-i\beta\theta}).
\end{aligned}$$

Doing the change of variables $r=R^{1/M}$ the above equation writes as the following differential equation

$$\frac{dR}{d\theta} = \frac{A_P(\theta)R^{P+1}+A_Q(\theta)R^{Q+1}}{1+B_P(\theta)R^{P}+B_Q(\theta)R^{Q}}, \tag{9}$$

where $A_P=Ma_P$, $A_Q=Ma_Q$, $B_P=b_P$ and $B_Q=b_Q$. The characterization of the centers of equation (7) is equivalent to find conditions that imply that $u(2\pi;\rho)\equiv\rho$, for ρ small enough, where $u(\theta;\rho)$ is the solution of equation (9) such that $u(0;\rho)=\rho$.

In fact, let us see that if

$$u(2\pi;\rho)=\rho+U\rho^S+\cdots, \quad U\neq 0,$$

then the first significant Poincaré–Lyapunov constant for equation (1) is

$$V_{(S-1)M+1}=\frac{U}{M}. \tag{10}$$

Let $r(\theta,\eta)$ be the solution of (8) satisfying $r(0,\eta)=\eta$. Since $R=r^M$, it holds that

$$\begin{aligned}
r(2\pi;\eta)=\sqrt[M]{u(2\pi;\eta^M)} &= \sqrt[M]{\eta^M+U\eta^{MS}+\cdots}=\eta\sqrt[M]{1+U\eta^{(S-1)M}+\cdots}\\
&=\eta\Big(1+\frac{U}{M}\eta^{(S-1)M}+\cdots\Big)=\eta+\frac{U}{M}\eta^{(S-1)M+1}+\cdots,
\end{aligned}$$

as we wanted to see.

Theorem 8. *Assume that $\alpha\beta\neq 0$ and $(\alpha-\beta)(\alpha+\beta)=0$. Then the first significant Poincaré–Lyapunov constant for equation (6) is*

$$V_{k+\ell+m+n-1}=2\pi\frac{(k-m)c_2}{\beta}.$$

Moreover, the origin is a center if and only if $c_2=0$ (reversible) or $k=m$ (Hamiltonian or Darboux).

Notice that $k+\ell+m+n-1 = 2(\ell+m)-1$ when $\alpha = \beta$ and $k+\ell+m+n-1 = 2(\ell+n)+1$ when $\alpha = -\beta$ and, as it must be, the Poincaré–Lyapunov constant has an odd subindex.

Proof of Theorem 8. A first step for proving the theorem will be to show that

$$u(2\pi;\rho) = \begin{cases} \rho + \dfrac{\pi M^2 c_2}{\beta}(P-Q)\rho^{P+Q+1} + \cdots & \text{if} \quad \alpha-\beta = 0, \\ \rho + \dfrac{\pi M c_2}{\beta}(M(P-Q)-2\beta)\rho^{P+Q+1} + \cdots & \text{if} \quad \alpha+\beta = 0, \end{cases} \tag{11}$$

where $u(\theta;\rho)$ denotes the solution of equation (9) such that $u(0;\rho) = \rho$.

Developing the right hand side of equation (9) in power series of R up to order $P+Q+1$ we obtain

$$\begin{aligned} &(A_P R^{P+1} + A_Q R^{Q+1})\big(1 - (B_P R^P + B_Q R^Q) + (B_P R^P + B_Q R^Q)^2 + \cdots\big) = \\ &(A_P R^{P+1} + A_Q R^{Q+1})\big(1 - B_P R^P - B_Q R^Q + B_P^2 R^{2P} - B_P^3 R^{3P} + \cdots \\ &+ (-1)^{K-1} B_P^{K-1} R^{(K-1)P} + \cdots\big) = \\ &A_P R^{P+1} - A_P B_P R^{2P+1} + A_P B_P^2 R^{3P+1} + \cdots + (-1)^{K-1} A_P B_P^{K-1} R^{KP+1} + \\ &A_Q R^{Q+1} - (A_P B_Q + A_Q B_P) R^{P+Q+1} + \cdots, \end{aligned} \tag{12}$$

where K is the maximal natural number satisfying $P(K-1) \le Q$.

There are only two cases to study: (i) $KP+1 < P+Q+1$ and (ii) $KP+1 = P+Q+1$.

First we study the case (i). In this case all the exponents are different because $(K-1)P+1 < Q+1 < KP+1 < P+Q+1$. We propose the solution $u(\theta;\rho)$ of (9) in power series of ρ with the same exponents that (12), that is

$$\begin{aligned} u(\theta;\rho) =& \rho + u_1(\theta)\rho^{P+1} + u_2(\theta)\rho^{2P+1} + \cdots + u_{K-1}(\theta)\rho^{(K-1)P+1} \\ &+ w_1(\theta)\rho^{Q+1} + u_K(\theta)\rho^{KP+1} + w_2(\theta)\rho^{P+Q+1}. \end{aligned}$$

Substituting this expression in equation (12) we obtain the recursive first order differential system

$$u'_{J+1}(\theta) = (-1)^J A_P(\theta) B_P^J(\theta) + \sum_{j=1}^{J} (-1)^{J-j}((J-j+1)P+1) A_P(\theta) B_P^{J-j}(\theta) u_j(\theta), \tag{13}$$

with $u_J(0) = 0$, for $J \in \{0,\ldots,K-1\}$ and

$$w'_1(\theta) = A_Q(\theta), \tag{14}$$

$$w'_2(\theta) = (P+1)A_P(\theta)w_1(\theta) + (Q+1)A_Q(\theta)u_1(\theta) - (A_P(\theta)B_Q(\theta) + A_Q(\theta)B_P(\theta)), \tag{15}$$

with $w_1(0) = w_2(0) = 0$.

