Limit cycles bifurcating from a degenerate center

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Abstract

We study the maximum number of limit cycles that can bifurcate from a degenerate center of a cubic homogeneous polynomial differential system. Using the averaging method of second order and perturbing inside the class of all cubic polynomial differential systems we prove that at most three limit cycles can bifurcate from the degenerate center. As far as we know this is the first time that a complete study up to second order in the small parameter of the perturbation is done for studying the limit cycles which bifurcate from the periodic orbits surrounding a degenerate center (a center whose linear part is identically zero) having neither a Hamiltonian first integral nor a rational one. This study needs many computations, which have been verified with the help of the algebraic manipulator Maple.

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1. Introduction

Hilbert in (16) asked for the maximum number of limit cycles which real polynomial differential systems in the plane of a given degree can have. This is actually the well known 16th Hilbert Problem, see for example the surveys (17; 18) and references therein. Recall that a limit cycle of a planar polynomial differential system is a periodic orbit of the system isolated in the set of all periodic orbits of the system.
Poincaré in (22) was the first to introduce the notion of a center for a vector field defined on the real plane. So according to Poincaré a center is a singular point surrounded by a neighborhood filled of periodic orbits with the unique exception of the singular point.

Consider the polynomial differential system
\[
\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),
\]
and as usually we denote by \( \dot{} = \frac{d}{dt} \). Assume that system (1) has a center located at the origin. Then after a linear change of variables and a possible scaling of time system (1) can be written in one of the following forms
\[
\begin{align*}
(A) \quad & \dot{x} = -y + F_1(x, y), \quad \dot{y} = x + F_2(x, y), \\
(B) \quad & \dot{x} = y + F_1(x, y), \quad \dot{y} = F_2(x, y), \\
(C) \quad & \dot{x} = F_1(x, y), \quad \dot{y} = F_2(x, y),
\end{align*}
\]
with \( F_1 \) and \( F_2 \) polynomials without constant and linear terms. When system (1) can be written into the form (A) we say that the center is of linear type. When system (1) can take the form (B) the center is nilpotent, and when system (1) can be transformed into the form (C) the center is degenerate.

Due to the difficulty of this problem mathematicians have consider simpler versions. Thus Arnold (1) considered the weakened 16th Hilbert Problem, which consists in determining an upper bound for the number of limit cycles which can bifurcate from the periodic orbits of a polynomial Hamiltonian center when it is perturbed inside a class of polynomial differential systems, see for instance (9) and the hundred of references quoted therein. It is known that in a neighborhood of a center always there is a first integral, see (21). When this first integral is not polynomial the computations become more difficult. Moreover, if the center is degenerate the computations become even harder.

In the literature we can basically find the following methods for studying the limit cycles that bifurcate from a center:

- The method that uses the Poincaré return map, like the articles (4; 8).
- The one that uses the Abelian integrals or Melnikov integrals (note that for systems in the plane the two notions are equivalent), see for example section 5 of Chapter 6 of (2) and section 6 of Chapter 4 of (15).
The one that uses the inverse integrating factor, see (11; 12; 13; 25).

The averaging theory (6; 14; 19; 23; 24).

The first two methods provide information about the number of limit cycles whereas the last two methods additionally give the shape of the bifurcated limit cycle up to any order in the perturbation parameter.

Almost all the papers studying how many limit cycles can bifurcate from the periodic orbits of a center, work with centers of linear type. There are very few papers studying this problem for nilpotent or degenerate centers. In fact, for degenerate centers as far as we known the bifurcation of limit cycles from the periodic orbits of a degenerate center only have been studying completely using formulas of first order in the small parameter of the perturbation. Here we will provide a complete study of this problem using formulas of second order, and as it occurs with the formulas of second order applied to linear centers that they provide in general more limit cycles than the formulas of first order, the same occurs for the formulas of second order applied to degenerate centers. Of course, the computations from first order to second order increases almost exponentially.

This paper deals with the weakened 16th Hilbert’s problem but perturbing non–Hamiltonian degenerate centers using the technique of the averaging method of second order, see (14), and Section 2 for a summary of the results that we need here.

Since we want to study the perturbation of a degenerate center with averaging of second order, from the homogeneous centers the first ones that are degenerate, are the cubic homogeneous centers, see for instance (7). In this class in (20) the authors studied the perturbation of the following cubic homogeneous center

\[
\dot{x} = -y(3x^2 + y^2), \quad \dot{y} = x(x^2 - y^2),
\]

inside the class of all cubic polynomial differential systems, using averaging theory of first order. Here we study this problem but using averaging theory of second order.

System (2) has a global center at the origin (i.e. all the orbits contained in \(\mathbb{R}^2 \setminus \{(0,0)\}\) are periodic), and it admits the non–rational first integral

\[
H(x, y) = (x^2 + y^2) \exp\left(-\frac{2x^2}{x^2 + y^2}\right).
\]
The limit cycles bifurcating from the periodic orbits of the global center (2) have already been studied in the following two results, see (20) and (5), respectively.

**Theorem 1.** We deal with differential system (2). Then the polynomial differential system

\[
\begin{align*}
\dot{x} &= -y(3x^2 + y^2) + \varepsilon \left( \sum_{0 \leq i+j \leq 3} a_{ij} x^i y^j \right), \\
\dot{y} &= x(x^2 - y^2) + \varepsilon \left( \sum_{0 \leq i+j \leq 3} b_{ij} x^i y^j \right),
\end{align*}
\]

has at most one limit cycle bifurcating from the periodic orbits of the center of system (2) using averaging theory of first order. Moreover, there are examples with 1 and 0 limit cycles.

**Proposition 2.** We consider the homogeneous polynomial differential system (2). Let \( P_i(x, y) \) and \( Q_i(x, y) \) for \( i = 1, 2 \) be polynomials of degree at most 3. Then for convenient polynomials \( P_i \) and \( Q_i \), the polynomial differential system

\[
\begin{align*}
\dot{x} &= -y(3x^2 + y^2) + \varepsilon P_1(x, y) + \varepsilon^2 P_2(x, y), \\
\dot{y} &= x(x^2 - y^2) + \varepsilon Q_1(x, y) + \varepsilon^2 Q_2(x, y),
\end{align*}
\]

has at first order averaging one limit cycle, and at second order averaging two limit cycles bifurcating from the periodic solutions of the global center (2).

Our main result is the following one and it do by first time the complete study of the averaging method of second order for a degenerate center having neither a Hamiltonian first integral nor a rational one.

**Theorem 3.** We consider the cubic homogeneous differential system (2). Then the perturbation of system (2) inside the class of all cubic polynomial systems

\[
\begin{align*}
\dot{x} &= -y(3x^2 + y^2) + \varepsilon \left( \sum_{0 \leq i+j \leq 3} a_{ij} x^i y^j \right) + \varepsilon^2 \left( \sum_{0 \leq i+j \leq 3} b_{ij} x^i y^j \right), \\
\dot{y} &= x(x^2 - y^2) + \varepsilon \left( \sum_{0 \leq i+j \leq 3} c_{ij} x^i y^j \right) + \varepsilon^2 \left( \sum_{0 \leq i+j \leq 3} d_{ij} x^i y^j \right),
\end{align*}
\]
has at most three limit cycles bifurcating from the periodic orbits of the center of system (2) using averaging theory of second order. Moreover, there are examples with 3, 2, 1 and 0 limit cycles.

The paper is organized as follows: In section 2 we present a summary of the averaging method of second order following (14). Next in section 3 we provide the proof of our main Theorem 3. In section 4 we provide three examples, of systems (3) with 0, 1, 2 and 3 limit cycles bifurcating from the degenerate center. At the end we present the Appendices A, B and C.

2. The averaging method of second order

In this section we present the averaging method of second order following (14). In that paper the averaging theory for differential equations of one variable is done up to any order in the small parameter of the perturbation. We consider the analytic differential equation

$$\frac{dr}{d\theta} = G_0(\theta, r) + \sum_{k \geq 1} \varepsilon^k G_k(\theta, r), \quad (4)$$

with $r \in \mathbb{R}$, $\theta \in S^1$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0$ a small positive real value, and the functions $G_k(\theta, r)$ are $2\pi$-periodic in the variable $\theta$. Note that for $\varepsilon = 0$ system (4) is unperturbed. Let $r_s(\theta, r_0)$ be the solution of system (4) with $\varepsilon = 0$ satisfying $r_s(0, r_0) = r_0$ and $r_s(\theta, r_0)$ is $2\pi$ periodic for $r_0 \in \mathcal{I}$ with $\mathcal{I}$ a real open interval. We are interested in the limit cycles of equation (4) which bifurcate from the periodic orbits of the unperturbed system with initial condition $r_0 \in \mathcal{I}$. So, we define by $r_\varepsilon(\theta, r_0)$ the solution of equation (4) satisfying $r_\varepsilon(0, r_0) = r_0$.

In what follows we denote by $u = u(\theta, r_0)$ the solution of the variational equation

$$\frac{\partial u}{\partial \theta} = \frac{\partial G_0}{\partial r}(\theta, r_s(\theta, r_0))u,$$

satisfying $u(0, r_0) = 1$. 

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We define
\[ u_1(\theta, r_0) = \int_0^{2\pi} G_1(\phi, r_s(\phi, r_0)) \frac{u(\phi, r_0)}{u(\theta, r_0)} d\phi = \int_0^{2\pi} G_1(w, r_s(w, r_0)) \frac{u(w, r_0)}{u(\theta, r_0)} dw, \]
\[ G_{10}(r_0) = 2\pi \int_0^{2\pi} G_1(\theta, r_s(\theta, r_0)) \frac{u(\theta, r_0)}{u(\theta, r_0)} d\theta, \]
\[ G_{20}(r_0) = \int_0^{2\pi} \left( \frac{G_2(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} + \frac{\partial G_1}{\partial r}(\theta, r_s(\theta, r_0)) u_1(\theta, r_0) + \frac{1}{2} \frac{\partial^2 G_0}{\partial r^2}(\theta, r_s(\theta, r_0)) u_1(\theta, r_0)^2 \right) d\theta. \]

In statement (b) of Corollary 5 of (14) it is proved the following result.

**Theorem 4.** Assume that the solution \( r_s(\theta, r_0) \) of the unperturbed equation (4) such that \( r_s(0, r_0) = r_0 \) is \( 2\pi \)-periodic for \( r_0 \in \mathcal{I} \) with \( \mathcal{I} \) a real open interval. If \( G_{10}(r_0) \) is identically zero in \( \mathcal{I} \) and \( G_{20}(r_0) \) is not identically zero in \( \mathcal{I} \), then for each simple zero \( r^* \in \mathcal{I} \) of \( G_{20}(r_0) = 0 \) there exists a periodic solution \( r_\varepsilon(\theta, r_0) \) of (4) such that \( r_\varepsilon(0, r_0) \to r^* \) when \( \varepsilon \to 0 \).

3. Proof of Theorem 3

System (2) in polar coordinates becomes
\[ \dot{r} = -2r^3 \cos \theta \sin \theta, \quad \dot{\theta} = r^2, \]
or equivalently,
\[ \frac{dr}{d\theta} = -2r \cos \theta \sin \theta, \]
and it has the solution \( r_s(\theta, r_0) = r_0 \exp(-\sin^2 \theta) \) satisfying that \( r_s(0, r_0) = r_0 \).

Now we perturb system (2) inside the class of all cubic polynomial differential systems as in (3). System (3) in polar coordinates give rise to the differential equation
\[ \frac{dr}{d\theta} = G_0(\theta, r) + \varepsilon G_1(\theta, r) + \varepsilon^2 G_2(\theta, r) + O(\varepsilon^3), \]
with
\[ G_0(\theta, r) = -2r \cos \theta \sin \theta, \]
\[ G_1(\theta, r) = g_{1,1}(\theta) \frac{1}{r^2} + g_{1,2}(\theta) \frac{1}{r} + g_{1,3}(\theta) + g_{1,4}(\theta) r, \]
\[ G_2(\theta, r) = g_{2,1}(\theta) \frac{1}{r^5} + g_{2,2}(\theta) \frac{1}{r^4} + g_{2,3}(\theta) \frac{1}{r^3} + g_{2,4}(\theta) \frac{1}{r^2} + g_{2,5}(\theta) \frac{1}{r} + g_{2,6}(\theta) + g_{2,7}(\theta) r, \]

where the expressions of the coefficients \( g_{1,i}(\theta) \) for \( i = 1, 2, 3, 4 \) and \( g_{2,j}(\theta) \) for \( j = 1, 2, \ldots, 7 \) are given in the Appendix A.

Additionally, we consider the variational equation
\[ \frac{\partial u}{\partial \theta} = \frac{\partial G_0}{\partial r}(\theta, r_s(\theta, r_0)), \]
and its solution \( u(\theta, r_0) \) satisfying \( u(0, r_0) = 1 \), namely \( u_s(\theta) = \exp(-\sin^2 \theta) \).

We define
\[ I_1 = \int_0^{2\pi} \exp(2 \sin^2 \theta) \cos^4 \theta \, d\theta = 3.572403292..., \]
\[ I_2 = \int_0^{2\pi} \exp(2 \sin^2 \theta) \cos^2 \theta \, d\theta = 5.985557563..., \]
\[ I_3 = \int_0^{2\pi} \exp(2 \sin^2 \theta) \, d\theta = 21.62373221... \]

**Lemma 5.** Consider \( I_1, I_2, I_3 \) defined in (7). Then for \( a_{10} = -(I_3 - 2I_1 + I_2)/(2I_1 - I_2)c_{01} \) and \( a_{30} = -2c_{03} - c_{21} \) we have that the function \( G_{10}(r_0) \) defined in (5) is identically zero.

**Proof.** We have
\[ \frac{G_1(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} = A(\theta) \frac{1}{r_0^2} + B(\theta) \frac{1}{r_0} + C(\theta) + D(\theta) r_0, \]
with
\[
A(\theta) = [\sin \theta (2 \cos^2 \theta + 1) c_{00} + \cos \theta (-1 + 2 \cos^2 \theta) a_{00}] e^{3 \sin^2 \theta},
\]
\[
B(\theta) = [\cos \theta \sin \theta (-1 + 2 \cos^2 \theta) a_{01} + \cos^2 \theta (-1 + 2 \cos^2 \theta) a_{10}
- (-\cos^2 \theta + 2 \cos^4 \theta - 1) c_{01} + \cos \theta \sin \theta (2 \cos^2 \theta + 1) c_{10}] e^{2 \sin^2 \theta},
\]
\[
C(\theta) = [-\cos \theta (-3 \cos^2 \theta + 1 + 2 \cos^4 \theta) a_{02} + \cos^3 \theta (-1 + 2 \cos^2 \theta) a_{20}
+ \cos^2 \theta \sin \theta (-1 + 2 \cos^2 \theta) a_{11} + \sin^3 \theta (2 \cos^2 \theta + 1) c_{02}
+ \cos^2 \theta \sin \theta (2 \cos^2 \theta + 1) c_{20} - \cos \theta (-\cos^2 \theta + 2 \cos^4 \theta - 1) c_{11}] e^{\sin^2 \theta},
\]
\[
D(\theta) = \cos \theta \sin^3 \theta (-1 + 2 \cos^2 \theta) a_{03} + \cos^4 \theta (-1 + 2 \cos^2 \theta) a_{30}
+ \cos^3 \theta \sin \theta (-1 + 2 \cos^2 \theta) a_{21} - \cos^2 \theta (-3 \cos^2 \theta + 1 + 2 \cos^4 \theta) a_{12}
+ (1 - 3 \cos^4 \theta + 2 \cos^6 \theta) c_{03} + \cos^3 \theta \sin \theta (2 \cos^2 \theta + 1) c_{30}
- \cos^2 \theta (-\cos^2 \theta + 2 \cos^4 \theta - 1) c_{21} + \cos \theta \sin^3 \theta (2 \cos^2 \theta + 1) c_{12}.
\]

Now
\[
G_{10}(r_0) = \frac{2\pi}{2} \int_0^{2\pi} \frac{G_1(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} d\theta
= \int_0^{2\pi} \left( A(\theta) \frac{1}{r_0^2} + B(\theta) \frac{1}{r_0} + C(\theta) + D(\theta) r_0 \right) d\theta,
\]
and considering the change of coordinates \( \theta = \phi + \pi \) in the interval \([0, 2\pi]\)
and the symmetries
\[
\sin(\theta + \pi) = -\sin \theta, \quad \cos(\theta + \pi) = -\cos \theta,
\]
we have that
\[
\int_0^{2\pi} A(\theta) d\theta = \int_{-\pi}^{\pi} A(\phi) d\phi = 0, \quad \int_0^{2\pi} C(\theta) d\theta = \int_{-\pi}^{\pi} C(\phi) d\phi = 0.
\]
So we have
\[
G_{10}(r_0) = \int_0^{2\pi} \frac{B(\theta)}{r_0} d\theta + \int_0^{2\pi} D(\theta) r_0 d\theta
= [(2a_{10} - 2c_{01}) I_1 + (c_{01} - a_{10}) I_2 + c_{01} I_3] \frac{1}{r_0} + \frac{\pi}{2} (2c_{03} + a_{30} + c_{21}) r_0,
\]
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and therefore $G_{10} \equiv 0$ if
$$a_{10} = \frac{I_3 - 2I_1 + I_2}{2I_1 - I_2} c_{01} = -17.65322447..c_{01}, \quad a_{30} = -2c_{03} - c_{21}.$$ 
This completes the proof of the lemma.

Now we have
$$\frac{G_2(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} = A_5(\theta) \frac{1}{r_0^5} + A_4(\theta) \frac{1}{r_0^4} + A_3(\theta) \frac{1}{r_0^3} + A_2(\theta) \frac{1}{r_0^2} + A_1(\theta) \frac{1}{r_0} + A_0(\theta) + \tilde{A}_1(\theta) r_0,$$
with $A_5(\theta), A_4(\theta), A_3(\theta), A_2(\theta), A_1(\theta), A_0(\theta), \tilde{A}_1(\theta)$ are given in the Appendix B.

