Singular solutions for a class of traveling wave equations arising in hydrodynamics *

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Abstract

We give an exhaustive characterization of singular weak solutions for ordinary differential equations of the form $\ddot{u}\,u+\frac{1}{2}\dot{u}^2+F'(u)=0$, where F is an analytic function. Our motivation stems from the fact that in the context of hydrodynamics several prominent equations are reducible to an equation of this form upon passing to a moving frame. We construct peaked and cusped waves, fronts with finite-time decay and compact solitary waves. We prove that one cannot obtain peaked and compactly supported traveling waves for the same equation. In particular, a peaked traveling wave cannot have compact support and vice versa. To exemplify the approach we apply our results to the Camassa-Holm equation and the equation for surface waves of moderate amplitude, and show how the different types of singular solutions can be obtained varying the energy level of the corresponding planar Hamiltonian systems.

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1 Introduction

In the present paper we propose to study certain types of weak solutions for ordinary differential equations (ODE) of the form

$$\ddot{u}\,u + \frac{1}{2}\dot{u}^2 + F'(u) = 0, \tag{1}$$

where F is an analytic function. Our motivation stems from the fact that a variety of model equations arising in the context of hydrodynamics, among them the well-known Camassa–Holm equation (cf. [2, 3, 9]) and the related equation for surface waves of moderate amplitude (cf. [4, 5, 6, 7, 10]), are reducible to an ODE of the form (1) upon passing to a moving frame. Owing to the fact that every solution of equation (1) may be interpreted as a traveling wave of a suitable underlying partial differential equation (PDE) we will call the solutions of (1) traveling waves.

The singular nature of Equation (1) accounts for the non-uniqueness of certain solutions, which we call *singular solutions*. These are in general weak solutions, but have stronger regularity than one would expect a priori: the solutions are analytic except for a countable number of points at which the equation is satisfied in the limit. Furthermore, equation (1) admits an order reduction which allows us to see that under certain conditions on F, the solutions are actually classical solutions of this reduced equation.

The main result of this paper consists in the exhaustive characterization of singular solutions of (1) from qualitative properties of the function $F(u)$. We show that equation (1) admits solutions with peaks and cusps, fronts with finite-time decay and solitary solutions with compact support. Furthermore, we find that one cannot obtain peaked and compactly supported solutions for the same F. In particular, a peaked solution cannot have compact support and vice versa. The characterization of classical solutions of (1) will be covered only very briefly for the convenience of the reader, since our main focus lies in the analysis of singular solutions.

We apply our results to the aforementioned nonlinear partial differential equations, and show how the different types of singular solutions are obtained varying the energy levels of the Hamiltonian planar differential system corresponding to (1). It lies beyond the scope of this paper to prove in full generality that every weak solution of (1) is also a weak traveling wave solution of an underlying PDE. For a discussion of this problem we refer the reader to [8] and [14], where it is shown that in the special case of the Camassa-Holm equation every weak solution of (1) is a weak traveling wave solution of the underlying PDE. Following similar steps the same result can be shown for the equation of surface waves of moderate amplitude.

The structure of the paper is as follows. In Section 2, we give the precise definitions of weak and singular solutions and provide a preliminary result on the non-uniqueness of solutions of (1). In Section 3 we introduce the notion of *elementary forms*, classical solutions of (1) defined on a subset of $\mathbb{R}$, from which we construct singular solutions. Furthermore, we discuss how the qualitative features of any traveling wave solution can be obtained from the properties of F. The main results of the paper, Propositions 6, 7, 11 and 12 are presented in Section 4 which is devoted to the complete characterization of singular solutions. In Section 5, we characterize the classical and singular traveling waves of the Camassa-Holm equation and the equation for surface waves of moderate amplitude in shallow water.

2 Weak and singular solutions

Our focus lies in the characterization of solutions which are not classical, so we require a weak formulation of (1). Keeping in mind that any solution of (1) can be interpreted as a traveling wave of an underlying PDE, we will consider only bounded solutions.

Definition 1. We say that a bounded function $u \in H^1_{loc}(\mathbb{R})$ is a *traveling wave solution (TWS)* if it satisfies (1) in the sense of distributions, i.e. if u satisfies

$$\int_{\mathbb{R}} (u^2)_t \phi_t + (u_t)^2 \phi - 2F'(u)\phi \, dt = 0, \tag{2}$$

for any test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. We say that u is a *strong TWS* if it satisfies (1) in the classical sense.

It turns out that the concept of weak solutions is quite crude. Indeed, if no further conditions are imposed it is possible to find a plethora of weak solutions of (1) giving rise to TWS with very complex shapes. For instance it is known that the Camassa–Holm equation can have TWS of the form $u = \varphi(t)$ such that some of its level sets $\{\varphi(t) = k\}$ are cantor sets, cf. [14]. In the present paper, however, we are interested in those solutions which fail to be strong TWS because of the singularity, but which still have a certain degree of regularity, and in fact are strong solutions of (1) except for a finite or a countable set of points (such solutions are, for instance, peaked or cusped waves, fronts with finite time decay, compact solitary waves and composite waves, see Section 4 for precise definitions). To see this, observe that equation (1) admits the order reduction

$$\frac{d}{dt}\Big(\frac{u\dot{u}^2}{2} + F(u)\Big) = \dot{u}\left(\ddot{u}\, u + \frac{1}{2}\dot{u}^2 + F'(u)\right),$$

which is equivalent to the fact that the planar differential system associated to (1) has a first integral. Therefore, any classical solution of (1) naturally satisfies

$$\frac{u\dot{u}^2}{2} + F(u) = h, \ \text{ for some constant } h \in \mathbb{R}. \tag{3}$$

The singularity of (3) leads to the existence of non constant solutions $u \in H^1_{loc}(\mathbb{R})$ of (2) which satisfy (3) except, perhaps, at a countable number of points where the derivative is not defined but the equation is still satisfied in the limit. This motivates the following definition:

Definition 2. Let $u(t) \in H^1_{loc}(\mathbb{R})$ be a non constant TWS of (1). Then $u(t)$ is called a

(a) *strong singular* TWS of (1) if u is a classical solution of (3) on $\mathbb{R}$.

(b) *weak singular* TWS of (1) if u is a classical solution of (3) on $\mathbb{R} \setminus \mathcal{S}$, where $\mathcal{S}$ is the set of countably many points t_k such that

$$\lim_{t \to t_k} \frac{u(t)\dot{u}(t)^2}{2} + F(u(t)) = h. \tag{4}$$

For the remainder of this paper we set $h_0 := F(0)$. The next result establishes the non-uniqueness of solutions of (3) for $h = h_0$. This will play a role in the construction of singular TWS of (1), see Section 4.

Lemma 3. *Consider equation* (3) *with* F *an analytic function such that* $u\,(h - F(u)) > 0$ *for* $u \in \mathcal{U} \setminus \{0\}$ *where* $\mathcal{U}$ *is a neighborhood of* 0.

(a) If $h \neq h_0$*, then equation* (3) *is not defined at* $u = 0$.

(b) If $h = h_0$ *then* $u \equiv 0$ *is a solution of equation* (3)*. Furthermore if* $F'(0) \neq 0$ *or* $F'(0) = 0$ *and* $F''(0) \neq 0$ *then equation* (3) *is not Lipschitz continuous in* $u = 0$.

Proof. Notice that equation (3) may be written as

$$\dot{u} = \pm\sqrt{2\,\frac{h - F(u)}{u}} =: v_h^{\pm}(u).$$

Observe that if $h \neq h_0$ then $\lim\limits_{u\to 0} v_h^{\pm}(u) = \pm\infty$, which proves statement (a). When $h = h_0$, $u \equiv 0$ is always a solution of (3). Note that

$$\frac{|v_{h_0}^{\pm}(u)|}{|u|} = \sqrt{-2\left(\frac{F'(0)}{u^2} + \frac{F''(0)}{2!\,u} + \frac{F^{(3)}(0)}{3!} + o(u)\right)}.$$

Hence, if $F'(0) \neq 0$ or $F'(0) = 0$ and $F''(0) \neq 0$ then $\lim\limits_{u\to 0} |v_{h_0}^{\pm}(u)|/|u| = +\infty$. Therefore it is not possible to find a constant $L > 0$ such that $|v_{h_0}^{\pm}(u)| \leq L|u|$ for $u \in \mathcal{U}$, and hence the right-hand side of the differential equation fails to be Lipschitz continuous in $u = 0$. ∎

Remark 4. Observe that $v_{h_0}^{\pm}(u)$ is Lipschitz continuous in $u = 0$ only if $F'(0) = F''(0) = 0$ and $F^{(3)}(0) \neq 0$. However, as a consequence of Propositions 6, 7 and 11, such F do not yield any singular solutions, and therefore they will not be considered in this paper.

