

Limit cycles bifurcating from a degenerate center

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Abstract

We study the maximum number of limit cycles that can bifurcate from a degenerate center of a cubic homogeneous polynomial differential system. Using the averaging method of second order and perturbing inside the class of all cubic polynomial differential systems we prove that at most three limit cycles can bifurcate from the degenerate center. As far as we know this is the first time that a complete study up to second order in the small parameter of the perturbation is done for studying the limit cycles which bifurcate from the periodic orbits surrounding a degenerate center (a center whose linear part is identically zero) having neither a Hamiltonian first integral nor a rational one. This study needs many computations, which have been verified with the help of the algebraic manipulator Maple.

Keywords: polynomial differential systems, centers, limit cycles, averaging theory.

2010 MSC: Primary 37G15, 34C07. Secondary 37C10, 34C05, 34A34

1. Introduction

Hilbert in (16) asked for the maximum number of limit cycles which real polynomial differential systems in the plane of a given degree can have. This is actually the well known *16th Hilbert Problem*, see for example the surveys (17; 18) and references therein. Recall that a *limit cycle* of a planar polynomial differential system is a periodic orbit of the system isolated in the set of all periodic orbits of the system.

Poincaré in (22) was the first to introduce the notion of a center for a vector field defined on the real plane. So according to Poincaré a *center* is a singular point surrounded by a neighborhood filled of periodic orbits with the unique exception of the singular point.

Consider the polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1)$$

and as usually we denote by $\cdot = d/dt$. Assume that system (1) has a center located at the origin. Then after a linear change of variables and a possible scaling of time system (1) can be written in one of the following forms

$$(A) \quad \begin{aligned} \dot{x} &= -y + F_1(x, y), \\ \dot{y} &= x + F_2(x, y), \end{aligned} \quad (B) \quad \begin{aligned} \dot{x} &= y + F_1(x, y), \\ \dot{y} &= F_2(x, y), \end{aligned} \quad (C) \quad \begin{aligned} \dot{x} &= F_1(x, y), \\ \dot{y} &= F_2(x, y), \end{aligned}$$

with F_1 and F_2 polynomials without constant and linear terms. When system (1) can be written into the form (A) we say that the center is of *linear type*. When system (1) can take the form (B) the center is *nilpotent*, and when system (1) can be transformed into the form (C) the center is *degenerate*.

Due to the difficulty of this problem mathematicians have consider simpler versions. Thus Arnold (1) considered the *weakened 16th Hilbert Problem*, which consists in determining an upper bound for the number of limit cycles which can bifurcate from the periodic orbits of a polynomial Hamiltonian center when it is perturbed inside a class of polynomial differential systems, see for instance (9) and the hundred of references quoted therein. It is known that in a neighborhood of a center always there is a first integral, see (21). When this first integral is not polynomial the computations become more difficult. Moreover, if the center is degenerate the computations become even harder.

In the literature we can basically find the following methods for studying the limit cycles that bifurcate from a center:

- The method that uses the Poincaré return map, like the articles (4; 8).
- The one that uses the Abelian integrals or Melnikov integrals (note that for systems in the plane the two notions are equivalent), see for example section 5 of Chapter 6 of (2) and section 6 of Chapter 4 of (15).

- The one that uses the inverse integrating factor, see (11; 12; 13; 25).
- The averaging theory (6; 14; 19; 23; 24).

The first two methods provide information about the number of limit cycles whereas the last two methods additionally give the shape of the bifurcated limit cycle up to any order in the perturbation parameter.

Almost all the papers studying how many limit cycles can bifurcate from the periodic orbits of a center, work with centers of linear type. There are very few papers studying this problem for nilpotent or degenerate centers. In fact, for degenerate centers as far as we know the bifurcation of limit cycles from the periodic orbits of a degenerate center only have been studying completely using formulas of first order in the small parameter of the perturbation. Here we will provide a complete study of this problem using formulas of second order, and as it occurs with the formulas of second order applied to linear centers that they provide in general more limit cycles than the formulas of first order, the same occurs for the formulas of second order applied to degenerate centers. Of course, the computations from first order to second order increases almost exponentially.

This paper deals with the weakened 16th Hilbert's problem but perturbing non-Hamiltonian degenerate centers using the technique of the averaging method of second order, see (14), and Section 2 for a summary of the results that we need here.

Since we want to study the perturbation of a degenerate center with averaging of second order, from the homogeneous centers the first ones that are degenerate, are the cubic homogeneous centers, see for instance (7). In this class in (20) the authors studied the perturbation of the following cubic homogeneous center

$$\dot{x} = -y(3x^2 + y^2), \quad \dot{y} = x(x^2 - y^2), \quad (2)$$

inside the class of all cubic polynomial differential systems, using averaging theory of first order. Here we study this problem but using averaging theory of second order.

System (2) has a global center at the origin (i.e. all the orbits contained in $\mathbb{R}^2 \setminus \{(0, 0)\}$ are periodic), and it admits the non-rational first integral

$$H(x, y) = (x^2 + y^2) \exp\left(-\frac{2x^2}{x^2 + y^2}\right).$$

The limit cycles bifurcating from the periodic orbits of the global center (2) have already been studied in the following two results, see (20) and (5), respectively.