We note that
$$\int_0^{2\pi} A_4(\theta) d\theta = 0, \quad \int_0^{2\pi} A_2(\theta) d\theta = 0, \quad \int_0^{2\pi} A_0(\theta) d\theta = 0,$$
because of the symmetries (8). So we have
$$\int_0^{2\pi} \frac{G_2(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} d\theta = \frac{1}{r_0^5} \int_0^{2\pi} A_5(\theta) d\theta + \frac{1}{r_0^4} \int_0^{2\pi} A_3(\theta) d\theta + \frac{1}{r_0^3} \int_0^{2\pi} A_1(\theta) d\theta + \left( \int_0^{2\pi} \tilde{A}_1(\theta) d\theta \right) r_0,$$
and we recall that the expressions of $A_5(\theta), A_3(\theta), A_1(\theta), \tilde{A}_1(\theta)$ are given in the Appendix B. We have
$$\int_0^{2\pi} A_5(\theta) d\theta = \left( -4 \int_0^{2\pi} e^{6\sin^2 \theta} \cos^4 \theta d\theta + 2 \int_0^{2\pi} e^{6\sin^2 \theta} \cos^2 \theta d\theta + \int_0^{2\pi} e^{6\sin^2 \theta} d\theta \right) a_{00} c_{00}$$
$$= 665.2264930..a_{00} c_{00},$$
$$\int_0^{2\pi} A_3(\theta) d\theta = 239.0000390..a_{01} c_{01} + 97.83745135..a_{00} c_{02} + 97.83745135..a_{02} c_{00}$$
$$- 257.2692783..c_{01} c_{10} + 12.93483815..a_{00} c_{20} - 7.99641945..a_{00} a_{11}$$
$$+ 12.93483815..a_{20} c_{00} - 28.92767705..c_{00} c_{11}.$$
\[ \int_{0}^{2\pi} A_1(\theta) d\theta = 1.159249021 \cdot b_{10} + 20.46448319 \cdot d_{01} + 2.318498043 \cdot a_{11}c_{11} \\
-4.73165232 \cdot c_{02}c_{11} + 2.318498043 \cdot a_{20}c_{02} + 2.318498045 \cdot a_{02}c_{20} \\
-3.761715750 \cdot c_{11}c_{20} + 14.47892563 \cdot a_{02}c_{02} - 1.253905250 \cdot a_{02}a_{11} \\
+ 0.189312456 \cdot a_{20}c_{20} + 0.094656226 \cdot a_{11}a_{20} + 0.647510463 \cdot a_{21}c_{01} \\
-5.110277230 \cdot c_{03}c_{10} + 2.318498043 \cdot a_{12}c_{10} + 2.22384182 \cdot a_{01}c_{21} \\
+ 36.6143963 \cdot a_{03}c_{01} - 3.95102821 \cdot c_{10}c_{21} - 1.25390525 \cdot a_{01}a_{12} \\
-45.6606188 \cdot c_{01}c_{12} - 7.1036910 \cdot c_{01}c_{30} + 14.28961317 \cdot a_{01}c_{03}, \\
\int_{0}^{2\pi} \tilde{A}_1(\theta) d\theta = -0.3926990817 \cdot c_{30}c_{21} + 0.1963495408 \cdot c_{30}a_{12} + 2.159844949 \cdot a_{03}c_{03} \\
-0.1963495408 \cdot a_{03}a_{12} - 0.7853981634 \cdot c_{12}c_{21} + 0.3926990817 \cdot a_{03}c_{21} \\
-1.178097245 \cdot c_{03}c_{12} + 0.3926990817 \cdot c_{12}a_{12} + 0.9817477042 \cdot c_{03}c_{30} \\
+ 1.570796327 \cdot d_{21} + 3.141592654 \cdot d_{03} + 1.570796327 \cdot b_{30}. \\
\]

Remark 6. (a) Looking at the expressions of \( A_3, A_1, \tilde{A}_1 \) in the Appendix B we can have the exact definition for the numerical coefficients which appear in the previous integrals. Thus for instance

\[ 239.0000390 \cdots = -\int_{0}^{2\pi} e^{4 \sin^2 \theta} \left( \cos^2 \theta - 1 \right) \left( 2 I_1 - 4 I_3 \cos^4 \theta + 2 I_3 \cos^2 \theta - I_2 \right) d\theta, \]

and \( I_1, I_2, I_3 \) satisfying relations (7).

(b) All the computations of this paper have been verified with the algebraic manipulator Maple.

We additionally have

\[ \frac{\partial G_1}{\partial r} (\theta, r_s(\theta, r_0)) = B_0(\theta) + \frac{B_1(\theta)}{r_0^2} + \frac{B_2(\theta)}{r_0^3}, \]

with

\[ B_0(\theta) = \cos^2 \theta \left( 1 + 2 \cos^2 \theta - 4 \cos^4 \theta \right) c_{21} \\
+ (-2 \cos^6 \theta + 1 - \cos^4 \theta) c_{03} \\
+ \cos \theta \left( \sin \theta \cos^2 \theta - 2 \cos^4 \theta \sin \theta + \sin \theta \right) c_{12} \\
+ \cos^4 \theta \left( \sin \theta + 2 \cos^2 \theta \sin \theta \right) c_{30} \\
+ \sin \theta \cos \theta \left( 3 \cos^2 \theta - 1 - 2 \cos^4 \theta \right) a_{03} \\
+ \cos^2 \theta \left( -1 + 3 \cos^2 \theta - 2 \cos^4 \theta \right) a_{12} \\
+ \sin \theta \cos^3 \theta \left( -1 + 2 \cos^2 \theta \right) a_{21}, \]

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\[ B_1(\theta) = \left( -1 - 18.65322447 \ldots \cos^2 \theta + 37.30644894 \ldots \cos^4 \theta \right) e^{2\sin^2 \theta} c_{01} \]
\[ - \sin \theta \cos \theta (2 \cos^2 \theta + 1) e^{2\sin^2 \theta} c_{10} \]
\[ + \sin \theta \cos \theta (-2 \cos^2 \theta + 1) e^{2\sin^2 \theta} c_{01}, \]
\[ B_2(\theta) = -2 \sin \theta (2 \cos^2 \theta + 1) c_{00} + 2 \left( -2 + e^{3\sin^2 \theta} \cos \theta \right) a_{00}. \]

Now we have
\[ \frac{G_1(w, r_0)}{u(w, r_0)} = \frac{C_2(w)}{r_0} + \frac{C_1(w)}{r_0} + C_0(w) + \tilde{C}_1(w) r_0, \]
with
\[ C_2(w) = e^{3\sin^2 w} [\cos w (2 \cos^2 w - 1) a_{00} + \sin w (1 + 2 \cos^2 w) c_{00}], \]
\[ C_1(w) = \left[ \cos w \sin w (-1 + 2 \cos^2 w) a_{01} \right. \]
\[ + \left( 1 + \frac{I_3}{2I_1 - I_2} (\cos^2 w - 2 \cos^4 w) \right) c_{01} \]
\[ + \cos w \sin w (1 + 2 \cos^2 w) c_{10} \right] e^{2\sin^2 w}, \]
\[ C_0(w) = [\cos^3 w (2 \cos^2 w - 1) a_{02} - \cos w (-3 \cos^2 w + 2 \cos^4 w + 1) a_{02} \]
\[ + \sin w \cos^2 w (2 \cos^2 w - 1) a_{11} + \sin w \cos^2 w (1 + 2 \cos^2 w) c_{20} \]
\[ + c_{02} \sin^3 w (1 + 2 \cos^2 w) - \cos w (2 \cos^4 w - 1 - \cos^2 w) c_{11}] e^{\sin^2 w}, \]
\[ \tilde{C}_1(w) = - (\cos^4 w + 2 \cos^6 w - 1) c_{03} + \sin^3 w \cos w (1 + 2 \cos^2 w) c_{12} \]
\[ - \cos^2 w (-1 - 2 \cos^2 w + 4 \cos^4 w) c_{21} + \cos^3 w \sin w (1 + 2 \cos^2 w) c_{30} \]
\[ + \sin^4 w \cos w (2 \cos^2 w - 1) a_{03} + \sin^2 w \cos^2 w (2 \cos^2 w - 1) a_{12} \]
\[ + \cos^3 w \sin w (2 \cos^2 w - 1) a_{21}. \]

Additionally, from (5) we obtain
\[ u_1(\theta, r_0) = \frac{1}{r_0^2} \int_0^\theta C_2(w) \, dw + \frac{1}{r_0} \int_0^\theta C_1(w) \, dw + \int_0^\theta C_0(w) \, dw + r_0 \int_0^\theta \tilde{C}_1(w) \, dw, \]
and so
\[ \frac{\partial G_1}{\partial r}(\theta, r, r_0) u_1(\theta, r_0) = s_5(\theta) \frac{1}{r_0} + s_4(\theta) \frac{1}{r_0} + s_3(\theta) \frac{1}{r_0} + s_2(\theta) \frac{1}{r_0} + s_1(\theta) \frac{1}{r_0} + s_0(\theta) + \tilde{s}_1(\theta) r_0, \]
\[ (10) \]
and the explicit expressions of $s_i(\theta)$ for $i = 0, 1, \cdots, 5$ and $\tilde{s}_1(\theta)$ are given in the Appendix C.

Since $\frac{\partial^2 G_0}{\partial r^2} = 0$ from (5) we have that

$$G_{20}(r_0) = \int_0^{2\pi} \left( \frac{G_2(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} + \frac{\partial G_1}{\partial r}(\theta, r_s(\theta, r_0)) u_1(\theta, r_0) \right) d\theta,$$

and we obtain

$$r_0^5 G_{20}(r_0) = v_6 r_0^6 + v_4 r_0^4 + v_2 r_0^2 + v_0,$$

with

$$v_6 = -0.3926990800 \cdots c_{21} c_{30} + 0.1963495397 \cdots a_{12} c_{30} + 2.159844949 \cdots a_{03} c_{03}\nonumber$$

$$-0.1963495365 \cdots a_{03} a_{12} - 0.7853981634 \cdots c_{12} c_{21} + 0.3926990817 \cdots a_{03} c_{21}\nonumber$$

$$-1.178097245 \cdots c_{03} c_{12} + 0.3926990817 \cdots a_{12} c_{12} + 0.9817477042 \cdots c_{03} c_{30}\nonumber$$

$$+ 1.570796327 \cdots d_{21} + 3.141592654 \cdots d_{03} + 1.570796327 \cdots b_{30},\nonumber$$

$$v_4 = -3.155691751 \cdots a_{21} c_{01} + 6.612510180 \cdots c_{03} c_{10} + 1.786201647 \cdots a_{12} c_{10}\nonumber$$

$$+ 2.413154277 \cdots a_{01} c_{21} + 38.88726613 \cdots a_{03} c_{01} - 1.253905255 \cdots c_{10} c_{21}\nonumber$$

$$- 1.206577137 \cdots a_{01} a_{12} - 68.30745733 \cdots c_{01} c_{10} - 38.88726635 \cdots c_{01} c_{30}\nonumber$$

$$+ 13.2734849 \cdots a_{01} c_{03} + 2.318498043 \cdots a_{11} c_{11} - 4.73165232 \cdots c_{02} c_{11}\nonumber$$

$$+ 2.318498043 \cdots a_{20} c_{02} + 2.318498045 \cdots a_{02} c_{20} - 3.761715750 \cdots c_{11} c_{20}\nonumber$$

$$+ 14.47892563 \cdots a_{02} c_{02} - 1.253905250 \cdots a_{02} a_{11} + 0.189312456 \cdots a_{20} c_{20}\nonumber$$

$$+ 0.094656226 \cdots a_{11} a_{20} + 1.159249021 \cdots b_{10} + 20.46448319 \cdots d_{01},\nonumber$$

$$v_2 = 95.95703341 \cdots a_{00} c_{02} + 105.7377762 \cdots a_{02} c_{00} + 239.0000390 \cdots a_{01} c_{01}\nonumber$$

$$- 7.649140220 \cdots a_{00} a_{11} + 16.68739736 \cdots a_{00} c_{20} - 110.9164314 \cdots c_{00} c_{11}\nonumber$$

$$- 257.2692783 \cdots c_{01} c_{10} - 0.000001 \cdots c_{01}^2 - 31.79348852 \cdots a_{20} c_{00},\nonumber$$

$$v_0 = 665.2264933 \cdots a_{00} c_{00}.\nonumber$$

We have that the coefficients $v_6, v_4, v_2, v_0$ are independent because $d_{03}$ only appears in $v_6$, $b_{10}$ only appears in $v_4$, $a_{00} c_{02}$ only appears in $v_2$, and $a_{00} c_{00}$ only appears in $v_0$.

Now we are going to use Descartes Theorem:

**Theorem 7** (Descartes Theorem). Consider the real polynomial $p(x) = a_i x^{i_1} + a_j x^{i_2} + \cdots + a_i x^{i_r}$ with $0 \leq i_1 < i_2 < \cdots < i_r$ and $a_{ij} \neq 0$ real
constants for \( j \in \{1, 2, \cdots, r\} \). When \( a_{ij}a_{ij+1} < 0 \), we say that \( a_{ij} \) and \( a_{ij+1} \) have a variation of sign. If the number of variations of signs is \( m \), then \( p(x) \) has at most \( m \) positive real roots. Moreover, it is always possible to choose the coefficients of \( p(x) \) in such a way that \( p(x) \) has exactly \( r - 1 \) positive real roots.

For a proof of Descartes Theorem see pages 82–83 of (3).

So from Descartes Theorem we can choose \( v_6, v_4, v_2, v_0 \) in order that the \( G_{20} \) has 3, 2, 1 or 0 real positive roots. This completes the proof of the first part of Theorem 3.

**Remark 8.** Again the exact definition for the numerical coefficients which appear in \( v_6, v_4, v_2 \) and \( v_0 \) are given in Appendices B and C. For instance

\[
v_0 = \int_0^{2\pi} A_5 d\theta + \int_0^{2\pi} s_{5,3} d\theta = \int_0^{2\pi} A_5 d\theta = 665.2264933...
\]

For completing the proof of Theorem 3 we shall provide examples of system (3) with 3, 2, 1 and 0 limit cycles. In fact, strictly speaking it is not necessary to provide examples with 3, 2, 1 and 0 limit cycles but we want to provide such examples.

4. Examples

**Example with 3 limit cycles**

In Figure 1 we see that for \( \varepsilon = 0.001 \) the system

\[
\dot{x} = -y(3x^2 + y^2) + \varepsilon + \varepsilon^2(3570.576292x - 752.8823806x^3)
= y(3x^2 + y^2) + 0.001 + 0.003570576292x - 0.0007528823806x^3,
\]

\[
\dot{y} = x(x^2 - y^2) + \varepsilon(1 - 37.74385845y^2)
= x(x^2 - y^2) + 0.001 - 0.03774385845y^2,
\]

has three limit cycles, since for system (11) we have

\[
G_{20}(r_0) = -1182.624878 r_0 + 4139.187071 \frac{1}{r_0} - 3621.788686 \frac{1}{r_0^3} + 665.2264933 \frac{1}{r_0^5},
\]

and from \( G_{20}(r_0) = 0 \) we obtain the three positive roots near to \( r_0 = 0.5, 1, 1.5 \).
\[ \varepsilon = 0 \quad \varepsilon = 0.001 \]

Figure 1: For \( \varepsilon = 0 \) we have the degenerate center of system (2), and for \( \varepsilon = 0.001 \) the perturbed system (11) has three limit cycles.

We have used the program P4 described in Chapters 9 and 10 of (10) for doing the phase portraits in the Poincaré disc which appear in this paper.

**Example with 2 limit cycles**

For \( \varepsilon = 0.001 \) the system

\[
\begin{align*}
\dot{x} &= -y(3x^2 + y^2) + \varepsilon(1 + y + x^2y) + \varepsilon^2(-856.6373973x + y^3) \\
&= -y(3x^2 + y^2) + 0.001 + 0.001y + 0.001x^2y - 0.0008566373973x + 0.000001y^3, \\
\dot{y} &= x(x^2 - y^2) + \varepsilon(1 + y^2) + \varepsilon^2(x + 73.80732101y^3) \\
&= x(x^2 - y^2) + 0.001 + 0.001y^2 + 0.000001x + 0.00007380732101y^3, \\
&= x(x^2 - y^2) + 0.010 + 0.010001 y + 0.005 xy - 0.000001 y^3,
\end{align*}
\]

(12)

gives

\[ G_{20}(r_0) = 231.8725375 r_0 - 993.0560642 \frac{1}{r_0} + 95.95703341 \frac{1}{r_0^3} + 665.2264933 \frac{1}{r_0^5}, \]

and \( G_{20}(r_0) = 0 \) has the two positive zeros \( r_0 = 1 \) and \( r_0 = 2 \). In Figure 2 we see the two limit cycles bifurcated from the degenerate center of the unperturbed system (12).

\[ \varepsilon = 0.001 \]

Figure 2: Two limit cycles bifurcate from the degenerate center of the unperturbed system (12).

**Example with 1 limit cycle**

For \( \varepsilon = 0.001 \) the system

\[
\begin{align*}
\dot{x} &= -y(3x^2 + y^2) + \varepsilon(1 - 176.5322447x) + \varepsilon^2(x + x^3) \\
&= -y(3x^2 + y^2) + 0.001 - 0.1765312447x + 0.000001 x^3, \\
\dot{y} &= x(x^2 - y^2) + \varepsilon(10 + 10y + 5xy) + \varepsilon^2(y - y^3) \\
&= x(x^2 - y^2) + 0.010 + 0.010001 y + 0.005 xy - 0.000001 y^3,
\end{align*}
\]

(13)

14
From relation $G_{20}(r_0) = 0$ we obtain $-1.097575824, 1.097575824, -5.725902515 - 5.148324797i$, $5.725902515 + 5.148324797i, -5.725902515 + 5.148324797i$. So only one limit cycle can bifurcate from a periodic orbit of the center of the unperturbed system (13), as we can see in Figure 3.

$\varepsilon = 0.001$

Figure 3: The limit cycle bifurcated from the degenerate center of the unperturbed system (13).