Lemma 9. *When $\alpha \ne 0$, for any given $K \in \mathbb{N}$, the solutions of the initial value problem (13) are $u_{J+1}(\theta) = d_{J+1}\sin^{J+1}(\alpha\theta)$ for some real constant d_J, for $J = 0,\ldots,K-1$.*

Proof. The first value problem is $u'_1(\theta) = A_P = M\cos(\alpha\theta)$, $u_1(0) = 0$. Hence the solution is $u_1(\theta) = \frac{M}{\alpha}\sin(\alpha\theta)$. The proof follows by an induction process. Assuming that $u_j(\theta) = d_j \sin^j(\alpha\theta)$, for $j \in 1,\ldots,J$, with $1 \le J < K-1$, then equation (13) becomes

$$\begin{aligned} u'_{J+1}(\theta) =& M\ \cos(\alpha\theta)\sin^J(\alpha\theta) + \\ & \sum_{j=1}^{J}(-1)^{J-j}((J-j+1)P+1)\, M\ \cos(\alpha\theta)\sin^{J-j}(\alpha\theta) d_j \sin^j(\alpha\theta) \\ =& M\big(1 + \sum_{j=1}^{J}(-1)^{J-j}((J-j+1)P+1)d_j\big)\cos(\alpha\theta)\sin^J(\alpha\theta) \\ =& (J+1)\,\alpha\, d_{J+1}\cos(\alpha\theta)\sin^J(\alpha\theta). \end{aligned}$$

Integrating the last expression and taking into account that $u_{J+1}(0) = 0$ the result follows. □

From Lemma 9 we obtain that $u_1(\theta) = \frac{M}{\alpha}\sin(\alpha\theta)$ and $u_{J+1}(2\pi) = 0$ for $J = 0, \dots, K-1$. Integrating equation (14) we obtain that $w_1(\theta) = \frac{M}{\beta}(c_1 \sin(\beta\theta) + c_2 \cos(\beta\theta))$ and $w_1(2\pi) = 0$. Integrating equation (15), under the hypotheses $\alpha+\beta = 0$ and $\alpha-\beta = 0$, and evaluating them at $\theta = 2\pi$ we obtain that $w_2(2\pi)$ is equal to the expressions given in (11) for each situation.

We continue with the second case (ii). Here there are only two pairs of equal exponents, $(K-1)P+1 = Q+1 < KP+1 = P+Q+1$. Therefore, the solution takes the form, up to order $P+Q+1$,

$$u(\theta;\rho) = \rho + u_1(\theta)\rho^{P+1} + u_2(\theta)\rho^{2P+1} + \cdots + u_{K-1}(\theta)\rho^{(K-1)P+1} + u_K(\theta)\rho^{KP+1} + \cdots .$$

Substituting this expression in equation (12) we obtain the recursive first order differential system

$$u'_{J+1}(\theta) = (-1)^J A_P(\theta)B_P^J(\theta) + \sum_{j=1}^{J}(-1)^{J-j}((J-j+1)P+1)A_P(\theta)B_P^{J-j}(\theta)u_j(\theta), \quad (16)$$

for $J = 0, \dots, K-3$ and

$$\begin{aligned} u'_{K-1}(\theta) =& A_Q(\theta) + (-1)^{K-2}A_P(\theta)B_P^{K-2}(\theta) && (17)\\ &+ \sum_{j=1}^{K-2}(-1)^{K-2-j}((K-2-j+1)P+1)A_P(\theta)B_P^{K-2-j}(\theta)u_j(\theta), \\ u'_K(\theta) =& (-1)^{K-1}A_P(\theta)B_P^{K-1}(\theta) && (18)\\ &+ \sum_{j=1}^{K-1}(-1)^{K-1-j}((K-1-j+1)P+1)A_P(\theta)B_P^{K-1-j}(\theta)u_j(\theta) \\ &+ (Q+1)A_Q(\theta)u_1(\theta) - (A_P(\theta)B_Q(\theta) + A_Q(\theta)B_P(\theta)), \end{aligned}$$

with $u_{J+1}(0) = 0$, $J \in \{0, \dots, K-1\}$. For $J = 0, \dots, K-3$, equations (16) and (13) coincide. Hence, from Lemma 9 we have that $u_{J+1}(2\pi) = 0$ for $J = 0, \dots, K-3$.

Notice that when $K = 2$ the expressions (16) and (17) are not well defined and have to be reinterpreted: equations (16) do not appear and equation (17) is $u'_1(\theta) = A_Q(\theta) + A_P(\theta)$. In fact, $K = 2$ corresponds to $k+\ell = m+n$ and $P = Q = 1$, i.e. the case of homogeneous nonlinearities in equation (6).