Theorem 1. *We deal with differential system (2). Then the polynomial differential system*

$$\begin{aligned}\dot{x} &= -y(3x^2 + y^2) + \varepsilon \left(\sum_{0 \leq i+j \leq 3} a_{ij} x^i y^j \right), \\ \dot{y} &= x(x^2 - y^2) + \varepsilon \left(\sum_{0 \leq i+j \leq 3} b_{ij} x^i y^j \right),\end{aligned}$$

has at most one limit cycle bifurcating from the periodic orbits of the center of system (2) using averaging theory of first order. Moreover, there are examples with 1 and 0 limit cycles.

Proposition 2. *We consider the homogeneous polynomial differential system (2). Let $P_i(x, y)$ and $Q_i(x, y)$ for $i = 1, 2$ be polynomials of degree at most 3. Then for convenient polynomials P_i and Q_i , the polynomial differential system*

$$\begin{aligned}\dot{x} &= -y(3x^2 + y^2) + \varepsilon P_1(x, y) + \varepsilon^2 P_2(x, y), \\ \dot{y} &= x(x^2 - y^2) + \varepsilon Q_1(x, y) + \varepsilon^2 Q_2(x, y),\end{aligned}$$

has at first order averaging one limit cycle, and at second order averaging two limit cycles bifurcating from the periodic solutions of the global center (2).

Our main result is the following one and it do by first time the complete study of the averaging method of second order for a degenerate center having neither a Hamiltonian first integral nor a rational one.

Theorem 3. *We consider the cubic homogeneous differential system (2). Then the perturbation of system (2) inside the class of all cubic polynomial systems*

$$\begin{aligned}\dot{x} &= -y(3x^2 + y^2) + \varepsilon \left(\sum_{0 \leq i+j \leq 3} a_{ij} x^i y^j \right) + \varepsilon^2 \left(\sum_{0 \leq i+j \leq 3} b_{ij} x^i y^j \right), \\ \dot{y} &= x(x^2 - y^2) + \varepsilon \left(\sum_{0 \leq i+j \leq 3} c_{ij} x^i y^j \right) + \varepsilon^2 \left(\sum_{0 \leq i+j \leq 3} d_{ij} x^i y^j \right),\end{aligned} \tag{3}$$

has at most three limit cycles bifurcating from the periodic orbits of the center of system (2) using averaging theory of second order. Moreover, there are examples with 3, 2, 1 and 0 limit cycles.

The paper is organized as follows: In section 2 we present a summary of the averaging method of second order following (14). Next in section 3 we provide the proof of our main Theorem 3. In section 4 we provide three examples, of systems (3) with 0, 1, 2 and 3 limit cycles bifurcating from the degenerate center. At the end we present the Appendices A, B and C.

2. The averaging method of second order

In this section we present the averaging method of second order following (14). In that paper the averaging theory for differential equations of one variable is done up to any order in the small parameter of the perturbation. We consider the analytic differential equation

$$\frac{dr}{d\theta} = G_0(\theta, r) + \sum_{k \geq 1} \varepsilon^k G_k(\theta, r), \quad (4)$$

with $r \in \mathbb{R}$, $\theta \in \mathbb{S}^1$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with ε_0 a small positive real value, and the functions $G_k(\theta, r)$ are 2π -periodic in the variable θ . Note that for $\varepsilon = 0$ system (4) is unperturbed. Let $r_s(\theta, r_0)$ be the solution of system (4) with $\varepsilon = 0$ satisfying $r_s(0, r_0) = r_0$ and $r_s(\theta, r_0)$ is 2π periodic for $r_0 \in \mathcal{I}$ with \mathcal{I} a real open interval. We are interested in the limit cycles of equation (4) which bifurcate from the periodic orbits of the unperturbed system with initial condition $r_0 \in \mathcal{I}$. So, we define by $r_\varepsilon(\theta, r_0)$ the solution of equation (4) satisfying $r_\varepsilon(0, r_0) = r_0$.

In what follows we denote by $u = u(\theta, r_0)$ the solution of the variational equation

$$\frac{\partial u}{\partial \theta} = \frac{\partial G_0}{\partial r}(\theta, r_s(\theta, r_0))u,$$

satisfying $u(0, r_0) = 1$.

We define

$$\begin{aligned}
u_1(\theta, r_0) &= \int_0^\theta \frac{G_1(\phi, r_s(\phi, r_0))}{u(\phi, r_0)} d\phi = \int_0^\theta \frac{G_1(w, r_s(w, r_0))}{u(w, r_0)} dw, \\
G_{10}(r_0) &= \int_0^{2\pi} \frac{G_1(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} d\theta, \\
G_{20}(r_0) &= \int_0^{2\pi} \left(\frac{G_2(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} + \frac{\partial G_1}{\partial r}(\theta, r_s(\theta, r_0)) u_1(\theta, r_0) \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2 G_0}{\partial r^2}(\theta, r_s(\theta, r_0)) u_1(\theta, r_0)^2 \right) d\theta.
\end{aligned} \tag{5}$$

In statement (b) of Corollary 5 of (14) it is proved the following result.