**Example with zero limit cycles**

Now for $\varepsilon = 0.001$ we consider system

$$
\begin{align*}
\dot{x} &= -y(3x^2 + y^2) + \varepsilon + \varepsilon^2 x \\
&= -y(3x^2 + y^2) + 0.001 + 0.000001 x, \\
\dot{y} &= x(x^2 - y^2) + \varepsilon(1 + y^2) + \varepsilon^2 y^3 \\
&= x(x^2 - y^2) + 0.001 + 0.001 y^2 + 0.000001 y^3,
\end{align*}
$$

with

$$
G_{20}(r_0) = 3.141592654 r_0 + 1.159249021 \frac{1}{r_0} + 95.95703341 \frac{1}{r_0^3} + 665.2264933 \frac{1}{r_0^5}.
$$

We have that $G_{20}(r_0) = 0$ has solutions $-2.116012294 - 1.570359831i, 2.116012294 + 1.570359831i, -2.095699520i, 2.095699520i, -2.116012294 + 1.570359831i, 2.116012294 - 1.570359831i$. So no limit cycles can bifurcate from the degenerate center, see also Figure 4.
Figure 4: No limit cycle bifurcates from the degenerate center of the unperturbed system (14).

5. Appendix A

\[ g_{1,1}(\theta) = \cos \theta (2 \cos^5 \theta - 1) a_{00} + \sin \theta (1 + 2 \cos^2 \theta)^2 c_{00}, \]

\[ g_{1,2}(\theta) = \sin^2 \theta (1 + 2 \cos^2 \theta) c_{01} + \sin \theta \cos \theta (1 + 2 \cos^2 \theta) c_{10} + \sin \theta \cos (2 \cos^2 \theta - 1) a_{01} + \cos^2 \theta (2 \cos^2 \theta - 1) a_{10}, \]

\[ g_{1,3}(\theta) = \sin^3 \theta (1 + 2 \cos^2 \theta) c_{02} + (2 \cos^5 \theta + \cos^3 \theta + \cos \theta) c_{11} + \cos^2 \theta \sin \theta (1 + 2 \cos^2 \theta) c_{20} + (2 \cos^3 \theta + 3 \cos^3 \theta - \cos \theta) a_{02} + \cos \theta \sin \theta (2 \cos^2 \theta - 1) a_{11} + \cos^3 \theta (2 \cos^2 \theta - 1) a_{20}, \]

\[ g_{1,4}(\theta) = \sin^4 \theta (1 + 2 \cos^2 \theta) c_{03} + \sin^3 \theta \cos \theta (1 + 2 \cos^2 \theta) c_{12} + \sin \theta \cos (2 \cos^2 \theta - 1) a_{03} + \sin \theta \cos^2 \theta (2 \cos^2 \theta - 1) a_{12} + \sin \theta \cos^3 \theta (2 \cos^2 \theta - 1) a_{21} + \cos^4 \theta (2 \cos^2 \theta - 1) a_{30}, \]

\[ g_{2,1}(\theta) = (2 \cos^2 \theta + 1 - 4 \cos^4 \theta) a_{00} a_{00} - \cos^2 \theta \cos \theta (2 \cos^2 \theta - 1) a_{00}^2 + \sin \theta \cos \theta (2 \cos^2 \theta - 1) a_{00}, \]

\[ g_{2,2}(\theta) = -\cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) c_{10} a_{00} - 2 \cos^2 \theta \sin \theta (2 \cos^2 \theta + 1) c_{10} a_{00} + 2 \cos^2 \theta \sin \theta (2 \cos^2 \theta - 1) a_{10} a_{00} - \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) c_{10} a_{00} - \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) c_{10} a_{00} - \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) c_{00} a_{10} + (4 \cos^3 \theta - 2 \cos \theta - 2 \cos^3 \theta) c_{01} c_{00} + (6 \cos^3 \theta - 4 \cos^3 \theta - 2 \cos \theta) a_{01} a_{00}, \]

\[ g_{2,3}(\theta) = (\cos^2 \theta + 4 \cos \theta - 6 \cos^2 \theta + 1) a_{00} a_{10} + (\cos^2 \theta + 4 \cos \theta - 6 \cos^2 \theta + 1) a_{00} a_{02} + (\cos^2 \theta + 4 \cos \theta - 6 \cos^2 \theta + 1) a_{02} a_{02} - c_{10} a_{00} \sin \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \]

\[ - c_{00} a_{10} \sin \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{20} a_{00} \sin \theta \cos^2 \theta (2 \cos^2 \theta + 1) + 2 a_{02} a_{00} \sin \theta \cos^3 \theta (2 \cos^2 \theta - 1) + 2 a_{02} a_{00} \sin \theta \cos \theta (2 \cos^2 \theta - 1) \]

\[ - 2 c_{02} a_{00} \sin \theta \cos \theta (2 \cos^2 \theta + 1) - c_{10} a_{01} \sin \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \]

\[ - c_{01} a_{10} \sin \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{20} a_{10} \cos \theta (2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \]

\[ + c_{00} a_{20} \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + (4 \cos^3 \theta - 2 \cos^2 \theta - 2 \cos^3 \theta) c_{11} a_{00} + (4 \cos^3 \theta - 2 \cos^2 \theta - 2 \cos^3 \theta) c_{01} a_{10} + (4 \cos^3 \theta - 2 \cos^2 \theta - 2 \cos^3 \theta) c_{10} a_{01} + a_{10} \sin \theta \cos \theta (2 \cos^2 \theta - 1) - c_{01} \sin \theta \cos \theta (2 \cos^2 \theta + 1) + a_{01} \sin \theta \cos \theta (2 \cos^2 \theta - 1) - c_{10} \sin \theta \cos \theta (2 \cos^2 \theta + 1), \]

\[ g_{2,4}(\theta) = -c_{00} a_{00} \sin^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{10} a_{00} \sin^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{00} a_{00} \sin^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{10} a_{00} \sin^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + a_{00} \sin \theta (2 \cos^2 \theta + 1) + c_{10} \cos \theta (2 \cos^2 \theta - 1) - c_{20} a_{10} \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \]

\[ - c_{00} a_{00} \sin \theta (2 \cos^2 \theta + 1) - c_{20} a_{10} \sin^3 \theta (2 \cos^2 \theta + 1) + 2 a_{12} a_{00} \sin \theta \cos^2 \theta (2 \cos^2 \theta - 1) + 2 a_{02} a_{00} \sin \theta \cos \theta (2 \cos^2 \theta - 1) + 2 a_{02} a_{01} \sin \theta \cos \theta (2 \cos^2 \theta - 1) \]

\[ - 2 c_{10} a_{00} \sin \theta \cos \theta (2 \cos^2 \theta + 1) - 2 c_{20} a_{10} \cos^3 \theta \sin \theta (2 \cos^2 \theta + 1) + 2 a_{10} a_{00} \sin \theta \cos \theta (2 \cos^2 \theta - 1), \]

\[ \varepsilon = 0.001 \]
\[ g_{2,5}(\theta) = -c_{10} a_{30} \cos^4 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{30} a_{10} \cos^4 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta), \]
\[ -c_{02} \sin^3 \theta \cos \theta (2 \cos^2 \theta + 1) - c_{11} \sin^3 \theta \cos^3 \theta (2 \cos^2 \theta + 1), \]
\[ -c_{20} \cos^4 \theta \sin \theta (2 \cos^2 \theta + 1) + a_{02} \sin^3 \theta \cos \theta (2 \cos^2 \theta - 1), \]
\[ + (-7 \cos^4 \theta + 4 \cos^2 \theta + 10 \cos^6 \theta) a_{01} c_{03} + (-7 \cos^4 \theta + 4 \cos^8 \theta + 10 \cos^6 \theta) a_{03} c_{01}, \]
\[ + (-7 \cos^4 \theta + 4 \cos^2 \theta + 10 \cos^6 \theta) a_{02} c_{02} + d_{10} \sin \theta \cos \theta (2 \cos^2 \theta + 1), \]
\[ + b_{10} \cos^2 \theta (2 \cos^2 \theta - 1) + d_{01} \sin^2 \theta \cos (2 \cos^2 \theta + 1) + a_{11} \sin^3 \theta \cos^3 (2 \cos^2 \theta - 1), \]
\[ - c_{01} a_{12} \sin \theta \cos \theta (2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{30} a_{01} \sin \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta), \]
\[ - c_{21} a_{10} \sin \cos^4 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{20} a_{11} \sin \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta), \]
\[ - c_{11} a_{20} \sin \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + 2 a_{11} a_{01} \sin \cos^3 \theta (2 \cos^2 \theta + 1), \]
\[ - c_{10} a_{21} \sin \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{01} a_{30} \sin \cos \theta (2 \cos^2 \theta - 1 + 4 \cos^4 \theta), \]
\[ - c_{12} a_{01} \sin \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{11} a_{02} \sin \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta), \]
\[ - c_{10} a_{03} \sin \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{03} a_{10} \sin \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta), \]
\[ - c_{02} a_{11} \sin \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + 2 a_{12} a_{01} \sin \cos^3 \theta (2 \cos^2 \theta - 1), \]
\[ + 2 a_{11} a_{02} \sin \cos \theta (2 \cos^2 \theta - 1) + a_{20} a_{02} \sin \cos \theta (2 \cos^2 \theta - 1), \]
\[ + 2 a_{20} a_{01} \sin \cos \theta (2 \cos^2 \theta - 1) + a_{20} a_{02} \sin \cos \theta (2 \cos^2 \theta - 1), \]
\[ + 2 a_{20} a_{01} \sin \cos \theta (2 \cos^2 \theta - 1) + 6 \cos^2 \theta - 4 \cos^4 \theta - 2 \cos^2 \theta - 1), \]
\[ + 2 a_{11} a_{02} \sin \cos \theta (2 \cos^2 \theta - 1) + a_{12} a_{01} \sin \cos \theta (2 \cos^2 \theta - 1), \]
\[ + 2 a_{12} a_{01} \sin \cos \theta (2 \cos^2 \theta - 1) + 6 \cos^2 \theta - 4 \cos^4 \theta - 2 \cos^2 \theta - 1), \]
\[\begin{align*}
+2a_{03}a_{02}\sin^6\theta\cos\theta \left(2\cos^2\theta - 1\right) - 2c_{03}a_{02}\sin^6\theta\cos\theta \left(2\cos^2\theta + 1\right) \\
+2a_{11}a_{03}\sin^4\theta\cos^2\theta \left(2\cos^2\theta - 1\right) - 2c_{11}a_{03}\sin^4\theta\cos^2\theta \left(2\cos^2\theta + 1\right) \\
-2c_{30}a_{02}\sin^6\theta\cos^4\theta \left(2\cos^2\theta + 1\right) + 2a_{12}a_{02}\sin^5\theta\cos^3\theta \left(2\cos^2\theta - 1\right) \\
- c_{11}a_{12}\sin^2\theta\cos^2\theta \left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) - c_{21}a_{02}\sin^3\theta\cos^2\theta \left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
+ b_{20}\cos^4\theta \left(2\cos^2\theta - 1\right) - 2c_{12}a_{02}\sin\theta\cos^3\theta \left(2\cos^2\theta + 1\right) \\
- c_{12}a_{11}\sin^2\theta\cos^2\theta \left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) - c_{30}a_{11}\cos\theta\sin\theta \left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
- c_{21}a_{20}\cos^3\theta\sin\theta \left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) + 2a_{30}a_{20}\cos^5\theta\sin\theta \left(2\cos^2\theta - 1\right) \\
- c_{03}a_{20}\sin\theta\cos\theta \left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) - c_{02}a_{21}\sin^3\theta\cos^2\theta \left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
+2a_{30}a_{02}\sin^5\theta\cos\theta \left(2\cos^2\theta - 1\right) - c_{03}a_{02}\sin^6\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
+ d_{20}\cos^4\theta\sin\theta \left(2\cos^2\theta + 1\right) + b_{11}\cos^3\theta\sin\theta \left(2\cos^2\theta - 1\right) \\
- c_{02}a_{03}\sin^3\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) + (\cos\theta + 10\cos^3\theta - 7\cos^5\theta - 4\cos^9\theta) c_{11}a_{03} \\
+ (\cos\theta + 10\cos^3\theta - 7\cos^5\theta - 4\cos^9\theta) c_{03}a_{11} + (\cos^3\theta - 6\cos^7\theta + 4\cos^9\theta + \cos^3\theta) c_{21}a_{11} \\
+ (\cos^3\theta - 6\cos^7\theta + 4\cos^9\theta + \cos^3\theta) c_{11}a_{21} + (\cos^5\theta - 6\cos^7\theta + 4\cos^9\theta + \cos^3\theta) c_{20}a_{12} \\
+ (\cos^3\theta - 6\cos^7\theta + 4\cos^9\theta + \cos^3\theta) c_{12}a_{20} + (\cos^5\theta - 6\cos^7\theta + 4\cos^9\theta + \cos^3\theta) c_{20}a_{30} \\
+ (\cos\theta + 10\cos^3\theta - 7\cos^5\theta - 4\cos^9\theta) c_{12}a_{02} + (\cos\theta + 10\cos^3\theta - 7\cos^5\theta - 4\cos^9\theta) c_{02}a_{12} \\
+ (\cos^3\theta - 6\cos^7\theta + 4\cos^9\theta + \cos^3\theta) c_{30}a_{02} + (-2\cos^3\theta + 6\cos^7\theta - 4\cos^9\theta) c_{21}a_{20} \\
+ (-2\cos^3\theta + 6\cos^7\theta - 4\cos^9\theta) c_{20}a_{30} + (-2\cos^3\theta + 6\cos^7\theta - 4\cos^9\theta) c_{12}c_{11} \\
+ (-4\cos^3\theta + 6\cos^7\theta - 2\cos^9\theta) a_{30}a_{11} + (-4\cos^3\theta + 6\cos^7\theta - 2\cos^9\theta) a_{21}a_{20} \\
+ (4\cos^3\theta - 2\cos^5\theta - 2\cos^7\theta) c_{30}c_{11} + (4\cos^3\theta - 2\cos^5\theta - 2\cos^7\theta) c_{21}c_{20}.
\end{align*}\]
6. Appendix B

We present the expressions of $A_4, A_2, A_0, A_5, A_3, A_1, \tilde{A}_1$ that appear in equation (9).

$$A_4(\theta) = e^5 \sin^2 \theta \left[ -\frac{(4a_0c_0I_0I_2 - 8I_1a_0c_0 - 8I_1c_00c_{10} - 4a_0c_0I_3 + 4c_00c_{10}I_2)\cos^4 \theta}{I_2 - 2I_1} 
- \frac{(-4I_1c_00c_{10} + 2a_00c_0I_3 - 2a_01c_0I_2 + 2c_00c_{10}I_2 + 4I_1a_0c_0I_2)}{I_2 - 2I_1} \right] \sin \theta$$

$$= \frac{(-a_01c_00I_2 + 2I_1a_0c_0 + 2I_1a_00c_0 - a_01c_00I_2)}{I_2 - 2I_1} \sin \theta$$

$$= \frac{(4c_00c_0I_3 + 4a_00c_0I_2 - 8I_1a_00c_0 - 8I_1a_01c_0 + 4a_00a_1I_2)(\cos \theta)^5}{I_2 - 2I_1}$$

$$= \frac{(12I_1a_00c_0I_3 + 4I_1a_00c_1I_2 - 2c_00c_0I_3 - 2a_00c_1I_2 - 6a_00a_0I_2)(\cos \theta)^3}{I_2 - 2I_1}$$

$$= \frac{(-2c_00c_0I_3 + 4c_00c_0I_2 - 4I_1a_00c_0 + 2a_00a_0I_2 + 2I_1a_00c_1 - c_00c_0I_3 - a_00c_1I_2)(\cos \theta)}{I_2 - 2I_1} \right]$$

$$A_2(\theta) = \left[ A_{2,7} \cos^7 \theta + A_{2,6} \cos^6 \theta \sin \theta + A_{2,5} \cos^5 \theta + A_{2,4} \cos^4 \theta \sin \theta + A_{2,3} \cos^3 \theta + A_{2,2} \cos^2 \theta \sin \theta \right] \exp(3\sin^2 \theta).$$

$$A_{2,7} = 4a_03a_{00} - 4a_21a_{00} - 4c_03a_{00} + 4c_12a_{00} - 4e_20a_{01} + 4a_02a_{01} + 4c_{11}a_{01} - 4c_{10}a_{20}$$

$$+ 4c_{10}a_{02} + 4a_{11}c_0I_2 + 4c_00c_{12} + 4c_03c_{00} + 8c_{21}c_{00} + 4c_01c_{20}I_3$$

$$= -4a_{12}a_{00} - 4c_03a_{00} - 8c_{21}a_{00} - 4a_{11}a_{01} - 4c_{20}a_{01} + 4c_02a_{01} - 4a_{20}c_0I_3$$

$$= \frac{+4a_{12}c_0I_3 - 4a_{11}c_{10} + 4a_{03}c_{00} - 4a_{21}c_{00} - 4c_{30}c_{00} + 4c_{12}c_{00}}{2I_1 - I_2}$$

$$A_{2,6} = -10a_04a_{00} + 6a_22a_{00} + 2c_03a_{00} - 6c_{12}a_{00} + 6e_20a_{01} - 10a_02a_{01} - 6c_{11}a_{01} + 2c_{10}a_{20}$$

$$- 6c_10a_{02} - 2c_01c_{11}I_3$$

$$= \frac{-6a_04a_{00} - 6c_{11}c_{03}I_3}{2I_1 - I_2} - 6c_{03}c_{00} - 2e_21c_{00} - 4c_{21}c_{00} - 2c_01c_{20}I_3}{2I_1 - I_2}$$

$$+ 6c_03c_{12}I_3 - 2c_11c_{10},$$

$$A_{2,5} = \frac{6a_04a_{02} - 2a_00c_{00} + 4a_00c_{21} + 6a_01a_{11} + 2a_01c_{20} - 6a_01c_{02} + 2a_02c_0I_3}{2I_1 - I_2}$$

$$- 6a_04c_0I_3 + 2a_{11}c_{10} - 6a_03c_{00} + 2a_{21}c_{00} - 2c_00c_{00} - 2c_{01}c_{12} - 2c_01c_{11}I_3$$

$$- 2c_10c_{20} - 2c_02c_{10},$$

$$A_{2,4} = 2b_00 + 8a_00a_{00} - 2a_00c_{21} + a_00c_{30} + a_00c_{12} - 2a_01c_{20} + 8a_01a_{02} + a_01c_{11} + a_02c_{10}$$

$$+ a_{02}c_{10} = \frac{(-2I_3 - I_2 + 2I_1)c_01a_{11}}{I_2 - 2I_1} + a_{12}c_{00} - 2c_00c_{03} - 3c_00c_{21}$$

$$= \frac{-I_2 + 2I_1 + I_3c_01c_{20}}{2I_1 - I_2} + \frac{(2I_1 - I_3 - I_2)c_02c_{01}}{I_2 - 2I_1} - 2c_{10}c_{11}.$$
\[ A_{2,2} = 2d_{00} - 2a_{00}a_{12} + a_{00}c_{03} + a_{00}c_{21} - 2a_{01}a_{11} + a_{01}c_{20} + a_{01}c_{02} + a_{20}c_{01} - (2I_3 - I_2 + 2I_1)c_{10}a_{20} - a_{11}c_{10} + 2a_{21} - 2c_{00}c_{12} - (2I_1 - I_2)c_{11}c_{01} - 2c_{02}c_{10}. \]

\[ A_{2,1} = -b_{00} - 2a_{00}a_{03} - a_{00}c_{02} - 2a_{01}a_{02} + a_{01}c_{11} + a_{01}c_{01} + a_{12}c_{20} - 2c_{00}c_{03} - (2I_1 - I_2)c_{01}a_{20}, \]

\[ A_{2,0} = a_{03}c_{00} + d_{00} + a_{00}c_{03} + a_{00}c_{02} + a_{01}c_{01}. \]