Integrating (17) we obtain $u_{K-1}(\theta) = d_{K-1}\sin^{K-1}(\alpha\theta) + \frac{M}{\beta}(c_1\sin(\beta\theta) + c_2\cos(\beta\theta))$. Consequently $u_{K-1}(2\pi) = 0$. Next, solving the last initial value problem (18) we get

$$u_K(\theta) = d_K \sin^K(\alpha\theta) + \int_0^\theta (\Delta_1(\theta) + \Delta_2(\theta) + \Delta_3(\theta))\, d\theta,$$

where

$$\begin{aligned} \Delta_1(\theta) &= (P+1)A_P(\theta)u_{K-1}(\theta), \\ \Delta_2(\theta) &= (Q+1)A_Q(\theta)u_1(\theta), \\ \Delta_3(\theta) &= -(A_P(\theta)B_Q(\theta) + A_Q(\theta)B_P(\theta)). \end{aligned}$$

Computing $\displaystyle\int_0^{2\pi} \Delta_i(\theta)\, d\theta$ for $i = 1,2,3$ and $\alpha \pm \beta = 0$, we obtain

$$\int_0^{2\pi}(P+1)M\cos(\alpha\theta)\Big(d_{K-1}\sin^{K-1}(\alpha\theta) + \frac{M}{\beta}(c_1\sin(\beta\theta) + c_2\cos(\beta\theta))\Big)\, d\theta = \frac{(P+1)M^2\pi}{\beta}c_2,$$

$$\int_0^{2\pi}(Q+1)\frac{M^2}{\alpha}\sin(\alpha\theta)\big(c_1\cos(\beta\theta)-c_2\sin(\beta\theta)\big)\,d\theta=-\frac{((K-1)P+1)M^2\pi}{\beta}c_2,$$

and

$$\int_0^{2\pi}-M\big(c_1\sin((\alpha+\beta)\theta)+c_2\cos((\alpha+\beta)\theta)\big)d\theta=\begin{cases}0 & \text{if } \alpha-\beta=0,\\ -2M\pi c_2 & \text{if } \alpha+\beta=0,\end{cases}$$

respectively. Finally,

$$u_K(2\pi)=\begin{cases}-\dfrac{M^2\pi c_2}{\beta}P(K-2) & \text{if } \alpha-\beta=0,\\ -\dfrac{M\pi c_2}{\beta}((K-2)MP+2\beta) & \text{if } \alpha+\beta=0.\end{cases}$$

These last values coincide with the values of (11) taking $Q=(K-1)P$ for the case (ii).

It turns out that, using that $PM=k+\ell-1$, $QM=m+n-1$, $\alpha=k-\ell-1$, $\beta=m-n-1$ and that either $\alpha-\beta=0$ or that $\alpha+\beta=0$, both expressions in (11) coincide, giving rise to

$$u(2\pi;\rho)=\rho+2\pi\frac{Mc_2(k-m)}{\beta}\rho^{P+Q+1}+\cdots.$$

Finally, using (10) with $S=P+Q+1$ and again that $PM=k+\ell-1$ and $QM=m+n-1$ we obtain that the first significant Poincaré–Lyapunov constant is $V_{k+\ell+m+n-1}=2\pi(k-m)c_2/\beta$, as we wanted to prove. This constant only vanishes when $c_2=0$ or $k=m$ giving rise to centers of type (c) or (d) in Theorem 1, respectively. This completes the proof of Theorem 8 and, consequently, Theorem 2(c).

3.5. **Case** (d) **in Theorem 2.** In this section we will describe the method that we use to prove that all the centers of the 2-parameter families given in Theorem 2(d) are the ones listed in Theorem 1.

We fix integer values of P, Q, p, and q satisfying the restrictions of the theorem. Let k,ℓ,m,n be such that $p\alpha+\beta q=0$ and $(k+\ell-1)Q-(m+n-1)P=0$. Notice that the latter condition implies that $k+\ell-1=PM$ and $m+n-1=QM$ for any $M\in\mathbb{N}$, because in Table 1 it is always satisfied that $\gcd(P,Q)=1$.

For these values, using the procedure described in Section 3.4, we get that the first significant center condition is of the form

$$u_{pP+|q|Q+1}(2\pi)=MD\pi\,\frac{C^{|q|}+(-1)^{p+|q|+1}\bar{C}^{|q|}}{i^{p+|q|+1}}E_L\left(\frac{M}{\beta}\right)$$

for a nonzero number $D\in\mathbb{Q}$ and a polynomial E_L of degree L with integer coefficients and such that $E_L(0)\neq 0$. Recall, that using (10), we get that this center condition corresponds to the Poincaré–Lyapunov constant $V_{p(k+\ell-1)+|q|(m+n-1)+1}$. When $C^{|q|}+(-1)^{p+|q|+1}\bar{C}^{|q|}=0$ the system always has a reversible center at the origin (see (4) and Theorem 1(c)) and we are done. Otherwise, $E_L(M/\beta)=0$. Since β and M are integer numbers, each rational root $s_j\in\mathbb{Q}$, of E_L, gives rise to a condition of the form $\beta=M/s_j$, which provides a new candidate to be a center.

Fixing each one of these rational roots we have got either a holomorphic center (Theorem 1(b)), or a Darboux center (Theorem 1(d)), or that we need to continue computing the next significant center condition. In this latter case, for all the studied equations, we have arrived to a reversible center at the origin.

For instance, when $P=1,Q=3$, according to Table 1, we have that $\mathcal{N}(1,3)=13$. This means that our study covers all $(p,q)\in\mathbb{N}\times\mathbb{Z}$ such that $p+3|q|\leq 13$. Our computations when $p+3|q|=13$ needed around one hour of CPU time with the software and computer characteristics explained in the introduction.

