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PIECEWISE SMOOTH DYNAMICAL SYSTEMS: NORMAL FORMS AND PERSISTENCE OF PERIODIC SOLUTIONS

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Abstract. We consider a n–dimensional piecewise smooth vector fields with two zones separated by a hyperplane Σ which admits an invariant hyperplane Ω transversal to Σ containing a period–annulus $\mathcal{A}$ fulfilled by crossing periodic solutions. For small discontinuous perturbations of these systems we develop a Melnikov–like function to control the persistence of periodic solutions contained in $\mathcal{A}$. When $n = 3$ we provide normal forms in the piecewise linear case. Finally we apply the Melnikov–like function to study discontinuous perturbations of the normal forms established.

1. Introduction

The study of the persistence of periodic orbits under small perturbations is a classical and important problem in the qualitative theory of vector fields. For smooth systems this problem was extensively studied at any dimension (see, for instance, the book [9] and the references therein). For nonsmooth systems some techniques to deal with this kind of problem have been recently developed (see, for instance, [18, 19]). However, due to the difficulty in applying these last ideas in higher dimensional systems, it has been considered, in general, for planar systems (see, for instance, [2, 20, 22, 23]). As far as we know there are only a few works dealing this problem in higher dimensional nonsmooth systems (see, for instance, [4]). Therefore, in this paper, our interest lies in studying the persistence of periodic orbits for n–dimensional piecewise smooth vector fields having a period–annulus of periodic solutions contained in an invariant hyperplane.

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Formally we consider the following (unperturbed) n–dimensional piecewise smooth vector field with two zones separated by the hyperplane $\Sigma = \{z = 0\}$

$$Z_0(\mathbf{x}, z) = \begin{cases} X_0^+(\mathbf{x}, z), & \text{if} \quad z > 0 \\ X_0^-(\mathbf{x}, z), & \text{if} \quad z < 0. \end{cases} \tag{1}$$

Here $(\mathbf{x}, z) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}$, and $X_0^\pm = (X_{0,1}^\pm, X_{0,2}^\pm, \dots, X_{0,n}^\pm)$. Throughout this paper, X_0^+ and X_0^- will be called, respectively, upper and lower vector fields. As the main hypothesis, we shall assume that there exists an invariant hyperplane $\Omega \subset \mathbb{R}^n$ transversal to Σ containing a period–annulus $\mathcal{A}$ fulfilled by crossing periodic solutions of (1) reaching Σ transversally (see Figure 1).

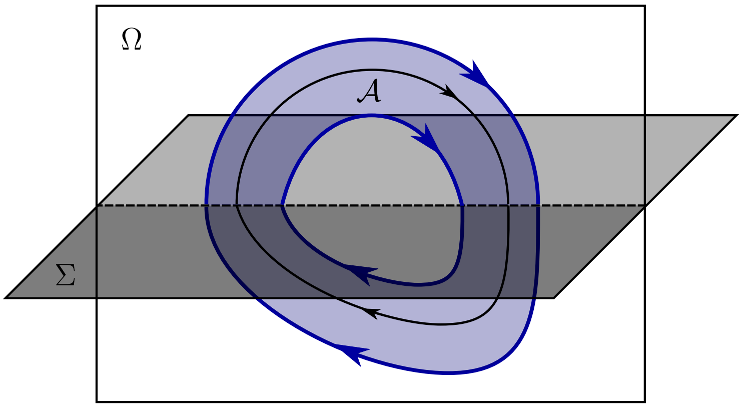

FIGURE 1. Representation of a n–dimensional piecewise smooth vector field with two zones separated by the hyperplane Σ having an invariant hyperplane Ω transversal to Σ which possess a period–annulus $\mathcal{A}$ fulfilled by crossing periodic solutions.

Particularly for $n = 3$, an interesting case to be considered is when the border of the period–annulus $\mathcal{A}$ contains the so called *Teixeira-singularity* (or just T–*singularity*), which represents a source of intricate and complex phenomena [7]. Roughly speaking, the T–singularity is a point where vector field (1) is tangent to both sides of the plane Σ and their orbits nearby return to Σ. In [25], for a particular definition of structural stability of nonsmooth systems, Teixeira showed that the T-singularity is not structural stable, and asymptotic stability is determined only under limited conditions of hyperbolicity. Recently, several other aspects of the dynamic of nonsmooth systems in the presence of a T-singularity has been studied, see, for instance, [6, 15, 16] and the references therein.

We aim to study the persistence of periodic orbits contained in $\mathcal{A}$ when (1) is perturbed inside the class of all piecewise smooth vector fields with two zones separated by the hyperplane Σ. A similar problem was addressed in [10] however for planar piecewise smooth systems. In this case the authors developed the Poincaré map in power series of some small parameter, and then a Melnikov–like function was obtained just by taking the first coefficient of this series. In dimension $n \geq 3$ the arguments are more delicate, and some recent techniques (see [21]) on Lyapunov–Schmidt reduction have to be used. For instance, in [17] the authors study the same problem using similar techniques however for 3D smooth system.

In smooth systems the normal form theory provides simplifications on the systems reducing the number of parameters and, eventually, simplifying the study of their dynamics. On the other hand, for nonsmooth systems formed by two regions separated by a hyperplane, more simplifications on the parameters can be done. In fact, piecewise continuous change of variables (smooth on each region determined by the hyperplane) is used to obtain normal forms for each one of the systems constituting the nonsmooth system. Some normal forms has been obtained in [12] for planar piecewise smooth systems separated by a straight line.

Taking all the comments above into account our first main result, Theorem A, develops a Melnikov–like function to control the persistence of periodic solutions of $\mathcal{A}$ for small discontinuous perturbation of system (1). Our second main result, Theorem B, provides normal forms for all 3D piecewise linear systems which admits an invariant plane Ω transveral to Σ fulfilled by periodic orbits. Finally, in section 4, we apply Theorem A, with $n = 3$, to study discontinuous perturbations of the normal forms established in section 3.

We point out the importance of nonsmooth systems, particularly piecewise smooth systems, as a source of new interesting phenomena in dynamics (see, for instance, [1, 24]), and their relevance in modelling real phenomena (see, for instance, [3, 8]).

2. Persistence of crossing periodic orbits

Consider the following discontinuous perturbation of system (1)

$$(2)\quad Z(x,\mathbf{y},z;\varepsilon)=\begin{cases} Z^+(x,\mathbf{y},z;\varepsilon)=X_0^+(x,\mathbf{y},z)+\varepsilon X_1^+(x,\mathbf{y},z) & \text{if} \quad z>0,\\ Z^-(x,\mathbf{y},z;\varepsilon)=X_0^-(x,\mathbf{y},z)+\varepsilon X_1^-(x,\mathbf{y},z) & \text{if} \quad z<0,\end{cases}$$

where $(x,\mathbf{y},z)\in D\subset\mathbb{R}\times\mathbb{R}^{n-2}\times\mathbb{R}$, $X_j=(X_{j,1},X_{j,2},\dots,X_{j,n})$ for $j=0,1$ and the set of discontinuity is given by the hyperplane $\Sigma=\{z=0\}$.

As usual the following open regions are distinguished in Σ:

(*i*) *Crossing Region:* $\Sigma^c=\{p\in\Sigma:\ (Z^+f)(p)(Z^-f)(p)>0\}$;
(*ii*) *Escaping Region:* $\Sigma^e=\{p\in\Sigma:\ (Z^+f)(p)>0,(Z^-f)(p)<0\}$;
(*iii*) *Sliding Region:* $\Sigma^s=\{p\in\Sigma:\ (Z^+f)(p)<0,(Z^-f)(p)>0\}$.

Here $(Z^{\pm}f)(p) = \langle Z^{\pm}(p), \nabla f(p)\rangle$.

A *crossing periodic orbit* of system (9) is a closed curve γ composed by trajectories of $Z^{\pm}$ having the same orientation such that $\gamma \cap \Sigma \subset \Sigma^c$.

We recall our main hypothesis:

(H) the hyperplane $\Omega = \{x = 0\}$ has a period–annulus $\mathcal{A} = \{(y, z) \in \Omega : r_0 \leq |(\mathbf{y}, z)| \leq r_1\}$ surrounding the origin and fulfilled by crossing periodic solutions of the unperturbed system $(\dot{x}, \dot{y}, \dot{z})^T = Z(x, y, z; 0)$ reaching Σ transversally.

Let $\varphi^{\pm}(t, x, \mathbf{y}, z; \varepsilon) = \big(\varphi_1^{\pm}(t, x, \mathbf{y}, z; \varepsilon), \varphi_2^{\pm}(t, x, \mathbf{y}, z; \varepsilon), \ldots, \varphi_n^{\pm}(t, x, \mathbf{y}, z; \varepsilon)\big)$ be the solutions of the systems $(\dot{x}, \dot{\mathbf{y}}, \dot{z})^T = Z^{\pm}(x, \mathbf{y}, z; \varepsilon)$ such that $\varphi^{\pm}(0, x, \mathbf{y}, z; \varepsilon) = (x, \mathbf{y}, z)$. From hypothesis, the solutions of (2), for $\varepsilon = 0$, contained in $\mathcal{A}$ reach transversally the set of discontinuity Σ. So for a small neighborhood $U \subset \mathbb{R}^n$ of $\mathcal{A}$ and $|\varepsilon| \neq 0$ small enough there exist a time $t^+(x, \mathbf{y}; \varepsilon) > 0$ (resp. $t^-(x, \mathbf{y}; \varepsilon) < 0$) such that an orbit of (2) starting in $(x, \mathbf{y}, 0) \in U \cap \Sigma$ returns, forward in time (resp. backward in time), to Σ, that is $\varphi_n^{\pm}(t^{\pm}(x, \mathbf{y}; \varepsilon), x, \mathbf{y}, 0; \varepsilon) = 0$.