Now we have

\[ A_0(\theta) = A_{0,0} \cos^9 \theta + A_{0,8} \cos^8 \theta + A_{0,7} \cos^7 \theta + A_{0,6} \cos^6 \theta + A_{0,5} \cos^5 \theta + A_{0,4} \cos^4 \theta + A_{0,3} \cos^3 \theta + A_{0,2} \cos^2 \theta + A_{0,1} \cos \theta + A_{0,0}, \]

with

\[ A_{0,0}(\theta) = -[\cdots] \]

\[ A_{0,8}(\theta) = \cdots \]

\[ A_{0,7}(\theta) = \cdots \]

\[ A_{0,6}(\theta) = \cdots \]

\[ A_{0,5}(\theta) = \cdots \]

\[ A_{0,4}(\theta) = \cdots \]

\[ A_{0,3}(\theta) = \cdots \]

\[ A_{0,2}(\theta) = \cdots \]

\[ A_{0,1}(\theta) = \cdots \]

\[ A_{0,0}(\theta) = \cdots \]
Additionally, we have

\[ A_3(\theta) = 2 \cos^2 \theta \sin \theta \left( -1 + 2 \cos^2 \theta \right) a_{00} + 2 \cos \theta \sin \theta \left( \cos^2 \theta - 1 \right) \left( -1 + 2 \cos^2 \theta \right) a_{00a_2} \]

\[ + 2 \cos^2 \theta \left( \cos^2 \theta - 1 \right) \left( -1 + 2 \cos^2 \theta \right) a_{00c_2} - \cos \theta \sin \theta \left( \cos^2 \theta - 1 \right) \left( -1 + 2 \cos^2 \theta \right) \left( 2 \cos^2 \theta - 1 \right) a_{00c_0} \]

\[ + \cos \theta \sin \theta \left( \cos^2 \theta - 1 \right) \left( -1 + 2 \cos^2 \theta \right) a_{00c_2} - \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) c_{11a_{00}} \]

\[ - \cos \theta \sin \theta \left( \cos^2 \theta - 1 \right) \left( -1 + 2 \cos^2 \theta \right) a_{00a_1} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) c_{10a_{01}} \]

\[ + \cos \theta \sin \theta \left( \cos^2 \theta - 1 \right) \left( -1 + 2 \cos^2 \theta \right) \left( 2 \cos^2 \theta - 1 \right) a_{00c_0} \]

\[ + 2 \cos^2 \theta \left( \cos^2 \theta - 1 \right) \left( 2 \cos^2 \theta + 1 \right) c_{00c_0} + 2 \cos \theta \sin \theta \left( \cos^2 \theta - 1 \right) \left( 2 \cos^2 \theta + 1 \right) c_{00c_0} \]

\[ + 2 \cos^2 \theta \left( \cos^2 \theta - 1 \right) \left( 2 \cos^2 \theta + 1 \right) c_{00c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 3 - 4 \cos^2 \theta \right) c_{10c_{21}} \]

\[ + 2 \cos^2 \theta \left( 2 \cos^2 \theta - 1 \right) \left( \cos^2 \theta - 1 \right) a_{02a_1} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + 2 \cos^2 \theta \left( \cos^2 \theta - 1 \right) \left( 2 \cos^2 \theta + 1 \right) c_{12a_{01}} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]

\[ + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} + \cos \theta \sin \theta \left( 4 \cos^2 \theta - 1 - 2 \cos^2 \theta \right) a_{02c_0} \]
\[ \tilde{}_1(\theta) = \cos \theta \sin \theta \left( \cos^4 \theta - \cos^2 \theta + 2 \cos^6 \theta + 3 \cos^8 \theta - 1 \right) \cos^2 \theta \\
- (4 \cos^6 \theta + 2 \cos^4 \theta - \cos^2 \theta - 1) (\cos^2 \theta - 1) a_{03} a_{03} \\
- \cos^3 \theta \sin \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{21} c_{30} + \cos^5 \theta (8 \cos^4 \theta - 3 - 4 \cos^2 \theta) c_{21} c_{30} \\
+ 2 \cos^4 \theta \left( \cos^2 \theta - \cos^2 \theta - 1 + 2 \cos^2 \theta \right) c_{03} c_{30} \\
+ \cos^3 \theta \sin \theta (2 \cos^2 \theta + 1) d_{30} + \cos^3 \theta \sin \theta (1 + 2 \cos^2 \theta) b_{12} \\
+ 2 c_{21}^2 \cos^5 \theta \sin \theta (4 \cos^3 \theta - 1 - 2 \cos^2 \theta) + a_{12}^2 \sin^2 \theta \cos^3 \theta (-1 + 2 \cos^2 \theta) \\
- \cos^3 \theta \sin \theta (2 \cos^2 \theta + 1) c_{30}^2 + (2 \cos^2 \theta + 1) (\cos^2 \theta - 1)^2 d_{03} \\
+ \cos^3 \theta \sin \theta (4 \cos^4 \theta + 8 \cos^6 \theta - 3 - 3 \cos^3 \theta) c_{03} c_{21} - \cos^2 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) d_{21} \\
- \cos^2 \theta (\cos^2 \theta - 1) (-1 + 2 \cos^2 \theta) b_{12} + \cos^4 \theta (-1 + 2 \cos^2 \theta) b_{30} \\
- \cos^3 \theta \sin \theta (-1 + 2 \cos^2 \theta) (\cos^2 \theta - 1) a_{03} a_{03} \\
+ \cos^3 \theta (\cos^2 \theta - 1) (8 \cos^4 \theta - 4 \cos^2 \theta - 1) a_{21} c_{21} \\
- \cos^3 \theta (\cos^2 \theta - 1) (8 \cos^4 \theta - 3 - 4 \cos^2 \theta) c_{12} c_{21} \\
- 2 \cos^2 \theta (-1 + 2 \cos^2 \theta) (\cos^2 \theta - 1) a_{03} a_{12} - \cos^2 \theta (8 \cos^4 \theta - 4 \cos^2 \theta - 1) (\cos^2 \theta - 1)^2 a_{03} c_{21} \\
- \cos^2 \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) (\cos^2 \theta - 1)^2 a_{12} c_{12} \\
+ \cos^2 \theta (\cos^2 \theta - 1) (4 \cos^6 \theta + 2 \cos^4 \theta - \cos^2 \theta - 1) a_{21} c_{03} \\
- \cos^3 \theta \sin \theta (\cos^2 \theta - 1) (-1 + 2 \cos^2 \theta) a_{21}^2 \\
- \cos^3 \theta \sin \theta (2 \cos^2 \theta + 1) (\cos^2 \theta - 1) c_{12}^2 + 2 \cos^3 \theta \sin \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) c_{12} c_{30} \\
+ 2 \cos^3 \theta \sin \theta (-1 + 2 \cos^2 \theta) (\cos^2 \theta - 1) a_{03} c_{21} \\
+ \cos^3 \theta \sin \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{03} c_{30} \\
+ \cos^3 \theta \sin \theta (\cos^2 \theta - 1) (8 \cos^4 \theta - 4 \cos^2 \theta - 1) a_{12} c_{21} \\
+ \cos^3 \theta \sin \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{21} c_{12} \\
- \cos^3 \theta \sin \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) (\cos^2 \theta - 1) a_{03} c_{12} \\
+ \cos^3 \theta \sin \theta (\cos^2 \theta - 1) (4 \cos^6 \theta + 2 \cos^4 \theta - \cos^2 \theta - 1) a_{12} c_{03}.

7. Appendix C

Here we present the explicit expressions of \( s_5(\theta), s_4(\theta), s_3(\theta), s_2(\theta), s_1(\theta), \tilde{s}_1(\theta) \) that appear in relation (10). Thus \( s_5(\theta) = s_{5,1}(\theta)c_{00}^2 + s_{5,2}(\theta)a_{00}^2 + s_{5,3}(\theta)a_{00}c_{00} \), with

\[
\begin{align*}
    s_{5,1}(\theta) &= -2 e^{3 \sin^2 \theta} \left( \int_0^\theta e^{3 \sin^2 w} \sin w \, dw + 2 \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w \, dw \right) \\
    &= (\sin \theta + 2 \sin \theta \cos^2 \theta), \\
    s_{5,2}(\theta) &= -2 e^{3 \sin^2 \theta} \left( -\int_0^\theta e^{3 \sin^2 w} \cos w \, dw + 2 \int_0^\theta e^{3 \sin^2 w} \cos^3 w \, dw \right) \\
    &= (-\cos \theta + 2 \cos^3 \theta),
\end{align*}
\]

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\[ s_{5,3}(\theta) = -2 e^3 \sin^2 \theta \left( \int_0^\theta e^3 \sin^2 w \sin w dw + 2 \int_0^\theta e^3 \sin^2 w \cos^2 w dw \right) \]
\[ \left( -\cos \theta + 2 \cos^3 \theta \right) \]
\[ -2 e^3 \sin^2 \theta \left( -\int_0^\theta e^3 \sin^2 w \cos w dw + 2 \int_0^\theta e^3 \sin^2 w \cos^3 w dw \right) \]
\[ (\sin \theta + 2 \sin \theta \cos^2 \theta) . \]

Additionally, we have
\[ s_4(\theta) = s_{4,1}(\theta)c_{00}c_{10} - s_{4,2}(\theta)a_{00}a_{01} - s_{4,3}(\theta)c_{00}c_{01} - s_{4,4}(\theta)c_{00}c_{10} - s_{4,5}(\theta)a_{00}c_{01} - s_{4,6}(\theta)a_{01}c_{00}, \]
and \( s_{4,i}(\theta) \) for \( i = 1 \ldots 6 \) are the following:

\[ s_{4,1}(\theta) = - \left( -\sin \theta e^{-\sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos w dw + 4 \cos^2 \theta \int_0^\theta e^3 \sin^2 w \sin w \cos w dw \right. \]
\[ + 2 \sin \theta e^{-\sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos^3 w dw + 4 \cos^2 \theta \sin \theta e^{-\sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos^3 w dw \]
\[ + 8 \cos^2 \theta \int_0^\theta e^2 \sin^2 w \cos^3 w \sin w dw - 2 \cos^2 \theta \sin \theta e^{-\sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos w dw \]
\[ - 4 \int_0^\theta e^2 \sin^2 w \cos^3 w \sin w dw - 2 \int_0^\theta e^2 \sin^2 w \sin w \cos w dw \right) \]
\[ e^3 \sin^2 \theta \cos \theta, \]

\[ s_{4,2}(\theta) = - \left( \sin \theta e^{-\sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos w dw - 2 \sin \theta e^{-\sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos^3 w dw \right. \]
\[ - 4 \cos^2 \theta \int_0^\theta e^2 \sin^2 w \sin w \cos w dw - 4 \int_0^\theta e^2 \sin^2 w \cos^3 w \sin w dw \]
\[ - 2 \cos^2 \theta \sin \theta e^{-\sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos w dw \]
\[ + 4 \cos^2 \theta \sin \theta e^{-\sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos^3 w dw + 2 \int_0^\theta e^2 \sin^2 w \sin w \cos w dw \]
\[ + 8 \cos^2 \theta \int_0^\theta e^2 \sin^2 w \cos^3 w \sin w dw \right) \]
\[ e^3 \sin^2 \theta \cos \theta, \]

\[ s_{4,3}(\theta) = - e^2 \sin^2 \theta \left( 8 e^3 \sin^2 \theta I_1 \cos^2 \theta \sin \theta \int_0^\theta e^2 \sin^2 w dw \right. \]
\[ + 2 I_2 \int_0^\theta e^2 \sin^2 w \sin w \cos^2 w dw + 2 I_1 \int_0^\theta e^3 \sin^2 w \sin w dw \]
\[ - 2 e^3 \sin^2 \theta I_2 \sin \theta \int_0^\theta e^2 \sin^2 w dw - 2 \cos^4 \theta I_3 \int_0^\theta e^3 \sin^2 w \sin w dw \]
\[ - 4 e^3 \sin^2 \theta I_2 \cos^2 \theta \sin \theta \int_0^\theta e^2 \sin^2 w dw + 2 e^3 \sin^2 \theta I_3 \sin \theta \int_0^\theta e^2 \sin^2 w \cos^2 w dw \]
\[ + 4 I_1 \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw - 4 e^3 \sin^2 \theta I_3 \sin \theta \int_0^\theta e^2 \sin^2 w \cos^4 w dw \]
\[ + 2 \cos^2 \theta I_3 \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw + 4 e^3 \sin^2 \theta I_3 \cos^2 \theta \sin \theta \int_0^\theta e^2 \sin^2 w \cos^2 w dw \]

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\[\begin{align*}
&+4e^{\sin^2 \theta} I_1 \sin \theta \int_0^\theta e^{2 \sin^2 w} dw + \cos^2 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \\
&-I_2 \int_0^\theta e^{3 \sin^2 w} \sin w \, dw - 4 \cos^4 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \\
&-8e^{\sin^2 \theta} I_3 \cos^2 \theta \sin \theta \int_0^\theta e^{2 \sin^2 w} \cos^2 w \, dw \right) \frac{1}{2I_1 - I_2}. \\
\end{align*}\]

\[s_{4,4}(\theta) = \left( -8e^{\sin^2 \theta} \cos^2 \theta \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w \, dw \\
+4e^{\sin^2 \theta} \cos^2 \theta \int_0^\theta e^{2 \sin^2 w} \sin w \, dw + 2 \left( \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \right) \cos \theta \\
+\left( \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \right) \cos \theta + 2e^{\sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \, dw \\
+4 \left( \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \right) \cos^3 \theta + 4e^{\sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w \, dw \\
+2 \left( \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \right) \cos^2 \theta \sin \theta,
\right)
\]

\[s_{4,5}(\theta) = -e^{2 \sin^2 \theta} \left( -2 I_1 \int_0^\theta e^{3 \sin^2 w} \sin w \, dw + 2 \cos^4 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \cos w \, dw \\
-2 I_2 \int_0^\theta e^{3 \sin^2 w} \cos^3 w \, dw - 4e^{\sin^2 \theta} I_3 \cos \theta \int_0^\theta e^{2 \sin^2 w} \cos^4 w \, dw \\
-2e^{\sin^2 \theta} I_3 \cos \theta \int_0^\theta e^{2 \sin^2 w} \cos^2 w \sin w \, dw - 4 \cos^4 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \\
+2e^{\sin^2 \theta} I_2 \cos \theta \int_0^\theta e^{2 \sin^2 w} \sin w \, dw + I_2 \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \\
-4e^{\sin^2 \theta} I_3 \cos^3 w \int_0^\theta e^{2 \sin^2 w} \sin w \, dw - 8e^{\sin^2 \theta} I_3 \cos^3 w \int_0^\theta e^{2 \sin^2 w} \cos^4 w \, dw \\
+4e^{\sin^2 \theta} I_3 \cos^3 w \int_0^\theta e^{2 \sin^2 w} \cos^2 w \sin w \, dw + 4 I_1 \int_0^\theta e^{3 \sin^2 w} \cos^3 w \, dw \\
+8e^{\sin^2 \theta} I_1 \cos \theta \int_0^\theta e^{2 \sin^2 w} \sin w \, dw + 2 \cos^2 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \\
- \cos^2 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \cos w \, dw - 4e^{\sin^2 \theta} I_1 \cos \theta \int_0^\theta e^{2 \sin^2 w} \sin w \, dw \right) \frac{1}{2I_1 - I_2},
\]

\[s_{4,6}(\theta) = - \left( 4 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w \, dw + 4 \cos^3 \theta e^{-\sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \\
+8 \cos^2 \theta \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w \, dw - 4 \cos^2 \theta \int_0^\theta e^{2 \sin^2 w} \sin w \, dw \\
+2 \cos^3 \theta e^{-\sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \sin w \, dw - \cos \theta e^{-\sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \\
- 2 \cos \theta e^{-\sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \sin w \, dw - 2 \int_0^\theta e^{2 \sin^2 w} \sin w \, dw \right) e^{3 \sin^2 \theta} \sin \theta.
\]
Now we have

\[
s_3(\theta) = s_{3,1}(\theta)a_{00}c_{02} + s_{3,2}(\theta)a_{01}c_{00} + s_{3,3}(\theta)a_{00}c_{01} + s_{3,4}(\theta)a_{01}c_{01} + s_{3,5}(\theta)a_{00}c_{20} + s_{3,6}(\theta)c_{00}c_{11} + s_{3,7}(\theta)c_{01}c_{10} + s_{3,8}(\theta)a_{00}a_{02} + s_{3,9}(\theta)a_{00}a_{20} + s_{3,10}(\theta)a_{00}c_{20} + s_{3,11}(\theta)a_{01}c_{10} + s_{3,12}(\theta)a_{11}c_{00} + s_{3,13}(\theta)c_{00}c_{02} + s_{3,14}(\theta)c_{00}c_{20} + s_{3,15}(\theta)a_{01}^2 + s_{3,16}(\theta)c_{10}^2 + s_{3,17}(\theta)c_{01}^2 + s_{3,18}(\theta)a_{20}c_{00},
\]

and \(s_{3,i}(\theta)\) for \(i = 1 \cdots , 18\) satisfying the following expressions:

\[
s_{3,1}(\theta) = -2e^3\sin^2\theta\cos\theta \left[ 4\int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w \, dw \right] \cos^2\theta - 2\left( \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw \right) \cos^2\theta - \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw \right],
\]

\[
s_{3,2}(\theta) = 2e^3\sin^7\theta \sin\theta \left[ 2\left( \int_0^\theta e^{\sin^2 w} \cos w \, dw \right) \cos^2\theta + 4\int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right] \cos^2\theta - 6\left( \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right) \cos^2\theta - 3\int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right],
\]

\[
s_{3,3}(\theta) = -e^2\sin^7\theta \left[ -2I_1 \cos\theta \sin\theta \int_0^\theta e^{\sin^2 w} \sin w \, dw + 4I_1 \int_0^\theta e^{\sin^2 w} \cos^3 w \sin w \, dw \right] - 2I_1 \int_0^\theta e^{\sin^2 w} \sin w \, dw + 4I_1 \int_0^\theta e^{\sin^2 w} \cos^3 w \sin w \, dw \right] \cos^3\theta + 2I_3 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right] \cos^3\theta - 2I_2 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + 2I_2 \int_0^\theta e^{\sin^2 w} \sin w \, dw \right] \cos^3\theta - 4I_3 \int_0^\theta e^{\sin^2 w} \sin w \, dw \right] \cos^3\theta - 2I_2 \int_0^\theta e^{\sin^2 w} \sin w \, dw \right] \cos^3\theta - 4I_3 \int_0^\theta e^{\sin^2 w} \sin w \, dw \right] \cos^3\theta - 2I_2 \int_0^\theta e^{\sin^2 w} \sin w \, dw \right] \cos^3\theta - 4I_3 \int_0^\theta e^{\sin^2 w} \sin w \, dw \right] \cos^3\theta \right]
\]

\[
\frac{1}{2I_1 - I_2},
\]

