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A COUNTEREXAMPLE TO A RESULT ON LOTKA–VOLTERRA SYSTEMS

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Abstract. In the article of "Acta Applicandae Mathematicae **23** (1991), 103–127" the authors claim the existence of a Hopf bifurcation which in general does not exist.

1. Introduction and the counterexample

In [1] the generalized Lotka–Volterra model of the form

$$\begin{aligned} \dot{x} &= x^{\hat{p}(\mu)} - x^{p(\mu)}y^{q(\mu)}, \\ \dot{y} &= C(\mu)\big(x^{p(\mu)}y^{q(\mu)} - y^{\hat{q}(\mu)}\big), \end{aligned} \tag{1}$$

with $C(\mu) > 0$ is analyzed. The authors claim to have the following result.

Theorem 1. *Assume that the functions $\hat{p}(\mu)$, $p(\mu)$, $\hat{q}(\mu)$, $q(\mu)$ and $C(\mu)$ are continuously differentiable and for admissible values of μ these functions are positive and satisfy*

$$\hat{p}q + p\hat{q} - \hat{p}\hat{q} > 0. \tag{2}$$

If, for some μ_0,

$$\hat{p}(\mu_0) - p(\mu_0) = C(\mu_0)\big(\hat{q}(\mu_0) - q(\mu_0)\big) \tag{3}$$

and

$$\frac{d}{d\mu}\Big(\hat{p} - p - C\big(\hat{q} - q\big)\Big)\Big|_{\mu=\mu_0} \neq 0, \tag{4}$$

then the system undergoes an Andronov–Hopf bifurcation at μ_0. Moreover, the bifurcation is supercritical, resp. subcritical according to

$$p(\hat{p} - p)(\hat{q} - 1)(\hat{q} + q) - q(\hat{q} - q)(\hat{p} - 1)(\hat{p} + p)\big|_{\mu=\mu_0} < 0,$$

resp. > 0.

Unfortunately there is a problem in the proof of Theorem 1 due to the following counterexample.

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Theorem 2. *Consider the polynomial differential system*

$$\begin{aligned}\dot{x} &= x - x^2y^2 = P(x,y),\\ \dot{y} &= (1+\mu)(x^2y^2 - y) = Q(x,y),\end{aligned} \tag{5}$$

with $1+\mu>0$*. System* (5) *satisfies all the assumptions of Theorem 1 with* $\mu=\mu_0=0$ *but it does not exhibit an Andronov–Hopf bifurcation.*

Proof. Comparing system (1) with system (5) we have

$$\hat{p}(\mu)=\hat{q}(\mu)=1, \quad p(\mu)=q(\mu)=2, \quad C(\mu)=1+\mu.$$

Then we have

$$\hat{p}q + p\hat{q} - \hat{p}\hat{q} = 3 > 0.$$

So, condition (2) holds.

Take $\mu_0=0$. Then, condition (3) is immediately satisfied, and for condition (4) we obtain

$$\frac{d}{d\mu}\Big(\hat{p}-p-C\big(\hat{q}-q\big)\Big)\Big|_{\mu=0} = 1 \neq 0.$$

In short, all the conditions of Theorem 1 are satisfied by system (5).

The unique equilibria of system (5) are the $(0,0)$ and $(1,1)$. Around the equilibrium point $(0,0)$ cannot exist periodic orbits because the straight lines $x=0$ and $y=0$ are invariant by the flow of system (5), i.e. they are formed by orbits of system (5). Therefore, if there are periodic orbits these must surround the equilibrium point $(1,1)$. We recall that in the bounded region limited by a periodic orbit of a differential system in the plane it must be an equilibrium point, see for instance Theorem 1.31 of [2].

We claim that the unique periodic orbits of systems (5) for all μ exist when $\mu=0$, and they are the periodic orbits surrounding the center $(1,1)$ of system (5) with $\mu=0$. Now we shall prove the claim. Clearly once the claim is proved it follows that system (5) cannot exhibit an Andronov–Hopf bifurcation.

System (5) with $\mu=0$ has the first integral $H=x+y+1/(xy)$, because the derivative of H on the orbits of system (5) with $\mu=0$ satisfies that

$$\frac{dH}{dt}=\frac{\partial H}{\partial x}(x-x^2y^2)+\frac{\partial H}{\partial y}(x^2y^2-y)=0.$$

Since the eigenvalues of the linear differential system (5) with $\mu=0$ at the equilibrium $(1,1)$ are $\pm\sqrt{3}\,i$, this equilibrium either is a focus or a center, but it cannot be a focus because the first integral H is defined at $(1,1)$. Hence, we have proved that the equilibrium $(1,1)$ for system

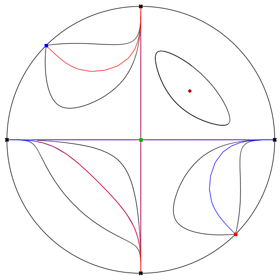

FIGURE 1. Phase portrait of system (5) with $\mu = 0$.

(5) with $\mu = 0$ is a center. Now we shall see that the periodic orbits of this center filled all the positive quadrant $Q = \{(x, y) : x > 0 \,\text{and}\, y > 0\}$. Assume that they do not filled all that quadrant. Since in that quadrant the unique equilibrium is the $(1, 1)$, the external boundary of the continuum set of periodic orbit surrounding the center $(1, 1)$ must be a periodic orbit γ, and after that orbit the nearby orbits must spiral. Consider the Poincaré map defined in an analytic transversal arc Σ to this periodic orbit γ. Since the flow of the polynomial differential system (5) with $\mu = 0$ is analytic, such a Poincaré map is analytic, but it is not possible that an analytic map of one variable be the identity on the piece of the arc Σ contained in the bounded region limited by γ, and different to the identity on the piece of the arc Σ outside the bounded region limited by γ. So such a last periodic orbit γ does not exist and the periodic orbits surrounding the center $(1, 1)$ filled all the positive quadrant Q. See a picture of the phase portrait of system (5) with $\mu = 0$ on the Poincaré disc in Figure 1, for more details on the Poincaré disc see Chapter 5 of [2].

For completing the proof of the claim we must show that system (5) with $\mu \neq 0$ has no periodic solutions surrounding the equilibrium $(1, 1)$. We shall use the Dulac criterium: Let P and Q be the polynomials defined in (5). *If there exists a C^1 function $B(x, y)$ in a simply connected region R such that $\partial(BP)/\partial x + \partial(BQ)/\partial y$ has constant sign and is not identically zero in any subregion of R, then system* (5) *does not have a periodic orbit lying entirely in R.* For a proof of this criterium see for instance Theorem 7.12 of [2].

Consider the function $B = 1/(x^2y^2)$ defined in the positive quadrant Q. Then

$$\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial x} = \frac{\mu}{x^2y^2} \neq 0 \text{ in } Q \text{ if and only if } \mu \neq 0.$$

Therefore, by the Dulac criterium, system (5) with $\mu \neq 0$ has no periodic solutions surrounding the equilibrium $(1,1)$, and this prevents the existence of a Hopf bifurcation. The proof of the claim and of Theorem 2 is completed. $\square$

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