Next, to illustrate in more detail the procedure, we study the cases where $P=1,Q=3$ and $p+3|q|=6$, that is $p=3$ and $q=\pm 1$. When $q=1$, the first significant center condition is

$$u_7(2\pi)=\frac{-iM\pi}{4\beta^3}(2\beta+3M)(\beta+3M)(\beta+6M)(C-\bar{C}).$$

When $C-\bar{C}=0$ we obtain a reversible center. Assume now that $C-\bar{C}\neq 0$. When $2\beta+3M=0$ the center is of Darboux type. In the other two cases, $\beta+3M=0$ and $\beta+6M=0$, the centers are reversible because the next significant center conditions are

$$u_{13}(2\pi)=\frac{-9iM\pi}{10}(C-\bar{C})C\bar{C} \quad \text{and} \quad u_9(2\pi)=\frac{7iM\pi}{512}(C-\bar{C})C\bar{C},$$

respectively. Similarly, when $q=-1$, we obtain

$$u_7(2\pi)=\frac{-iM\pi}{4\beta^3}(\beta-3M)(\beta+3M)(\beta-6M)(C-\bar{C}).$$

When $C-\bar{C}=0$ we obtain again a reversible center. When $C-\bar{C}\neq 0$, the case $\beta-3M=0$ gives a holomorphic center, the case $\beta+3M=0$ is a Darboux center, and when $\beta-6M=0$ we have once more a reversible center because the next significant center condition is

$$u_9(2\pi)=\frac{7iM\pi}{512}(C-\bar{C})C\bar{C}.$$

Notice that the second significant center condition depends on the relation between β and M. In this case, it is either $u_9(2\pi)$ or $u_{13}(2\pi)$.

To have an idea of the computational effort needed to solve the center-focus problem in this case (d) we show in Table 2 the time needed for each fixed couple $P\leq Q$, $\gcd(P,Q)=1$ when $(p,q)\in\mathbb{N}\times\mathbb{Z}$ are such that $pP+|q|Q\leq\mathcal{N}(P,Q)$.

$P\backslash Q$	1	2	3	4	5	6
1	8 (263 h)	10 (4.5 h)	13 (8.5 h)	13 (2.8 h)	15 (2.2 h)	15 (2.5 h)
2	-	-	19 (60.1 h)	-	19 (0.6 h)	-
3	-	-	-	23 (3.3 h)	23 (0.6 h)	-

TABLE 2. Values of $\mathcal{N}(P,Q)$ and, in brackets, the total CPU time needed in each case to solve the center-focus problem.

In all the studied cases the polynomial E_L has some rational roots. In most of the cases, all them are rational and simple. Anyway, in very few cases there is a multiple rational root, for instance when $P=1,Q=2$, $2\alpha-3\beta=0$ or when $P=3,Q=5$, $\alpha-3\beta=0$. Sometimes E_L has an irreducible factor without rational roots, for example when $P=1,Q=2$, $4\alpha\pm 3\beta=0$ or when $P=1,Q=3$, $4\alpha\pm 3\beta=0$.

As we have already explained, when the polynomial E_L vanishes for some rational value it is necessary to go further in the computation of the center conditions. This second condition associated to a given rational root always has been enough to solve the center-focus problem. But, a priori, we do not know how far we need to go to reach this second significant center condition. In all the cases given in Table 1 the second significant condition is either $u_{(p+2)P+|q|Q+1}(2\pi)$ or $u_{pP+(|q|+2)Q+1}(2\pi)$.

Nevertheless, we present two cases not covered by Table 1, for which the second center condition is none of both conditions, illustrating that the question of knowing a priori how far we need to go to get all the significative center conditions is very intricate.

The first case is for $P = 3$, $Q = 14$ and $2\alpha + 3\beta = 0$. Following the procedure described in Section 3.4 we obtain, using $p = 2$, $q = 3$, that the first significant center condition is

$$u_{pP+|q|Q+1}(2\pi) = u_{49}(2\pi) = \frac{M\pi}{8\beta^4}(5\beta + 22M)(\beta - 2M)(\beta + 14M)(\beta + 2M)(C^3 + \bar{C}^3).$$

The first root, $5\beta + 22M = 0$, corresponds with a Darboux case. The other three roots correspond with reversible cases because the next center conditions are

$$\begin{aligned} u_{55}(2\pi) &= \frac{6919M\pi}{14}(C^3 + \bar{C}^3), \\ u_{55}(2\pi) &= -\frac{4617M\pi}{67228}(C^3 + \bar{C}^3), \\ u_{61}(2\pi) &= \frac{81M\pi}{50}(C^3 + \bar{C}^3), \end{aligned}$$

respectively, but notice that $(p+2)P + |q|Q + 1 = 55$ and $pP + (|q|+2)Q + 1 = 77$. When $M = 1$ this family corresponds with the exponents $(k, \ell, m, n) = (4, 0, 7, 8)$ and is also considered in the next section. The total CPU time needed to finish this special family has been of 11 hours.