Theorem 4. *Assume that the solution $r_s(\theta, r_0)$ of the unperturbed equation (4) such that $r_s(0, r_0) = r_0$ is 2π -periodic for $r_0 \in \mathcal{I}$ with \mathcal{I} a real open interval. If $G_{10}(r_0)$ is identically zero in \mathcal{I} and $G_{20}(r_0)$ is not identically zero in \mathcal{I} , then for each simple zero $r^* \in \mathcal{I}$ of $G_{20}(r_0) = 0$ there exists a periodic solution $r_\varepsilon(\theta, r_0)$ of (4) such that $r_\varepsilon(0, r_0) \rightarrow r^*$ when $\varepsilon \rightarrow 0$.*

3. Proof of Theorem 3

System (2) in polar coordinates becomes

$$\dot{r} = -2r^3 \cos \theta \sin \theta, \quad \dot{\theta} = r^2, \tag{6}$$

or equivalently,

$$\frac{dr}{d\theta} = -2r \cos \theta \sin \theta,$$

and it has the solution $r_s(\theta, r_0) = r_0 \exp(-\sin^2 \theta)$ satisfying that $r_s(0, r_0) = r_0$.

Now we perturb system (2) inside the class of all cubic polynomial differential systems as in (3). System (3) in polar coordinates give rise to the differential equation

$$\frac{dr}{d\theta} = G_0(\theta, r) + \varepsilon G_1(\theta, r) + \varepsilon^2 G_2(\theta, r) + O(\varepsilon^3),$$

with

$$G_0(\theta, r) = -2r \cos \theta \sin \theta,$$

$$G_1(\theta, r) = g_{1,1}(\theta) \frac{1}{r^2} + g_{1,2}(\theta) \frac{1}{r} + g_{1,3}(\theta) + g_{1,4}(\theta) r,$$

$$G_2(\theta, r) = g_{2,1}(\theta) \frac{1}{r^5} + g_{2,2}(\theta) \frac{1}{r^4} + g_{2,3}(\theta) \frac{1}{r^3} + g_{2,4}(\theta) \frac{1}{r^2} \\ + g_{2,5}(\theta) \frac{1}{r} + g_{2,6}(\theta) + g_{2,7}(\theta) r,$$

where the expressions of the coefficients $g_{1,i}(\theta)$ for $i = 1, 2, 3, 4$ and $g_{2,j}(\theta)$ for $j = 1, 2, \dots, 7$ are given in the Appendix A.

Additionally, we consider the variational equation

$$\frac{\partial u}{\partial \theta} = \frac{\partial G_0}{\partial r}(\theta, r_s(\theta, r_0)),$$

and its solution $u(\theta, r_0)$ satisfying $u(0, r_0) = 1$, namely $u_s(\theta) = \exp(-\sin^2 \theta)$.

We define

$$I_1 = \int_0^{2\pi} \exp(2 \sin^2 \theta) \cos^4 \theta \, d\theta = 3.572403292\dots, \\ I_2 = \int_0^{2\pi} \exp(2 \sin^2 \theta) \cos^2 \theta \, d\theta = 5.985557563\dots, \quad (7) \\ I_3 = \int_0^{2\pi} \exp(2 \sin^2 \theta) \, d\theta = 21.62373221\dots$$

Lemma 5. Consider I_1, I_2, I_3 defined in (7). Then for $a_{10} = -(I_3 - 2I_1 + I_2)/(2I_1 - I_2)c_{01}$ and $a_{30} = -2c_{03} - c_{21}$ we have that the function $G_{10}(r_0)$ defined in (5) is identically zero.

Proof. We have

$$\frac{G_1(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} = A(\theta) \frac{1}{r_0^2} + B(\theta) \frac{1}{r_0} + C(\theta) + D(\theta) r_0,$$

with

$$A(\theta) = [\sin \theta (2 \cos^2 \theta + 1) c_{00} + \cos \theta (-1 + 2 \cos^2 \theta) a_{00}] e^{3 \sin^2 \theta},$$

$$B(\theta) = [\cos \theta \sin \theta (-1 + 2 \cos^2 \theta) a_{01} + \cos^2 \theta (-1 + 2 \cos^2 \theta) a_{10} - (-\cos^2 \theta + 2 \cos^4 \theta - 1) c_{01} + \cos \theta \sin \theta (2 \cos^2 \theta + 1) c_{10}] e^{2 \sin^2 \theta},$$

$$C(\theta) = [-\cos \theta (-3 \cos^2 \theta + 1 + 2 \cos^4 \theta) a_{02} + \cos^3 \theta (-1 + 2 \cos^2 \theta) a_{20} + \cos^2 \theta \sin \theta (-1 + 2 \cos^2 \theta) a_{11} + \sin^3 \theta (2 \cos^2 \theta + 1) c_{02} + \cos^2 \theta \sin \theta (2 \cos^2 \theta + 1) c_{20} - \cos \theta (-\cos^2 \theta + 2 \cos^4 \theta - 1) c_{11}] e^{\sin^2 \theta},$$

$$D(\theta) = \cos \theta \sin^3 \theta (-1 + 2 \cos^2 \theta) a_{03} + \cos^4 \theta (-1 + 2 \cos^2 \theta) a_{30} + \cos^3 \theta \sin \theta (-1 + 2 \cos^2 \theta) a_{21} - \cos^2 \theta (-3 \cos^2 \theta + 1 + 2 \cos^4 \theta) a_{12} + (1 - 3 \cos^4 \theta + 2 \cos^6 \theta) c_{03} + \cos^3 \theta \sin \theta (2 \cos^2 \theta + 1) c_{30} - \cos^2 \theta (-\cos^2 \theta + 2 \cos^4 \theta - 1) c_{21} + \cos \theta \sin^3 \theta (2 \cos^2 \theta + 1) c_{12}.$$