3 Traveling wave solutions from qualitative properties of F

The planar system associated to equation (1) is given by

$$\begin{cases} \dot{u} = v \\ u\,\dot{v} = -F'(u) - \frac{1}{2}\,v^2. \end{cases} \tag{5}$$

A straightforward computation shows that system (5) possesses the first integral

$$H(u, v) = \frac{u\,v^2}{2} + F(u), \tag{6}$$

whose *energy* $H(u, v) = h$ is therefore constant along solutions. We will study how the TWS of (1) can be obtained from the orbits of the associated system (5), and how these orbits depend on the qualitative properties of F. Our study resembles the study of conservative systems of one degree of freedom [1, §12]. For such systems, the qualitative features of the phase portrait are in correspondence with the ones of the potential function. Analogously, we find that the qualitative features of the function F characterize the phase portrait of (5) and therefore the types of TWS that (1) may exhibit. In contrast to potential systems, the Hamiltonian (6) has no purely kinetic term, which leads to the presence of a singular line in the phase portrait and the existence of solutions defined on subsets of $\mathbb{R}$. Without loss of generality we consider the system only in the half-plane $(\{u > 0\} \cup \{u = 0\}) \times \mathbb{R}$, which is not restrictive since our analysis is of a qualitative nature. It is worth mentioning however that in some applications it might be interesting to consider solutions in $\{u \leq 0\}$.

The solutions of equation (1) are associated to the orbits of (5), which correspond to the level sets of the energy $H(u, v) = h$, i.e. they lie on curves of the form $\gamma_h = \{H = h\}$ which are composed of two symmetric branches $(u, v_h^{\pm}(u))$, where

$$v_h^{\pm}(u) = \pm\sqrt{2\,\frac{h - F(u)}{u}}. \tag{7}$$

Therefore, solutions of equation (1) corresponding to energy h exist if there is a non-empty set $I \subseteq \{u \geq 0\}$ such that $F(u) - h < 0$ for all $u \in I$. Recall that our interest lies in the analysis of bounded solutions, which correspond to branches of the curves γ_h that are bounded in the u-direction. These branches are defined if and only if either there exist $m_1, m_2 > 0$ such that $F(m_1) = F(m_2) = h$ and $F(u) < h$ for all $u \in (m_1, m_2)$, or there exists $m > 0$ such that $F(m) = h$ and $F(u) < h$ for all $u \in (0, m)$. The latter case corresponds to a type of strong solutions of (1) whose maximal domain of definition is a (finite) subset of $\mathbb{R}$. We call these solutions *elementary forms*, which play an important role in our construction of singular TWS. The former case corresponds to classical smooth solutions which are uniquely defined on $\mathbb{R}$. Observe that possible oscillations of F in (m_1, m_2) or $(0, m)$ affect the solutions of (1) only at the level of their convexity.

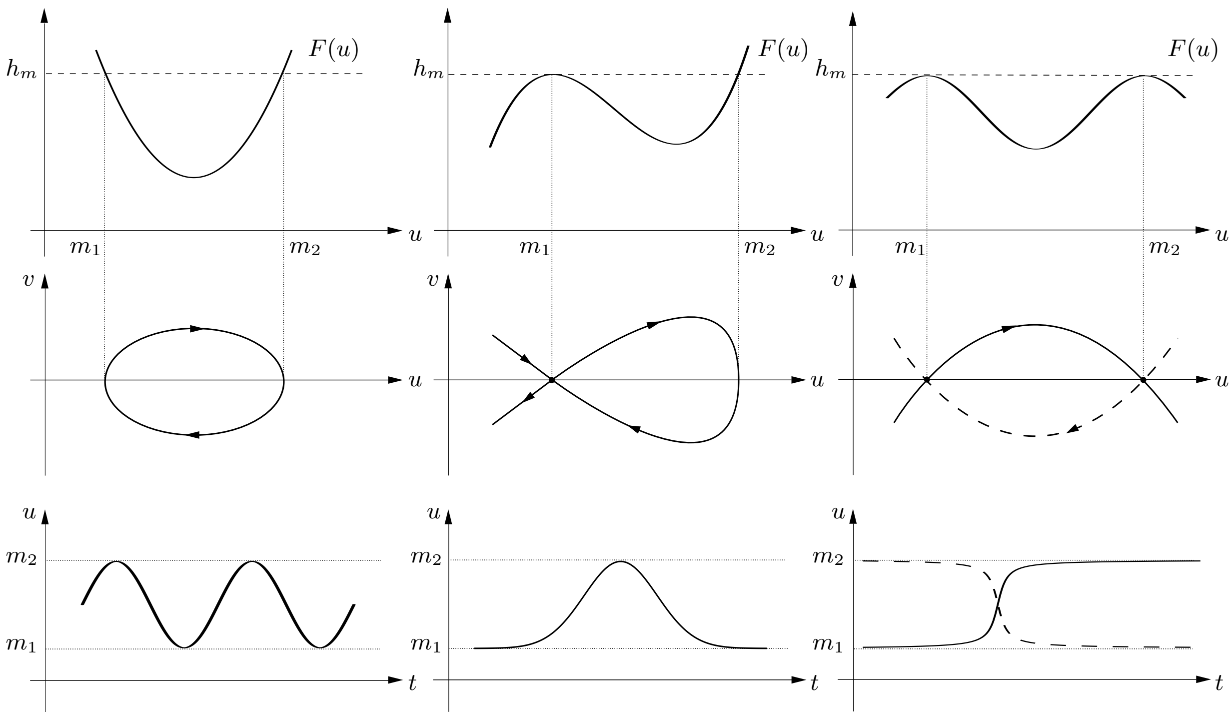
3.1 Smooth TWS on $\mathbb{R}$

In this subsection we briefly summarize how smooth TWS are characterized in terms of qualitative aspects of F. Assume that there exist $m_1, m_2 > 0$ such that $F(m_1) = F(m_2) = h_m$ and $F(u) < h_m$ for all $u \in (m_1, m_2)$. Then the two branches of γ_{h_m} given by $(u, v_{h_m}^{\pm}(u))$ are defined for $u \in [m_1, m_2]$ and coincide at the points $p_1 = (m_1, 0)$ and $p_2 = (m_2, 0)$.

Observe that the critical points of system (5) in $\{u > 0\} \times \mathbb{R}$ are of the form $(u, 0)$ where $F'(u) = 0$. Hence a point $p_i = (m_i, 0)$ will be a critical point if $F'(m_i) = 0$ and a regular point otherwise. We distinguish between the following cases (see Figure 1):

(A) If $F'(m_i) \neq 0$ for $i = 1, 2$, then the points p_1 and p_2 are regular, and the two branches $(u, v_{h_m}^{\pm}(u))$ give rise to an isolated closed curve, which is a periodic orbit of (5). This periodic orbit corresponds to a strong smooth periodic solution $u(t)$ of equation (1). Observe that since $v_{h_m}^{+}(u)$ and $v_{h_m}^{-}(u)$ are not multivalued functions, the closed curve has no lobes and therefore $u(t)$ reaches a unique local minimum at m_1 and a unique local maximum at m_2 in each period. Furthermore, this periodic solution is symmetric with respect to its local minima and maxima.

(B) If $F'(m_1) = 0$ and $F'(m_2) \neq 0$ then p_1 is a critical point and p_2 is a regular one. In this case, the branches $(u, v_{h_m}^{\pm}(u))$ yield a homoclinic loop giving rise to a smooth solitary wave solution of (1) which has a unique maximum at m_2 and which decays



(A) (B) (C)

Figure 1: Global strong TWS of (1). (A) Periodic wave; (B) Solitary wave or pulse; (C) Smooth front.

exponentially on either side of the maximum such that $\lim_{t\to\pm\infty} u(t) = m_1$ (recall that since p_1 is a critical point, the time that the orbit of (5) takes to leave from or to reach this point is infinite). Furthermore, this solitary wave solution is symmetric with respect to its maximum. If $F'(m_1) \neq 0$ and $F'(m_2) = 0$ then there appears a solitary wave of (1) with a unique minimum at m_1 and such that $\lim_{t\to\pm\infty} u(t) = m_2$.

(C) If $F'(m_1) = F'(m_2) = 0$ then p_1 and p_2 are critical points, and the branches $(u, v^{\pm}_{h_m}(u))$ yield a heteroclinic loop connecting these points in an infinite time. This pair of connecting orbits gives rise to a smooth front decaying from m_2 to m_1 such that $\lim_{t\to-\infty} u(t) = m_2$ and $\lim_{t\to+\infty} u(t) = m_1$, and another smooth front increasing from m_1 to m_2 such that $\lim_{t\to-\infty} u(t) = m_1$ and $\lim_{t\to+\infty} u(t) = m_2$.

3.2 Elementary forms: smooth TWS on a subset of $\mathbb{R}$.

In this subsection we discuss two types of elementary forms. The first type, studied in cases (a) and (b) below, are the ones associated to the curves γ_{h_0} which correspond to the energy level $h_0 = F(0)$ and which intersect the singular line $\{u = 0\} \times \mathbb{R}$ at a finite point. The second type are the ones associated to curves γ_h that are unbounded in the component v and are arbitrarily close to the singular line $\{u = 0\} \times \mathbb{R}$ at infinity. They are studied in the case (c). In both cases, it will be a key step to study whether the orbits of (5) approach the singular line $u = 0$ in finite time. To this end we prove the following auxiliary result.

Lemma 5. *Let F be an analytic function. Suppose that there exist $\varepsilon > 0$ and $m > 0$ such that $F(m) = F(0)$ and $F(u) < F(0)$ for $u \in (0, \varepsilon) \cup (m - \varepsilon, m)$.*

(i) Let $n \in \mathbb{N}$ be the lowest order such that $F^{(n)}(0) \neq 0$. If $n \leq 2$ then

$$\int_0^{\varepsilon} \sqrt{\frac{u}{F(0) - F(u)}} du \tag{8}$$

is convergent. Otherwise the integral diverges.