The solutions $\varphi^{\pm}(t, x, \mathbf{y}, z; \varepsilon)$ can be expressed in power series of ε, that is

$$\varphi_i^{\pm}(t, x, \mathbf{y}, z; \varepsilon) = \psi_{0,i}^{\pm}(t, x, \mathbf{y}, z) + \varepsilon\psi_{1,i}^{\pm}(t, x, \mathbf{y}, z) + \mathcal{O}(\varepsilon^2), \quad i = 1, 2, \ldots, n$$

such that $\varphi^{\pm}(t, x, \mathbf{y}, z; \varepsilon) = \psi_0^{\pm}(t, x, \mathbf{y}, z) + \varepsilon\psi_1^{\pm}(t, x, \mathbf{y}, z)$ and $\psi_0^{\pm}(t, x, \mathbf{y}, z)$ are the solutions of the unperturbed systems $(\dot{x}, \dot{\mathbf{y}}, \dot{z})^T = Z^{\pm}(x, \mathbf{y}, z; 0)$. The times $t^{\pm}$, introduced above, can also be written as power series in ε, that is

$$t^{\pm}(x, \mathbf{y}; \varepsilon) = \tau_0^{\pm}(x, \mathbf{y}) + \varepsilon\tau_1^{\pm}(x, \mathbf{y}) + \mathcal{O}(\varepsilon^2).$$

However here we only have to assume that the expressions $\sigma_j^{\pm}(\mathbf{y}) = \tau_j^{\pm}(0, \mathbf{y})$, $j = 0, 1$, are explicitly known. In fact, the next result gives the expressions of $\sigma_1^{\pm}$ and $\psi_1^{\pm}$ in terms of the solutions $\psi_0^{\pm}$ of the unperturbed systems.

Proposition 1. *Let $p = (x, \mathbf{y}, z) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}$, and denote $Y^{\pm}(t, p_0) = D_p\psi_0^{\pm}(t, p_0)$ the derivative of $\psi_0^{\pm}(t, p)$ with respect to the initial condition p evaluated at $p_0 = (x_0, \mathbf{y}_0, z_0)$. Then the following equalities hold:*

$$\psi_1^{\pm}(t, p) = Y^{\pm}(t, p) \int_0^t Y^{\pm}(s, p)^{-1} X_1^{\pm}\left(\psi_0^{\pm}(s, p)\right) ds, \tag{3}$$

$$\sigma_1^{\pm}(\mathbf{y}) = -\frac{\psi_{1,n}^{\pm}(\sigma_0^{\pm}(\mathbf{y}), 0, \mathbf{y}, 0)}{X_{0,n}^{\pm}\left(\psi_0^{\pm}(\sigma_0^{\pm}(\mathbf{y}), 0, \mathbf{y}, 0)\right)}. \tag{4}$$

Proof. Computing the derivative in the variable t of both sides of the equality $\varphi^{\pm}(t, x, \mathbf{y}, z; \varepsilon) = \psi_0^{\pm}(t, x, \mathbf{y}, z) + \varepsilon\psi_1^{\pm}(t, x, \mathbf{y}, z) + \mathcal{O}(\varepsilon^2)$ we obtain

$$X_0^{\pm}\left(\varphi^{\pm}(t, p; \varepsilon)\right) + \varepsilon X_1^{\pm}\left(\varphi^{\pm}(t, p; \varepsilon)\right) = X_0\left(\psi_0^{\pm}(t, p)\right) + \varepsilon\frac{\partial\psi_1^{\pm}}{\partial t}(t, p) + \mathcal{O}(\varepsilon^2).$$

Developing the left hand side of the above equation in power series of ε, and studying the coefficient of ε we obtain

$$\frac{\partial \psi_1^{\pm}}{\partial t}(t,p) = DX_0^{\pm}\left(\psi_0^{\pm}(t,p)\right)\psi_1^{\pm}(t,p) + X_1^{\pm}\left(\psi_0^{\pm}(t,p)\right).$$

Moreover $\psi_0^{\pm}(0,p) = 0$. Hence the solution of the above differential equation is given by (3).

To see equality (4) we first develop the equation $\varphi_n^{\pm}(t^{\pm}(0,\mathbf{y};\varepsilon),0,\mathbf{y},0;\varepsilon) = 0$ in power series of ε, being $t^{\pm}(0,\mathbf{y};\varepsilon) = \sigma_0^{\pm}(\mathbf{y}) + \varepsilon\sigma_1^{\pm}(\mathbf{y}) + \mathcal{O}(\varepsilon)$. After that $\sigma_1^{\pm}$ can be isolated as (4). This concludes the proof of the proposition. □

Now let $\mathcal{V} = \{\nu \in \mathbb{R}^{n-2} : (0,\nu,0) \in U\}$. We define a Melnikov–like function $\mathcal{M} : \mathcal{V} \to \mathbb{R}^{n-2}$ as

$$\mathcal{M}(\nu) = \Lambda(\nu) - \frac{\lambda_1(\nu)}{\omega_1(\nu)}\Omega(\nu), \tag{5}$$

where

(6)

$$\begin{aligned}
\lambda_i(\nu) = \; & X_{0,i}^{+}(\psi_0^{+}(\sigma_0^{+}(\nu),0,\nu,0))\sigma_1^{+}(\nu) - X_{0,i}^{-}(\psi_0^{-}(\sigma_0^{-}(\nu),0,\nu,0))\sigma_1^{-}(\nu) \\
& + \psi_{1,i}^{+}(\sigma_0^{+}(\nu),0,\nu,0) - \psi_{1,i}^{-}(\sigma_0^{-}(\nu),0,\nu,0), \\
\omega_i(\nu) = \; & \frac{\partial \psi_{0,i}^{+}}{\partial x}(\sigma_0^{+}(\nu),0,\nu,0) - \frac{X_{0,i}^{+}\left(\psi_0^{+}(\sigma_0^{+}(\nu),0,\nu,0)\right)}{X_{0,n}^{+}(\psi_0^{+}(\sigma_0^{+}(\nu),0,\nu,0))}\frac{\partial \psi_{0,n}^{+}}{\partial x}(\sigma_0^{+}(\nu),0,\nu,0) \\
& - \frac{\partial \psi_{0,i}^{-}}{\partial x}(\sigma_0^{-}(\nu),0,\nu,0) + \frac{X_{0,i}^{-}\left(\psi_0^{-}(\sigma_0^{+}(\nu),0,\nu,0)\right)}{X_{0,n}^{-}(\psi_0^{-}(\sigma_0^{-}(\nu),0,\nu,0))}\frac{\partial \psi_{0,n}^{-}}{\partial x}(\sigma_0^{-}(\nu),0,\nu,0),
\end{aligned}$$

for $i = 1,2,\ldots,n-1$, and

$$\Lambda(\nu) = \left(\lambda_2(\nu),\lambda_3(\nu),\ldots,\lambda_{n-1}(\nu)\right),$$

$$\Omega(\nu) = \left(\omega_2(\nu),\omega_3(\nu),\ldots,\omega_{n-1}(\nu)\right).$$

Let $J_{\mathcal{M}}(\nu)$ denote the determinant of the Jacobian matrix of $\mathcal{M}$ evaluated at ν. The next theorem is the main result of this section.

Theorem A. *In addition to hypothesis* (H) *we assume that* $\omega_1(\nu) \neq 0$ *for every* $\nu \in \mathcal{V}$. *Then for each* $\nu^* \in \mathcal{V}$ *such that* $\mathcal{M}(\nu^*) = 0$ *and* $J_{\mathcal{M}}(\nu^*) \neq 0$, *there exists a crossing periodic solution* $\phi(t,\varepsilon)$ *of system* (2) *such that* $\phi(0,\varepsilon) \to (0,\nu^*,0) \in \mathcal{A}$ *when* $\varepsilon \to 0$. *Moreover if* $M(\nu) \neq 0$ *for every* $\nu \in \mathcal{V}$ *then there are no crossing periodic solutions bifurcating from* $\mathcal{A}$ *for* $|\varepsilon| \neq 0$ *sufficiently small.*

To prove Theorem A we shall use a version (see [21]) of the so called Lyapunov–Schmidt reduction for finite dimensional function (see [5]). In what follows, for positive integers $k < d$, the functions $\xi : \mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^k$ and $\xi^{\perp} : \mathbb{R}^k \times \mathbb{R}^{d-k} \to$

$\mathbb{R}^{d-k}$ will denote the projections onto the first k coordinates and onto the last $d-k$ coordinates, respectively. A point $\zeta \in \mathbb{R}^d$ is denoted by $\zeta = (a, b) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$.

Lemma 2 ([21]). *Assume that $k \le d$ are positive integers. Let $\mathcal{D}$ and V be open bounded subsets of $\mathbb{R}^d$ and $\mathbb{R}^k$, respectively. Let g_0, g_1 and $\beta : \overline{V} \to \mathbb{R}^{d-k}$ be $\mathcal{C}^2$ functions, consider $g : \mathcal{D} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^d$ as*

$$g(\zeta, \varepsilon) = g_0(\zeta) + \varepsilon g_1(\zeta) + \mathcal{O}(\varepsilon^2),$$

and take $\mathcal{Z} = \{\zeta_\nu = (\nu, \beta(\nu)) : \nu \in \overline{V}\} \subset \mathcal{D}$. We denote by Γ_ν the upper right corner $k \times (d-k)$ matrix of $D\, g_0(\zeta_\nu)$, and by Δ_ν the lower right corner $(d-k) \times (d-k)$ matrix of $D\, g_0(\zeta_\nu)$. Assume that for each $\zeta_\nu \in \mathcal{Z}$, $\det(\Delta_\nu) \neq 0$ and $g_0(\zeta_\nu) = 0$. We define the bifurcation function $f_1 : \overline{V} \to \mathbb{R}^k$ as

$$f_1(\nu) = \; -\Gamma_\nu \Delta_\nu^{-1} \xi^\perp g_1(\zeta_\nu) + \xi g_1(\zeta_\nu). \tag{7}$$

If there exists $\nu^* \in V$ such that $f_1(\nu^*) = 0$ and $J_{f_1}(\nu^*) \neq 0$, then there exists ν_ε such that $g(\zeta_{\nu_\varepsilon}, \varepsilon) = 0$ and $\zeta_{\nu_\varepsilon} \to \zeta_{\nu^*}$ when $\varepsilon \to 0$.