Now

$$G_{10}(r_0) = \int_0^{2\pi} \frac{G_1(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} d\theta = \int_0^{2\pi} \left(A(\theta) \frac{1}{r_0^2} + B(\theta) \frac{1}{r_0} + C(\theta) + D(\theta) r_0 \right) d\theta,$$

and considering the change of coordinates $\theta = \phi + \pi$ in the interval $[0, 2\pi]$ and the symmetries

$$\sin(\theta + \pi) = -\sin \theta, \quad \cos(\theta + \pi) = -\cos \theta, \quad (8)$$

we have that

$$\int_0^{2\pi} A(\theta) d\theta = \int_{-\pi}^{\pi} A(\phi) d\phi = 0, \quad \int_0^{2\pi} C(\theta) d\theta = \int_{-\pi}^{\pi} C(\phi) d\phi = 0.$$

So we have

$$\begin{aligned} G_{10}(r_0) &= \int_0^{2\pi} \frac{B(\theta)}{r_0} d\theta + \int_0^{2\pi} D(\theta) r_0 d\theta \\ &= [(2a_{10} - 2c_{01})I_1 + (c_{01} - a_{10})I_2 + c_{01}I_3] \frac{1}{r_0} + \frac{\pi}{2} (2c_{03} + a_{30} + c_{21}) r_0, \end{aligned}$$

and therefore $G_{10} \equiv 0$ if

$$a_{10} = -\frac{I_3 - 2I_1 + I_2}{2I_1 - I_2}c_{01} = -17.65322447..c_{01}, \quad a_{30} = -2c_{03} - c_{21}.$$

This completes the proof of the lemma. \square

Now we have

$$\frac{G_2(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} = A_5(\theta) \frac{1}{r_0^5} + A_4(\theta) \frac{1}{r_0^4} + A_3(\theta) \frac{1}{r_0^3} + A_2(\theta) \frac{1}{r_0^2} + A_1(\theta) \frac{1}{r_0} + A_0(\theta) + \tilde{A}_1(\theta) r_0,$$

with $A_5(\theta), A_4(\theta), A_3(\theta), A_2(\theta), A_1(\theta), A_0(\theta), \tilde{A}_1(\theta)$ are given in the Appendix B.

We note that

$$\int_0^{2\pi} A_4(\theta) d\theta = 0, \quad \int_0^{2\pi} A_2(\theta) d\theta = 0, \quad \int_0^{2\pi} A_0(\theta) d\theta = 0,$$

because of the symmetries (8). So we have

$$\int_0^{2\pi} \frac{G_2(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} d\theta = \frac{1}{r_0^5} \int_0^{2\pi} A_5(\theta) d\theta + \frac{1}{r_0^3} \int_0^{2\pi} A_3(\theta) d\theta + \frac{1}{r_0} \int_0^{2\pi} A_1(\theta) d\theta + \left(\int_0^{2\pi} \tilde{A}_1(\theta) d\theta \right) r_0, \quad (9)$$

and we recall that the expressions of $A_5(\theta), A_3(\theta), A_1(\theta), \tilde{A}_1(\theta)$ are given in the Appendix B. We have

$$\begin{aligned} \int_0^{2\pi} A_5(\theta) d\theta &= \left(-4 \int_0^{2\pi} e^{6 \sin^2 \theta} \cos^4 \theta d\theta + 2 \int_0^{2\pi} e^{6 \sin^2 \theta} \cos^2 \theta d\theta + \int_0^{2\pi} e^{6 \sin^2 \theta} d\theta \right) a_{00} c_{00} \\ &= 665.2264930..a_{00} c_{00}, \\ \int_0^{2\pi} A_3(\theta) d\theta &= 239.0000390..a_{01} c_{01} + 97.83745135..a_{00} c_{02} + 97.83745135..a_{02} c_{00} \\ &\quad - 257.2692783..c_{01} c_{10} + 12.93483815..a_{00} c_{20} - 7.99641945..a_{00} a_{11} \\ &\quad + 12.93483815..a_{20} c_{00} - 28.92767705..c_{00} c_{11}, \end{aligned}$$

$$\begin{aligned}
\int_0^{2\pi} A_1(\theta) d\theta &= 1.159249021..b_{10} + 20.46448319..d_{01} + 2.318498043..a_{11}c_{11} \\
&\quad -4.73165232..c_{02}c_{11} + 2.318498043..a_{20}c_{0,2} + 2.318498045..a_{02}c_{20} \\
&\quad -3.761715750..c_{11}c_{20} + 14.47892563..a_{02}c_{02} - 1.253905250..a_{02}a_{11} \\
&\quad +0.189312456..a_{20}c_{20} + 0.094656226..a_{11}a_{20} + 0.647510463..a_{21}c_{01} \\
&\quad -5.110277230..c_{03}c_{10} + 2.318498043..a_{12}c_{10} + 2.22384182..a_{01}c_{21} \\
&\quad +36.6143963..a_{03}c_{01} - 3.95102821..c_{10}c_{21} - 1.25390525..a_{01}a_{12} \\
&\quad -45.6606188..c_{01}c_{12} - 7.1036910..c_{01}c_{30} + 14.28961317..a_{01}c_{03}, \\
\int_0^{2\pi} \tilde{A}_1(\theta) d\theta &= -0.3926990817..c_{30}c_{21} + 0.1963495408..c_{30}a_{12} + 2.159844949..a_{03}c_{03} \\
&\quad -0.1963495408..a_{03}a_{12} - 0.7853981634..c_{12}c_{21} + 0.3926990817..a_{03}c_{21} \\
&\quad -1.178097245..c_{03}c_{12} + 0.3926990817..c_{12}a_{12} + 0.9817477042..c_{03}c_{30} \\
&\quad +1.570796327..d_{21} + 3.141592654..d_{03} + 1.570796327..b_{30}.
\end{aligned}$$