(ii) If $F'(m) > 0$, then

$$\int_{m-\varepsilon}^{m} \sqrt{\frac{u}{F(0) - F(u)}} du \tag{9}$$

is convergent. Otherwise the integral diverges.

Proof. In both cases we will apply the following criterium: let $f(x)$ be an unbounded function at $x = c$ such that $f(x) \geq 0$ and $\lim_{x \to c} f(x)|x - c|^k = A$, where $A \neq \infty$ and $A \neq 0$. Then, for $k < 1$ the integral $\int_a^c f(x)dx$ is convergent, whereas for $k \geq 1$ it is divergent. To prove (i), set

$$f(u) := \sqrt{\frac{u}{F(0) - F(u)}},$$

and $n \in \mathbb{N}$ such that $F^{(n)}(0) < 0$ and $F^{(k)}(0) = 0$ for $k < n$. Hence

$$f(u) = \sqrt{\frac{-n!}{F^{(n)}(0)u^{n-1} + o(u^n)}},$$

and

$$\lim_{u \to 0^+} f(u)|u|^{\frac{n-1}{2}} = \sqrt{\frac{-n!}{F^{(n)}(0)}} =: K, \text{ where } 0 \neq K \neq \infty.$$

Therefore, using the criterium stated above, for $n \leq 2$ the integral (8) converges and otherwise it diverges. To prove (ii) let

$$f(u) := \sqrt{\frac{u}{F(m) - F(u)}},$$

and $n \geq 1$ such that $F^{(n)}(m) < 0$ and $F^{(k)}(m) = 0$ for $k < n$. Then

$$f(u) = \sqrt{\frac{-n!\, u}{F^{(n)}(m)(u - m)^n + o((u - m)^{n+1})}},$$

and therefore

$$\lim_{u \to 0^+} f(u)|u - m|^{\frac{n}{2}} = \sqrt{\frac{-n!\, m}{F^{(n)}(m)}} =: K, \text{ where } 0 \neq K \neq \infty.$$

Hence for $n = 1$ the integral (9) converges and otherwise it diverges. ∎

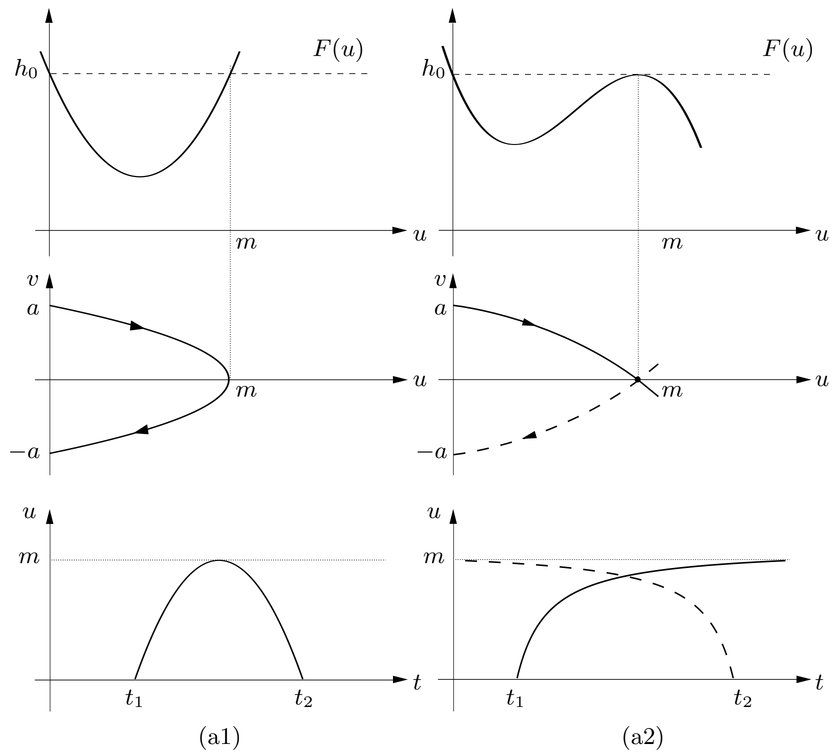


Figure 2: Strong solutions of (1) defined in a subset of $\mathbb{R}$ with energy $h = h_0$ when $F'(0) \neq 0$. These provide the elementary forms to construct peaked periodic and solitary TWS.

We start with a discussion of solutions associated to the energy level $h = h_0$. Suppose that there exists $m > 0$ such that $F(m) = h_0$ and $F(u) < h_0$ for all $u \in (0, m)$. In this case, the curves γ_{h_0} intersect the singular line $\{u = 0\}$ at a finite point, and the branches $(u, v^{\pm}_{h_0}(u))$ are defined for $u \in [0, m]$, and coincide at the point $p = (m, 0)$. As in the discussion for smooth solutions in Section 3.1 we consider the following cases:

(a) Suppose that $F'(0) \neq 0$. Then,

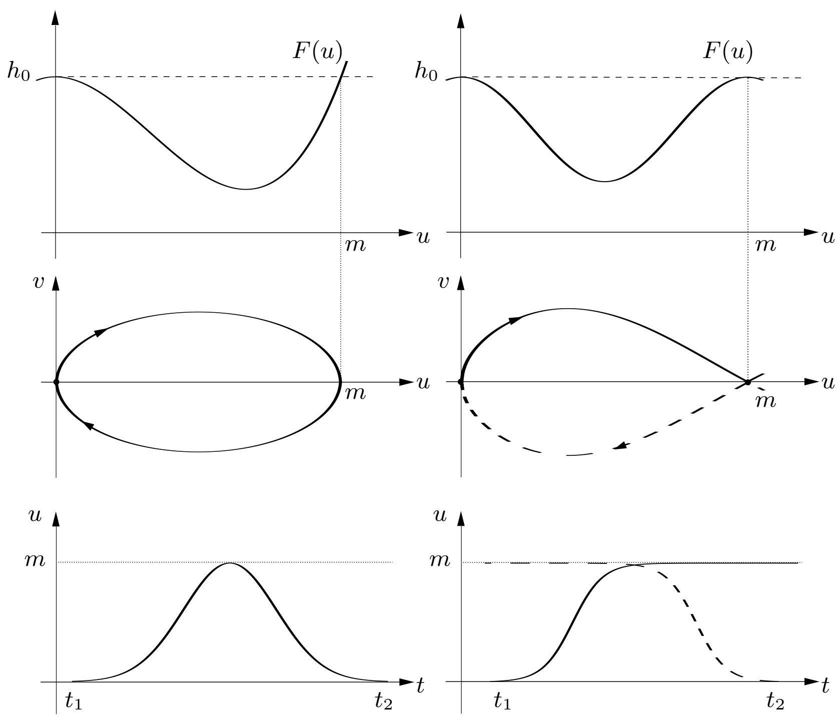
$$\lim_{u \to 0^+} v^{\pm}_{h_0}(u) = \lim_{u \to 0^+} \pm\sqrt{2\,\frac{F(0) - F(u)}{u}} = \pm\sqrt{-2F'(0)} =: \pm a \neq \pm\infty.$$

and therefore the branches $(u, v^{\pm}_{h_0}(u))$ intersect the singular line $\{u = 0\}$ at two distinct points $(0, \pm a)$. Moreover, they reach the singular line in finite time, since Lemma 5 (i) guarantees that the orbit γ_{h_0} of system (5) connects the point $(0, +a)$ with the point $(\bar{u}, v^{+}_{h_0}(\bar{u}))$ in *finite time* $\Delta T(\bar{u})$. Indeed, by direct integration of system (5) with $v = v^{+}_{h_0}(u)$, and recalling that $F'(0) < 0$, we have

$$\Delta T(\bar{u}) = \int_0^{\bar{u}} \frac{du}{v} = \int_0^{\bar{u}} \sqrt{\frac{u}{2\,(h_0 - F(u))}}\, du = \frac{1}{\sqrt{2}} \int_0^{\bar{u}} \sqrt{\frac{u}{F(0) - F(u)}}\, du < \infty. \quad (10)$$

(a1) If $F'(m) \neq 0$, then p is a regular point and there is an isolated orbit of (5) connecting the points $(0, +a)$ and $(0, -a)$, see Figure 2 (a1). By the above arguments, this orbit connects these points in a finite time, hence it gives rise to a strong solution $u(t)$ of (1) defined for some finite interval (t_1, t_2). As in the smooth cases, this isolated orbit defines a curve with no lobes and therefore $u(t)$ has a unique maximum m at $t = (t_1 + t_2)/2$. This solution is symmetric with respect its maximum due to the symmetry of the curve γ_{h_0}.

(a2) If $F'(m) = 0$, then p is a critical point and the branch $(u, v^+_{h_0}(u))$ defines an orbit leaving its α-limit $(+a, 0)$ in finite time and reaching its ω-limit p in infinite time. This orbit corresponds to a strong solution $u_+(t)$ of (1) defined for some interval $(t_1, +\infty)$, $t_1 \in \mathbb{R}$, such that $\lim_{t\to t_1^+} u_+(t) = 0$, $\lim_{t\to t_1^+} \dot{u}_+(t) = a$ and $\lim_{t\to\infty} u_+(t) = m$, Figure 2 (a2). Analogously, the branch $(u, v^-_{h_0}(u))$ gives an orbit such that each point on the orbit connects with the point p in an infinite time, and with $(-a, 0)$ in a finite time. This gives a strong solution $u_-(t)$ of (1) defined for some interval $(-\infty, t_2)$, $t_2 \in \mathbb{R}$, such that $\lim_{t\to t_2^-} u_-(t) = 0$, $\lim_{t\to t_2^-} \dot{u}_-(t) = -a$ and $\lim_{t\to-\infty} u_-(t) = m$. Notice that $u_+(t_1 - t) = u_-(t_1 - t)$ when $t_1 = t_2$ due to the symmetry of system (5).