For a proof of Lemma 2 see [21].

Proof of Theorem A. For $(x, \mathbf{y}) \in \mathbb{R}^2$ such that $(x, \mathbf{y}, 0) \in U$ we define the map $\delta(x, \mathbf{y}; \varepsilon) = \big(\delta_1(x, \mathbf{y}; \varepsilon), \delta_2(x, \mathbf{y}; \varepsilon), \ldots, \delta_n(x, \mathbf{y}; \varepsilon)\big)$ as

$$\delta_i(x, \mathbf{y}; \varepsilon) = \varphi_i^+(t^+(x, \mathbf{y}; \varepsilon), x, \mathbf{y}, 0; \varepsilon) - \varphi_i^-(t^-(x, \mathbf{y}; \varepsilon), x, \mathbf{y}, 0; \varepsilon), \quad i = 1, 2, \ldots, n.$$

So $\delta_i(x, \mathbf{y}; \varepsilon) = \delta_i^0(x, \mathbf{y}) + \varepsilon \delta_i^1(x, \mathbf{y}) + \mathcal{O}(\varepsilon^2)$, where

$$\begin{aligned} \delta_i^0(x, \mathbf{y}) = \;& \psi_{0,i}^+(\tau_0^+(x, \mathbf{y}), x, \mathbf{y}, 0) - \psi_{0,i}^-(\tau_0^-(x, \mathbf{y}), x, \mathbf{y}, 0), \\ \delta_i^1(x, \mathbf{y}) = \;& X_{0,i}^+(\psi_0^+(\tau_0^+(x, \mathbf{y}), x, \mathbf{y}, 0))\tau_1^+(x, \mathbf{y}) - X_{0,i}^-(\psi_0^-(\tau_0^-(x, \mathbf{y}), x, \mathbf{y}, 0))\tau_1^-(x, \mathbf{y}) \\ & + \psi_{1,i}^+(\tau_0^+(x, \mathbf{y}), x, \mathbf{y}, 0) - \psi_{1,i}^-(\tau_0^-(x, \mathbf{y}), x, \mathbf{y}, 0). \end{aligned} \tag{8}$$

Here $\mathcal{O}$ is one of the *Landau's symbols*, that is $g(\varepsilon) = \mathcal{O}(\varepsilon^\ell)$ for some positive integer ℓ if there exists constants $\varepsilon_1 > 0$ and $M > 0$ such that $|g(\varepsilon)| \le M|\varepsilon^\ell|$ for $-\varepsilon_1 < \varepsilon < \varepsilon_1$.

Clearly $\delta(x_\varepsilon, \mathbf{y}_\varepsilon, \varepsilon) = 0$ for some $(x_\varepsilon, \mathbf{y}_\varepsilon, 0) \in U$ if and only if the solution of system (2) passing through $(x_\varepsilon, \mathbf{y}_\varepsilon, 0)$ is periodic.

In the sequel, for the purpose of proving Theorem A, we identify the elements of Lemma 2. Take $d = n-1$, $k = n-2$, $\mathcal{D} = \{(\mathbf{a}, b) \in \mathbb{R}^{n-2} \times \mathbb{R} : (b, \mathbf{a}, 0) \in U\}$, $V = \{\nu \in \mathbb{R}^{n-2} : r_0 < |\nu - \nu^*| < r_1\}$, $\beta = 0$, $\zeta_\nu = (\nu, 0)$, $g_0(\mathbf{a}, b) = \big(\underline{\delta}^0(b, \mathbf{a}), \delta_1^0(b, \mathbf{a})\big)$, $g_1(\mathbf{a}, b) = \big(\underline{\delta}^1(b, \mathbf{a}), \delta_1^1(b, \mathbf{a})\big)$, with $\underline{\delta}^j = (\delta_2^j, \delta_3^j, \ldots, \delta_n^j)$ for $j = 0, 1$, and $\mathcal{Z} = \{\zeta_\nu = (\nu, \beta(\nu)) = (\nu, 0) : \nu \in \overline{V}\} \subset \mathcal{D}$.

From (8), $g_1(\zeta_\nu) = \big(\Lambda(\nu), \lambda_1(\nu)\big)$. Now for each $\nu \in \overline{V}$ the solution of system (2), for $\varepsilon = 0$, passing through $(0, \nu, 0) \in \mathcal{A}$ is periodic. Therefore $\big(\delta_1^0(0,\nu), \delta_2^0(0,\nu)\big) = \delta(0,\nu,0) = (0,0)$ and $g_0(\zeta_\nu) = g_0(\nu, 0) = \big(\delta_2^0(0,\nu), \delta_1^0(0,\nu)\big) = (0,0)$. Moreover

$$Dg_0(\nu,0) = \begin{pmatrix} \dfrac{\partial \underline{\delta}^0}{\partial y}(0,\nu) & \dfrac{\partial \underline{\delta}^0}{\partial x}(0,\nu) \\ \dfrac{\partial \delta_1^0}{\partial y}(0,\nu) & \dfrac{\partial \delta_1^0}{\partial x}(0,\nu) \end{pmatrix}.$$

So $\Delta_\nu = \dfrac{\partial \delta_1^0}{\partial x}(0,\nu) = \omega_1(\nu)$ and $\Gamma_\nu = \dfrac{\partial \underline{\delta}^0}{\partial x}(0,\nu) = \Omega(\nu)$. Indeed

$$\begin{aligned} \frac{\partial \delta_i^0}{\partial x}(0,\nu) = \;& X_{0,i}^+\big(\psi_0^+(\sigma_0^+(\nu),0,\nu,0)\big)\frac{\tau_0^+}{\partial x}(0,\nu) - X_{0,i}^-\big(\psi_0^-(\sigma_0^+(\nu),0,\nu,0)\big)\frac{\tau_0^-}{\partial x}(0,\nu) \\ & + \frac{\partial \psi_{0,i}^+}{\partial x}(\sigma_0^+(\nu),0,\nu,0) - \frac{\partial \psi_{0,i}^-}{\partial x}(\sigma_0^-(\nu),0,\nu,0), \end{aligned}$$

and computing implicitly the derivative in the variable ε of the equality $\varphi_n^\pm(t^\pm(x,y;\varepsilon),x,y,0;\varepsilon) = 0$ we get

$$\frac{\partial \tau_0^\pm}{\partial x}(0,\nu) = -\frac{\dfrac{\partial \psi_{0,n}^\pm}{\partial x}(\tau_0^\pm(0,\nu),0,\nu,0)}{X_{0,n}^\pm(\psi_0^\pm(\tau_0^\pm(0,\nu),0,\nu,0))}.$$

Therefore

$$\begin{aligned} \frac{\partial \delta_i^0}{\partial x}(0,\nu) = \;& \frac{\partial \psi_{0,i}^+}{\partial x}(\sigma_0^+(\nu),0,\nu,0) - \frac{X_{0,i}^+\big(\psi_0^+(\sigma_0^+(\nu),0,\nu,0)\big)}{X_{0,n}^+(\psi_0^+(\sigma_0^+(\nu),0,\nu,0))}\frac{\partial \psi_{0,n}^+}{\partial x}(\sigma_0^+(\nu),0,\nu,0) - \\ & \frac{\partial \psi_{0,i}^-}{\partial x}(\sigma_0^-(\nu),0,\nu,0) + \frac{X_{0,i}^-\big(\psi_0^-(\sigma_0^+(\nu),0,\nu,0,0)\big)}{X_{0,n}^-(\psi_0^-(\sigma_0^-(\nu),0,\nu,0))}\frac{\partial \psi_{0,n}^-}{\partial x}(\sigma_0^-(\nu),0,\nu,0) \\ = \;& \omega_i(\nu), \end{aligned}$$

for $i = 1, 2, \ldots, n$. Now we compute the function (7) as

$$f_1(\nu) = -\Gamma_\nu \Delta_\nu^{-1} \xi^\perp g_1(\zeta_\nu) + \xi g_1(\zeta_\nu) = -\frac{\lambda_1(\nu)}{\omega_1(\nu)}\Omega(\nu) + \Lambda(\nu) = \mathcal{M}(\nu).$$

Since $\omega_1(\nu) \neq 0$, the proof of this theorem follows by applying Lemma 2. □

3. Normal forms of piecewise linear systems

Consider the following piecewise linear differential system

$$Z(x,y,z) = \begin{cases} X^+(x,y,z), & \text{if } z > 0 \\ X^-(x,y,z), & \text{if } z < 0, \end{cases} \tag{9}$$

$(x, y, z) \in \mathbb{R}^3$. The discontinuity set is given by $\Sigma = \{z = 0\}$. Here $X^\pm = (X_1^\pm, X_2^\pm, X_3^\pm)$, where

$$X_j^\pm(x, y, z) = a_j^\pm x + b_j^\pm y + c_j^\pm z + d_j^\pm,$$

for $j \in \{1, 2, 3\}$. Note that $\Sigma = f^{-1}(0)$, where $f(x, y, z) = z$. Moreover, we denote $\Sigma^+ = \{p \in \mathbb{R}^3; f(p) > 0\}$ and $\Sigma^- = \{p \in \mathbb{R}^3; f(p) < 0\}$.