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\[
\begin{align*}
\hat{s}_{3,4}(\theta) &= 2 e^{3 \sin^2 \theta} \cos \theta \left[ 2 \int_0^\theta e^{\sin^2 w \cos^2 w \sin w} \cos^2 w \right. \\
&\left. - \int_0^\theta e^{\sin^2 w \cos^2 w \sin w} + 2 \int_0^\theta e^{\sin^2 w \cos^4 w \sin w} \right] \cos \theta, \\
\hat{s}_{3,5}(\theta) &= -2 e^{3 \sin^2 \theta} \cos \theta \left[ 2 \left( \int_0^\theta e^{\sin^2 w \cos^2 w \sin w} \cos^2 w \right) \right. \\
&\left. - \int_0^\theta e^{\sin^2 w \cos^2 w \sin w} - 2 \int_0^\theta e^{\sin^2 w \cos^4 w \sin w} \right] \cos \theta, \\
\hat{s}_{3,6}(\theta) &= -2 e^{3 \sin^2 \theta} \sin \theta \left[ \int_0^\theta e^{\sin^2 w \cos w} - 2 \int_0^\theta e^{\sin^2 w \cos^3 w \sin w} \right. \\
&\left. + \int_0^\theta e^{\sin^2 w \cos^3 w \sin w} + 2 \left( \int_0^\theta e^{\sin^2 w \cos w} \cos^2 w \right) \right. \\
&\left. + 2 \left( \int_0^\theta e^{\sin^2 w \cos^3 w \sin w} \cos^2 w \right) \cos^2 \theta \right] \sin \theta, \\
\hat{s}_{3,7}(\theta) &= -e^{2 \sin^2 \theta} \left[ 2 L_1 \cos \theta \sin \theta \int_0^\theta e^{\sin^2 w \sin w} \cos \theta \right. \\
&\left. + 2 L_1 \int_0^\theta e^{\sin^2 w \sin w} \cos \theta \sin \theta \right] \int_0^\theta e^{\sin^2 w \sin w} \cos \theta \\
&+ L_3 \cos \theta \sin \theta \int_0^\theta e^{\sin^2 w \cos w} - L_2 \cos \theta \sin \theta \int_0^\theta e^{\sin^2 w \cos w} \\
&- 2 L_2 \cos \theta \sin \theta \int_0^\theta e^{\sin^2 w \cos w} \cos \theta \sin \theta \left( \int_0^\theta e^{\sin^2 w \sin w} \cos \theta \right) \cos \theta \\
&- 2 L_2 \int_0^\theta e^{\sin^2 w \cos^3 w \sin w} \cos \theta - 2 L_2 \cos \theta \sin \theta \left( \int_0^\theta e^{\sin^2 w \cos^3 w \sin w} \cos \theta \right) \cos \theta \\
&+ 2 L_3 \cos \theta \left( \int_0^\theta e^{\sin^2 w \cos^3 w \sin w} \cos \theta \right) \cos \theta + 2 L_3 \left( \int_0^\theta e^{\sin^2 w \cos^3 w \sin w} \cos \theta \right) \cos \theta \\
&- 4 L_3 \left( \int_0^\theta e^{\sin^2 w \cos^3 w \sin w} \cos \theta \right) \cos \theta \right] \left. \frac{1}{2 I_1 - I_2} \cdot \right]
\end{align*}
\]
\[ s_{3,8}(\theta) = -2 e^{3 \sin^2 \theta} \cos \theta \left[ -3 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - 2 \left( \int_0^\theta e^{\sin^2 w} \cos w \, dw \right) \cos^2 \theta \right. \]
\[ + 6 \left( \int_0^\theta e^{\sin^2 w} \cos^2 w \, dw \right) \cos^2 \theta + 2 \left( \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw \right) \cos^2 \theta \]
\[ + \int_0^\theta e^{\sin^2 w} \cos w \, dw - 4 \left( \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw \right) \cos^2 \theta \left. \right], \]
\[ s_{3,9}(\theta) = 2 e^{3 \sin^2 \theta} \cos \theta \left[ 2 \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw - 4 \left( \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw \right) \cos^2 \theta \right. \]
\[ - \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + 2 \left( \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right) \cos^2 \theta \left. \right], \]
\[ s_{3,10}(\theta) = -2 e^{3 \sin^2 \theta} \cos \theta \left[ 2 \left( \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right) \cos^2 \theta \right. \]
\[ + 2 \left( \int_0^\theta e^{\sin^2 w} \cos w \, dw \right) \cos^2 \theta - \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right. \]
\[ - 4 \left( \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw \right) \cos^2 \theta - \int_0^\theta e^{\sin^2 w} \cos w \, dw + 2 \left( \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw \right) \cos^2 \theta \left. \right], \]
\[ s_{3,11}(\theta) = 2 e^{2 \sin^2 \theta} \sin \theta \cos \theta \left[ -4 \left( \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w \, dw \right) \cos^2 \theta \right. \]
\[ + \int_0^\theta e^{2 \sin^2 w} \sin w \, \cos w \, dw \left. \right], \]
\[ s_{3,12}(\theta) = 2 e^{3 \sin^2 \theta} \sin \theta \left[ 2 \left( \int_0^\theta e^{\sin^2 w} \cos^3 w \sin w \, dw \right) \cos^2 \theta \right. \]
\[ + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw - 2 \left( \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \right) \cos^2 \theta \right. \]
\[ - 4 \left( \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \right) \cos^2 \theta \left. \right], \]
\[ s_{3,13}(\theta) = -2 e^{3 \sin^2 \theta} \sin \theta \left[ 2 \left( \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w \, dw \right) \cos^2 \theta \right. \]
\[ + \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw + 4 \left( \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w \, dw \right) \cos^2 \theta \right. \]
\[ + 2 \left( \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw \right) \cos^2 \theta \left. \right]. \]
\[ s_{3,14}(\theta) = -2 e^3 \sin^7 \theta \sin \theta \left[ 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw + 4 \left( \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \right) \cos^2 \theta + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw + 2 \left( \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw \right) \cos^2 \theta \right], \]

\[ s_{3,15}(\theta) = e^2 \sin^2 \theta \sin \theta \cos \theta \left[ (-4 \int_0^\theta e^2 \sin^2 w \cos^3 w \sin w \, dw) \cos^2 \theta - 4 \int_0^\theta e^2 \sin^2 w \sin w \cos w \, dw + 2 \left( \int_0^\theta e^2 \sin^2 w \sin w \cos w \, dw \right) \cos^2 \theta + 2 \left( \int_0^\theta e^2 \sin^2 w \cos^3 w \sin w \, dw \right) \cos^2 \theta \right], \]

\[ s_{3,16}(\theta) = -e^2 \sin^2 \theta \sin \theta \cos \theta \left[ 4 \left( \int_0^\theta e^2 \sin^2 w \cos^3 w \sin w \, dw \right) \cos^2 \theta + \int_0^\theta e^2 \sin^2 w \sin w \cos w \, dw + 2 \left( \int_0^\theta e^2 \sin^2 w \sin w \cos w \, dw \right) \cos^2 \theta + 2 \left( \int_0^\theta e^2 \sin^2 w \cos^3 w \sin w \, dw \right) \cos^2 \theta \right], \]

\[ s_{3,17}(\theta) = -e^2 \sin^7 \theta \left[ -4 I_1 I_2 \int_0^\theta e^2 \sin^2 w \, dw - 2 I_3^2 \left( \int_0^\theta e^2 \sin^2 w \cos^4 w \, dw \right) \cos^2 \theta - I_2 I_3 \int_0^\theta e^2 \sin^2 w \cos^2 w \sin w \, dw + I_2^3 \int_0^\theta e^2 \sin^2 w \, dw + 2 I_1 I_3 \int_0^\theta e^2 \sin^2 w \cos^2 w \sin w \, dw + 2 I_2 I_3 \int_0^\theta e^2 \sin^2 w \cos^4 w \, dw + 2 I_3^2 \left( \int_0^\theta e^2 \sin^2 w \cos^2 w \, dw \right) \cos^2 \theta - I_2 I_3 \left( \int_0^\theta e^2 \sin^2 w \cos^2 w \, dw \right) \cos^2 \theta + 4 I_1 \int_0^\theta e^2 \sin^2 w \cos^4 w \, dw + 4 I_2 I_3 \left( \int_0^\theta e^2 \sin^2 w \cos^2 w \, dw \right) \cos^4 \theta - 4 I_1 I_3 \left( \int_0^\theta e^2 \sin^2 w \cos^2 w \, dw \right) \cos^4 \theta + 4 I_3^2 \left( \int_0^\theta e^2 \sin^2 w \cos^4 w \, dw \right) \cos^4 \theta \right] \frac{1}{(-I_2 + 2 I_1)^2}, \]

\[ s_{3,18}(\theta) = 2 e^3 \sin^3 \theta \sin \theta \left[ -2 \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw + \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + \left( \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right) \cos^2 \theta - 4 \left( \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw \right) \cos^2 \theta \right]. \]
Now we have

\[
s_2(\theta) = s_{2,1}(\theta) a_{00c30} + s_{2,2}(\theta) a_{20c10} + s_{2,3}(\theta) a_{00c12} + s_{2,4}(\theta) a_{02c10} + s_{2,5}(\theta) a_{12c00} + s_{2,6}(\theta) a_{00a21} + s_{2,7}(\theta) a_{01a20} + s_{2,8}(\theta) a_{01c11} + s_{2,9}(\theta) a_{11c01} + s_{2,10}(\theta) c_{00c21} + s_{2,11}(\theta) c_{01c20} + s_{2,12}(\theta) c_{10c11} + s_{2,13}(\theta) a_{00a03} + s_{2,14}(\theta) a_{01a02} + s_{2,15}(\theta) c_{00c03} + s_{2,16}(\theta) c_{01c02} + s_{2,17}(\theta) a_{00a12} + s_{2,18}(\theta) a_{03c00} + s_{2,19}(\theta) a_{11c10} + s_{2,20}(\theta) a_{01c02} + s_{2,21}(\theta) a_{01c20} + s_{2,22}(\theta) a_{01a11} + s_{2,23}(\theta) a_{06c03} + s_{2,24}(\theta) a_{06c21} + s_{2,25}(\theta) a_{02c01} + s_{2,26}(\theta) a_{20c01} + s_{2,27}(\theta) a_{21c00} + s_{2,28}(\theta) c_{00c12} + s_{2,29}(\theta) c_{00c30} + s_{2,30}(\theta) c_{01c11} + s_{2,31}(\theta) c_{02c10} + s_{2,32}(\theta) c_{10c20},
\]

and \( s_{2,i}(\theta) \) for \( i = 1, \ldots, 32 \) satisfying the following expressions:

\[
s_{2,1}(\theta) = \frac{1}{6} \left( 12 \cos^2 \theta \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos^3 w \cos w dw - 12 \cos^4 \theta \sin \theta \left( \int_0^\theta e^3 \sin^2 w \cos w dw \right) e^{-3 \sin^2 \theta} + 2 \cos^6 \theta - 14 \cos^2 \theta - 6 \cos^2 \theta \sin \theta \left( \int_0^\theta e^3 \sin^2 w \cos w dw \right) e^{-3 \sin^2 \theta} + 24 \cos^4 \theta \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos^3 w \cos w dw - 3 \cos^4 \theta + 7 + 8 \cos^8 \theta \right) e^3 \sin^2 \theta \cos \theta,
\]

\[
s_{2,2}(\theta) = -2 \int_0^\theta e^3 \sin^2 w \cos^3 w dw + 2 \left( \int_0^\theta e^3 \sin^2 w \cos^3 w dw \right) \cos^2 \theta - 4 \left( \int_0^\theta e^3 \sin^2 w \cos^3 w dw \right) \cos^2 \theta \int_0^\theta e^3 \sin^2 w \cos w dw - 9 \cos^4 \theta
\]

\[
s_{2,3}(\theta) = -6 \left( -12 \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos^3 w dw - 9 \cos^4 \theta \right)
\]

\[
s_{2,4}(\theta) = -4 \left( \int_0^\theta e^3 \sin^2 w \cos^3 w dw \right) \cos^2 \theta + 3 \left( \int_0^\theta e^3 \sin^2 w \cos^3 w dw \right) \cos^2 \theta + 6 \left( \int_0^\theta e^3 \sin^2 w \cos^3 w dw \right) \cos^2 \theta - \int_0^\theta e^3 \sin^2 w \cos w dw - 2 \left( \int_0^\theta e^3 \sin^2 w \cos w dw \right) \cos^2 \theta - 2 \int_0^\theta e^3 \sin^2 w \cos^3 w dw \right) e^3 \sin^2 \theta \cos \theta,
\]

\[
s_{2,5}(\theta) = -\frac{1}{3} \left( 6 \cos^4 \theta \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw + 4 e^3 \sin^2 \theta \cos^7 \theta - 18 \left( \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw \right) \cos^2 \theta - 6 e^3 \sin^2 \theta \cos^5 \theta - 9 \left( \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw \right) \cos^2 \theta + 2 e^3 \sin^2 \theta \cos \theta + 3 \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw \right) \cos^2 \theta + 12 \cos^4 \theta \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw + 6 \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw \right) \cos^2 \theta,
\]

\]

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\[ s_{2,6}(\theta) = \frac{1}{6} \left(-12 \cos^2 \theta \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos^3 w \, dw + 1 - 2 \cos^2 \theta \right. \\
-10 \cos^6 \theta - 12 \cos^4 \theta \sin \theta \left( \int_0^\theta e^3 \sin^2 w \cos w \, dw \right) e^{-3 \sin^2 \theta} + 3 \cos^4 \theta \\
+24 \cos^4 \theta \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^3 \sin^2 w \cos^3 w \, dw \\
\left. + 6 \cos^2 \theta \sin \theta \left( \int_0^\theta e^3 \sin^2 w \cos w \, dw \right) e^{-3 \sin^2 \theta} + 8 \cos^8 \theta \right) e^\sin^2 \theta \cos \theta, \\
\]

\[ s_{2,7}(\theta) = \left( 2 \int_0^\theta e^\sin^2 w \cos^5 w \, dw - 4 \left( \int_0^\theta e^\sin^2 w \cos^5 w \, dw \right) \cos^2 \theta \\
- \int_0^\theta e^\sin^2 w \cos^3 w \, dw + 2 \left( \int_0^\theta e^\sin^2 w \cos^3 w \, dw \right) \cos^2 \theta \sin \theta \cos \theta, \\
\]

\[ s_{2,8}(\theta) = \left( -2 \left( \int_0^\theta e^\sin^2 w \cos^3 w \, dw \right) \cos^2 \theta + 2 \left( \int_0^\theta e^\sin^2 w \cos w \, dw \right) \cos^2 \theta - \int_0^\theta e^\sin^2 w \cos^3 w \, dw \\
-4 \left( \int_0^\theta e^\sin^2 w \cos^3 w \, dw \right) \cos^2 \theta - \int_0^\theta e^\sin^2 w \cos^5 w \, dw + 2 \int_0^\theta e^\sin^2 w \cos^5 w \, dw \right) e^\sin^2 \theta \sin \theta \cos \theta, \\
\]

\[ s_{2,9}(\theta) = e^\sin^2 \theta \left( -I_2 \int_0^\theta e^\sin^2 w \cos^2 w \sin w \, dw + 2 I_1 \int_0^\theta e^\sin^2 w \cos^2 w \sin w \, dw \\
-4 I_1 \int_0^\theta e^\sin^2 w \cos^4 w \sin w \, dw - 2 I_3 \cos^2 \theta \int_0^\theta e^\sin^2 w \cos^4 w \sin w \, dw \\
+ I_3 \cos^2 \theta \int_0^\theta e^\sin^2 w \cos^2 w \sin w \, dw + 2 I_2 \int_0^\theta e^\sin^2 w \cos^4 w \sin w \, dw \\
+ 4 I_3 \cos^4 \theta \int_0^\theta e^\sin^2 w \cos^4 w \sin w \, dw + 2 I_2 \int_0^\theta e^\sin^2 w \cos^4 w \sin w \, dw \right) \frac{1}{2 I_4 - I_2}, \\
\]