(b1) (b2)

Figure 3: Strong solutions of (1) defined in a subset of $\mathbb{R}$ with energy $h = h_0$ when $F'(0) = 0$. These provide elementary forms to construct TWS of class $\mathcal{C}^1(\mathbb{R})$ (with compact support).

(b) Suppose that $F'(0) = 0$. Then,

$$\lim_{u\to 0^+} v^\pm_{h_0}(u) = \pm\sqrt{-2F'(0)} = 0.$$

Hence the branches $(u, v^\pm_{h_0}(u))$ are defined for $u \in [0, m]$, and coincide in $(0, 0)$ and $p = (m, 0)$.

(b1) If $F'(m) \neq 0$, then p is a regular point. Hence, there is an isolated orbit of (5) in $\{u > 0\} \times \mathbb{R}$ whose α and ω–limit is $(0,0)$, Figure 3 (b1). Lemma 5 (i) and (ii) ensures that the orbit of (5) connects $(0,0)$ with $(m,0)$ in finite time if $F''(0) < 0$ and $F'(m) > 0$. Therefore, it corresponds to a strong solution $u(t)$ of (1) defined in a finite interval (t_1, t_2), such that $\lim_{t\to t_1^+} u(t) = \lim_{t\to t_2^-} u(t) = 0$ and $\lim_{t\to t_1^+} \dot{u}(t) = \lim_{t\to t_2^-} \dot{u}(t) = 0$. In addition, $u(t)$ reaches a unique local maximum m at $t = (t_1 + t_2)/2$ and is symmetric with respect to this maximum due to the symmetry of system (5).

(b2) If $F'(m) = 0$, then p is a critical point and there are two orbits with energy h_0 connecting $(0,0)$ with p, cf. Figure 3 (b2). In view of Lemma 5, any point of this orbit connects with $(0,0)$ in finite time, thus the heteroclinic orbit connecting $(0,0)$ with p through the branch $(u, v^+_{h_0}(u))$ corresponds to a strong solution $u_+(t)$ of (1) defined in (t_1,∞), $t_1 \in \mathbb{R}$, such that $\lim_{t\to t_1^+} u_+(t) = \lim_{t\to t_1^+} \dot{u}_+(t) = 0$ and $\lim_{t\to\infty} u(t) = m$. The branch $(u, v^-_{h_0}(u))$ corresponds to a strong solution $u_-(t)$ defined in $(-\infty, t_2)$ with $t_2 \in \mathbb{R}$ such that $\lim_{t\to t_2^-} u_-(t) = \lim_{t\to t_1^+} \dot{u}_-(t) = 0$ and $\lim_{t\to-\infty} u(t) = m$. Notice that $u_+(t_1 + t) = u_-(t_1 - t)$ when $t_1 = t_2$ due to the symmetry of system (5).

The second type of solutions are the ones associated to curves γ_h which are defined for $u \in (0, m]$ but do not tend to a finite point on the singular line $\{u = 0\} \times \mathbb{R}$, hence they satisfy

$$\lim_{u\to 0^+} v^\pm_h = \pm\infty.$$

Such curves appear if and only if there exists $m > 0$ such that $F(u) < h_m = F(m)$ for all $u \in [0, m)$, see Figure 4. In addition, observe that for $0 < \bar{u} < m$

$$\Delta T(\bar{u}) = \int_0^{\bar{u}} \frac{du}{v} = \frac{1}{\sqrt{2}} \int_0^{\bar{u}} \sqrt{\frac{u}{F(m) - F(u)}} du < \infty, \tag{11}$$

since $F(m) - F(u) > 0$ for $u \in [0, m)$. Consequently, the orbits of (5) passing through the points $(\bar{u}, v^\pm_{h_m}(\bar{u}))$ are unbounded in the component v but aproach the singular line $u = 0$ at infinity in finite time. Analogously to the previous cases, we distinguish between two scenarios:

(c1) If $F'(m) \neq 0$ then the branches $(u, v^\pm_{h_m}(u))$ coincide at the regular point $p = (m, 0)$. Thus there is an unbounded orbit of (5) tending to infinity in the v-direction in finite time, Figure 4 (c1). It corresponds to a strong solution $u(t)$ of (1) defined in a finite interval (t_1, t_2) such that $\lim_{t\to t_1^+} u(t) = \lim_{t\to t_2^-} u(t) = 0$ and $\lim_{t\to t_1^+} \dot{u}(t) = +\infty$, $\lim_{t\to t_2^-} \dot{u}(t) = -\infty$. This solution has a unique maximum m at $t = (t_1 + t_2)/2$ and it is symmetric with respect this point.

(c2) If $F'(m) = 0$ then the branches $(u, v^\pm_{h_m}(u))$ coincide at the critical point $p = (m, 0)$ and there are two unbounded orbits of (5) tending to infinity in the v-direction in finite time, cf. Figure 4 (c2), but any point in the orbit takes an infinite time to reach the singular point p. The orbit defined by the branch $(u, v^+_{h_m}(u))$ corresponds to a strong solution $u_+(t)$ of (1) defined for $t \in (t_1, \infty)$, $t_1 \in \mathbb{R}$, such that

$\lim_{t\to t_1^+} u_+(t) = 0$, $\lim_{t\to t_1^+} \dot{u}_+(t) = +\infty$, and $\lim_{t\to\infty} u_+(t) = m$. Similarly, the orbit defined by the branch $(u, v^-_{h_m}(u))$ corresponds to a strong solution $u_-(t)$ of (1) defined for $t \in (-\infty, t_2)$, $t_2 \in \mathbb{R}$, such that $\lim_{t\to-\infty} u_-(t) = m$ and $\lim_{t\to t_2^-} u_-(t) = 0$, $\lim_{t\to t_2^-} \dot{u}_-(t) = -\infty$. Notice that as before, $u_+(t_1 + t) = u_-(t_1 - t)$ when $t_1 = t_2$ due to the symmetry of system (5).

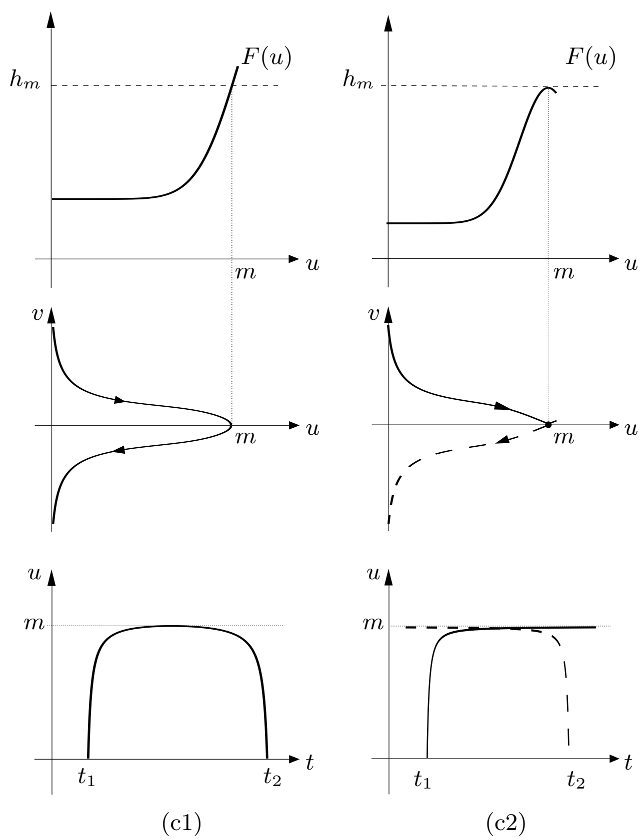


Figure 4: Strong solutions of (1) defined in a subset of $\mathbb{R}$ associated to curves γ_h with unbounded component v near the singular line $\{u = 0\} \times \mathbb{R}$. They provide elementary forms to construct TWS with cusps.

4 Singular traveling wave solutions

This section is devoted to the characterization of singular TWS of equation (1). We are interested in singular TWS such that either $v_h^{\pm}(0)$ is well-defined for $h = h_0$, or $\lim_{u\to 0^+} v_h^{\pm}(u) = \pm\infty$ for $h \neq h_0$.

4.1 Peaked waves

A TWS of (1) given by a function $u : \mathbb{R} \to \mathbb{R}$ is called *peaked* if it is smooth except at a finite or countable number of points (*peaks*) $\mathcal{S} = \{t_k \in \mathbb{R},\ k \in \mathbb{Z}\}$ where

$$0 \neq \lim_{t \to t_k^+} u'(t) = -\lim_{t \to t_k^-} u'(t) \neq \pm\infty,$$

The main result of this section is the following characterization:

Proposition 6. *The equation* (1) *has peaked TWS if and only if*

- $F'(0) < 0$ *and*
- *there exists* $m > 0$ *such that* $F(m) = F(0)$ *and* $F(u) < F(0)$ *for* $u \in (0, m)$.