Remark 6. (a) Looking at the expressions of A_3, A_1, \tilde{A}_1 in the Appendix B we can have the exact definition for the numerical coefficients which appear in the previous integrals. Thus for instance

$$239.0000390\dots = - \int_0^{2\pi} \frac{e^{4 \sin^2 \theta} (\cos^2 \theta - 1) (2 I_1 - 4 I_3 \cos^4 \theta + 2 I_3 \cos^2 \theta - I_2)}{2 I_1 - I_2} d\theta,$$

and I_1, I_2, I_3 satisfying relations (7).

(b) All the computations of this paper have been verified with the algebraic manipulator Maple.

We additionally have

$$\frac{\partial G_1}{\partial r}(\theta, r_s(\theta, r_0)) = B_0(\theta) + \frac{B_1(\theta)}{r_0^2} + \frac{B_2(\theta)}{r_0^3},$$

with

$$\begin{aligned}
B_0(\theta) &= \cos^2 \theta (1 + 2 \cos^2 \theta - 4 \cos^4 \theta) c_{21} \\
&\quad + (-2 \cos^6 \theta + 1 - \cos^4 \theta) c_{03} \\
&\quad + \cos \theta (\sin \theta \cos^2 \theta - 2 \cos^4 \theta \sin \theta + \sin \theta) c_{12} \\
&\quad + \cos^3 \theta (\sin \theta + 2 \cos^2 \theta \sin \theta) c_{30} \\
&\quad + \sin \theta \cos \theta (3 \cos^2 \theta - 1 - 2 \cos^4 \theta) a_{03} \\
&\quad + \cos^2 \theta (-1 + 3 \cos^2 \theta - 2 \cos^4 \theta) a_{12} \\
&\quad + \sin \theta \cos^3 \theta (-1 + 2 \cos^2 \theta) a_{21},
\end{aligned}$$

$$\begin{aligned}
B_1(\theta) = & (-1 - 18.65322447... \cos^2 \theta + 37.30644894... \cos^4 \theta) e^{2 \sin^2 \theta} c_{01} \\
& - \sin \theta \cos \theta (2 \cos^2 \theta + 1) e^{2 \sin^2 \theta} c_{10} \\
& + \sin \theta \cos \theta (-2 \cos^2 \theta + 1) e^{2 \sin^2 \theta} a_{01},
\end{aligned}$$

$$B_2(\theta) = -2 \sin \theta (2 \cos^2 \theta + 1) c_{00} + 2 \left(-2 + e^{3 \sin^2 \theta} \cos \theta \right) a_{00}.$$

Now we have

$$\frac{G_1(w, r_s(w, r_0))}{u(w, r_0)} = \frac{C_2(w)}{r_0^2} + \frac{C_1(w)}{r_0} + C_0(w) + \tilde{C}_1(w) r_0,$$

with

$$C_2(w) = e^{3 \sin^2 w} [\cos w (2 \cos^2 w - 1) a_{00} + \sin w (1 + 2 \cos^2 w) c_{00}],$$

$$\begin{aligned}
C_1(w) = & \left[\cos w \sin w (-1 + 2 \cos^2 w) a_{01} \right. \\
& + \left(1 + \frac{I_3}{2I_1 - I_2} (\cos^2 w - 2 \cos^4 w) \right) c_{01} \\
& \left. + \cos w \sin w (1 + 2 \cos^2 w) c_{10} \right] e^{2 \sin^2 w},
\end{aligned}$$

$$\begin{aligned}
C_0(w) = & [\cos^3 w (2 \cos^2 w - 1) a_{20} - \cos w (-3 \cos^2 w + 2 \cos^4 w + 1) a_{02} \\
& + \sin w \cos^2 w (2 \cos^2 w - 1) a_{11} + \sin w \cos^2 w (1 + 2 \cos^2 w) c_{20} \\
& + c_{02} \sin^3 w (1 + 2 \cos^2 w) - \cos w (2 \cos^4 w - 1 - \cos^2 w) c_{11}] e^{\sin^2 w},
\end{aligned}$$

$$\begin{aligned}
\tilde{C}_1(w) = & -(\cos^4 w + 2 \cos^6 w - 1) c_{03} + \sin^3 w \cos w (1 + 2 \cos^2 w) c_{12} \\
& - \cos^2 w (-1 - 2 \cos^2 w + 4 \cos^4 w) c_{21} + \cos^3 w \sin w (1 + 2 \cos^2 w) c_{30} \\
& + \sin^3 w \cos w (2 \cos^2 w - 1) a_{03} + \sin^2 w \cos^2 w (2 \cos^2 w - 1) a_{12} \\
& + \cos^3 w \sin w (2 \cos^2 w - 1) a_{21}.
\end{aligned}$$

Additionally, from (5) we obtain

$$u_1(\theta, r_0) = \frac{1}{r_0^2} \int_0^\theta C_2(w) dw + \frac{1}{r_0} \int_0^\theta C_1(w) dw + \int_0^\theta C_0(w) dw + r_0 \int_0^\theta \tilde{C}_1(w) dw,$$

and so

$$\frac{\partial G_1}{\partial r}(\theta, r_s(\theta, r_0)) u_1(\theta, r_0) = s_5(\theta) \frac{1}{r_0^5} + s_4(\theta) \frac{1}{r_0^4} + s_3(\theta) \frac{1}{r_0^3} + s_2(\theta) \frac{1}{r_0^2} + s_1(\theta) \frac{1}{r_0} + s_0(\theta) + \tilde{s}_1(\theta) r_0, \quad (10)$$

and the explicit expressions of $s_i(\theta)$ for $i = 0, 1, \dots, 5$ and $\tilde{s}_1(\theta)$ are given in the Appendix C.