Firstly we assume the existence of a plane Ω transversal to Σ which is invariant through the flow of system (9). Without loss of generality, after a linear change of variable, we may assume that $\Omega = \{x = 0\}$, equivalently $b_1^\pm = c_1^\pm = d_1^\pm = 0$.

In what follows we assume, without loss of generality, that system (9) has a center at the origin turning the orbits around it in counterclockwise sense, that is $b_3^+ > 0$ and $b_3^- > 0$. Therefore $(0, y, 0) \in \Sigma^c$, for $y \neq 0$ small enough, and $d_3^\pm = 0$. Now $(0, y, 0) \in \Sigma^c$ if and only if

$$(X^+ f)(0, y, 0)(X^- f)(0, y, 0) = b_3^+ b_3^- y^2 > 0.$$

In order to simplify the computations it is convenient to make a continuous piecewise linear change of variables which transforms system (9) into a simple form. This change of variables is a homeomorphism h which keeps invariant the discontinuity set Σ, the plane $\Omega = \{x = 0\}$ and the sets Σ^+ and Σ^-. Furthermore, h restricted to $\Sigma^+ \cup \Sigma^-$ will be a topological equivalence between the vector fields. More precisely, following closely the ideas of [12], we have the next result.

Proposition 3. *Consider system* (9) *with* $b_1^\pm = c_1^\pm = d_1^\pm = d_3^\pm = 0$, $b_3^+ > 0$ *and* $b_3^- > 0$. *Then the homeomorphism* $(u, v, w) = h(x, y, z)$ *given by*

$$(u, v, w)^T = M^\pm (x, y, z)^T \quad \text{if} \quad (x, y, z) \in \Sigma^\pm \cup \Sigma,$$

where

$$M^\pm = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{c_3^\pm}{b_3^\pm} \\ 0 & 0 & \frac{1}{b_3^\pm} \end{pmatrix},$$

transforms system (9) *into the normal form*

$$\dot{u}, \dot{v}, \dot{w})^T = Y^\pm(u, v, w) = A^\pm (u, v, w)^T + (0, d_2^\pm, 0)^T, \tag{10}$$

where

$$A^\pm = \begin{pmatrix} a_1^\pm & 0 & 0 \\ \frac{a_2^\pm b_3^\pm + a_3^\pm c_3^\pm}{b_3^\pm} & b_3^\pm + c_3^\pm & -(b_2^\pm c_3^\pm - c_2^\pm b_3^\pm) \\ \frac{a_3^\pm}{b_3^\pm} & 1 & 0 \end{pmatrix}.$$

The discontinuity set Σ *and the plane* $\{x = 0\}$ *are invariant by* h. *Moreover the following statements hold.*

(*i*) *h transforms the crossing and sliding sets, tangency points and boundary equilibria from the original system* (9) *into sets and points of same type for system* (10)*;*

(*ii*) *h is a topological equivalence between systems* (9) *and* (10) *for all their orbits which do not have points in common with* $\Sigma^e \cup \Sigma^s$*;*

(*iii*) *h preserves the sets* Σ^e *and* Σ^s*, that is* $h(\Sigma^e)$ *and* $h(\Sigma^s)$ *are the escaping and sliding regions of system* (10)*, respectively.*

Proof. With straightforward computations system (9) becomes system (10). Obviously the set Σ and the plane $\{x = 0\}$ are invariant by h. The statements (*i*), (*ii*) and (*iii*) follow immediately from the equalities

$$Y^{\pm}f(h(x,y,0)) = Y^{\pm}f(u,v,0) = \frac{1}{b_3^{\pm}}X^{\pm}f(u,v,0),$$

because $Y^+f \cdot Y^-f|_\Sigma$ differs from $X^+f \cdot X^-f|_\Sigma$ by a positive constant. Clearly, orbits totally contained in $\Sigma^{\pm}$ are transformed in a homeomorphic way. Furthermore, the topological equivalence is not lost at the crossing set because orbits arriving at the crossing set are continued by the natural concatenation. □

A point $p \in \Sigma$ where the flow of $X^+|_\Omega$ (resp. $X^-|_\Omega$) has a parabolic contact with Σ of the form $X^{\pm}f(p) = 0$ and $(X^{\pm})^2f(p) \neq 0$, is called *fold* point of X^+ (resp. $X^-|_\Omega$). Moreover, when the orbit of $X^+|_\Omega$ (resp. $X^-|_\Omega$) through p is locally contained in Σ^- (resp. Σ^+), that is $(X^+)^2f(p) < 0$ (resp. $(X^+)^2f(p) > 0$), the fold point p is called *invisible*, otherwise, it is called *visible*.

A point $p \in \Omega \cap \Sigma$ is a *center* of system (9) if there exists a neighborhood V of p such that $V \cap \Omega \setminus \{p\}$ is fulfilled by crossing periodic orbit of system (9). This means that

$$(X^+f)(q)(X^-f)(q) > 0$$

for all $q \in (V \cap \{x = 0\}) \cap \Sigma \setminus \{p\}$, and $(X^+f)(p) = (X^-f)(p) = 0$. For system (9) there are three different types of centers in a point $p \in \Sigma$: *focus-focus* (FF), *focus-parabolic* (FP) and *parabolic-parabolic* (PP). Now we give a briefly description of each one below.

In the focus-focus type both systems $X^{\pm}|_\Omega$ have a singular point at p which has eigenvalues with non-zero imaginary part. In the focus-parabolic (resp. parabolic-focus) type the system $X^+|_\Omega$ (resp. $X^-|_\Omega$) has a singular point having eigenvalues with non-zero imaginary part, while the system $X^-|_\Omega$ (resp. $X^+|_\Omega$) has an invisible fold. Finally, in the parabolic-parabolic type both systems $X^{\pm}|_\Omega$ have an invisible fold at p.

In order to state thhe main result of this section, which classify the centers of system (9), we need some preliminary notations: $\eta^{\pm} = b_2^{\pm} + c_3^{\pm}$, $\chi^{\pm} = \sqrt{-[(b_2^{\pm} + c_3^{\pm})^2 - 4(b_2^{\pm}c_3^{\pm} - c_2^{\pm}b_3^{\pm})]}$, $\gamma^{\pm} = \eta^{\pm}/\chi^{\pm}$ provided $\chi^{\pm} \neq 0$, and $S^{\pm} =$

$(b_1^\pm)^2+(c_1^\pm)^2+(d_1^\pm)^2+(d_3^\pm)^2$. We note that $S^\pm=0$ if and only if $b_1^\pm=c_1^\pm=d_1^\pm=d_3^\pm=0$.

Theorem B. *System* (9) *has a center in the plane* $\Omega=\{x=0\}$ *of type:*

(*i*) *focus-focus if and only if* $S^\pm=0$, $b_3^+b_3^->0$, $d_2^\pm=0$, $\chi^\pm>0$ *and* $\gamma^++\gamma^-=0$*;*
(*ii*) *focus-parabolic* (*resp. parabolic-focus*) *if and only if* $S^\pm=0$, $b_3^+b_3^->0$, $d_2^+=0$, $d_2^->0$ (*resp.* $d_2^-=0$, $d_2^+<0$), $\chi^+>0$ (*resp.* $\chi^->0$) *and* $\eta^\pm=0$*;*
(*iii*) *parabolic-parabolic if and only if* $S^\pm=0$, $b_3^+b_3^->0$, $d_2^+<0$, $d_2^->0$, *either* $\eta^\pm=0$ *or* $\eta^+\neq0$, $\eta^-d_2^+=\eta^+d_2^-$ *and* $(b_2^-c_3^--c_2^-b_3^-)(d_2^+)^2=(b_2^+c_3^+-c_2^+b_3^+)(d_2^-)^2$.

Remark 4. *A point* $p\in\Sigma$ *is a T-singularity of vector field associated to system* (9)*, if* p *is an invisible fold of both vector fields* $X^\pm$ *and the fold-curves* $X^\pm f=0$ *are transversal at* p*. Therefore in addition to the hypotheses of statement* (*iii*)*, the parabolic–parabolic center is a T-singularity if and only if* $a_3^-b_3^+-a_3^+b_3^-\neq0$.

The proof of Theorem B follows the same ideas of [14] for proving their Theorem 3.1. Indeed our proof could be obtained by applying their theorem to the restriction of system (9) to Ω. So a major part of it could be omitted here. However, since the conditions on the original parameters that characterize the types of centers are not given explicitly in [14], and also for sake of completeness, we prefer to write the whole proof here.

In order to prove Theorem B we shall apply the normal form (10). To do that the next lemma is needed.