These solutions are either

(i) peaked periodic, with period $T = \dfrac{2}{\sqrt{2}} \displaystyle\int_0^m \sqrt{\frac{u}{F(0) - F(u)}}\,du$*, if and only if in addition* $F'(m) \neq 0$*, or*

(ii) peaked solitary if and only if in addition $F'(m) = 0$.

These solutions are weak singular TWS and they are analytic except for a discontinuity in the first derivative at the peaks. Furthermore, the peaked solitary waves are symmetric with respect to their unique maximum and decay exponentially to zero at infinity. Peaked periodic solutions have a unique maximum and minimum per period and are symmetric with respect to these local extrema.

Proof. Peaked TWS are compositions of the elementary forms studied in the case (a) of Section 3.2. They appear as a consequence of the existence of values $t \in \mathbb{R}$ such that the solutions of (1) reach $u = 0$ with non-vanishing one-sided derivative. These solutions are associated to the integral curves of (5) with energy $h = h_0$. The branches $(u, v_{h_0}^{\pm}(u))$ must be defined and bounded, intersecting the singular line $\{u = 0\}$ at finite points $(0, +a)$ and $(0, -a)$ different from $(0, 0)$, so that the corresponding solutions reach 0 with non-vanishing derivative. Therefore, there must exist $m > 0$ such that $F(u) < F(0)$ for $u \in (0, m)$ and $F(m) = F(0)$, and additionally $F'(0) < 0$. These two necessary conditions are also sufficient for the existence of peaks, as we will see in the construction of solutions in the proofs of (i) and (ii).

Suppose that $F'(m) \neq 0$, so $(m, 0)$ is a regular point and therefore, by the symmetry of system (5) and by equation (10), there is an orbit of system (5) connecting $(0, +a)$ with $(0, -a)$ in finite time. This means that there exist $t_1, t_2 < \infty$ such that the first component of the solution of system (5), $u(t)$, is a strong solution of equation (1) (and equation (3) with $h = h_0$) with maximal interval of definition given by $t \in (t_1, t_2)$. This solution is the elementary form studied in the case (a1) in Section 3.2, hence $\lim_{t \to t_1^+} u(t) = 0$ and $\lim_{t \to t_1^+} \dot{u}(t) = a$. By symmetry, $\lim_{t \to t_2^-} u(t) = 0$, $\lim_{t \to t_2^-} \dot{u}(t) = -a$ and $u\left(\frac{t_1+t_2}{2}\right) = m$. Lemma 3 allows us to extend the above solution continuously to $\mathbb{R}$ by gluing together copies of it defined in the intervals (t_k, t_{k+1}) with $k \in \mathbb{Z}$, where $t_k = t_1 + k(t_2 - t_1)$. This leads to the function

$$\tilde{u}(t) = \begin{cases} u\left(t - (k-1)(t_2 - t_1)\right) & \text{for } t \in (t_k, t_{k+1}), \\ 0 & \text{for } t = t_k, \end{cases} \tag{12}$$

which is periodic with period

$$T = t_2 - t_1 = \frac{2}{\sqrt{2}} \int_0^m \sqrt{\frac{u}{F(0) - F(u)}}\, du.$$

By Lemma 5 (ii) the above integral is well defined, since $F'(m) \neq 0$. Notice also that $\lim_{t \to t_k} \tilde{u}(t) = 0$ and $\lim_{t \to t_k^{\pm}} \dot{\tilde{u}}(t) = \pm a$, so $\tilde{u}(t)$ is peaked periodic. Furthermore, the solution attains a unique maximum m at $t = (t_k + t_{k+1})/2$ and is symmetric with respect to it on each period. Observe that $\tilde{u} \in H^1_{loc}(\mathbb{R})$. Indeed, the function $\tilde{u}$ is bounded and analytic in $\mathbb{R} \setminus \mathcal{S}$ where $\mathcal{S} = \{t_k,\, k \in \mathbb{Z}\}$, and since $\lim_{t \to t_k} (\dot{\tilde{u}}(t))^2 = a^2$, also $|\dot{\tilde{u}}|^2$ is bounded in $\mathbb{R} \setminus \mathcal{S}$. Recalling in addition that $\tilde{u}(t)$ is a strong solution of (1) on each interval (t_k, t_{k+1}), cf. (a1) in Section 3.2, we get that it is a TWS in the sense of Definition 1. By construction $\tilde{u}(t)$ satisfies equation (3) with $h = h_0$ except at the points in $\mathcal{S}$ where the equation holds in the limit (4). So according to Definition 2, it is a weak singular TWS of (1). Finally, observe that the solution $\tilde{u}$ constructed above is the only possible continuous continuation of $u(t)$ on $\mathbb{R}$.

Suppose now that $F'(m) = 0$, so $(m, 0)$ is a critical point of (5). In this case there exist two orbits connecting $(m, 0)$ with $(0, +a)$ and $(0, -a)$, respectively, cf. the elementary forms of (a2) in Section 3.2. Again, Lemma 5 (i) guarantees that any point on these orbits reaches $(0, \pm a)$ in finite time, but it takes an infinite time to reach $(m, 0)$. So there exist strong solutions $u_{\pm}(t)$ of (1) defined on (t_1, ∞) and $(-\infty, t_2)$, respectively. The non-uniqueness of solutions then allows us to choose $t_2 = t_1$ to construct the function

$$u(t) = \begin{cases} u_-(t) & \text{for } t \in (-\infty, t_1], \\ u_+(t) & \text{for } t \in (t_1, +\infty), \end{cases}$$

which is a peaked solitary wave defined on $\mathbb{R}$. The same arguments as before show that u is in $H^1_{loc}(\mathbb{R})$, it is a weak solution of (1), and in particular a weak singular TWS of (1). Again, u is the only possible continuous continuation of u_- and u_+ onto $\mathbb{R}$. ∎

4.2 Solitary waves with compact support and associated composite waves

A TWS of (1) given by a function $u : \mathbb{R} \to \mathbb{R}$ has *compact support* if there exist $-\infty < t_1 < t_2 < \infty$ such that u is constant on $\mathbb{R} \setminus [t_1, t_2]$ and u is non constant on $[t_1, t_2]$. We call $[t_1, t_2]$ the support of u. A solitary wave with compact support, or simply *compact solitary wave*, is a continuous TWS which has compact support and a unique extremum. We say that a TWS $\tilde{u}(t)$ is a composite wave associated to a compact solitary wave $u(t)$ with support $[t_1, t_2]$, or simply a *composite wave of* $u(t)$, if it is obtained by gluing together copies of one compact solitary wave in such a way that the supports of each copy do not overlap. More precisely, $\tilde{u}(t)$ is a composite wave of a solitary wave $u(t)$ if there exists a collection of intervals $I_k = (t_1 - a_k(t_2 - t_1), t_2 - a_k(t_2 - t_1))$ for $a_k \in \mathbb{R}$ and $k \in K$, where K is either $\mathbb{Z}$ or a finite collection of indices, with $I_k \cap I_j \neq \emptyset$ for all $k \neq j$, such that

$$\tilde{u}(t) = \begin{cases} u(t + a_k(t_2 - t_1)) & \text{if } t \in I_k \text{ for some } k \in K, \\ 0 & \text{if } t \in \mathbb{R} \setminus \cup_{k \in K} I_k. \end{cases}$$

The following result characterizes the compact solitary waves and composite waves.

Proposition 7. *The equation* (1) *has compact solitary TWS and their associated composite solutions if and only if*

- $F'(0) = 0$ *and* $F''(0) < 0$*, and*
- *there exists* $m > 0$ *such that* $F(m) = F(0)$*,* $F(u) < 0$ *for* $u \in (0, m)$ *and* $F'(m) \neq 0$*.*

These solutions are either compact solitary waves, composite multi-bump solutions with compact support or composite waves with non-compact support that can be either aperiodic or periodic with arbitrary period. These solutions are strong singular TWS, they are $\mathcal{C}^1(\mathbb{R})$ *and piecewise analytic. Furthermore, the compact solitary waves are symmetric with respect to their unique maximum.*

Proof. Compact solitary TWS appear when there exist values $t \in \mathbb{R}$ such that the solutions of (1) reach $u = 0$ with vanishing one-sided derivative. These solutions are associated to the integral curves of (5) with energy $h = h_0$, and as before, the branches $(u, v_{h_0}^{\pm}(u))$ must be defined and bounded such that they intersect the singular line $\{u = 0\} \times \mathbb{R}$ at $(0, 0)$ in finite time. This last fact guarantees that the corresponding solution of (1) reaches $u = 0$ with vanishing derivative in finite time.

As studied in the case (b) of Section 3.2, the existence and boundedness of such curves is guaranteed if there exists $m > 0$ such that $F(u) < F(0)$ for $u \in (0, m)$ and $F(m) = F(0)$. The intersection of the curves $(u, v_{h_0}^{\pm}(u))$ with the singular line $\{u = 0\} \times \mathbb{R}$ at $(0, 0)$ is ensured if $F'(0) = 0$, since in this case

$$\lim_{u \to 0^+} v_{h_0}^{\pm}(u) = \lim_{u \to 0^+} \pm \sqrt{\frac{-2\, F''(u) u^2 + o(u^3)}{u}} = 0. \tag{13}$$

By Lemma 5 (i) and (ii), the orbits of (5) connect $(0, 0)$ with $(m, 0)$ in finite time if $F''(0) < 0$ and $F'(m) > 0$. Under these conditions there exist $t_1, t_2 \in \mathbb{R}$ such that the first component of the solution of system (5), $u(t)$, is a strong solution of (1) (and also (3) with $h = h_0$) in the maximal interval of definition $t \in (t_1, t_2)$. Moreover, $\lim_{t \to t_1^+} u(t) = 0$ and by symmetry $\lim_{t \to t_2^-} u(t) = 0$ and $u\left(\frac{t_1 + t_2}{2}\right) = m$ is the unique maximum. This solution $u(t)$ is the elementary form that appears in the case (b1) of Section 3.2.