The second case corresponds to $P = 5$, $Q = 11$ and $\alpha+5\beta = 0$. Then, the first significant center condition is

$$u_{pP+|q|Q+1}(2\pi) = u_{61}(2\pi) = \frac{3iM\pi}{16\beta^5}(\beta+M)(\beta-9M)(\beta-19M)(\beta-4M)(\beta-M)(C^5-\bar{C}^5).$$

Fixing the exponents $(k, \ell, m, n) = (1, 5, 7, 5)$ we are in the above situation. For these values we have that $u_{61}(2\pi) = 0$ because $\beta = M = 1$, situation that corresponds with the last root of E_L. The next center condition is

$$u_{81}(2\pi) = \frac{77039000187043840i\pi}{147}(C^5 - \bar{C}^5).$$

This last example neither satisfies our first guess about which is the next significant center condition because $(p + 2)P + |q|Q + 1 = 71$ and $pP + (|q| + 2)Q + 1 = 83$.

3.6. **Case (e) in Theorem 2.** As in Section 3.4 it suffices to study the center problem for equation (6),

$$\dot{z} = iz + z^k\bar{z}^\ell + Cz^m\bar{z}^n,$$

for some $0 \neq C \in \mathbb{C}$, with $\alpha\beta \neq 0$ and $d = k + \ell = m + n$.

We start computing the first significant Poincaré–Lyapunov constant. If this constant decides the center-focus problem we stop our computations. Otherwise we compute the second significant constant. As in case (d), for all the k, ℓ, m and n that we have considered these two constants decide the center-focus problem and always give a reversible center.

As an example, Table 3 shows the significant constants when $k+\ell = m+n = 5$. Notice that, by symmetry we can assume without loss of generality, that $k > m$.

To have an idea of the difficulty of the computations we comment that when d is even and $d \leq 34$ we have needed 20 days of CPU time. Similarly, when d is odd and $d \leq 57$ we have used 19 days of CPU time. In fact, the necessary time to finish each degree seems to increase exponentially: the time to finish all the even degrees lower or equal than 30 is 6 days but for the cases of degrees 32 and 34 the needed time has been of 5 and 8 days, respectively.

The reason why we can go further in the odd case is well understood: the significant Poincaré–Lyapunov constants are $V_{d+j(d-1)}$ with $j \in \mathbb{N} \cup \{0\}$ when d is odd and $j \in 2\mathbb{N}$ when d is even. See also Corollary 16 in the appendix. Therefore, in general to have the same number of significant constants in the even case we need to go further in the computations. In particular the time to finish all the odd degrees smaller or equal than

(k,ℓ)	(m,n)	
$(5,0)$	$(4,1)$	$V_{21}=\frac{32}{3}\pi(4C\bar{C}-1)(C^2+\bar{C}^2)$,
		$V_{29}=-\frac{672\pi}{5}(C^2+\bar{C}^2)$
$(5,0)$	$(3,2)$	$V_5=2\pi(C+\bar{C})$
$(5,0)$	$(2,3)$	$V_{13}=-48\pi(C^2+\bar{C}^2)$
$(5,0)$	$(1,4)$	$V_9=8i\pi(C-\bar{C})$
$(5,0)$	$(0,5)$	$V_{21}=160\pi(C^2+\bar{C}^2)$
$(4,1)$	$(3,2)$	$V_5=2\pi(C+\bar{C})$
$(4,1)$	$(2,3)$	$V_9=8i\pi(C-\bar{C})$
$(4,1)$	$(1,4)$	$V_{21}=32\pi(3C\bar{C}-4)(C+\bar{C})$,
		$V_{29}=\frac{11776\pi}{45}(C+\bar{C})$
$(4,1)$	$(0,5)$	$V_{17}=\frac{64i\pi}{3}(C-\bar{C})$
$(3,2)$	(m,n)	$\alpha=0$
$(2,3)$	$(1,4)$	$V_{13}=16\pi(C+\bar{C})$
$(2,3)$	$(0,5)$	$V_{17}=-160i\pi(C-\bar{C})$
$(1,4)$	$(0,5)$	$V_{21}=\frac{800}{3}\pi(C^2+\bar{C}^2)$

TABLE 3. The case $d=k+\ell=m+n=5$. First two significant Poincaré–Lyapunov constants.

31 is 1.4 hours but we need around 6 days for finishing the even degrees smaller or equal than 30.

As it can be seen in Table 3 the maximum order of a weak-focus depends on k,ℓ,m and n. Using the notions introduced in Section 3.4, the homogeneous case corresponds to $P=Q=1$ and there exist two coprime numbers p and q, $(p,q)\in\mathbb{N}\times\mathbb{Z}$, such that $p\alpha+q\beta=0$. Then the maximum order of a weak-focus corresponds to $j=p+|q|-1$ or $j=p+|q|+3$ in all the above expressions of V_N. This fact has been checked in all the homogeneous systems presented in our study, but we can not predict, a priori, neither which of both situations appears, nor if there are other possible values for N. Moreover, in almost all cases, with only one Poincaré–Lyapunov constant the procedure finishes and then $j=p+|q|-1$. But, sometimes two are necessary and, when this happens, $j=p+|q|+3$.

The computations done in this section provide concrete examples of simple polynomial systems of degree d with a weak-focus of very high order. This problem has been already studied in [2, 16, 23]. For general even degree d, systems with a weak-focus of order d^2-d and d^2-1 are given in [2] and [16, 23], respectively. When the degree d is odd, there are examples of systems with weak-foci of order $(d^2-1)/2$, see again [16] and [23]. We notice that in the first paper the equation has nonhomogeneous nonlinearities while in the second one the nonlinearities are homogeneous. Moreover, examples with a nonlinearity formed by three complex monomials and providing weak-foci of a higher order, $(d^2+d-2)/2$, are presented in [23] when d is odd and $d\le 19$.