Lemma 5 (See [11])**.** *Consider the equation*

$$\frac{dy}{dx}=ax+by+G(x,y),\qquad a\neq0, \tag{11}$$

where G *is of class* C^4 *at the origin satisfying*

$$G(x,y)=cx^2+dxy+ey^2+fx^3+gx^2y+hx^4+\mathcal{O}(x^4+y^2),$$

for (x,y) *near the origin. Let* $y=Y(x)$ *be the solution of* (11) *satisfying*

$$\begin{aligned} &Y(-\rho)=Y(\sigma)=0 &&\text{for}\quad -\rho<0<\sigma,\\ &aY(x)<0 &&\text{for}\quad -\rho<x<\sigma. \end{aligned}$$

Then for $\rho>0$ *small*

$$\sigma=\rho+\mu\rho^2+\mu^2\rho^3+k\rho^4+\mathcal{O}(\rho^5),$$

where

$$\begin{aligned} \mu &= \frac{2}{3}\left(b - \frac{c}{a}\right), \\ k &= \frac{10}{11}\mu^3 + \frac{\mu}{5}d + \frac{2}{15}L, \\ L &= \frac{bc^2}{a^2} - \frac{2c^3}{a^3} - 2ae - \frac{2bf}{a} + \frac{5cf}{a^2} + g - \frac{3h}{a}. \end{aligned}$$

Proof of Theorem B. If system (9) has a center in the plane $\{x = 0\}$ it satisfies the conditions $b_1^\pm = c_1^\pm = d_1^\pm = d_3^\pm = 0$ and $b_3^+ b_3^- > 0$. Conveniently we assume the counterclockwise direction for the flow, i.e. $b_3^- > 0$ and $b_3^+ > 0$. Hence, by Proposition 3 we write system (9) in the form (10). Denoting the dependent variables of system (10) by x, y, z, system (10), restricted to the plane $\{x = 0\}$, reads

$$\begin{aligned} \dot{y} &= (b_3^\pm + c_3^\pm)y - (b_2^\pm c_3^\pm - c_2^\pm b_3^\pm)z + d_2^\pm, \\ \dot{z} &= y, \end{aligned} \tag{12}$$

for $z \gtrless 0$, where $(0, y, z) \in \Sigma^\pm \cup \Sigma$.

Now we start the proof of (i). In this case the upper and lower systems have a singular point at the origin, i.e. $d_2^\pm = 0$, with eigenvalues given by $\eta^\pm + i\chi^\pm$ and $\eta^\pm - i\chi^\pm$, where $\chi^\pm > 0$. Let $\varphi^\pm(t, (0, r, 0)) = (\varphi_1^\pm(t, (0, r, 0)), \varphi_2^\pm(t, (0, r, 0)))$ be the flow of system (12) where

$$\begin{aligned} \varphi_1^\pm(t, (0, r, 0)) &= re^{\frac{\eta^\pm}{2}t}\left[\cos\left(\frac{\chi^\pm}{2}t\right) + \gamma^\pm \sin\left(\frac{\chi^\pm}{2}t\right)\right], \\ \varphi_2^\pm(t, (0, r, 0)) &= \frac{2r}{\chi^\pm}e^{\frac{\eta^\pm}{2}t}\sin\left(\frac{\chi^\pm}{2}t\right). \end{aligned}$$

Therefore the half Poincaré maps of upper and lower systems are given by

$$\pi^\pm(r) = \varphi_1^\pm\left(\frac{\pi}{\chi^\pm}, (0, r, 0)\right) = -re^{\gamma^\pm \pi},$$

where $\dfrac{\pi}{\chi^\pm}$ is such that $\varphi_2^\pm\left(\dfrac{\pi}{\chi^\pm}, (0, r, 0)\right) = 0$. Hence the Poincaré map is

$$\pi(r) = \pi^- \circ \pi^+(r) = re^{(\gamma^- + \gamma^+)\pi}.$$

Thus we have a center if $\gamma^- + \gamma^+ = 0$, and statement (i) is proved.

Under the assumptions of statement (ii) the upper system (12) has a singular point at the origin, i.e. $d_2^+ = 0$ and the eigenvalues of Jacobian matrix are $\eta^+ \pm i\chi^+$ with $\chi^+ > 0$. Analogous to the previous case the half Poincaré map of the upper system is $\pi^+(r) = -re^{\gamma^+\pi}$. On the other hand, the lower system (12) has an invisible fold at the origin, i.e. $d_2^- > 0$. Consider the differential equation

$$\frac{dz}{dy} = \frac{y}{(b_3^- + c_3^-)y - (b_2^- c_3^- - c_2^- b_3^-)z + d_2^-}, \tag{13}$$

associated to the lower system (12). Since $d_3^- > 0$ we can expand the right hand side of (13) around $y = 0$ and obtain the following expression

$$\begin{aligned} \frac{dz}{dy} &= \frac{1}{d_2^-}y - \frac{(b_3^- + c_3^-)}{(d_2^-)^2}y^2 + \frac{(b_2^- c_3^- - c_2^- b_3^-)}{(d_2^-)^2}yz + \frac{(b_3^- + c_3^-)}{(d_2^-)^2}y^3 \\ &\quad - \frac{2(b_3^- + c_3^-)(b_2^- c_3^- - c_2^- b_3^-)}{(d_2^-)^3}y^2 z - \frac{(b_3^- + c_3^-)^3}{(d_2^-)^4}y^4 + \mathcal{O}(y^4 + z^2). \end{aligned} \tag{14}$$

Let $z = z(y)$ be the solution of equation (14) with initial condition $z(-r) = 0$. Then $z(\pi^-(-r)) = 0$ and by Lemma 5

$$\pi^-(-r) = r + \mu^- r^2 + (\mu^-)^2 r^3 + k^- r^4 + \mathcal{O}(r^5),$$

where

$$\begin{aligned} \mu^- &= \frac{2}{3}\frac{b_2^- + c_3^-}{d_2^-}, \\ L^- &= \frac{-2(b_2^- c_3^- - c_2^- b_3^-)(b_2^- + c_3^-)}{(d_2^-)^3}, \\ k^- &= \frac{2}{3}\frac{(b_2^- + c_3^-)}{d_2^-}\left[\frac{40}{99}\left(\frac{b_2^- + c_3^-}{d_2^-}\right)^2 - \frac{1}{5}\frac{(b_2^- c_3^- - c_2^- b_3^-)}{(d_2^-)^2}\right] \\ &= \mu^-\left[\frac{40}{99}\left(\frac{b_2^- + c_3^-}{d_2^-}\right)^2 - \frac{1}{5}\frac{(b_2^- c_3^- - c_2^- b_3^-)}{(d_2^-)^2}\right]. \end{aligned} \tag{15}$$

Thus the Poincaré map is

$$\pi(r) = \pi^- \circ \pi^+(r) = \pi^-(-re^{\gamma^+\pi}) = e^{\gamma^+\pi}r + \mu^- e^{2\gamma^+\pi}r^2 + \mathcal{O}(r^3).$$

Hence a necessary condition for having a center at the origin is $e^{\gamma^+\pi} = 1$ and $\mu^- e^{2\gamma^+\pi} = 0$, i.e. $\gamma^+ = \mu^- = 0$, and so $\alpha^\pm = 0$. In this case we get $\pi^+(r) = r$, and the lower system (12) becomes

$$\begin{aligned} \dot{y} &= b_3^-(b_2^- + c_2^-)z + d_2^-, \\ \dot{z} &= y, \end{aligned}$$

which is invariant under the change of variables $(y, z, t) \mapsto (-y, z, -t)$. Therefore $\pi^-(-r) = r$ and so $\pi(r) = r$, i.e. the origin is a center.

Now the case parabolic–focus can be reduced to the case focus–parabolic trough the change of variables $(x, y, t) \mapsto (-x, -y, t)$. Hence statement (ii) is proved.

Finally we prove statement (iii). Then the lower and the upper systems (12) have an invisible fold at the origin, i.e. $d_2^+ < 0$ and $d_2^- > 0$. As in the previous statement applying Lemma 5, we obtain

$$(\pi^+)^{-1}(-r) = r + \mu^+ r^2 + (\mu^+)^2 r^3 + k^+ r^4 + \mathcal{O}(r^5),$$

where μ^+, L^+ and k^+ are the same as in (15) just changing the sign "$-$" by "$+$". Hence for $r > 0$ we obtain

$$\pi^-(-r) - (\pi^+)^{-1}(-r) = (\mu^- - \mu^+)r^2 + [(\mu^-)^2 - (\mu^+)^2]r^3 + (k^- - k^+)r^4 + \mathcal{O}(r^5).$$

Then, a necessary condition for having a center at the origin are $\mu^- - \mu^+ = 0$ and $k^- - k^+ = 0$. Observe that the first condition implies that $d_2^+\eta^- = d_2^-\eta^+$. Now if $\eta^+ = 0$ we have $\eta^- = 0$ and $k^+ = k^- = 0$. For $\eta^+ \neq 0$ the condition $k^+ - k^- = 0$ implies that $(d_2^+)^2(b_2^-c_3^- - c_2^-b_3^-) = (d_2^-)^2(b_2^+c_3^+ - c_2^+b_3^+)$. In the first case, i.e. $\eta^+ = 0$, system (12) becomes

$$\begin{aligned} \dot{y} &= -(b_2^\pm c_3^\pm - c_2^\pm b_3^\pm)z + d_2^\pm, \\ \dot{z} &= y, \end{aligned}$$

which is invariant under the change of variables $(y, z, t) \mapsto (-y, z, -t)$. Therefore $(\pi^+)^{-1}(-r) = \pi^-(-r) = r$ and so $\pi^- - (\pi^+)^{-1} \equiv 0$, i.e. the origin is a center.