In particular, the above conditions imply that equation (1) is not unique at $u = 0$, and that $u(t) \equiv 0$ is also a solution. Therefore it is possible to obtain a continuous continuation of $u(t)$ with compact support in $\mathbb{R}$. We define the function

$$\tilde{u}(t) = \begin{cases} u(t) & \text{for } t \in (t_1, t_2), \\ 0 & \text{for } t \in \mathbb{R} \setminus (t_1, t_2). \end{cases}$$

Observe that $\tilde{u}(t) \in H^1_{loc}(\mathbb{R})$ and it is analytic in $\mathbb{R} \setminus \{t_1, t_2\}$. Furthermore, $\lim_{t \to t_1^+} \dot{u}(t) = 0$ and $\lim_{t \to t_2^-} \dot{u}(t) = 0$ in view of (13). Hence $\tilde{u}(t) \in \mathcal{C}^1(\mathbb{R})$, and by construction it is a strong singular TWS of (1). In a similar way as above, we can also construct the following composite waves: multi-bump waves with compact support by gluing together a finite number of copies of $u(t)$, and waves with non-compact support that can be either aperiodic or periodic with arbitrary period. All these solutions are strong singular TWS of (1), they are piecewise analytic and at most $\mathcal{C}^1(\mathbb{R})$, cf. Remark 9 below.

The sufficiency part of the proof follows from the fact that if F satisfies the stated conditions, then it is possible to find the desired solutions following the above construction.

∎

As a direct consequence of Propositions 6 and 7, we obtain that it is not possible to find a peaked wave with compact support.

Corollary 8. *The equation* (1) *cannot admit peaked and compactly supported TWS for the same F. In particular, a peaked TWS can not have compact support, and conversely, a TWS with compact support can not have peaks.*

Remark 9. In view of Lemma 5, the conditions of Proposition 7 guarantee the existence of a homoclinic orbit of system (5) where every point on the orbit is connected to $(0,0)$ in a finite time. They also account for the fact that the compactly supported solutions are at most $\mathcal{C}^1$. Indeed, notice that under the conditions of Proposition 7 we have that

$$\dot{u}(t) = v_{h_0}^{\pm} = \pm\sqrt{-2F''(u)u + o(u^2)}$$

and

$$\begin{aligned}\ddot{u}(t) &= \pm\frac{-2F^{(3)}(u)u\dot{u} - 2F''(u)\dot{u} + o(u)\dot{u}}{2(\pm\dot{u})}\\ &= -F^{(3)}(u)u - F''(u) + o(u).\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{t\to t_1^-}\ddot{u}(t) = -F''(u) \neq 0 = \lim_{t\to t_1^+}\ddot{u}(t)\\ \lim_{t\to t_2^+}\ddot{u}(t) = -F''(u) \neq 0 = \lim_{t\to t_1^-}\ddot{u}(t),\end{aligned} \tag{14}$$

and hence the continuation to $u(t) \equiv 0$ is not $\mathcal{C}^2$ since $F''(u) < 0$. If we were to demand a $\mathcal{C}^2$-continuation, this would necessarily require the second derivative of F to vanish. As a consequence, the solutions would loose the compactness property, since in view of Lemma 5 the existence time of the loop would become infinite.

Remark 10. Observe that a different kind of composition of locally defined solutions may also be obtained under conditions stated in Proposition 7 but with $F'(0) < 0$. In this case

$$\lim_{u\to 0^+} v_{h_0}^{\pm}(u) = \lim_{u\to 0^+} \pm\sqrt{\frac{-2\,F'(u)u + o(u^2)}{u}} = \pm\sqrt{-2F'(0)} =: \pm a \neq 0,$$

and hence $\lim_{t\to t_1^+}\dot{u}(t) = a$ and $\lim_{t\to t_2^-}\dot{u}(t) = -a$. However, $u(t) \equiv 0$ is not a solution of (1) under this assumption on F and therefore the only possible continuous continuation is given by the periodic function (12). But this is not a composition of a compact solitary wave, i.e not a composite wave as defined at the beginning of this section.

4.3 Fronts with finite-time decay and plateau-shaped waves

Proposition 11. *The equation* (1) *has fronts, solitary and plateau-shaped singular TWS solutions if and only if*

- $F'(0) = 0$ *and* $F''(0) < 0$, *and*
- *there exists* $m > 0$ *such that* $F(m) = F(0)$, $F(u) < 0$ *for* $u \in (0, m)$ *and* $F'(m) = 0$.

The solitary waves are strong solutions of (1) *and therefore analytic. The fronts and plateau-shaped waves are strong singular TWS which are at most* $\mathcal{C}^1$ *and piecewise analytic. Moreover, the fronts have finite-time decay on one side.*

Proof. Under the present hypothesis, there exist orbits of system (5) corresponding to the branches $(u, v^{\pm}_{h_0}(u))$, which connect $(0,0)$ with the critical point $(m, 0)$. By Lemma 5, any point on these orbits is connected to $(0,0)$ in finite time and to $(m, 0)$ in infinite time. They yield two strong solutions $u_{\pm}(t)$ with maximal interval of definition (t_1, ∞) and $(-\infty, t_2)$, respectively, which are given by the elementary forms of the case (b2) of Section 3.2. By gluing them together with the solution $u(t) \equiv 0$ we get the following solutions: fronts given by

$$\tilde{u}_-(t) = \begin{cases} u_-(t) & \text{for } t \in (-\infty, t_1), \\ 0 & \text{for } t \in [t_1, \infty), \end{cases} \quad \text{and} \quad \tilde{u}_+(t) = \begin{cases} 0 & \text{for } t \in (-\infty, t_2], \\ u_+(t) & \text{for } t \in (t_2, \infty). \end{cases}$$

Setting $t_1 = t_2$ we get a solitary wave given by

$$u(t) = \begin{cases} u_-(t) & \text{for } t \in (-\infty, t_1), \\ 0 & \text{for } t = t_1, \\ u_+(t) & \text{for } t \in (t_1, \infty), \end{cases}$$

which is symmetric with respect to its unique maximum. Choosing $t_1 < t_2$ we get a plateau-shaped solitary wave given by

$$u(t) = \begin{cases} u_-(t) & \text{for } t \in (-\infty, t_2), \\ 0 & \text{for } t \in [t_2, t_1], \\ u_+(t) & \text{for } t \in (t_1, \infty), \end{cases}$$

The front and plateau-shaped solutions are analytic in $\mathbb{R} \setminus \{t_i\}$ and satisfy equation (3) for $h = h_0$, hence they are strong singular TWS of (1). Furthermore, they are at most $\mathcal{C}^1(\mathbb{R})$ for the same reason explained in Remark 9. The solitary wave solutions, however, are in fact strong solutions of (1) on $\mathbb{R}$ since $\ddot{u}(t)$ is defined everywhere, cf. (14), and therefore they are analytic on $\mathbb{R}$. ∎

We emphasize that the fronts described above are not the classical smooth fronts with exponential decay on both ends, but they decay in finite time on one end.

4.4 Cusped waves

A TWS of (1) given by a function $u : \mathbb{R} \to \mathbb{R}$ is called *cusped* if it is smooth except at a finite or countable number of points (*cusps*) $\mathcal{S} = \{t_k \in \mathbb{R},\, k \in \mathbb{Z}\}$ where

$$\lim_{t \to t_k^+} u'(t) = - \lim_{t \to t_k^-} u'(t) = \pm\infty.$$

Proposition 12. *The equation* (1) *has cusped TWS if and only if there exists* $m > 0$ *such that* $F(m) - F(u) > 0$ *for all* $u \in [0, m)$*. These solutions are either*

(i) cusped periodic with period $T = \frac{2}{\sqrt{2}} \int_0^m \sqrt{\frac{u}{F(m) - F(u)}} du$ *if and only if in addition* $F'(m) \neq 0$*, or*

(ii) cusped solitary if and only if in addition $F'(m) = 0$*.*

These solutions are weak singular TWS and they are analytic except for a discontinuity in the first derivative at the peaks. Furthermore, the cusped solitary waves are symmetric with respect to their unique maximum and decay exponentially to zero at infinity. Cusped periodic solutions have a unique maximum and minimum per period and are symmetric with respect to these local extrema.

Proof. Cusped TWS correspond to orbits of (5) which are defined by the curves $(u, v_h^{\pm}(u))$ with $h \neq h_0$ satisfying

$$\lim_{u \to 0^+} v_h^{\pm}(u) = \lim_{u \to 0^+} \pm \sqrt{2 \frac{h - F(u)}{u}} = \pm \infty.$$

Therefore, a necessary condition for the appearance of cusps is that there exists $m > 0$ such that $F(m) - F(u) > 0$ for $u \in [0, m)$. In particular, this implies that $h_m := F(m) > F(0)$. As shown in the case (c) of Section 3.2 the orbits of (5) passing through the points $(\bar{u}, v_{h_m}^{\pm}(\bar{u}))$ are unbounded in the component v but approach the singular line $u = 0$ at infinity in a finite time. We will see in the construction of solutions in the proofs of (i) and (ii) below that the necessary condition deduced above is also sufficient for the existence of cusped TWS.