Studying all the families of statement (e) in Theorem 2 we select the two concrete families with higher order weak-focus to extend our computations only for them up to odd degree $d\le 87$ using 19 days more of CPU time. We remark that the cases of degree 87 needed almost 2 days of CPU time each. All these computations are summarized in the next result.

Proposition 10. *Consider the differential equations*

$$\dot{z} = iz + z^d + C_1 z^{d-1}\bar{z} \quad \text{and} \quad \dot{z} = iz + z^{d-1}\bar{z} + C_2 z\bar{z}^{d-1}$$

with $C_1^{d_1} - (-1)^d \bar{C}_1^{d_1} \neq 0$ *and* $C_2^{d_2} - (-1)^d \bar{C}_2^{d_2} \neq 0$ *where*

$$\begin{cases} d_1 = \frac{d-1}{2}, d_2 = \frac{d-3}{2} & \text{for } d \text{ odd,} \\ d_1 = d-1, d_2 = d-3 & \text{for } d \text{ even.} \end{cases}$$

Then the origin is a weak-focus of order $(d^2+d-2)/2$ *for any odd* $5 \leq d \leq 87$ *and of order* $d^2 - d$ *for any even* $d \leq 34$.

As far as we know, our result for d odd and $21 \leq d \leq 87$ provides the highest known orders for weak-foci of planar polynomial equation of any of these degrees. The orders of the weak-foci obtained when d is even coincide with the ones given in [2].

3.7. **Case (f) in Theorem 2.** As in the previous section, we can restrict our attention to solve the center problem for equation (6):

$$\dot{z} = iz + z^k \bar{z}^\ell + C z^m \bar{z}^n,$$

for some $0 \neq C \in \mathbb{C}$ and $\alpha\beta \neq 0$. Moreover, because of the results of previous section, we only need to consider the nonhomogeneous nonlinearities case: $5 \leq Z \leq 20$, with $Z = k+\ell+m+n$, $k+\ell < m+n$, and fix k, ℓ, m, and n.

We follow the next procedure. If the fixed values k, ℓ, m, and n satisfy one of the conditions:

(i) Case $\ell = n = 0$ (holomorphic center); or
(ii) Case $k = m$ and $\alpha \neq 0$ (possible Hamiltonian or Darboux center);

there is nothing to be computed, because we already know the centers by using the previous results. Otherwise, we start computing the first significant Poincaré–Lyapunov constant. In all cases of our study this constant decides the center-focus problem and gives a reversible center.

To illustrate the procedure, in Table 4 we show the results for the simplest case $Z = 5$, $k+\ell = 2 < 3 = m+n$. We comment that the CPU time needed to study all $5 \leq Z \leq 18$ is of around 34 hours while only for the case $Z = 19$ is of around 22 days. All the cases when $Z = 20$ except $(k, \ell, m, n) = (8, 0, 1, 11)$ have needed 10 days of CPU time. The study of the case $(8, 0, 1, 11)$ has exhausted the RAM memory of our computer. We have been able to finish it, after three weeks of computation, by using a new and more powerful Xeon computer (CPU E5-2650, 2.60 GHz, RAM 128 Gb). We remark that half of the RAM memory has been necessary to finish this special case.

Fixing the values k, ℓ, m, n associated to equation (2), we define $s = \gcd(|\alpha|, |\beta|)$ and $M = \gcd(|k+\ell-1|, |m+n-1|)$. Then $\alpha = -\operatorname{sgn}(\beta) qs$, $\beta = \operatorname{sgn}(\beta) ps$, $k+\ell-1 = PM$, and $m+n-1 = QM$, for some coprime pairs p, q and P, Q. Moreover, the relation $p\alpha + q\beta = 0$ is satisfied with $p \in \mathbb{N}$ and $q \in \mathbb{Z}$.

The total number of cases satisfying $5 \leq Z \leq 20$ and $k+\ell < m+n$ are 4416 and 3463 (78.4%) are of reversible type. These are the cases for which the computation of the Poincaré–Lyapunov constant is necessary. Using again the algorithm given in [10] and described in the appendix, we have found that the first significant Poincaré–Lyapunov constant, V_N, corresponds to

$$N \in \{N_1, N_1 + 2PM, N_1 + 2QM\}, \quad \text{where} \quad N_1 = pPM + |q|QM + 1 \tag{19}$$

in 82.3%, 13.5% and 4.1% of the cases, respectively. Only the cases $(k, \ell, m, n) = (1, 5, 7, 5)$ for $Z = 18$ and $(k, \ell, m, n) = (4, 0, 7, 8)$ for $Z = 19$ do not satisfy (19). For the first case $\alpha = -5$, $\beta = 1$, $P = 5$, $Q = 11$, $p = 1$, $q = 5$, $M = 1$, and consequently the corresponding three values are 61, 71 and 83, but the first significant Poincaré–Lyapunov constant is V_{81} as we have mentioned in Section 3.5. The second case corresponds to $\alpha = 3$, $\beta = -2$,