In the second case the upper and lower systems (12) become the systems

$$\begin{aligned} \dot{y} &= (b_3^+ + c_3^+)y - (b_2^+c_3^+ - c_2^+b_3^+)z + d_2^+, \\ \dot{z} &= y, \end{aligned} \tag{16}$$

when $(0, y, z) \in \Sigma^+ \cup \Sigma$, and

$$\begin{aligned} \dot{y} &= \frac{d_2^-}{d_2^+}(b_3^+ + c_3^+)y - \left(\frac{d_2^-}{d_2^+}\right)^2 (b_2^+c_3^+ - c_2^+b_3^+)z + d_2^-, \\ \dot{z} &= y, \end{aligned} \tag{17}$$

when $(0, y, z) \in \Sigma^- \cup \Sigma$, respectively. Note that by the change of variables

$$(y, z, t) \mapsto \left(y, \frac{d_2^-}{d_2^+}z, \frac{d_2^-}{d_2^+}t\right), \tag{18}$$

system (16) becomes system (17). Hence if

$$\varphi^-(t, (0, -r, 0)) = (\varphi_1^-(t, (0, -r, 0)), \varphi_2^-(t, (0, -r, 0))$$

is the solution of system (17) through the point $(0, -r, 0)$, then

$$\pi^-(-r) = \varphi_1^-(t^-(-r), (0, -r, 0)),$$

where $t^-(-r)$ is such that $\varphi_2^-(t^-(-r),(0,-r,0))=0$. Thus, using the inverse of the change of variables (18), it follows that

$$\begin{aligned}\varphi^+(t,(0,-r,0)) &= (\varphi_1^+(t,(0,-r,0)),\varphi_2^+(t,(0,-r,0)))\\ &= \left(\varphi_1^-\left(\frac{d_2^+}{d_2^-}t,(0,-r,0)\right),\frac{d_2^+}{d_2^-}\varphi_2^-\left(\frac{d_2^+}{d_2^-}t,(0,-r,0)\right)\right),\end{aligned}$$

is the solution of system (16) through the point $(0,-r,0)$. Therefore

$$(\pi^+)^{-1}(-r)=\varphi_1^+(-t^+(-r),(0,-r,0)),$$

where $t^+(-r)$ is such that $\varphi_2^+(-t^+(-r),(0,-r,0))=0$. Then $d_2^+t^+(-r)=-d_2^-t^-(-r)$ and $(\pi^+)^{-1}(-r)=\pi^-(-r)$, i.e. the origin is a center. □

4. Perturbation of the normal forms

In this section we use the theory developed in section 2 to study the persistence of crossing periodic solutions of the normal forms given by Theorem B, when it is perturbed inside the class of piecewise linear differential systems with two zones separated by the plane $\Sigma=\{z=0\}$.

4.1. **Parabolic–Parabolic Center.** We consider the unperturbed systems

$$X_0^a(x,y,z)=\big(x,-z-\mathrm{sign}(z),x+y\big)\quad\text{and}\quad X_0^b(x,y,z)=\big(x,z-\mathrm{sign}(z),x+y\big).$$

From statement (iii) of Theorem B it is easy to see that the origin is a parabolic–parabolic center of the unperturbed systems $(\dot{x},\dot{y},\dot{z})^T=X_0^a(x,y,z)$ and $(\dot{x},\dot{y},\dot{z})^T=X_0^b(x,y,z)$. Indeed for both systems $S^\pm=0$, $b_3^+b_3^-=1>0$, $d_2^+=-1<0$, $d_2^-=1>0$, and $\eta^\pm=0$. The singular points of the upper and lower vector fields associated to X_0^a are invisible centers and so the period–annulus $\mathcal{A}$ is all the plane $\{x=0\}$, see Figure 2. Now, the singular points of the upper and lower vector fields associated to X_0^b are visible saddles and so the period–annulus $\mathcal{A}$ is bounded by the separatrices of the saddles, see Figure 3.

The theory developed in the previous sections allows us to study the limit cycles of the following perturbed system

$$(19)\qquad X^a(x,y,z;\varepsilon):\begin{cases}\begin{aligned}&\dot{x}=x+\varepsilon\big(\alpha_0^++\alpha_2^+y+\alpha_3^+z\big)\\&\dot{y}=-z-1+\varepsilon\beta_2^+y\\&\dot{z}=x+y+\varepsilon\big(\kappa_0^++\kappa_3^+z\big)\end{aligned} & \text{if}\quad z>0,\\[2ex]\begin{aligned}&\dot{x}=x+\varepsilon\big(\alpha_0^-+\alpha_2^-y+\alpha_3^-z\big)\\&\dot{y}=-z+1+\varepsilon\big(\beta_0^-+\beta_2^-y+\beta_3^-z\big)\\&\dot{z}=x+y+\varepsilon\big(\kappa_0^-+\kappa_2^-y+\kappa_3^-z\big)\end{aligned} & \text{if}\quad z<0.\end{cases}$$

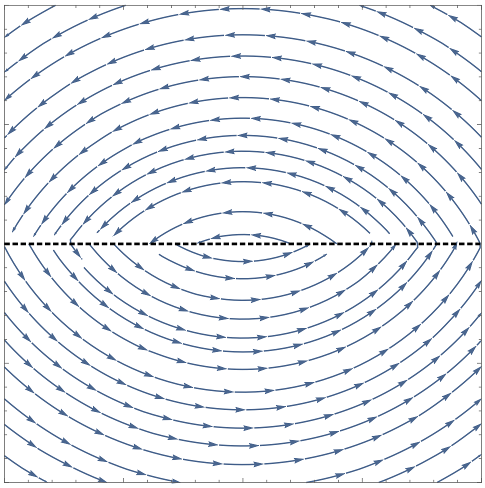

FIGURE 2. Phase Portrait of unperturbed vector field X_0^a.

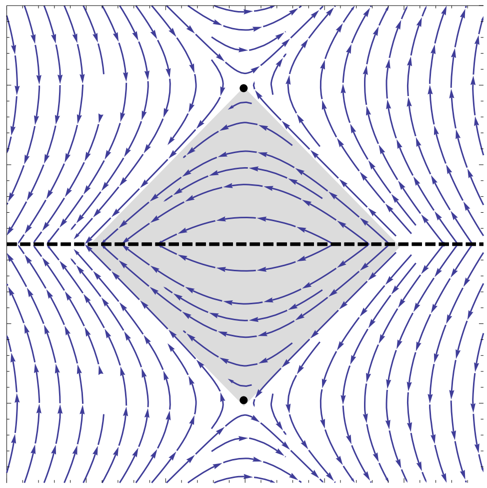

FIGURE 3. Phase Portrait of unperturbed vector field X_0^b.

$$X^b(x,y,z;\varepsilon) : \begin{cases} \dot{x} = x + \varepsilon\big(\alpha_0^+ + \alpha_2^+ y + \alpha_3^+ z\big) & \\ \dot{y} = z - 1 + \varepsilon\beta_2^+ y & \text{if} \quad z > 0, \\ \dot{z} = x + y + \varepsilon\big(\kappa_0^+ + \kappa_3^+ z\big) & \\ \dot{x} = x + \varepsilon\big(\alpha_0^- + \alpha_2^- y + \alpha_3^- z\big) & \\ \dot{y} = z + 1 + \varepsilon\big(\beta_0^- + \beta_2^- y + \beta_3^- z\big) & \text{if} \quad z < 0. \\ \dot{z} = x + y + \varepsilon\big(\kappa_0^- + \kappa_2^- y + \kappa_3^- z\big) & \end{cases} \tag{20}$$

Note that, by Remark 4, the point

$$\left(-\varepsilon\kappa_0^+ + \frac{\kappa_0^- - \kappa_0^+}{\kappa_2^-}, \frac{\kappa_0^+ - \kappa_0^-}{\kappa_2^-}, 0\right)$$

is a T–singularity for both vector fields X_0^a and X_0^b provided that $\varepsilon \neq 0$ and $\kappa_2^- \neq 0$.

In terms of system (2) we are assuming that the perturbations of X_0^a and X_0^b are given by

$$X_1^+(x,y,z) = \big(\alpha_0^+ + \alpha_2^+ y + \alpha_3^+ z, \beta_2^+ y, \kappa_0^+ + \kappa_3^+ z\big),$$
$$X_1^-(x,y,z) = \big(\alpha_0^- + \alpha_2^- y + \alpha_3^- z, \beta_0^- + \beta_2^- y + \beta_2^- z, \kappa_0^- + \kappa_2^- y + \kappa_3^- z\big).$$

One can also take $X_1^{\pm}$ as a general linear function. However after the computations it can be seen that only the parameters considered above actually play a roll.

We have the following result.

Proposition 6. *There exist parameters $\alpha_i^{\pm}$, β_i^-, κ_i^-, $i = 1, 2, 3$, β_2^+, κ_0^+, and κ_3^+ such that system* (19) *admits, for $|\varepsilon| \neq 0$ small enough, at least* 8 *limit cycles converging, when ε goes to* 0*, to some of the periodic orbits contained in $\{x = 0\}$.*

Proposition 7. *There exist parameters $\alpha_i^{\pm}$, β_i^-, κ_i^-, $i = 1, 2, 3$, β_2^+, κ_0^+, and κ_3^+ such that system* (20) *admits, for $|\varepsilon| \neq 0$ small enough, at least* 8 *limit cycles converging, when ε goes to* 0*, to some of the periodic orbits contained in $\{x = 0\}$.*

Proof of Proposition 6. In order to prove this proposition we have to identify the elements of Theorem A. Computing the solutions of $(\dot{x}, \dot{y}, \dot{z})^T = X_0^a(x, y, z)$ we obtain

$$\psi_0^+(t, 0, \nu, 0) = \Big(0, \nu\cos t - \sin t, \nu\sin t + \cos t - 1\Big),$$

$$\psi_0^-(t, 0, \nu, 0) = \Big(0, \nu\cos t + \sin t, \nu\sin t - \cos t + 1\Big),$$