(i) If $F'(m) \neq 0$ then the point $(m, 0)$ connecting the two branches $(u, v_{h_m}^{\pm}(u))$ is regular, so by equation (11) there exist $t_1, t_2 < \infty$ such that the first component of the solution of system (5) $u(t)$, is a strong solution of (1) (and also of (3) with $h = h_m$), with maximal interval of definition $t \in (t_1, t_2)$, which is given by the elementary form that appears in case (c1) of Section 3.2. Notice that $u(t) \equiv 0$ is not a solution of (3) for $h = h_m > 0$. Therefore, the only possible continuous continuation of $u(t)$ preserving the energy is given by the periodic function

$$\tilde{u}(t) = \begin{cases} u\left(t - (k-1)(t_2 - t_1)\right) & \text{for } t \in (t_k, t_{k+1}), \\ 0 & \text{for } t = t_k, \end{cases}$$

where $t_k = t_1 + k(t_2 - t_1)$ for $k \in \mathbb{Z}$, which is periodic with period

$$T = t_2 - t_1 = \frac{2}{\sqrt{2}} \int_0^m \sqrt{\frac{u}{F(m) - F(u)}} du.$$

Observe now that integrating system (5), and denoting $u_\varepsilon := u(t_1 + \varepsilon)$ we get,

$$\int_{t_1}^{t_1 + \varepsilon} \dot{u}^2 dt = \int_0^{u_\varepsilon} v^2 \frac{du}{v} = \int_0^{u_\varepsilon} \sqrt{\frac{2(F(m) - F(u))}{u}} du.$$

Since $F(m) - F(u) > 0$ for all $u \in [0, u_\varepsilon]$ and recalling the criterion used in Lemma 5, we find that $\lim_{u \to 0^+} \sqrt{(F(m) - F(u))/u} \cdot \sqrt{u} = A$ with $A \neq 0$ and $A \neq \infty$. Therefore,

$$\int_0^{u_\varepsilon} \sqrt{\frac{2(F(m) - F(u))}{u}} du < \infty,$$

and hence $\int_{t_1}^{t_1+\varepsilon} \dot{u}^2 dt < \infty$. Analogously $\int_{t_2-\varepsilon}^{t_2} \dot{u}^2 dt < \infty$ and hence $\int_{t_k-\varepsilon}^{t_k+\varepsilon} (\dot{\tilde{u}})^2 dt$ is convergent for every t_k so that for each compact $K \subset \mathbb{R}$ we have

$$\int_K (\dot{\tilde{u}})^2 dt < \infty.$$

Since $\tilde{u}(t)$ is bounded as well, this solution is in $H^1_{loc}(\mathbb{R})$, and analytic on $\mathbb{R} \setminus \mathcal{S}$ where $\mathcal{S} = \{t_k \in \mathbb{R},\, k \in \mathbb{Z}\}$. Finally observe that $\tilde{u}(t)$ is a weak singular TWS of (1), because

$$\lim_{t \to t_k} \frac{u(\dot{u})^2}{2} + F(u) = u\,\frac{F(m) - F(u)}{u} + F(u) = F(m) = h_m.$$

(ii) If $F'(m) = 0$ then the orbits corresponding to $(u, v^{\pm}_{h_m})$ approach the singular line $\{u = 0\}$ in finite time, but it takes an infinite time to reach $(m, 0)$. So there exist two strong solutions of (1) $u_\pm(t)$ given by the elementary forms of the case (c2) of Section 3.2, defined on the maximal intervals $(t_1, +\infty)$ and $(-\infty, t_2)$, respectively. The only way to construct a continuous continuation in $\mathbb{R}$ preserving the energy is by choosing $t_2 = t_1$ and gluing together the corresponding solutions $u_\pm$. These considerations lead us to define the function

$$\tilde{u}(t) = \begin{cases} u_-(t) & \text{for } t \in (-\infty, t_1), \\ 0 & \text{for } t = t_1 \\ u_+(t) & \text{for } t \in (t_1, +\infty), \end{cases}$$

which is a cusped solitary TWS defined in $\mathbb{R}$. The same arguments as in the proof of statement (a) show that $\tilde{u}$ is a weak singular TWS of (1). ∎

4.5 Exhaustivity of the characterization

Observe that the elementary forms presented in Section 3.2 capture all the strong solutions of equation (1) reaching or tending to $\{u = 0\}$, whose maximal interval of definition is not $\mathbb{R}$. In addition, the singular solutions described in the preceding Section 4 cover all possible continuous extensions to $\mathbb{R}$ on the same energy level, using these elementary forms and the constant function $u(t) \equiv 0$ whenever it is a solution. In consequence, our characterization of singular TWS for equation (1) given in Propositions 6, 7, 11 and 12 is exhaustive.

5 Application to shallow water equations

The aim of this section is to demonstrate the applicability of the propositions developed in the preceding sections. We exemplify our approach by studying the equation for surface waves of moderate amplitude in shallow water and the Camassa-Holm equation. In particular, we show how the different types of singular TWS can be obtained varying the energy of the corresponding Hamiltonian systems. This approach may be applied to study singular

TWS of a variety of other equations, for example a class of nonlinear wave equations related to the inviscid Burgers' equation and Camassa-Holm equation studied in [15], the family of equations analyzed in [16], and a generalization of the Camassa-Holm equation studied in [17]. In the latter paper, the authors conclude with a conjecture on the non-existence of peaked solitary solutions when a certain parameter becomes non-positive. In view of the results in Section 4 we are able to give an affirmative answer.

5.1 Surface waves of moderate amplitude in shallow water

In this section we study singular TWS of the equation for surface waves of moderate amplitude in shallow water,

$$u_t + u_x + 6uu_x - 6u^2u_x + 12u^3u_x + u_{xxx} - u_{xxt} + 14uu_{xxx} + 28u_xu_{xx} = 0, \qquad (15)$$

which was first derived by Johnson [12], whose considerations were extended by Constantin and Lannes [4]. We refer to [11] for a first study of smooth solitary waves and to [10] for a more extensive characterization of TWS of equation (15). We introduce the traveling wave Ansatz $u(x,t) = u(x - c\,t)$ and integrate once to obtain

$$u''\Big(u + \frac{1+c}{14}\Big) + \frac{1}{2}(u')^2 + K + (1-c)u + 3u^2 - 2u^3 + 3u^4 = 0, \qquad (16)$$

for some constant $K \in \mathbb{R}$. Notice that after the change of variables $u \mapsto u - \frac{1+c}{14}$ the above equation is of the form (1), with F a suitable polynomial in u depending on the parameters c and K. The relation between the parameters and the qualitative properties of F is studied in detail in [10]. In particular, it is observed that F has either no extremum or there are two extrema which we denote by p_1 (the local maximum) and p_2 (the local minimum of F) such that $p_1 < p_2$. Let $h_i = F(p_i)$ for $i = 1, 2$. We distinguish different cases depending on the position of p_1 and p_2 with respect to $u = 0$, and the sign of $h_0 - h_2$. Taking into account these cases and the characterization given in Propositions 6, 7 and 12 we obtain the following types of bounded singular TWS for equation (15) varying the energy. We point out that, as a consequence of Corollary 8, compact solitary waves and peaked waves cannot coexist within a single case. To give an example, Figure 5 shows the different TWS of equation (15) that appear for different energy levels when $0 < p_1 < p_2$ and $h_2 > h_0$.

Energy/Case	$0 < p_1 < p_2$ and $h_2 > h_0$	Energy/Case	$0 < p_1 < p_2$ and $h_2 < h_0$
$h > h_1$	cusped periodic	$h > h_1$	cusped periodic
$h = h_1$	cusped & smooth solitary	$h = h_1$	cusped & smooth solitary
$h_1 > h > h_2$	cusped & smooth periodic	$h_1 > h > h_0$	cusped & smooth periodic
$h = h_2$	cusped periodic & constant	$h_0 \geq h > h_2$	smooth periodic
$h_2 > h > h_0$	cusped periodic	$h = h_2$	constant
$h \leq h_0$		$h \leq h_2$	

Energy/Case	$0 < p_1 < p_2$ and $h_2 = h_0$	Energy/Case	$0 = p_1 < p_2$
$h > h_1$	cusped periodic	$h > h_1$	cusped periodic
$h = h_1$	cusped & smooth solitary	$h = h_1 = h_0$	compact solitary & composite
$h_1 > h > h_0 = h_2$	cusped & smooth periodic	$h_1 > h > h_2$	smooth periodic
$h = h_0$	constant	$h = h_2$	constant
$h < h_0$		$h < h_2$	

Energy/Case	$p_1 < 0 < p_2$	$p_1 < p_2 = 0$	$p_1 < p_2 < 0$
$h > h_0$	cusped periodic	cusped periodic	cusped periodic
$h = h_0$	peaked periodic	constant	
$h_0 > h > h_2$	smooth periodic		
$h = h_2$	constant		
$h < h_2$			