Since $\frac{\partial^2 G_0}{\partial r^2} = 0$ from (5) we have that

$$G_{20}(r_0) = \int_0^{2\pi} \left(\frac{G_2(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} + \frac{\partial G_1}{\partial r}(\theta, r_s(\theta, r_0)) u_1(\theta, r_0) \right) d\theta,$$

and we obtain

$$r_0^5 G_{20}(r_0) = v_6 r_0^6 + v_4 r_0^4 + v_2 r_0^2 + v_0,$$

with

$$\begin{aligned} v_6 = & -0.3926990800 \cdots c_{21} c_{30} + 0.1963495397 \cdots a_{12} c_{30} + 2.159844949 \cdots a_{03} c_{03} \\ & -0.1963495365 \cdots a_{03} a_{12} - 0.7853981634 \cdots c_{12} c_{21} + 0.3926990817 \cdots a_{03} c_{21} \\ & -1.178097245 \cdots c_{03} c_{12} + 0.3926990817 \cdots a_{12} c_{12} + 0.9817477042 \cdots c_{03} c_{30} \\ & +1.570796327 \cdots d_{21} + 3.141592654 \cdots d_{03} + 1.570796327 \cdots b_{30}, \end{aligned}$$

$$\begin{aligned} v_4 = & -3.155691751 \cdots a_{21} c_{01} + 6.612510180 \cdots c_{03} c_{10} + 1.786201647 \cdots a_{12} c_{10} \\ & +2.413154277 \cdots a_{01} c_{21} + 38.88726613 \cdots a_{03} c_{01} - 1.253905255 \cdots c_{10} c_{21} \\ & -1.206577137 \cdots a_{01} a_{12} - 68.30745733 \cdots c_{01} c_{12} - 38.88726635 \cdots c_{01} c_{30} \\ & +13.27234849 \cdots a_{01} c_{03} + 2.318498043 \cdots a_{11} c_{11} - 4.73165232 \cdots c_{02} c_{11} \\ & +2.318498043 \cdots a_{20} c_{02} + 2.318498045 \cdots a_{02} c_{20} - 3.761715750 \cdots c_{11} c_{20} \\ & +14.47892563 \cdots a_{02} c_{02} - 1.253905250 \cdots a_{02} a_{11} + 0.189312456 \cdots a_{20} c_{20} \\ & +0.094656226 \cdots a_{11} a_{20} + 1.159249021 \cdots b_{10} + 20.46448319 \cdots d_{01}, \end{aligned}$$

$$\begin{aligned} v_2 = & 95.95703341 \cdots a_{00} c_{02} + 105.7377762 \cdots a_{02} c_{00} + 239.0000390 \cdots a_{01} c_{01} \\ & -7.649140220 \cdots a_{00} a_{11} + 16.68739736 \cdots a_{00} c_{20} - 110.9164314 \cdots c_{00} c_{11} \\ & -257.2692783 \cdots c_{01} c_{10} - 0.000001 \cdots c_{01}^2 - 31.79348852 \cdots a_{20} c_{00}, \end{aligned}$$

$$v_0 = 665.2264933 \cdots a_{00} c_{00}.$$

We have that the coefficients v_6, v_4, v_2, v_0 are independent because d_{03} only appears in v_6 , b_{10} only appears in v_4 , $a_{00} c_{02}$ only appears in v_2 , and $a_{00} c_{00}$ only appears in v_0 .

Now we are going to use Descartes Theorem:

Theorem 7 (Descartes Theorem). *Consider the real polynomial $p(x) = a_{i_1} x^{i_1} + a_{i_2} x^{i_2} \cdots + a_{i_r} x^{i_r}$ with $0 \leq i_1 < i_2 < \cdots < i_r$ and $a_{i_j} \neq 0$ real*

constants for $j \in \{1, 2, \dots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m , then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $r - 1$ positive real roots.

For a proof of Descartes Theorem see pages 82–83 of (3).

So from Descartes Theorem we can choose v_6, v_4, v_2, v_0 in order that the G_{20} has 3, 2, 1 or 0 real positive roots. This completes the proof of the first part of Theorem 3.

Remark 8. Again the exact definition for the numerical coefficients which appear in v_6, v_4, v_2 and v_0 are given in Appendices B and C. For instance

$$v_0 = \int_0^{2\pi} A_5 d\theta + \int_0^{2\pi} s_{5,3} d\theta = \int_0^{2\pi} A_5 d\theta = 665.2264933\dots$$

For completing the proof of Theorem 3 we shall provide examples of system (3) with 3, 2, 1 and 0 limit cycles. In fact, strictly speaking it is not necessary to provide examples with 3, 2, 1 and 0 limit cycles but we want to provide such examples.