Expanding the solutions of system (19) in power series of ε we get

$$\begin{aligned}
\psi_{1,1}^+(t, 0, \nu, 0) = \;& \big(\alpha_2^+ + \alpha_3^+ + (\alpha_2^+ - \alpha_3^+)\nu\big)\frac{\sin t}{2} - \big(\alpha_2^+ - \alpha_3^+ - (\alpha_2^+ - \alpha_3^+)\nu\big)\frac{\cos t}{2} \\
& + \big(2\alpha_0^+ + (\alpha_2^+ + -\alpha_3^+)(\nu - 1)\big)\frac{e^t}{2} - \alpha_0^+ - \alpha_3^+, \\
\psi_{1,2}^+(t, 0, \nu, 0) = \;& \big(2\alpha_0^+ - 2(\beta_2^+ + \kappa_3^+)t - \alpha_2^+(2 - t(\nu - 1)) + 2(\beta_2^+ - \kappa_3^+)\nu \\
& - \alpha_3^+(2 - t - (t + 2)\nu)\big)\frac{\sin t}{4} - \big(2\alpha_0^+ - 4(\kappa_0^+ - \kappa_3^+) \\
& - 2(\beta_2^+ + \kappa_3^+)t\nu - \alpha_3^+(3 + t - (t - 1)\nu)\big)\frac{\cos t}{4} \\
& - \big(\alpha_0^+ + (\alpha_2^+ + \alpha_3^+)(\nu - 1)\big)\frac{e^t}{4} + \alpha_0^+ - \alpha_3^+ - \kappa_0^+ + \kappa_3^+, \\
\psi_{1,3}^+(t, 0, \nu, 0) = \;& -\big(2(\alpha_0^+ - \alpha_3^+) + 2(\beta_2^+ - 2\kappa_0^+ + \kappa_3^+) - (\alpha_2^+ + \alpha_3^+)t \\
& - (\alpha_2^+ - \alpha_3^+ + 2(\beta_2^+ + \kappa_3^+))t\nu\big)\frac{\sin t}{4} - \big(2\alpha_0^+ - \alpha_2^+ - \alpha_3^+ \\
& + (\alpha_2^+ + \alpha_3^+)(t + 1)\nu - (\alpha_2^+ - \alpha_3^+ + 2(\beta_2^+ + \kappa_3^+))t\big)\frac{\cos t}{4} \\
& + \big(2\alpha_0^+ + (\alpha_2^+ + \alpha_3^+)(\nu - 1)\big)\frac{e^t}{4},
\end{aligned}$$

and

$$
\begin{aligned}
\psi^-_{1,1}(t,0,\nu,0) = \ & \big(\alpha_2^-(\nu-1)-\alpha_3^-(\nu+1)\big)\frac{\sin t}{2} - \big(\alpha_2^- - \alpha_3^- + (\alpha_2^- + \alpha_3^-)\nu\big)\frac{\cos t}{2} \\
& \big(2\alpha_0^- + (\alpha_2^- + \alpha_3^-)(1+\nu)\big)\frac{e^t}{2} - 2(\alpha_0^- + \alpha_3^-), \\
\psi^-_{1,2}(t,0,\nu,0) = \ & \big(2\alpha_0^- + \alpha_2^-(2+t+t\nu) + \alpha_3^-(2-t+(2+t)\nu) \\
& +2(2\beta_0^- + \beta_3^- + (\beta_2^- + \kappa_3^-)t + (\beta_2^- - \kappa_3^-)\nu + \beta_3^- t\nu \\
& -\kappa_2^-(1+t\nu))\big)\frac{\sin t}{4}
\end{aligned}
$$

$$
\begin{aligned}
& -\big(2\alpha_0^- - 4(\kappa_0^- + \kappa_3^-) - \alpha_2^-(1+t(-1+\nu)+\nu) \\
& +2t(\beta_3^- - \kappa_2^- - (\beta_2^- + \kappa_3^-)\nu) + \alpha_3^-(3+t+(-1+t)\nu)\big)\frac{\cos t}{4} \\
& -\big(2\alpha_0^- + (\alpha_2^- + \alpha_3^-)(1+\nu)\big)\frac{e^t}{4} + \alpha_0^- + \alpha_3^- - \kappa_0^- - \kappa_3^-, \\
\psi^-_{1,3}(t,0,\nu,0) = \ & -\big(2\alpha_0^- + 2(\beta_0^- + 2\kappa_0^- + \kappa_3^-) + \alpha_2^- t - 2(-\beta_3^- + \kappa_2^-)t - \\
& (2(\beta_3^- + \kappa_2^-) - (\alpha_2^- + 2(\beta_2^- + \kappa_3^-))t)\nu + \alpha_3^-(2+t+t\nu)\big)\frac{\sin t}{4} \\
& -\big(2\alpha_0^- + 4(\beta_0^- + \beta_3^-) + \alpha_2^-(1+t)(1+\nu) + \alpha_3^-(1+t(-1+\nu) \\
& +\nu) + 2t(\beta_2^- + \kappa_3^+ + (\beta_2^- + \kappa_3^- + (\beta_3^- - \kappa_2^-)\nu)\big)\frac{\cos t}{4} \\
& \big(2\alpha_0^- + \alpha_2^- + \alpha_3^- + (\alpha_2^- + \alpha_3^-)\nu\big)\frac{e^t}{4} + \beta_0^- + \beta_3^-.
\end{aligned}
$$

Analogously we compute $\sigma_0^{\pm}(t,0,\nu,0) = \pm 2\arctan\nu$,

$$
\begin{aligned}
\sigma_1^+(\nu) = \ & \big(2\alpha_0^+ + (\alpha_2^+ + \alpha_3^+)(\nu-1)\big)\frac{e^{2\arctan\nu}}{4\nu} + \big(\alpha_2^+ - \alpha_3^+ + 2(\beta_2^+ + \kappa_3^+) \\
& +(\alpha_2^+ + \alpha_3^+)\nu\big)\frac{\arctan\nu}{2\nu} + \frac{1}{4(\nu^2+1)\nu}\big(\alpha_3^+ + \alpha_2^+(\nu-1)^2(\nu+1) \\
& -2\alpha_0^+(1-(\nu-2)\nu) - (4(\beta_2^+ - 2\kappa_0^+ + \kappa_3^+) - \alpha_3^+(3+(\nu-1)\nu))\nu\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_1^-(\nu) = \;& \big(2\alpha_0^- + (\alpha_2^- + \alpha_3^-)(\nu+1)\big)\frac{e^{-2\arctan\nu}}{4\nu} + \big(\alpha_2^- - \alpha_3^- + 2(\beta_2^- + \kappa_3^-) \\
& -(\alpha_2^- + \alpha_3^- + 2(\beta_2^- - \kappa_2^-))\nu\big)\frac{\arctan\nu}{2\nu} - \big(\alpha_3^- - \alpha_2^-(\nu-1)(\nu+1)^2 \\
& +2\alpha_0^-(1-(\nu+2)\nu) + (4(\beta_2^- + 2\kappa_0^- + \kappa_3^- - (2\beta_0^- + \beta_3^- - \kappa_2^-)\nu) \\
& -\alpha_3^-(3+(\nu+1)\nu))\nu\big)\frac{1}{4(\nu^2+1)\nu}.
\end{aligned}
$$

Using the formulae (6) we conclude

$$
\begin{aligned}
\omega_1(\nu) = \;& 2\sinh(2\arctan\nu), \\
\omega_2(\nu) = \;& \frac{1}{\nu} + \frac{e^{-2\arctan\nu}(\nu-1)}{2\nu} - \frac{e^{2\arctan\nu}(1+\nu)}{2\nu},
\end{aligned}
$$

$$
\begin{aligned}
\lambda_1(\nu) = \;& \big(2\alpha_0^+ + (\alpha_2^+ + \alpha_3^+)(\nu-1)\big)\frac{e^{2\arctan\nu}}{2} \\
& +\big(2\alpha_0^- + (\alpha_2^- + \alpha_3^-)(1+\nu)\big)\frac{e^{-2\arctan\nu}}{2} \\
& +\frac{1}{2}\big(-2\alpha_0^- - \alpha_0^- - \alpha_3^- - 2\alpha_0^+ + \alpha_2^+ + \alpha_3^+ + (\alpha_2^- + \alpha_3^- + \alpha_2^+ + \alpha_3^+)\nu)\big), \\
\lambda_2(\nu) = \;& -\frac{\big(2\alpha_0^+ + (\alpha_2^+ + \alpha_3^+)(-1+\nu)\big)(1+\nu)}{4\nu}e^{2\arctan\nu} \\
& -\frac{(1+\nu)\big(2\alpha_0^- + (\alpha_2^- + \alpha_3^-)(1+\nu)\big)}{4\nu}e^{-2\arctan\nu} \\
& +\frac{1}{2\nu}\big(-\alpha_2^- + \alpha_3^- - \alpha_2^+ + \alpha_3^+ - 2(\beta_2^- + \beta_2^+ + \kappa_3^- + \kappa_3^+) \\
& +2(\alpha_2^- + \alpha_3^- + 2\beta_3^- - 2\kappa_2^-)\nu \\
& +(\alpha_2^- - \alpha_3^- - \alpha_2^+ + \alpha_3^+ + 2(\beta_2^- - \beta_2^+ + \kappa_3^- - \kappa_3^+))\nu^2\big)\arctan\nu \\
& +\frac{P(\nu)}{4(\nu+\nu^3)},
\end{aligned}
$$

where

$$
\begin{aligned}
P(\nu) = \; & 2\alpha_0^- + \alpha_2^- + \alpha_3^- + 2\alpha_0^+ - \alpha_2^+ - \alpha_3^+ + 2\big(- \alpha_0^- + \alpha_2^- - \alpha_3^- + 3\alpha_0^+ \\
& +2(-\alpha_3^+ + \beta_2^- + \beta_2^+ + 2\kappa_0^- + \kappa_3^- - 2\kappa_0^+ + \kappa_3^+)\big)\nu \\
& -2\big(3\alpha_0^- + 2\alpha_2^- + 2\alpha_3^- - \alpha_0^+ + \alpha_2^+ + \alpha_3^+ + 8\beta_0^- + 4\beta_3^- - 4\kappa_2^-\big)\nu^2 \\
& +2\big(3\alpha_0^- - \alpha_2^- + \alpha_3^- + 3\alpha_0^+ - 2(\alpha_3^+ + \beta_2^- - \beta_2^+ + 2\kappa_0^- + \kappa_3^- \\
& +2\kappa_0^+ - \kappa_3^+)\big)\nu^3 - \big(\alpha_2^- + \alpha_3^- + \alpha_2^+ + \alpha_3^+\big)\nu^4.
\end{aligned}
$$