\[ s_{2,10}(\theta) = \left( \frac{1}{3} \left( \int_0^\theta e^\sin^2 w \sin w \, dw \right) \cos^2 \theta - 2 e^\sin^2 \theta \cos \theta - 3 \left( \int_0^\theta e^\sin^2 w \sin w \, dw \right) \cos^2 \theta - 2 e^\sin^2 \theta \cos^2 \theta \\
-6 \left( \int_0^\theta e^\sin^2 w \sin w \, dw \right) \cos^2 \theta - 12 \left( \int_0^\theta e^\sin^2 w \sin w \, dw \right) \cos^2 \theta - 6 e^\sin^2 \theta \cos^2 \theta \\
+ 8 e^\sin^2 \theta \cos \theta + 12 \cos^4 \theta \left( \int_0^\theta e^\sin^2 w \sin w \, dw + 24 \cos^4 \theta \int_0^\theta e^\sin^2 w \sin w \, dw \right), \\
\]

\[ s_{2,11}(\theta) = e^\sin^2 \theta \left( 2 I_1 \int_0^\theta e^\sin^2 w \cos^2 w \sin w \, dw + 2 I_3 \cos^2 \theta \int_0^\theta e^\sin^2 w \cos^4 w \sin w \, dw \\
- I_2 \int_0^\theta e^\sin^2 w \cos^2 w \sin w \, dw + I_3 \cos^2 \theta \int_0^\theta e^\sin^2 w \cos^2 w \sin w \, dw \\
- 2 I_3 \cos^2 \theta \int_0^\theta e^\sin^2 w \cos^2 w \sin w \, dw - 4 I_3 \cos^4 \theta \int_0^\theta e^\sin^2 w \cos^4 w \sin w \, dw \\
- 2 I_3 \int_0^\theta e^\sin^2 w \cos^2 w \sin w \, dw + 4 I_1 \int_0^\theta e^\sin^2 w \cos^4 w \sin w \, dw \right) \frac{1}{2 I_4 - I_2}. \\
\]
\[ s_{2,12}(\theta) = - \left( \int_0^\theta \sin^2 w \cos w \, dw - 2 \int_0^\theta \sin^2 w \cos^5 w \, dw + \int_0^\theta \cos^2 w \, dw \right) \cos^2 \theta + 2 \left( \int_0^\theta \sin^2 w \cos^3 w \, dw \right) \sin^2 \theta \]
\[ + 4 \left( \int_0^\theta \sin^3 w \cos^5 w \, dw \right) \cos^2 \theta \cos \theta \sin \theta, \]

\[ s_{2,13}(\theta) = - \frac{1}{6} \left( -12 \cos^4 \theta \sin \theta \int_0^\theta \sin^2 w \cos w \, dw - 6 \sin \theta \int_0^\theta \sin^2 w \cos w \, dw \right) \cos \theta, \]

\[ s_{2,14}(\theta) = - \left( -3 \int_0^\theta \sin^2 w \cos^3 w \, dw - 2 \left( \int_0^\theta \sin^2 w \cos w \, dw \right) \cos^2 \theta + 6 \left( \int_0^\theta \sin^2 w \cos^3 w \, dw \right) \cos^2 \theta \right) \cos \theta, \]

\[ s_{2,15}(\theta) = -2 \int_0^\theta \sin^2 w \cos \theta - \frac{1}{3} \int_0^\theta \sin^2 w \cos^2 w \, dw - \int_0^\theta \sin^2 w \cos^2 w \, dw - 4 \int_0^\theta \sin^2 w \cos^2 w \, dw - 3 \cos^2 \theta \cos^2 w \, dw \]
\[ + 2 \int_0^\theta \sin^2 w \cos^2 w \, dw \]
\[ + \frac{10}{3} \int_0^\theta \sin^2 w \cos^2 w \, dw - \frac{4}{3} \cos^2 \theta \cos^2 w \, dw - \frac{1}{3} \cos^2 \theta \cos^2 w \, dw \]

\[ s_{2,16}(\theta) = - \frac{1}{4} \left( -2 I_2 \int_0^\theta \sin^2 w \cos^2 w \, dw + 2 I_3 \cos^2 \theta \int_0^\theta \sin^2 w \cos^2 w \, dw \right) \cos \theta \sin \theta \]
\[ + I_1 \int_0^\theta \sin^2 w \cos^2 w \, dw + 4 I_1 \int_0^\theta \sin^2 w \cos^2 w \, dw - 4 I_3 \cos^4 \theta \int_0^\theta \sin^2 w \cos^2 w \, dw \]
\[ + I_3 \cos^2 \theta \int_0^\theta \sin^2 w \cos^2 w \, dw - I_2 \int_0^\theta \sin^2 w \cos^2 w \, dw - 2 I_4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^2 w \, dw \]
\[ + \frac{1}{2} \int_0^\theta \sin^2 w \cos^2 w \, dw - \frac{1}{2} I_1 - I_2, \]

\[ s_{2,17}(\theta) = \frac{1}{3} \left( -6 \int_0^\theta \sin^2 w \cos^3 w \, dw + 3 \int_0^\theta \sin^2 w \cos^3 w \, dw + 2 \int_0^\theta \sin^2 w \cos^3 w \, dw \right) \cos \theta \sin \theta \]
\[ + 6 \cos^4 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw + 4 \int_0^\theta \sin^2 w \cos^3 w \, dw - 6 \int_0^\theta \sin^2 w \cos^3 w \, dw \]
\[ - 9 \int_0^\theta \sin^2 w \cos^3 w \, dw + (18 \cos^2 \theta - 12 \cos^4 \theta) \int_0^\theta \sin^2 w \cos^3 w \, dw \right) \cos^2 \theta., \]
\[ s_{2.18}(\theta) = -\frac{1}{6} \left( 8 e^3 \sin^2 \theta \cos^8 \theta - 14 e^3 \sin^2 \theta \cos^6 \theta - e^3 \sin^2 \theta + 12 \cos^5 \theta \int_0^\theta e^3 \sin^2 w \sin w \, dw 
+ 24 \cos^5 \theta \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w \, dw + 4 e^3 \sin^2 \theta \cos^3 \theta - 36 \cos^3 \theta \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w \, dw 
- 18 \cos^3 \theta \int_0^\theta e^3 \sin^2 w \sin w \, dw + 6 \cos \theta \int_0^\theta e^3 \sin^2 w \sin w \, dw 
+ 12 \cos \theta \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w + 3 e^3 \sin^2 \theta \cos^4 \theta \right) \sin \theta, \]
\[ s_{2.19}(\theta) = \left( 2 \int_0^\theta e^3 \sin^2 w \cos^2 w \sin w \, dw \right) \cos^2 \theta + \int_0^\theta e^3 \sin^2 w \cos^2 w \sin w \, dw 
- 2 \int_0^\theta e^3 \sin^2 w \cos^4 w \sin w \, dw - 4 \left( \int_0^\theta e^3 \sin^2 w \cos^4 w \sin w \cos \theta \, dw \right) e^2 \sin^2 \theta \cos \theta \sin \theta, \]
\[ s_{2.20}(\theta) = -\left( 4 \int_0^\theta e^3 \sin^2 w \sin^3 w \cos^2 w \, dw \right) \cos^2 \theta - 2 \int_0^\theta e^3 \sin^2 w \sin^3 w \cos^2 w \, dw 
- \int_0^\theta e^3 \sin^2 w \sin^3 w \sin w \, dw + 2 \left( \int_0^\theta e^3 \sin^2 w \sin^3 w \cos \theta \, dw \right) e^2 \sin^2 \theta \cos \theta \sin \theta, \]
\[ s_{2.21}(\theta) = -\left( 2 \int_0^\theta e^3 \sin^2 w \cos^2 w \sin w \, dw \right) \cos^2 \theta + 4 \left( \int_0^\theta e^3 \sin^2 w \cos^4 w \sin w \, dw \right) \cos \theta 
- \int_0^\theta e^3 \sin^2 w \cos^2 w \sin w \, dw - 2 \int_0^\theta e^3 \sin^2 w \cos^4 w \sin w \, dw \right) e^2 \sin^2 \theta \cos \theta \sin \theta, \]
\[ s_{2.22}(\theta) = \left( 2 \int_0^\theta e^3 \sin^2 w \cos^2 w \sin w \, dw \right) \cos^2 \theta - \int_0^\theta e^3 \sin^2 w \cos^2 w \sin w \, dw 
+ 2 \int_0^\theta e^3 \sin^2 w \cos^4 w \sin w \, dw - 4 \left( \int_0^\theta e^3 \sin^2 w \cos^4 w \sin w \cos \theta \, dw \right) e^2 \sin^2 \theta \cos \theta \sin \theta, \]
\[ s_{2.23}(\theta) = 2 e^3 \sin^\theta \cos^\theta \sin \theta + 4 e^3 \sin^2 \theta \cos^8 \theta \sin \theta + 2 \int_0^\theta e^3 \sin^2 w \cos^3 w \, dw + \frac{8}{3} e^3 \sin^2 \theta \cos^4 \theta \sin \theta 
- 2 \cos^4 \theta \int_0^\theta e^3 \sin^2 w \cos \theta \, dw - 2 e^3 \sin^2 \theta \cos^2 \theta \sin \theta + 2 \cos^6 \theta \int_0^\theta e^3 \sin^2 w \cos w \, dw 
- 4 \cos^6 \theta \int_0^\theta e^3 \sin^2 w \cos \theta \, dw - \int_0^\theta e^3 \sin^2 w \cos w \, dw + \cos^4 \theta \int_0^\theta e^3 \sin^2 w \cos w \, dw, \]
\[ s_{2.24}(\theta) = \frac{1}{3} \left( 6 \int_0^\theta e^3 \sin^2 w \cos^3 w \, dw + 12 \cos^4 \theta \int_0^\theta e^3 \sin^2 w \cos w \, dw + 12 \cos^2 \theta \int_0^\theta e^3 \sin^2 w \cos^3 w \, dw 
- 3 \int_0^\theta e^3 \sin^2 w \cos w \, dw + 8 e^3 \sin^2 \theta \cos^6 \theta \sin \theta - 24 \cos^4 \theta \int_0^\theta e^3 \sin^2 w \cos^3 w \, dw 
- 6 \cos^2 \theta \int_0^\theta e^3 \sin^2 w \cos w \, dw - 2 e^3 \sin^2 \theta \cos^2 \theta \sin \theta \right) \cos^2 \theta, \]
\[ s_{2,25}(\theta) = -e^2 \sin^2 \theta \left( 3 I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + 2 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w \, dw \right) \]

\[ + 6 I_1 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - 4 I_1 \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw + 4 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \]

\[ + 2 I_2 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + I_2 \int_0^\theta e^{\sin^2 w} \cos w \, dw - 6 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \]

\[ - 2 I_1 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw - 2 I_1 \int_0^\theta e^{\sin^2 w} \cos w \, dw \]

\[ - 3 I_2 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w \, dw \left( \frac{1}{2I_1 - I_2} \right) \]

\[ s_{2,26}(\theta) = e^2 \sin^2 \theta \left( -2 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + 4 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw \right) \]

\[ - I_2 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - 2 I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw + 2 I_2 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \]

\[ + 2 I_1 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - 4 I_1 \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw + I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \]

\[ \left( \frac{1}{2I_1 - I_2} \right) \]

\[ s_{2,27}(\theta) = \frac{1}{6} \left( -6 \cos^3 \theta \int_0^\theta e^{\sin^2 w} \sin w \, dw + 12 \cos^5 \theta \int_0^\theta e^{\sin^2 w} \sin w \, dw - 3 e^3 \sin^5 w \cos^4 \theta \right) \sin \theta, \]

\[ - 2 e^3 \sin^2 \theta \cos^2 \theta - 2 e^3 \sin^2 \theta \cos^6 \theta - e^3 \sin^5 \theta + 24 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \]

\[ - 12 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw + 8 e^3 \sin^2 \theta \cos^8 \theta \right) \sin \theta, \]

\[ s_{2,28}(\theta) = -\frac{1}{6} \left( 5 e^3 \sin^3 \theta + 24 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \right) \]

\[ - 12 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw - 12 \cos \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \]

\[ - 6 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw - 15 e^3 \sin^7 \theta \cos^4 \theta - 6 \cos \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \]

\[ + 4 e^3 \sin^2 \theta \cos^2 \theta + 8 e^3 \sin^2 \theta \cos^8 \theta + 12 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \]

\[ - 2 e^3 \sin^2 \theta \cos^6 \theta \right) \sin \theta, \]

\[ s_{2,29}(\theta) = \frac{1}{6} \left( -14 e^3 \sin^2 \theta \cos^2 \theta + 10 e^3 \sin^2 \theta \cos^6 \theta + 12 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \right) \]

\[ + 6 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw + 12 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw - 8 e^3 \sin^2 \theta \cos^8 \theta \]

\[ + 3 e^3 \sin^2 \theta \cos^4 \theta - 7 e^3 \sin^2 \theta + 24 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \, dw \right) \sin \theta, \]
\[ s_{2,30}(\theta) = -e^2 \sin^2 \theta \left( -4 I_1 \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw - 2 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w \, dw \right) \]
\[ + 2 I_1 \int_0^\theta e^{\sin^2 w} \cos w \, dw - 2 I_2 \int_0^\theta e^{\sin^2 w} \cos^3 \, dw - 2 I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w \, dw \]
\[ + 2 I_1 \int_0^\theta e^{\sin^2 w} \cos^3 \, dw + I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w \, dw \]
\[ - I_2 \int_0^\theta e^{\sin^2 w} \cos w \, dw + 4 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 \, dw \right) \frac{1}{2 I_1 - I_2}, \]

\[ s_{2,31}(\theta) = - \left( 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w \, dw + \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw \right) \frac{1}{2} \sin \theta \cos \theta \sin \theta, \]

\[ s_{2,32}(\theta) = - \left( 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \, dw + 4 \left( \int_0^\theta e^{\sin^2 w} \cos^4 w \, dw \right) \frac{1}{2} \sin \theta \cos \theta \sin \theta \right) \]
\[ + \int_0^\theta e^{\sin^2 w} \cos^2 w \, dw \left( \frac{1}{2} \sin \theta \cos \theta \sin \theta \right) \]

We also have

\[ s_1(\theta) = s_{1,1}(\theta) a_{01} c_{10} + s_{1,2}(\theta) a_{01} c_{03} + s_{1,3}(\theta) a_{01} c_{05} + s_{1,4}(\theta) a_{01} c_{30} + s_{1,5}(\theta) c_{10} c_{21} + s_{1,6}(\theta) a_{01} c_{12} + s_{1,7}(\theta) c_{01} c_{12} + s_{1,8}(\theta) a_{01} c_{11} + s_{1,9}(\theta) c_{03} c_{10} + s_{1,10}(\theta) a_{12} c_{10} + s_{1,11}(\theta) c_{01} c_{21} + s_{1,12}(\theta) c_{10} c_{12} + s_{1,13}(\theta) a_{21} c_{10} + s_{1,14}(\theta) a_{01} c_{21} + s_{1,15}(\theta) c_{10} c_{30} + s_{1,16}(\theta) a_{01} c_{21} + s_{1,17}(\theta) a_{01} c_{03} + s_{1,18}(\theta) a_{01} c_{12} + s_{1,19}(\theta) a_{12} c_{03} + s_{1,20}(\theta) c_{01} c_{03} + s_{1,21}(\theta) a_{21} c_{01}, \]

and \( s_{1,i}(\theta) \) for \( i = 1 \ldots 21 \) are given in the following expressions:

\[ s_{1,1}(\theta) = - \frac{1}{12} \left( 24 \int_0^\theta e^{\sin^2 w} \sin w \, dw \right) \cos^4 \theta - e^2 \sin^2 \theta + 12 \int_0^\theta e^{\sin^2 w} \sin w \, dw \]
\[ + 4 e^2 \sin^2 \theta \cos^2 \theta - 36 \left( \int_0^\theta e^{\sin^2 w} \sin w \, dw \right) \cos^2 \theta - 72 \left( \int_0^\theta e^{\sin^2 w} \cos^3 w \sin w \, dw \right) \cos^2 \theta \]
\[ + 24 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + 8 e^2 \sin^2 \theta \cos^8 \theta + 48 \left( \int_0^\theta e^{\sin^2 w} \cos^3 w \sin w \, dw \right) \cos^4 \theta \]
\[ + 3 e^2 \sin^2 \theta \cos^4 \theta - 14 e^2 \sin^2 \theta \cos^4 \theta \right) \cos \theta \sin \theta, \]
\[ s_{1,2}(\theta) = \frac{1}{3} e^2 \sin^2 \theta \cos^6 \theta + 2 \cos^6 \theta \int_0^\theta e^2 \sin^2 w \sin w \cos wdw - \frac{2}{3} e^2 \sin^2 \theta \cos^{10} \theta \\
-2 \left( \int_0^\theta e^2 \sin^3 w \cos^3 w \sin wdw \right) \cos^4 \theta - \int_0^\theta e^2 \sin^2 w \sin w \cos wdw \\
+ \left( \int_0^\theta e^2 \sin^2 w \sin w \cos wdw \right) \cos^4 \theta + \frac{7}{3} e^2 \sin^2 \theta \cos^4 \theta - 4 \cos^6 \theta \int_0^\theta e^2 \sin^2 w \cos^3 w \sin wdw \\
- e^2 \sin^2 \theta \cos^2 \theta - \frac{1}{3} e^2 \sin^2 \theta \cos^8 \theta + 2 \int_0^\theta e^2 \sin^2 w \cos^3 w \sin wdw, \]

\[ s_{1,3}(\theta) = e^2 \sin^2 \theta (\cos \theta - 1) (\cos \theta + 1) (-8 \cos^8 \theta I_3 + 14 \cos^6 \theta I_3 - 7 \cos^4 \theta I_3 + 8 \cos^4 \theta I_1 \\
-4 \cos^4 \theta I_2 + 24 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \cos^2 wdw \\
+ 48 I_1 \cos^4 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \sin \cos wdw - 48 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \cos^4 wdw \\
- 24 I_2 \cos^4 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \cos^2 wdw + \cos^2 \theta I_3 - 10 \cos^2 \theta I_1 + 5 \cos^2 \theta I_2 \\
+ 12 I_2 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \cos^2 wdw - 12 I_3 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \cos^2 wdw \\
- 24 I_1 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \cos^2 wdw \\
+ 24 I_3 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \cos^4 wdw - I_2 + 2 I_1) \frac{1}{12 (\cos^2 \theta - 1)}, \]