4. Examples

Example with 3 limit cycles

In Figure 1 we see that for $\varepsilon = 0.001$ the system

$$\begin{aligned} \dot{x} &= -y(3x^2 + y^2) + \varepsilon + \varepsilon^2(3570.576292x - 752.8823806x^3) \\ &= y(3x^2 + y^2) + 0.001 + 0.003570576292x - 0.0007528823806x^3, \\ \dot{y} &= x(x^2 - y^2) + \varepsilon(1 - 37.74385845y^2) \\ &= x(x^2 - y^2) + 0.001 - 0.03774385845y^2, \end{aligned} \tag{11}$$

has three limit cycles, since for system (11) we have

$$G_{20}(r_0) = -1182.624878 r_0 + 4139.187071 \frac{1}{r_0} - 3621.788686 \frac{1}{r_0^3} + 665.2264933 \frac{1}{r_0^5},$$

and from $G_{20}(r_0) = 0$ we obtain the three positive roots near to $r_0 = 0.5, 1, 1.5$.

$$\varepsilon = 0 \qquad \varepsilon = 0.001$$

Figure 1: For $\varepsilon = 0$ we have the degenerate center of system (2), and for $\varepsilon = 0.001$ the perturbed system (11) has three limit cycles.

We have used the program P4 described in Chapters 9 and 10 of (10) for doing the phase portraits in the Poincaré disc which appear in this paper.

Example with 2 limit cycles

For $\varepsilon = 0.001$ the system

$$\begin{aligned} \dot{x} &= -y(3x^2 + y^2) + \varepsilon(1 + y + x^2y) + \varepsilon^2(-856.6373973x + y^3) \\ &= -y(3x^2 + y^2) + 0.001 + 0.001y + 0.001x^2y - 0.0008566373973x + 0.000001y^3, \\ \dot{y} &= x(x^2 - y^2) + \varepsilon(1 + y^2) + \varepsilon^2(x + 73.80732101y^3) \\ &= x(x^2 - y^2) + 0.001 + 0.001y^2 + 0.000001x + 0.00007380732101y^3, \end{aligned} \tag{12}$$

gives

$$G_{20}(r_0) = 231.8725375 r_0 - 993.0560642 \frac{1}{r_0} + 95.95703341 \frac{1}{r_0^3} + 665.2264933 \frac{1}{r_0^5},$$

and $G_{20}(r_0) = 0$ has the two positive zeros $r_0 = 1$ and $r_0 = 2$. In Figure 2 we see the two limit cycles bifurcated from the degenerate center of the unperturbed system (12).

$$\varepsilon = 0.001$$

Figure 2: Two limit cycles bifurcate from the degenerate center of the unperturbed system (12).

Example with 1 limit cycle

For $\varepsilon = 0.001$ the system

$$\begin{aligned} \dot{x} &= -y(3x^2 + y^2) + \varepsilon(1 - 176.5322447x) + \varepsilon^2(x + x^3) \\ &= -y(3x^2 + y^2) + 0.001 - 0.1765312447x + 0.000001x^3, \\ \dot{y} &= x(x^2 - y^2) + \varepsilon(10 + 10y + 5xy) + \varepsilon^2(y - y^3) \\ &= x(x^2 - y^2) + 0.010 + 0.010001y + 0.005xy - 0.000001y^3, \end{aligned} \tag{13}$$

has

$$G_{20}(r_0) = -1.570796327r_0 + 21.62373221\frac{1}{r_0} - 5545.821670\frac{1}{r_0^3} + 6652.264933\frac{1}{r_0^5}.$$

From relation $G_{20}(r_0) = 0$ we obtain $-1.097575824, 1.097575824, -5.725902515 - 5.148324797i, 5.725902515 + 5.148324797i, -5.725902515 + 5.148324797i, 5.725902515 - 5.148324797i$. So only one limit cycle can bifurcate from a periodic orbit of the center of the unperturbed system (13), as we can see in Figure 3.

$$\varepsilon = 0.001$$

Figure 3: The limit cycle bifurcated from the degenerate center of the unperturbed system (13).

Example with zero limit cycles

Now for $\varepsilon = 0.001$ we consider system

$$\begin{aligned} \dot{x} &= -y(3x^2 + y^2) + \varepsilon + \varepsilon^2 x \\ &= -y(3x^2 + y^2) + 0.001 + 0.000001x, \\ \dot{y} &= x(x^2 - y^2) + \varepsilon(1 + y^2) + \varepsilon^2 y^3 \\ &= x(x^2 - y^2) + 0.001 + 0.001y^2 + 0.000001y^3, \end{aligned} \tag{14}$$

with

$$G_{20}(r_0) = 3.141592654r_0 + 1.159249021\frac{1}{r_0} + 95.95703341\frac{1}{r_0^3} + 665.2264933\frac{1}{r_0^5}.$$

We have that $G_{20}(r_0) = 0$ has solutions $-2.116012294 - 1.570359831i, 2.116012294 + 1.570359831i, -2.095699520i, 2.095699520i, -2.116012294 + 1.570359831i, 2.116012294 - 1.570359831i$. So no limit cycles can bifurcate from the degenerate center, see also Figure 4.

$$\varepsilon = 0.001$$

Figure 4: No limit cycle bifurcates from the degenerate center of the unperturbed system (14).