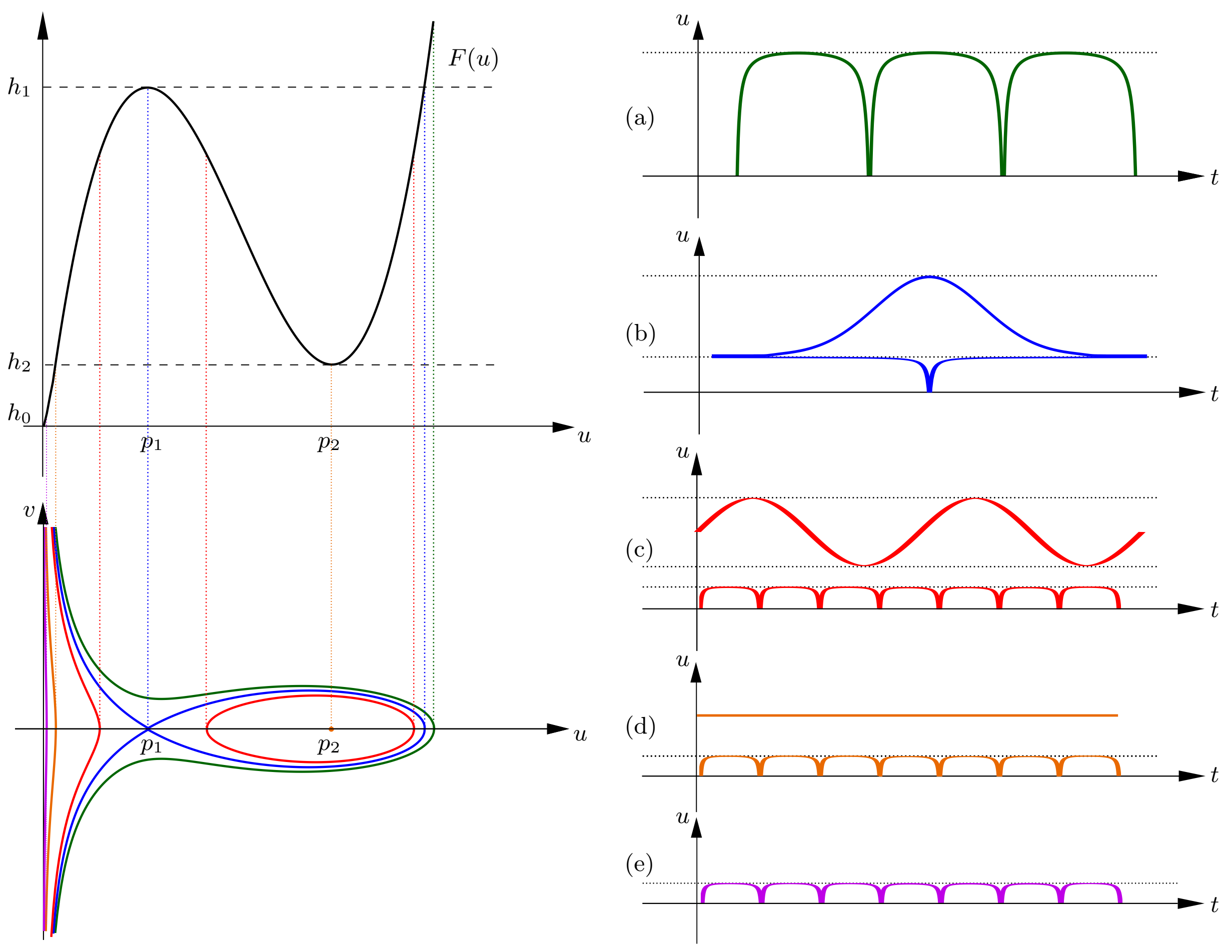


Figure 5: Singular TWS of equation (15) varying the energy level h in the case $0 < p_1 < p_2$ and $h_2 > h_0$. (a) cusped periodic waves for $h > h_1$; (b) cusped and smooth solitary waves for $h = h_1$; (c) cusped and smooth periodic waves for $h_1 > h > h_2$; (d) cusped periodic waves and constant solutions for $h = h_2$; (e) cusped periodic waves for $h_2 > h > h_0$.

5.2 The Camassa-Holm Equation

In the present section we will study singular TWS of the Camassa-Holm equation (CH)

$$u_t + 2\kappa\, u_x - u_{txx} + 3\, u\, u_x = 2\, u_x u_{xx} + u\, u_{xxx}, \tag{17}$$

for $x \in \mathbb{R}$, $t > 0$ and $\kappa \in \mathbb{R}$, which was introduced in the context of water waves by Camassa and Holm [2]. For a classification of weak traveling wave solutions of the Camassa–Holm equation we refer to [13]. Proceeding as in the previous section we introduce the traveling wave Ansatz $u(x,t) = u(x - c\,t)$. Integrating once equation (17) takes the form

$$u''(u-c) + \frac{(u')^2}{2} + r + (c - 2\kappa)\, u - \frac{3}{2}u^2 = 0,$$

where r is a constant of integration. The change of variables

$$w = u - c,$$

transforms the above equation to the form (1) with

$$F(w) = A\, w + B\, w^2 - \frac{1}{2}w^3, \tag{18}$$

where $A = r - 2\kappa c - \frac{1}{2}c^2$ and $B = -(c+\kappa)$. $F(w)$ is a third order polynomial which satisfies $F(0) = 0$, it has at most three roots

$$w = 0 \text{ and } w = B \pm \sqrt{B^2 + 2A},$$

and at most two extrema

$$p_i = \frac{2B + (-1)^i\sqrt{4B^2 + 6A}}{3}, \;\; i = 1, 2.$$

We may assume that $B \geq 0$ since otherwise the change of variables $(\hat{w}, \hat{v}) = -(w, v)$ yields this situation (this is equivalent to considering system (5) only for $u \geq 0$). Note that F does not have any extremum when $4B^2 + 6A \leq 0$, and it has two distinct extrema otherwise. In the latter case, p_1 is the local minimum and $p_2 > 0$ the local maximum (with $F''(p_2) < 0$). We denote $h_1 = F(p_1)$ and $h_2 = F(p_2)$, and distinguish between the following cases:

Case		
(i)	$A > 0$	$p_1 < 0 < p_2 \quad h_1 < 0 < h_2$
(ii)	$A = 0, \quad B > 0$	$p_1 = 0 < p_2 \quad h_1 = 0 < h_2$
(iii)	$A < 0, \quad 4B^2 + 6A > 0, \quad B^2 + 2A > 0$	$0 < p_1 < p_2 \quad h_1 < 0 < h_2$
(iv)	$A < 0, \quad 4B^2 + 6A > 0, \quad B^2 + 2A = 0$	$0 < p_1 < p_2 \quad h_1 < h_2 = 0$
(v)	$A < 0, \quad 4B^2 + 6A > 0, \quad B^2 + 2A < 0$	$0 < p_1 < p_2 \quad h_1 < h_2 < 0$
(vi)	$A < 0, \quad 4B^2 + 6A \leq 0$	

Taking into account the cases described above and the classification given in Propositions 6 and 12 we obtain the following types of singular TWS varying the energy level h:

Energy/Case	(i)	(ii)	(iii)	(iv)
$h < h_1$				
$h = h_1$			constant	constant
$h_1 < h < 0$			smooth periodic	smooth periodic
$h = 0$		constant	peaked periodic	peaked solitary
$0 < h < h_2$	cusped periodic	cusped periodic	cusped periodic	
$h = h_2$	cusped solitary	cusped solitary	cusped solitary	
$h > h_2$				

Energy/Case	(v)
$h < h_1$	
$h = h_1$	constant
$h_1 < h < h_2 < 0$	smooth periodic
$h = h_2$	smooth solitary
$h > h_2$	

5.3 Generalized Camassa-Holm Equation

In [17] the authors study peaked solitary and periodic cusped traveling wave solutions of a generalization of the CH equation of the form

$$u_t + 2\kappa\, u_x - u_{txx} + a\, u\, u_x = 2\, u_x u_{xx} + u\, u_{xxx}, \tag{19}$$

where $a \in \mathbb{R}$ is an additional parameter. At the end of their paper they state the following conjecture: "If the parameter $a \leq 0$, then equation (19) has no peaked solitary wave solution". Using the approach developed in the preceding sections it is easy to see that this assertion is true. Proceeding as with the CH above we introduce the traveling wave Ansatz and integrate once to find that after the change of variables $w = u - c$ we obtain

$$w''w + \frac{(w')^2}{2} + F'(w) = 0,$$

which is an equation of the form (1) with

$$F(w) = A\, w + B\, w^2 - \frac{a}{6} w^3,$$

where $A = r + (1 - \frac{a}{2})c^2 - 2\kappa c$ and $B = (1 - a)\frac{c}{2} - \kappa$. Our analysis (cf. Proposition 6) shows that an equation of this form has peaked solitary TWS if and only if $F'(0) < 0$ and there exists $m > 0$ such that $F(m) = F(0)$ with $F(u) < F(0)$ for $u \in (0, m)$ and $F'(m) = 0$. Assuming that $a < 0$, we see that $F(w) \to \pm\infty$ as $w \to \pm\infty$. This contradicts the conditions for the existence of peaked solitary solutions stated above. Indeed, if equation (19) had peaked solitary TWS, then $F'(0) = A < 0$. Thus, F would have a maximum to the left and a minimum to the right of $w = 0$. Hence there exists $m > 0$ such that $F(m) = F(0)$, but $F'(m) \neq 0$ since F has at most two extrema. For $a = 0$ the situation is similar. This shows that the conjecture is true.

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