\[ s_{1,4}(\theta) = -\frac{1}{12} \left( 24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \sin w \cos wdw - 7 + 3 \cos^4 \theta \\
- 8 \cos^8 \theta + 14 \cos^6 \theta - 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^3 w \cos^3 w \sin wdw \\
- 2 \cos^6 \theta + 12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \sin w \cos wdw \\
- 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^2 \sin^2 w \cos^3 w \sin wdw \right) e^2 \sin^2 \theta \sin \theta \cos \theta, \]

\[ s_{1,5}(\theta) = \frac{1}{3} \left[ - e^2 \sin^2 \theta \cos^2 \theta + 12 \left( \int_0^\theta e^2 \sin^2 w \sin w \cos wdw \right) \cos^4 \theta \\
+ 24 \left( \int_0^\theta e^2 \sin^2 w \cos^3 w \sin wdw \right) \cos^4 \theta - 3 \int_0^\theta e^2 \sin^2 w \sin w \cos wdw \\
- 6 \left( \int_0^\theta e^2 \sin^2 w \sin w \cos wdw \right) \cos^2 \theta - 6 \int_0^\theta e^2 \sin^2 w \cos^3 w \sin wdw \\
- 12 \left( \int_0^\theta e^2 \sin^2 w \cos^3 w \sin wdw \right) \cos^2 \theta + 4 e^2 \sin^2 \theta \cos^8 \theta \\
- 3 e^2 \sin^2 \theta \cos^4 \theta \right] \cos^2 \theta, \]
\[ s_{1,6}(\theta) = -\frac{1}{3} \left( 12 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w \cos \theta \cos^2 \theta + 9 \left( \int_0^\theta e^{2 \sin^2 w} \sin w \cos \theta \cos^2 \theta + 3 \left( \int_0^\theta e^{2 \sin^2 w} \sin w \cos \theta \cos^2 \theta \right) + 6 \left( \int_0^\theta e^{2 \sin^2 w} \sin w \cos \theta \cos^2 \theta \right) \cos^2 \theta - 6 \left( \int_0^\theta e^{2 \sin^2 w} \sin w \cos \theta \cos^2 \theta \right) \cos^2 \theta \right) \right) + 6 \left( \int_0^\theta e^{2 \sin^2 w} \sin w \cos \theta \cos^2 \theta \right) \cos^2 \theta + 24 \int e^{2 \sin^2 w} \sin w \cos \theta \cos^2 \theta \right) \cos^2 \theta, \]

\[ s_{1,7}(\theta) = -\frac{1}{12} e^{2 \sin^2 \theta} \left( 12 I_2 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 12 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w - 24 I_1 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w - 24 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w \right) \cos^2 \theta, \]

\[ s_{1,8}(\theta) = \frac{1}{12} e^{2 \sin^2 \theta} \left( -4 \cos^6 \theta I_2 - 48 I_3 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w - \cos^3 \theta I_1 + 14 \cos^4 \theta I_2 - 24 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w \right) \cos^2 \theta, \]
\[
\begin{align*}
\mathcal{I}_1 &= - \left( \int_0^\theta e^2 \sin^2 w \sin w \cos w \, dw \right) \cos^4 \theta + \int_0^\theta e^2 \sin^2 w \sin w \cos w \, dw \\
&- \int_0^\theta e^2 \sin^2 \theta \cos^8 \theta - 2 \cos^6 \theta \int_0^\theta e^2 \sin^2 w \sin w \cos w \, dw \right) \cos^4 \theta \\
&+ \int_0^\theta e^2 \sin^2 w \sin w \cos^2 w \, dw \\
&- 4 \cos^6 \theta \int_0^\theta e^2 \sin^2 w \sin w \cos w \, dw - 2 \left( \int_0^\theta e^2 \sin^2 \theta \cos^4 \theta \right) \cos^4 \theta \\
&\int_0^\theta e^2 \sin^2 w \cos^2 w \, dw \\
&+ 5/3 e^2 \sin^2 \theta \cos^4 \theta + 2 \int_0^\theta e^2 \sin^2 w \cos^3 w \sin w \, dw \\
&- 4 \cos^6 \theta \int_0^\theta e^2 \sin^2 w \sin w \cos w \, dw - 2 \left( \int_0^\theta e^2 \sin^2 \theta \cos^4 \theta \right) \cos^4 \theta \\
&\int_0^\theta e^2 \sin^2 w \cos^2 w \, dw \\
&+ e^2 \sin^2 \theta \cos^2 \theta - 2 \left( \int_0^\theta e^2 \sin^2 w \cos^3 w \sin w \, dw \right) \cos^4 \theta, \\
\end{align*}
\]
\[ s_{1,12}(\theta) = -\frac{1}{12} \left( -12 e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w \sin w \cos w dw - 15 \cos^4 \theta - 2 \cos^6 \theta \\
+ 24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w \sin w \cos w dw - 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \sin w dw \\
- 24 e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \sin w + 8 \cos^8 \theta + 4 \cos^2 \theta \\
+ 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \sin w - 12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w dw + 5 \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \]

\[ s_{1,13}(\theta) = \frac{1}{12} \left( -12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w dw \\
+ 24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w dw + 8 \cos^8 \theta - 3 \cos^3 \theta \\
- 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \sin w + 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \sin w - 1 - 2 \cos^2 \theta - 2 \cos^6 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \]

\[ s_{1,14}(\theta) = \frac{1}{12} \left( 1 + 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \sin w + 8 \cos^8 \theta \\
- 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \sin w - 10 \cos^6 \theta \\
- 24 \cos \theta^4 e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w dw - 2 \cos^2 \theta \\
+ 12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w dw + 3 \cos^4 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \]

\[ s_{1,15}(\theta) = \frac{1}{12} \left( 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \sin w + 10 \cos^6 \theta \\
+ 24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w dw \\
+ 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \sin w + 3 \cos^4 \theta + 12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w dw + 8 \cos^8 \theta \\
- 14 \cos^2 \theta - 7 \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta. \]
\[ s_{1.16}(\theta) = -\frac{1}{3} \left( -6 \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \, dw - 12 \left( \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \, dw \right) \right) \cos^4 \theta \\
\quad + e^{2 \sin^2 \theta} \cos^2 \theta + 4 e^{2 \sin^2 \theta} \cos^8 \theta - 12 \left( \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \, dw \right) \cos^2 \theta \\
\quad + 24 \left( \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \, dw \right) \cos^4 \theta - 4 e^{2 \sin^2 \theta} \cos^6 \theta \\
\quad + 6 \left( \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w \, dw \right) \cos^2 \theta - e^{2 \sin^2 \theta} \cos^4 \theta \\
\quad + 3 \left( \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w \, dw \right) \cos^2 \theta, \]

\[ s_{1.17}(\theta) = -\frac{1}{12} \left( 48 \cos^4 \theta e^{2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \, dw \\
\quad - 12 e^{2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w \, dw - 24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w \, dw \\
\quad + 36 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w \, dw + 8 \cos^8 \theta + 1 + 21 \cos^4 \theta \\
\quad + 24 e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \, dw - 72 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \, dw - 8 \cos^2 \theta \\
\quad - 22 \cos^6 \theta) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \]

\[ s_{1.18}(\theta) = -\frac{1}{12} \left( -24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w \, dw \\
\quad - 24 e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \, dw + 16 \cos^2 \theta - 5 \\
\quad + 12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w \, dw \\
\quad - 9 \cos^4 \theta - 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \, dw - 10 \cos^6 \theta \\
\quad + 12 e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \sin w \cos w \, dw + 8 \cos^8 \theta \\
\quad + 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_{0}^{\theta} e^{2 \sin^2 w} \cos^3 w \sin w \, dw \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta. \]
\begin{align*}
\text{s}_{1.19}(\theta) &= -\frac{1}{3} e^{2 \sin^2 \theta} \cos^2 \theta \left( 2 \cos \theta \sin \theta I_1 - 18 I_1 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw \right) \\
&\quad - \cos \theta \sin \theta I_2 - 3 \cos^5 \theta \sin \theta I_3 + \cos^3 \theta \sin \theta I_2 \\
&\quad + 9 I_2 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw + 18 I_3 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 \, dw \\
&\quad + \cos^3 \theta \sin \theta I_3 - 6 I_2 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw \\
&\quad + 6 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 \, dw + 6 I_1 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw \\
&\quad + 2 \cos \theta^7 \sin \theta I_3 - 3 I_2 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw - 9 I_3 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 \, dw \\
&\quad - 12 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 \, dw + 12 I_1 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw \\
&\quad - 2 \cos^3 \theta \sin \theta I_1 - 6 I_3 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 \, dw + 3 I_3 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 \, dw \right) \frac{1}{2 I_1 - I_2},
\end{align*}

\begin{align*}
\text{s}_{1.20}(\theta) &= -\frac{1}{3} e^{2 \sin^2 \theta} \left( -6 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 \, dw \\
&\quad + 2 \cos^3 \theta \sin \theta I_2 + 12 I_1 \cos^6 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw \\
&\quad - 6 I_2 \cos^6 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw - 3 I_2 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw \\
&\quad - 3 \cos^3 \theta \sin \theta I_3 + 12 I_3 \cos^6 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 \, dw \\
&\quad + 6 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 \, dw - 6 I_1 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw + \cos^5 \theta \sin \theta I_2 \\
&\quad + 4 \cos^5 \theta \sin \theta I_3 + 3 \cos \theta \sin \theta I_2 + 3 \cos^7 \theta \sin \theta I_3 \\
&\quad - 3 I_3 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 \, dw + 3 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 \, dw \\
&\quad + 3 I_2 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \, dw - 6 \cos \theta \sin \theta I_1 - 2 \cos^5 \theta \sin \theta I_1 \\
&\quad + 6 I_3 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 \, dw - 4 \cos^3 \theta \sin \theta I_1 + 2 I_3 \cos^9 \theta \sin \theta \\
&\quad + 6 I_3 \cos^9 \theta \sin \theta \int_0^\theta e^{2 \sin^2 w} \cos^2 \, dw \right) \frac{1}{2 I_1 - I_2},
\end{align*}
\[ s_{1,21}(\theta) = \frac{1}{12} e^{2 \sin \theta^2} \left( 2 \cos^3 \theta I_3 - 48 I_3 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w \cos^4 w} dw \right) \]

\[ -3 \cos^6 \theta I_3 - 4 \cos^6 \theta I_2 + I_2 - 12 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w \cos^2 w} dw \]

\[ -6 \cos^4 \theta I_1 - 8 I_3 \cos^{10} \theta - 2 I_2 + 24 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w \cos^4 w} dw \]

\[ -24 I_2 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w \cos^2 w} dw + 12 I_2 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w \cos^4 w} dw \]

\[ -24 I_1 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w \cos^2 w} dw + 10 \cos^8 \theta I_3 \]

\[ + 8 \cos^6 \theta I_1 - \cos^2 \theta I_3 + 24 I_4 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w \cos^2 w} dw \]

\[ + 3 \cos^4 \theta I_2 + 48 I_4 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w \cos^2 w} dw \] \[ \cdot \frac{1}{2 I_1 - I_2}. \]

Now we present the expression of \( s_0(\theta) \).

\[ s_0(\theta) = s_{0,1}(\theta) c_{12} c_{11} + s_{0,2}(\theta) a_{20} c_{03} + s_{0,3}(\theta) a_{20} c_{03} + s_{0,4}(\theta) a_{21} c_{02} + s_{0,5}(\theta) a_{21} c_{02} + s_{0,6}(\theta) a_{21} c_{20} + s_{0,7}(\theta) a_{02} c_{30} + s_{0,8}(\theta) a_{02} c_{03} + s_{0,9}(\theta) a_{02} c_{03} + s_{0,10}(\theta) a_{02} c_{21} + s_{0,11}(\theta) a_{02} c_{11} + s_{0,12}(\theta) a_{03} c_{02} + s_{0,13}(\theta) a_{03} c_{20} + s_{0,14}(\theta) a_{11} c_{12} + s_{0,15}(\theta) a_{11} c_{12} + s_{0,16}(\theta) a_{11} c_{30} + s_{0,17}(\theta) a_{12} c_{20} + s_{0,18}(\theta) a_{11} c_{12} + s_{0,19}(\theta) a_{03} c_{20} + s_{0,20}(\theta) a_{02} c_{21} + s_{0,21}(\theta) a_{11} c_{21} + s_{0,22}(\theta) c_{20} c_{30} + s_{0,23}(\theta) c_{11} c_{03} + s_{0,24}(\theta) c_{02} c_{03} + s_{0,25}(\theta) c_{11} c_{21} + s_{0,26}(\theta) c_{11} c_{21} + s_{0,27}(\theta) c_{02} c_{21} + s_{0,28}(\theta) a_{02} c_{12} + s_{0,29}(\theta) a_{02} c_{03} + s_{0,30}(\theta) a_{02} c_{03} + s_{0,31}(\theta) a_{02} c_{21} + s_{0,32}(\theta) a_{02} c_{21} + s_{0,33}(\theta) c_{20} c_{03} + s_{0,34}(\theta) c_{02} c_{21} + s_{0,35}(\theta) c_{03} c_{20} + s_{0,36}(\theta) c_{11} c_{12} + s_{0,37}(\theta) a_{20} c_{30} + s_{0,38}(\theta) a_{12} c_{20} + s_{0,39}(\theta) a_{20} c_{12} + s_{0,40}(\theta) a_{11} c_{03} + s_{0,41}(\theta) a_{12} c_{02} + s_{0,42}(\theta) a_{03} c_{11}. \]

and \( s_{0,i}(\theta) \) for \( i = 1 \cdots 42 \) are the following:

\[ s_{0,1}(\theta) = -\cos^2 \theta \left( 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w \cos w} dw + \int_0^\theta e^{\sin^2 w \cos w} dw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w \cos w} dw \right) \]

\[ -4 \cos^4 \theta \int_0^\theta e^{\sin^2 w \cos^5 w} dw - 2 \int_0^\theta e^{\sin^2 w \cos^5 w} dw + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w \cos^5 w} dw \]

\[ + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w \cos^3 w} dw + \int_0^\theta e^{\sin^2 w \cos^3 w} dw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w \cos^3 w} dw \), \]

\[ s_{0,2}(\theta) = -4 \cos^6 \theta \int_0^\theta e^{\sin^2 w \cos^5 w} dw + 2 \int_0^\theta e^{\sin^2 w \cos^5 w} dw - 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w \cos^5 w} dw \]

\[ + 2 \cos^6 \theta \int_0^\theta e^{\sin^2 w \cos^3 w} dw - \int_0^\theta e^{\sin^2 w \cos^3 w} dw + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w \cos^3 w} dw \), \]

\[ s_{0,3}(\theta) = \cos^2 \theta \left( 2 \int_0^\theta e^{\sin^2 w \cos^5 w} dw + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w \cos^5 w} dw - 8 \cos^4 \theta \int_0^\theta e^{\sin^2 w \cos^5 w} dw \right) \]

\[ - \int_0^\theta e^{\sin^2 w \cos^3 w} dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w \cos^3 w} dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w \cos^3 w} dw \), \]

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\[ s_{0,4}(\theta) = \cos^3 \theta \sin \theta \left( 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos w \, dw - \int_0^\theta \sin^2 w \cos \theta \, dw - 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw \right) + 2 \int_0^\theta \sin^2 w \cos^5 w \, dw + 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw - \int_0^\theta \sin^2 w \cos^3 w \, dw, \]

\[ s_{0,5}(\theta) = \cos^3 \theta \sin \theta \left( 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w dw - \int_0^\theta \sin^2 w \cos^3 w dw \right) + 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^2 w \, dw - 2 \int_0^\theta \sin^2 w \cos^2 w \, dw, \]

\[ s_{0,6}(\theta) = \cos^3 \theta \sin \theta \left( 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^4 w \, dw - \int_0^\theta \sin^2 w \cos^4 w \, dw \right) + 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^4 w \, dw - 2 \int_0^\theta \sin^2 w \cos^4 w \, dw, \]

\[ s_{0,7}(\theta) = \cos^3 \theta \sin \theta \left( \int_0^\theta \sin^2 w \cos^3 w \, dw + 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw \right) + 2 \int_0^\theta \sin^2 w \cos^3 w \, dw + 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw, \]

\[ s_{0,8}(\theta) = - \cos^2 \theta \left( 6 \cos^4 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw + 3 \int_0^\theta \sin^2 w \cos^3 w \, dw - 9 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw \right) - 4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw - 2 \int_0^\theta \sin^2 w \cos^5 w \, dw + 6 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw - 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw - 3 \int_0^\theta \sin^2 w \cos^5 w \, dw, \]

\[ s_{0,9}(\theta) = - 6 \cos^6 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw + 3 \int_0^\theta \sin^2 w \cos^3 w \, dw - 3 \cos^4 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw + 4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw - 2 \int_0^\theta \sin^2 w \cos^5 w \, dw + 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw + 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw, \]

\[ s_{0,10}(\theta) = - \cos^2 \theta \left( -3 \int_0^\theta \sin^2 w \cos^3 w \, dw - 6 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw + 12 \cos^4 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw \right) + 2 \int_0^\theta \sin^2 w \cos^5 w \, dw + 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw - 8 \cos^4 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw + 4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw - 2 \int_0^\theta \sin^2 w \cos^5 w \, dw + 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw - 4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw, \]

\[ s_{0,11}(\theta) = \cos \theta \sin \theta \left( 2 \cos^4 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w \, dw + \int_0^\theta \sin^2 w \cos^2 w \sin w \, dw \right) - 3 \cos^2 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w \, dw - 4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w \, dw + 2 \int_0^\theta \sin^2 w \cos^4 w \, dw + 6 \cos^2 \theta \int_0^\theta \sin^2 w \cos^4 w \, dw, \]
\[ s_{0,12}(\theta) = - \cos \theta \sin \theta \left( 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw + \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw \right) \]
\[ - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w \, dw \]
\[ + 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w \, dw - 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w \, dw \right), \]
\[ s_{0,13}(\theta) = - \cos \theta \sin \theta \left( 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw \right) \]
\[ - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \sin w \, dw \]
\[ + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw - 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \right), \]
\[ s_{0,14}(\theta) = - \cos^3 \theta \sin \theta \left( 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw \right) \]
\[ - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \right), \]
\[ s_{0,15}(\theta) = \cos \theta \sin \theta \left( - \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw \right) \]
\[ + 2 \cos^3 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \sin w \, dw \]
\[ + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \right), \]
\[ s_{0,16}(\theta) = - \cos^3 \theta \sin \theta \left( \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw \right) \]
\[ - 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \right), \]
\[ s_{0,17}(\theta) = \cos^2 \theta \left( - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw \right) \]
\[ + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \]
\[ - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right), \]
\[ s_{0,18}(\theta) = \cos^2 \theta \left( 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw \right)
\[ + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw \]
\[ - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw - 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \]
\[ + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \right), \]