Now expanding the function $\mathcal{M}(\nu)$ (see (5)) in power series of ν we get

$$\mathcal{M}(\nu) = \sum_{i=0}^{8} C_i \nu^i + \mathcal{O}(\nu^9).$$

Here the parameters C_i, $i = 0, \dots 8$, are linear combinations of the parameters $\alpha_i^{\pm}$, β_i^-, κ_i^-, $i = 1, 2, 3$, β_2^+, κ_0^+, and κ_3^+. Moreover they are linearly independent. So we can choose them in order to obtain 8 positive simple zeros of the equation $\mathcal{M}(\nu) = 0$. The proof of this theorem follows by applying Theorem A. $\square$

Proof of Proposition 7. The proof of this proposition is analogously the to proof of Proposition 6, noting that the solutions of $(\dot{x}, \dot{y}, \dot{z})^T = X_0^b(x, y, z)$ are

$$
\begin{aligned}
\psi_0^+(t, 0, \nu, 0) &= \Big(0, \nu \cosh t - \sinh t, \nu \sinh t - \cosh t + 1\Big), \\
\psi_0^-(t, 0, \nu, 0) &= \Big(0, \nu \cosh t + \sinh t, \nu \sinh t + \cosh t - 1\Big),
\end{aligned}
$$

and $\sigma_0^+(t, 0, \nu, 0) = \log\left(\dfrac{\nu + 1}{\nu - 1}\right)$ and $\sigma_0^-(t, 0, \nu, 0) = \log\left(\dfrac{1 - \nu}{1 + \nu}\right)$. Here again, expanding the function $\mathcal{M}(\nu)$ (see (5)) in power series of ν we get

$$\mathcal{M}(\nu) = \sum_{i=0}^{8} C_i \nu^i + \mathcal{O}(\nu^9),$$

where the parameters C_i, $i = 0, \dots 8$, are independent linear combinations of the parameters $\alpha_i^{\pm}$, β_i^-, κ_i^-, $i = 1, 2, 3$, β_2^+, κ_0^+, and κ_3^+. $\square$

4.2. **Focus–Focus Center.** We consider the unperturbed systems $X_0^c(x, y, z) = \big(x, -z + \text{sign}(z)y, x + y\big)$. From statement *(ii)* of Theorem B it is easy to see that the origin is a focus–focus center of the unperturbed system $(\dot{x}, \dot{y}, \dot{z})^T = X_0^c(x, y, z)$ in such way that their periodic orbits fulfil all the plane $\{x = 0\}$ (see Figure 4). Indeed for both systems $S^{\pm} = 0$, $b_3^+ b_3^- = 1 > 0$, $d_2^+ = d_2^- = 0$, $\chi^{\pm} = \sqrt{3}$, $\eta^{\pm} = \pm 1$ and then $\gamma_1 + \gamma_2 = 0$.

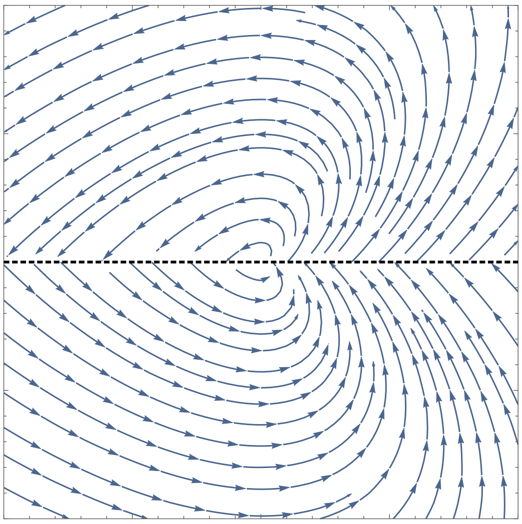

FIGURE 4. Phase Portrait of unperturbed vector fields X_0^c.

We study the limit cycles of the following perturbed system

$$X^c(x,y,z;\varepsilon):\begin{cases}\begin{array}{ll}\dot{x}=x+\varepsilon\big(\alpha_0^+ +\alpha_1^+x+\alpha_2^+y+\alpha_3^+z\big) & \\ \dot{y}=-z+y+\varepsilon\big(\beta_0^+ +\beta_1^+x+\beta_2^+y+\beta_3^+z\big) & \text{if}\quad z>0,\\ \dot{z}=x+y+\varepsilon\big(\kappa_0^+ +\kappa_1^+x+\kappa_2^+y+\kappa_3^+z\big) & \\ \dot{x}=x+\varepsilon\big(\alpha_0^- +\alpha_1^-x+\alpha_2^-y+\alpha_3^-z\big) & \\ \dot{y}=-z-y+\varepsilon\big(\beta_0^- +\beta_1^-x+\beta_2^-y+\beta_3^-z\big) & \text{if}\quad z<0.\\ \dot{z}=x+y+\varepsilon\big(\kappa_0^- +\kappa_1^-x+\kappa_2^-y+\kappa_3^-z\big) & \end{array}\end{cases}\tag{21}$$

So we have the following result.

Proposition 8. *If $B\neq 0$ then, for $|\varepsilon|\neq 0$ small enough, there is a unique crossing periodic solution $\phi(t,\varepsilon)$ of system (21) bifurcating from periodic orbits contained in $\{x=0\}$, such that*

$$\phi(0,\varepsilon)\to\left(0\,,\,-\frac{\sqrt{3}\left(1+e^{\frac{\pi}{\sqrt{3}}}\right)A}{2B}\,,\,0\right),\quad \textit{when}\quad \varepsilon\to 0,$$

where

$$
\begin{aligned}
A &= \alpha_0^- \left(3 + 4e^{-\frac{2\pi}{\sqrt{3}}}\left(e^{\frac{\pi}{\sqrt{3}}} - 1\right) - 2\,\mathrm{sech}\left(\frac{\pi}{\sqrt{3}}\right)\right) \\
&\quad + \alpha_0^+ - 2\alpha_0^+\,\mathrm{sech}\left(\frac{\pi}{\sqrt{3}}\right) - 3(\beta_0^- + \beta_0^+ + 2\kappa_0^-) \quad \textit{and} \\
B &= \frac{2(\alpha_2^- + \alpha_3^- + 3(\alpha_2^+ + \alpha_3^+))}{\sqrt{3}\left(e^{\frac{\pi}{\sqrt{3}}} - 1\right)} + \frac{\left(\tanh\left(\frac{\pi}{\sqrt{3}}\right) - 1\right)(\alpha_2^- + \alpha_3^- + 3(\alpha_2^+ + \alpha_3^+))}{\sqrt{3}} \\
&\quad - \frac{2e^{-\sqrt{3}\pi}(\alpha_2^- + \alpha_3^-)}{\sqrt{3}} + 2\pi\alpha_2^- - \pi\alpha_3^- + 2\sqrt{3}\alpha_2^+ - 2\pi\alpha_2^+ + 2\sqrt{3}\alpha_3^+ - \pi\alpha_3^+ \\
&\quad + 3\pi\beta_3^- - 2\pi\beta_2^+ - \pi\beta_3^+ - 3\pi\kappa_2^- + 3\pi\kappa_3^- + \pi\kappa_2^+ - \pi\kappa_3^+.
\end{aligned}
$$

Moreover if $A \neq 0$ and $B = 0$ then there are no crossing periodic solutions bifurcating from periodic orbits contained in $\{x = 0\}$.

It is worth to say that even assuming $B \neq 0$ and $|\varepsilon| \neq 0$ small enough this proposition does not provide the global uniqueness of crossing periodic solutions of system (21) close to the plane $\{x = 0\}$. Indeed some crossing periodic solutions may bifurcate purely from infinite (see [13]).

Proof of Proposition 8. The proof of the first part of this proposition is analogously to the proof of Proposition 6, noting that the solutions of $(\dot{x}, \dot{y}, \dot{z})^T = X_0^c(x, y, z)$ are

$$
\begin{aligned}
\psi_0^+(t, 0, \nu, 0) &= \left(0, \nu e^{\frac{t}{2}}\left(\frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}t}{2}\right) + \cos\left(\frac{\sqrt{3}t}{2}\right)\right), \frac{2\nu e^{\frac{t}{2}}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)\right), \\
\psi_0^-(t, 0, \nu, 0) &= \left(0, \nu e^{-\frac{t}{2}}\left(\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}t}{2}\right)\right), \frac{2\nu e^{-\frac{t}{2}}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)\right),
\end{aligned}
$$

and $\sigma_0^\pm(t, 0, \nu, 0) = \pm\dfrac{2\pi}{\sqrt{3}}$. Moreover, in this case, the Melnikov-like function (5) is linear, i.e. $\mathcal{M}(\nu) = C_0 + C_1\nu$, where

$$
C_0 = \frac{1 + e^{\frac{\pi}{\sqrt{3}}}}{2}A \quad \text{and} \quad C_1 = \frac{e^{\frac{\pi}{\sqrt{3}}}}{3\sqrt{3}}B.
$$

Clearly if $A \neq 0$ and $B = 0$, then $\mathcal{M}(\nu) = C_0 \neq 0$ for all $\nu \in \mathcal{V}$. It concludes this proof. □

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