(k,ℓ)	(m,n)	
(k,ℓ)	$(2,1)$	Reversible ($\beta=0$)
$(2,0)$	$(3,0)$	Holomorphic ($\ell=n=0$)
$(2,0)$	$(1,2)$	Reversible, $V_9=\frac{8\pi}{3}C\bar{C}(C+\bar{C})$
$(2,0)$	$(0,3)$	Reversible, $V_{11}=\frac{28\pi}{15}C\bar{C}(C+\bar{C})$
$(1,1)$	$(3,0)$	Reversible, $V_5=-4\pi(C+\bar{C})$
$(1,1)$	$(1,2)$	Darboux ($k=m=1$)
$(1,1)$	$(0,3)$	Reversible, $V_7=-\frac{9\pi}{2}(C+\bar{C})$
$(0,2)$	$(3,0)$	Reversible, $V_9=8\pi(C^3+\bar{C}^3)$
$(0,2)$	$(1,2)$	Reversible, $V_9=\frac{16\pi}{3}(C^3+\bar{C}^3)$
$(0,2)$	$(0,3)$	Hamiltonian ($k=m=0$)

TABLE 4. The case $Z=5$, where $k+\ell=2<3=m+n$. Center types and first significant Poincaré–Lyapunov constant.

$P=3$, $Q=14$, $p=2$, $q=3$, and $M=1$. Here, looking at (19), the expected values of N would be $49,55,77$ but the good one is 61, as we have also explained in Section 3.5.

The highest computational difficulties are due to some special cases that need a big amount of CPU time and memory to finish and a big number of Poincaré–Lyapunov constants. For $Z\le 20$, the cases that has needed the biggest CPU time has been $(k,\ell,m,n)=(7,0,0,12)$ and $(k,\ell,m,n)=(8,0,1,11)$. The first one used the total amount of RAM and 9 days of CPU time and we get

$$V_{167}=-\frac{971613257193767158730304329526047869011276857344400\pi}{9763161303355126061593487476473}(C^6+\bar{C}^6)C\bar{C},$$

where $\alpha=q=6$, $\beta=-p=-13$, $P=6$, $Q=11$, and $M=1$. For the second one,

$$V_{177}=\frac{1908426733410273629377078177782119201310244864i\pi}{1022687776531655325}(C^7-\bar{C}^7)C\bar{C},$$

where $\alpha=q=7$, $\beta=-p=-11$, $P=7$, $Q=11$, and $M=1$. In particular this is also one case with the highest order weak-focus when $Z\le 20$.

Nevertheless, it is worth mentioning that in our study, for a given total degree d, the order of the highest order weak-focus obtained when the nonlinearities are homogeneous (case (e)) is bigger than the corresponding one obtained in this case, where the nonlinearities are nonhomogeneous.

In Table 5 we have indicated a case (k,ℓ,m,n), for each $5\le Z\le 19$, that presents the highest order weak-focus and, generally, it needs the biggest computational effort. We remark that in this table the time used for solving the case $Z=19$ is even bigger than expected because the computations exhaust the RAM memory and therefore the procedure has needed to use virtual memory. Moreover, we have not added the case $Z=20$ because we have needed to use a different computer to finish it. The exponential growth of the CPU time is an indication of the difficulties to go further using this approach.

In all the presented cases, the significant Poincaré–Lyapunov constant V_N needed to finish the study satisfies $N\le pP+(|q|+2)Q+1$, value that corresponds to the maximum of the three values of (19). Moreover, it writes as

$$V_N=D\pi\,\frac{C^{|q|}+(-1)^{p+|q|+1}\bar{C}^{|q|}}{i^{p+|q|+1}}(C\bar{C})^s,$$

for some non zero rational constant D and $s\in\{0,1\}$. This result supports once more the fact that all centers are as we expect, see (4).

Z	Total CPU time in sec.	(k, ℓ, m, n)	N in V_N	CPU time in sec.
5	0.2	(2,0,0,3)	11	0.1
6	1.0	(2,0,0,4)	15	0.2
7	2.5	(3,0,0,4)	23	0.5
8	3.3	(2,0,0,6)	23	1.0
9	14.5	(3,0,0,6)	35	2.5
10	24.7	(4,0,0,6)	47	7.7
11	78.8	(5,0,1,5)	51	7.2
12	74.1	(4,0,1,7)	57	21.6
13	794.1	(5,0,0,8)	79	130.5
14	529.5	(6,0,1,7)	85	152.4
15	3 639.5	(7,0,1,7)	99	399.1
16	7 009.4	(6,0,0,10)	119	3 207.6
17	36 487.3	(8,0,1,8)	129	4 799.0
18	73 592.9	(8,0,1,9)	145	28 643.6
19	1 883 865.5	(7,0,0,12)	167	769 368.2

TABLE 5. Second column shows the computation time needed to solve the complete case $Z = k + \ell + m + n$. Third and four columns show some values (k, ℓ, m, n) with the highest order weak-foci and the value N of its corresponding significant Poincaré–Lyapunov constant. To solve these cases, the biggest CPU time is usually needed.

4. Appendix. The computation of the Poincaré–Lyapunov constants

The algorithm that we use for proving cases (e) and (f) of Theorem 2 relies on the following theoretical result, that is proved in [10], and which in turn is based on the results of [7]. It is also used by the software P4 (Polynomial Planar Phase Portraits), introduced in [6, Sec. 9-10].