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\[ s_{0,19}(\theta) = \cos \theta \sin \theta \left( -4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right) , \]

\[ s_{0,20}(\theta) = \cos^3 \theta \sin \theta \left( 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - 3 \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w \, dw + \int_0^\theta e^{\sin^2 w} \cos w \, dw \right) , \]

\[ s_{0,21}(\theta) = \cos^2 \theta \left( - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw + 4 \cos^3 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - 8 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \right) , \]

\[ s_{0,22}(\theta) = \cos^3 \theta \sin \theta \left( \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w \, dw + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w \, dw \right) , \]

\[ s_{0,23}(\theta) = -2 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos w \, dw + \int_0^\theta e^{\sin^2 w} \cos w \, dw - \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w \, dw + 4 \cos^3 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw , \]

\[ s_{0,24}(\theta) = -2 \left( \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw \right) \cos^6 \theta + \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw - \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw - 4 \left( \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw \right) \cos^6 \theta + 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw - 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \, dw , \]

\[ s_{0,25}(\theta) = -\cos^2 \theta \left( - \int_0^\theta e^{\sin^2 w} \cos w \, dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w \, dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w \, dw + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw - 8 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w \, dw - \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w \, dw \right) . \]
\[ s_{0,20}(\theta) = -\cos \theta \sin \left( -\cos^2 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w \,dw - \int_0^\theta \sin^2 w \cos^2 w \sin w \,dw \\
+ 2 \cos^4 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w \,dw - 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^4 w \sin w \,dw - 2 \int_0^\theta \sin^2 w \cos^4 w \sin w \,dw \\
+ 4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^4 w \sin w \,dw \right), \]

\[ s_{0,27}(\theta) = -\cos \theta \sin \left( -\cos^2 \theta \int_0^\theta \sin^2 w \sin^3 w \,dw - \int_0^\theta \sin^3 w \sin^3 w \,dw \\
+ 2 \cos^4 \theta \int_0^\theta \sin^2 w \sin^3 w \,dw - 2 \cos^2 \theta \int_0^\theta \sin^2 w \sin^2 w \cos^2 w \,dw - 2 \int_0^\theta \sin^2 w \sin^3 w \cos^2 w \,dw \\
+ 4 \cos^4 \theta \int_0^\theta \sin^2 w \sin^3 w \cos^2 w \,dw \right), \]

\[ s_{0,28}(\theta) = -\cos \theta \sin \left( -3 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \,dw - 3 \int_0^\theta \sin^2 w \cos^3 w \,dw \\
+ 6 \cos^4 \theta \int_0^\theta \sin^2 w \cos^3 w \,dw + 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \,dw \\
+ 2 \int_0^\theta \sin^2 w \cos^5 w \,dw - 4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^5 w \,dw + \cos^2 \theta \int_0^\theta \sin^2 w \cos w \,dw \\
+ \int_0^\theta \sin^2 w \cos w \,dw - 2 \cos^4 \theta \int_0^\theta \sin^2 w \cos w \,dw \right), \]

\[ s_{0,29}(\theta) = \cos^3 \theta \sin \left( 3 \int_0^\theta \sin^2 w \cos^3 w \,dw + 6 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \,dw - 2 \int_0^\theta \sin^2 w \cos^5 w \,dw \\
- 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \,dw - \int_0^\theta \sin^2 w \cos w \,dw - 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos w \,dw \right), \]

\[ s_{0,30}(\theta) = -\cos \theta \sin \left( 6 \cos^4 \theta \int_0^\theta \sin^2 w \cos^3 w \,dw + 3 \int_0^\theta \sin^2 w \cos^3 w \,dw \\
- 9 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \,dw - 4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^5 w \,dw - 2 \int_0^\theta \sin^2 w \cos^5 w \,dw \\
+ 6 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \,dw - 2 \cos^4 \theta \int_0^\theta \sin^2 w \cos w \,dw - \int_0^\theta \sin^2 w \cos w \,dw \\
+ 3 \cos^2 \theta \int_0^\theta \sin^2 w \cos w \,dw \right), \]

\[ s_{0,31}(\theta) = -\cos^3 \theta \sin \left( -4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \,dw + 2 \int_0^\theta \sin^2 w \cos^5 w \,dw \\
+ 2 \cos^3 \theta \int_0^\theta \sin^2 w \cos^3 w \,dw - \int_0^\theta \sin^2 w \cos^3 w \,dw \right), \]

\[ s_{0,32}(\theta) = \cos^3 \theta \sin \left( \int_0^\theta \sin^2 w \cos w \,dw + 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos w \,dw - 2 \int_0^\theta \sin^2 w \cos^5 w \,dw \\
- 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \,dw + \int_0^\theta \sin^2 w \cos^3 w \,dw + 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \,dw \right). \]
$s_{0,33}(\theta) = -\cos^2 \theta \left( -\int_0^\theta \cos^2 w \sin w \, dw - 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w \, dw \\
+ 4 \cos^3 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w \, dw - 2 \int_0^\theta \sin^2 w \cos^4 w \sin w \, dw \\
- 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^4 w \sin w + 8 \cos^4 \theta \int_0^\theta \sin^2 w \cos^4 w \sin w \, dw \right),$

$s_{0,34}(\theta) = -\cos^2 \theta \left( -\int_0^\theta \sin^3 w \, dw - 2 \cos^2 \theta \int_0^\theta \sin^3 w \, dw \\
+ 4 \cos^3 \theta \int_0^\theta \sin^3 w \cos^2 w \sin w \, dw - 2 \int_0^\theta \sin^3 w \cos^4 w \sin w \, dw \\
- 4 \cos^2 \theta \int_0^\theta \sin^3 w \cos^2 w \sin w + 8 \cos^4 \theta \int_0^\theta \sin^3 w \cos^2 w \sin w \, dw \right),$

$s_{0,35}(\theta) = -2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w \, dw + \cos^4 \theta \int_0^\theta \sin^2 w \cos^4 w \sin w \, dw \\
- \cos^4 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w - 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^4 w \sin w \, dw \\
+ 2 \int_0^\theta \sin^2 w \cos^4 w \sin w \, dw - 2 \cos^4 \theta \int_0^\theta \sin^2 w \cos^4 w \sin w \, dw,$

$s_{0,36}(\theta) = -\cos \theta \sin \theta \left( -\cos^2 \theta \int_0^\theta \sin^2 w \cos w \, dw - \int_0^\theta \sin^2 w \cos w \, dw \\
+ 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos w \, dw + 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw + 2 \int_0^\theta \sin^2 w \cos^5 w \, dw \\
- 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw - \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw - \int_0^\theta \sin^2 w \cos^3 w \, dw \\
+ 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw \right),$

$s_{0,37}(\theta) = -\cos^3 \theta \sin \theta \left( -2 \int_0^\theta \sin^2 w \cos^5 w \, dw - 4 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw \\
+ \int_0^\theta \sin^2 w \cos^3 w \, dw + 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw \right),$

$s_{0,38}(\theta) = -\cos^2 \theta \left( 2 \cos^3 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w \, dw + \int_0^\theta \sin^2 w \cos^2 w \sin w \, dw \\
- 3 \cos^2 \theta \int_0^\theta \sin^2 w \cos^2 w \sin w \, dw + 4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^4 w \sin w \, dw \\
+ 2 \int_0^\theta \sin^2 w \cos^4 w \sin w \, dw - 6 \cos^2 \theta \int_0^\theta \sin^2 w \cos^4 w \sin w \, dw \right),$

$s_{0,39}(\theta) = \cos \theta \sin \theta \left( 2 \cos^2 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw + 2 \int_0^\theta \sin^2 w \cos^5 w \, dw - 4 \cos^4 \theta \int_0^\theta \sin^2 w \cos^5 w \, dw \\
- \cos^2 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw - \int_0^\theta \sin^2 w \cos^3 w \, dw + 2 \cos^4 \theta \int_0^\theta \sin^2 w \cos^3 w \, dw \right).$
\[
\begin{align*}
s_{0,40}(\theta) &= 2 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin wdw - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin wdw + \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin wdw \nonumber \\
&\quad - 4 \cos^3 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin wdw + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin wdw - 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin wdw, \\
s_{0,41}(\theta) &= - \cos^2 \theta \left( 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 wdw + \int_0^\theta e^{\sin^2 w} \sin^3 wdw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 wdw 
\right) + (4 \cos^4 \theta + 2 - 6 \cos^2 \theta) \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw, \\
s_{0,42}(\theta) &= - \cos \theta \sin \theta \left( 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos wdw + \int_0^\theta e^{\sin^2 w} \cos wdw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos wdw 
\right) + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 wdw + \int_0^\theta e^{\sin^2 w} \cos^3 wdw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 wdw \right). \\
\end{align*}
\]

Now we have
\[
\tilde{s}_1(\theta) = \tilde{s}_{1,1}(\theta)c_{03}c_{12} + \tilde{s}_{1,2}(\theta)c_{03}c_{30} + \tilde{s}_{1,3}(\theta)a_{03}c_{03} + \tilde{s}_{1,4}(\theta)a_{21}c_{03} + \tilde{s}_{1,5}(\theta)c_{03}c_{12} + \tilde{s}_{1,6}(\theta)c_{12}a_{21} + \tilde{s}_{1,7}(\theta)c_{03}a_{21} + \tilde{s}_{1,8}(\theta)a_{03}a_{21} + \tilde{s}_{1,9}(\theta)a_{03}c_{30} + \tilde{s}_{1,10}(\theta)a_{03}c_{12} + \tilde{s}_{1,11}(\theta)a_{12}c_{03} + \tilde{s}_{1,12}(\theta)a_{12}c_{21} + \tilde{s}_{1,13}(\theta)c_{03}c_{21} + \tilde{s}_{1,14}(\theta)a_{21}a_{12} + \tilde{s}_{1,15}(\theta)a_{21}a_{12} + \tilde{s}_{1,16}(\theta)c_{12}a_{12} + \tilde{s}_{1,17}(\theta)c_{03}a_{21} + \tilde{s}_{1,18}(\theta)c_{03}a_{12} + \tilde{s}_{1,19}(\theta)c_{03}c_{21} + \tilde{s}_{1,20}(\theta)c_{03}a_{21} + \tilde{s}_{1,21}(\theta)c_{12}c_{21} + \tilde{s}_{1,22}(\theta)c_{12}c_{21} + \tilde{s}_{1,23}(\theta)c_{21} + \tilde{s}_{1,24}(\theta)c_{21} + \tilde{s}_{1,25}(\theta)a_{03} + \tilde{s}_{1,26}(\theta)c_{03} + \tilde{s}_{1,27}(\theta)a_{21} + \tilde{s}_{1,28}(\theta)c_{12},
\]
and \(\tilde{s}_{1,i}(\theta)\) for \(i = 1, \ldots, 28\) satisfying the following expressions:

\[
\begin{align*}
\tilde{s}_{1,1}(\theta) &= - \frac{1}{12} (16 \cos^8 \theta + 34 \cos^6 \theta + 37 \cos^4 \theta + 8 \cos^2 \theta - 5) (\cos^2 \theta - 1) ^2, \\
\tilde{s}_{1,2}(\theta) &= \frac{1}{12} (\cos^2 \theta - 1) (16 \cos^{10} \theta + 38 \cos^8 \theta + 53 \cos^6 \theta + 15 \cos^4 \theta - 7 \cos^2 \theta - 7), \\
\tilde{s}_{1,3}(\theta) &= - \frac{1}{12} (16 \cos^8 \theta + 14 \cos^6 \theta + 15 \cos^4 \theta - 16 \cos^2 \theta + 1) (\cos^2 \theta - 1)^2, \\
\tilde{s}_{1,4}(\theta) &= \frac{1}{12} (\cos^2 \theta - 1) (16 \cos^{10} \theta + 18 \cos^8 \theta + 19 \cos^6 \theta - 15 \cos^4 \theta - 1), \\
\tilde{s}_{1,5}(\theta) &= \frac{1}{12} \cos \theta \sin^5 \theta (2 \cos^2 \theta + 1) (8 \cos^4 \theta + 12 \cos^2 \theta + 7), \\
\tilde{s}_{1,6}(\theta) &= \frac{1}{12} \cos \theta \sin^5 \theta (16 \cos^6 \theta + 12 \cos^4 \theta - 2 \cos^2 \theta + 1), \\
\tilde{s}_{1,7}(\theta) &= \frac{1}{6} \cos^3 \theta \sin^3 \theta (8 \cos^6 \theta + 8 \cos^4 \theta + 5 \cos^2 \theta - 3), \\
\tilde{s}_{1,8}(\theta) &= \frac{1}{12} \cos \theta \sin^5 \theta (-1 + 2 \cos^2 \theta) (8 \cos^4 \theta + 1), \\
\tilde{s}_{1,9}(\theta) &= \frac{1}{12} \cos \theta \sin^5 \theta (16 \cos^6 \theta + 12 \cos^4 \theta + 6 \cos^2 \theta - 7), \\
\tilde{s}_{1,10}(\theta) &= \frac{1}{6} \cos \theta \sin^7 \theta (8 \cos^4 \theta + 4 \cos^2 \theta - 3), \\
\tilde{s}_{1,11}(\theta) &= - \frac{4}{3} \cos^3 \theta \sin^3 \theta (\cos^6 \theta + \cos^4 \theta + \cos^2 \theta - 1), \\
\tilde{s}_{1,12}(\theta) &= - \frac{1}{3} \cos^5 \theta \sin^3 \theta (4 \cos^4 \theta - \cos^2 \theta - 1), \\
\tilde{s}_{1,13}(\theta) &= \frac{2}{3} \cos^3 \theta \sin \theta (-5 \cos^2 \theta - 2 + 5 \cos^6 \theta + 4 \cos^4 \theta + 4 \cos^8 \theta), 
\end{align*}
\]
\[ \tilde{s}_{1,14}(\theta) = \frac{-1}{12} \sin^2 \theta \cos^2 \theta \left( 32 \cos^8 \theta - 4 \cos^6 \theta - 6 \cos^4 \theta - 3 \cos^2 \theta - 1 \right), \]
\[ \tilde{s}_{1,15}(\theta) = \frac{1}{12} \sin^4 \theta \cos^2 \theta \left( -1 + 2 \cos^2 \theta \right) \left( 8 \cos^4 \theta + \cos^2 \theta + 1 \right), \]
\[ \tilde{s}_{1,16}(\theta) = \frac{1}{12} \sin^6 \theta \cos^2 \theta \left( 16 \cos^6 \theta + 10 \cos^2 \theta - 5 \right), \]
\[ \tilde{s}_{1,17}(\theta) = \frac{1}{12} \sin^4 \theta \cos^2 \theta \left( 32 \cos^6 \theta - 12 \cos^4 \theta - 6 \cos^2 \theta + 1 \right), \]
\[ \tilde{s}_{1,18}(\theta) = \frac{1}{12} \sin^6 \theta \cos^2 \theta \left( 8 \cos^2 \theta - 1 \right) \left( -1 + 2 \cos^2 \theta \right), \]
\[ \tilde{s}_{1,19}(\theta) = \frac{1}{12} \sin^2 \theta \cos^2 \theta \left( 32 \cos^6 \theta + 36 \cos^6 \theta + 14 \cos^4 \theta - 21 \cos^2 \theta - 7 \right), \]
\[ \tilde{s}_{1,20}(\theta) = \frac{1}{12} \sin^4 \theta \cos^2 \theta \left( 16 \cos^6 \theta + 14 \cos^4 \theta + 7 \cos^2 \theta - 7 \right), \]
\[ \tilde{s}_{1,21}(\theta) = \frac{1}{12} \sin^4 \theta \cos^2 \theta \left( 32 \cos^6 \theta + 28 \cos^4 \theta - 10 \cos^2 \theta - 5 \right), \]
\[ \tilde{s}_{1,22}(\theta) = \frac{1}{12} \cos^8 \theta \sin^5 \theta \left( -1 + 2 \cos^2 \theta \right), \]
\[ \tilde{s}_{1,23}(\theta) = \frac{1}{3} \cos^5 \theta \sin \theta \left( 2 \cos^2 \theta + 1 \right) \left( 4 \cos^4 \theta - 1 - 2 \cos^2 \theta \right), \]
\[ \tilde{s}_{1,24}(\theta) = \frac{1}{12} \cos^3 \theta \sin^3 \theta \left( 2 \cos^2 \theta + 1 \right) \left( 4 \cos^4 \theta + 7 \cos^2 \theta + 7 \right), \]
\[ \tilde{s}_{1,25}(\theta) = \frac{1}{12} \cos \theta \sin^7 \theta \left( 2 \cos \theta + 1 \right) \left( 2 \cos \theta - 1 \right) \left( -1 + 2 \cos^2 \theta \right), \]
\[ \tilde{s}_{1,26}(\theta) = \frac{1}{12} \cos \theta \sin \theta \left( \cos^4 \theta + 2 \cos^2 \theta + 3 \right) \left( 2 \cos^6 \theta - 1 + \cos^4 \theta \right), \]
\[ \tilde{s}_{1,27}(\theta) = \frac{1}{12} \cos^3 \theta \sin^3 \theta \left( -1 + 2 \cos^2 \theta \right) \left( 4 \cos^4 \theta + \cos^2 \theta + 1 \right), \]
\[ \tilde{s}_{1,28}(\theta) = \frac{1}{12} \cos \theta \sin^2 \theta \left( 4 \cos^2 \theta + 5 \right) \left( 2 \cos^2 \theta + 1 \right). \]