Theorem 11. *Consider the differential equation*

$$\dot{z} = iz + \sum_{j=2}^{\infty} F_j(z, \bar{z}), \tag{20}$$

or the equivalent expression

$$dH(z, \bar{z}) + \omega_1(z, \bar{z}) + \omega_2(z, \bar{z}) + \cdots = 0,$$

where $H(z, \bar{z}) = \frac{1}{2} z\bar{z}$ *and* $\omega_j(z, \bar{z}) = 2\,\mathrm{Im}(F_{j+1}(z, \bar{z})d\bar{z})$, *for all* $j \in \mathbb{N}$. *If* $V_2 = V_3 = \cdots = V_{N-1} = 0$ *then its* N*-th Poincaré–Lyapunov constant is*

$$V_N = \frac{1}{2^{\frac{N+1}{2}}} \frac{1}{\rho^{\frac{N+1}{2}}} \int_{H=\rho} \sum_{j=1}^{N-1} \omega_j h_{N-1-j},$$

where $h_0 \equiv 1$ *and* h_m, *for all* $m = 1, \ldots, N-2$, *are polynomials in two variables defined recurrently by*

$$d\left(\sum_{j=1}^{m} \omega_j h_{m-j}\right) = -d\left(h_m dH\right).$$

To apply the above result for obtaining V_N we need to compute the given integral and to obtain the functions h_j. The first goal is solved by the following lemma.

Lemma 12. *Let $\omega = 2\,\mathrm{Im}(W(z,\bar z)d\bar z)$ be a polynomial 1-form. Then*

$$\int_{H=\rho}\omega = -4\pi i\sum_k \mathrm{coeff}\left(\mathrm{Re}\left(\frac{\partial W(z,\bar z)}{\partial z}\right), z^k\bar z^k\right)\left(\frac{(2\rho)^{k+1}}{k+1}\right),$$

where $\mathrm{coeff}(f, z^k\bar z^k)$ *denotes the coefficient of the monomial* $z^k\bar z^k$ *of* f *for any* k.

To achieve the second goal and to facilitate the implementation of Theorem 11 we introduce some operators.

Definition 13. *Let $\mathcal{P}$ be the set of all complex polynomials in $z,\bar z$ variables and set $G(z,\bar z)=\sum g_{k,\ell}z^k\bar z^\ell \in \mathcal{P}$ of degre n. We consider, for each $j\geq 2$, the following operators $\mathcal{G},\mathcal{F},\mathcal{F}_j:\mathcal{P}\longrightarrow\mathcal{P}$ and $\mathcal{H}_j:\mathcal{P}\longrightarrow\mathbb{R}$,*

$$\mathcal{G}(G)=\sum_{k\neq\ell}\frac{2}{k-\ell}g_{k,\ell}z^k\bar z^\ell,\quad \mathcal{F}(G)=-\,\mathrm{Im}\left(\mathcal{G}\left(\frac{\partial G(z,\bar z)}{\partial z}\right)\right),$$

$$\mathcal{F}_j(G)=\mathcal{F}(F_jG),\qquad \mathcal{H}_j(G)=-\frac{1}{(2\rho)^{\frac{n+j+1}{2}}}\int_{H=\rho}\mathrm{Im}\,(F_jGd\bar z),$$

where F_j are given in (20)*. Notice that they are not defined on the whole space $\mathcal{P}$.*

Next lemma shows how to find the functions h_j.

Lemma 14. *Let $\omega = 2\,\mathrm{Im}(W(z,\bar z)d\bar z)$ be a polynomial 1-form such that $\int_{H=\rho}\omega\equiv 0$. Then, the polynomial*

$$h(z,\bar z)=2\,\mathcal{G}\left(\mathrm{Re}\left(\frac{\partial W(z,\bar z)}{\partial z}\right)\right),$$

is such that $d\omega = d(hdH)$.

In [5], the Poincaré–Lyapunov constants are written in term of some "words". The procedure suggested by Theorem 11 gives a diferent expression of the Poincaré–Lyapunov constants in terms of words, where the "words" are given by the homogeneous components of the differential equation (20). Next two results are the ones implemented to find the centers of families (e) and (f) of Theorem 2, respectively.

Theorem 15. *Assume that for equation* (20) *the first $N-2$ Poincaré–Lyapunov constants vanish ($V_2=V_3=\cdots=V_{N-1}=0$). Then its N-th Poincaré–Lyapunov constant is*

$$V_N=\sum_{k=2}^{N}\mathcal{H}_k\left(\sum_{(m_1,\ldots,m_s)\in S_{N-k}}\mathcal{F}_{m_1}\left(\mathcal{F}_{m_2}\left(\cdots\left(\mathcal{F}_{m_s}\left(1\right)\right)\right)\right)\right),$$

where $S_0=\{(1)\}$ and then the second summation reduces to 1 and otherwise

$$S_\ell=\bigcup_{s\in\mathbb{N}^+}\left\{(m_1,\ldots,m_s)\in\left(\mathbb{N}^+\setminus\{1\}\right)^s\ \text{such that}\ \sum_{j=1}^{s}(m_j-1)=\ell\right\}.$$

In the particular case that the nonlinearities of (20) are homogeneous, next corollary simplifies the expression of the V_N given above.

Corollary 16. *Let $F_d(z,\bar z)$ be a homogeneous polynomial of degree d. Then, the only significant Poincaré–Lyapunov constants of the equation $\dot z = iz+F_d(z,\bar z)$ are*

$$V_{d+j(d-1)}=\mathcal{H}_d(\mathcal{F}_d(\mathcal{F}_d(\overset{j}{\cdots}(\mathcal{F}_d(1)))))):=\mathcal{H}_d(\mathcal{F}_d^j(1)),$$

where j is any natural number (resp. any odd natural number) when d is odd, (resp. even).

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