5. Appendix A

$$\begin{aligned}
g_{1,1}(\theta) &= \cos \theta (2 \cos^5 \theta - 1) a_{00} + \sin \theta (1 + 2 \cos^2 \theta)^2 c_{00}, \\
g_{1,2}(\theta) &= \sin^2 \theta (1 + 2 \cos^2 \theta) c_{01} + \sin \theta \cos \theta (1 + 2 \cos^2 \theta) c_{10} \\
&\quad + \sin \theta \cos \theta (2 \cos^2 \theta - 1) a_{01} + \cos^2 \theta (2 \cos^2 \theta - 1) a_{10}, \\
g_{1,3}(\theta) &= \sin^3 \theta (1 + 2 \cos^2 \theta) c_{02} + (-2 \cos^5 \theta + \cos^3 \theta + \cos \theta) c_{11} \\
&\quad + \cos^2 \theta \sin \theta (1 + 2 \cos^2 \theta) c_{20} + (-2 \cos^5 \theta + 3 \cos^3 \theta - \cos \theta) a_{02} + \\
&\quad \cos^2 \theta \sin \theta (2 \cos^2 \theta - 1) a_{11} + \cos^3 \theta (2 \cos^2 \theta - 1) a_{20}, \\
g_{1,4}(\theta) &= \sin^4 \theta (1 + 2 \cos^2 \theta) c_{03} + \sin^3 \theta \cos \theta (1 + 2 \cos^2 \theta) c_{12} \\
&\quad + \sin^2 \theta \cos^2 \theta (1 + 2 \cos^2 \theta) c_{21} + \sin \theta \cos^3 \theta (1 + 2 \cos^2 \theta) c_{30} \\
&\quad + \sin^3 \theta \cos \theta (2 \cos^2 \theta - 1) a_{03} + \sin^2 \theta \cos^2 \theta (2 \cos^2 \theta - 1) a_{12} \\
&\quad + \sin \theta \cos^3 \theta (2 \cos^2 \theta - 1) a_{21} + \cos^4 \theta (2 \cos^2 \theta - 1) a_{30}, \\
g_{2,1}(\theta) &= (2 \cos^2 \theta + 1 - 4 \cos^4 \theta) a_{00} c_{00} - c_{00}^2 \sin \theta \cos \theta (2 \cos^2 \theta + 1) c_{00}^2 \\
&\quad + \sin \theta \cos \theta (2 \cos^2 \theta - 1) a_{00}^2, \\
g_{2,2}(\theta) &= -\cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) c_{10} a_{00} - 2 \cos^2 \theta \sin \theta (2 \cos^2 \theta + 1) c_{10} c_{00} \\
&\quad + 2 \cos^2 \theta \sin \theta (2 \cos^2 \theta - 1) a_{10} a_{00} - \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) c_{01} a_{00} \\
&\quad - \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) c_{00} a_{01} - \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) c_{00} a_{10} \\
&\quad + (4 \cos^5 \theta - 2 \cos \theta - 2 \cos^3 \theta) c_{01} c_{00} + (6 \cos^3 \theta - 4 \cos^5 \theta - 2 \cos \theta) a_{01} a_{00}, \\
g_{2,3}(\theta) &= (\cos^2 \theta + 4 \cos^6 \theta - 6 \cos^4 \theta + 1) a_{01} c_{01} + (\cos^2 \theta + 4 \cos^6 \theta - 6 \cos^4 \theta + 1) a_{00} c_{02} \\
&\quad + (\cos^2 \theta + 4 \cos^6 \theta - 6 \cos^4 \theta + 1) a_{02} c_{00} - c_{11} a_{00} \sin \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
&\quad - c_{00} a_{11} \sin \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - 2 c_{20} c_{00} \sin \theta \cos^3 \theta (2 \cos^2 \theta + 1) \\
&\quad + 2 a_{20} a_{00} \sin \theta \cos^3 \theta (2 \cos^2 \theta - 1) + 2 a_{02} a_{00} \sin^3 \theta \cos \theta (2 \cos^2 \theta - 1) \\
&\quad - 2 c_{02} c_{00} \sin^3 \theta \cos \theta (2 \cos^2 \theta + 1) - c_{10} a_{01} \sin \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
&\quad - c_{01} a_{10} \sin \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{20} a_{00} \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
&\quad - c_{00} a_{20} \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + (-4 \cos^6 \theta + 6 \cos^4 \theta - 2 \cos^2 \theta) a_{11} a_{00} \\
&\quad + (4 \cos^6 \theta - 2 \cos^2 \theta - 2 \cos^4 \theta) c_{11} c_{00} - c_{10} a_{10} \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
&\quad + (-4 \cos^6 \theta + 6 \cos^4 \theta - 2 \cos^2 \theta) a_{10} a_{01} + (4 \cos^6 \theta - 2 \cos^2 \theta - 2 \cos^4 \theta) c_{10} c_{01} \\
&\quad + a_{10}^2 \sin \theta \cos^3 \theta (2 \cos^2 \theta - 1) - c_{01}^2 \sin^3 \theta \cos \theta (2 \cos^2 \theta + 1) \\
&\quad + a_{01}^2 \sin^3 \theta \cos \theta (2 \cos^2 \theta - 1) - c_{10}^2 \sin \theta \cos^3 \theta (2 \cos^2 \theta + 1), \\
g_{2,4}(\theta) &= -c_{03} a_{00} \sin^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{00} a_{03} \sin^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
&\quad - c_{30} a_{00} \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{00} a_{30} \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
&\quad + d_{00} \sin \theta (2 \cos^2 \theta + 1) + b_{00} \cos \theta (2 \cos^2 \theta - 1) - c_{20} a_{10} \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
&\quad - c_{10} a_{20} \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{02} a_{01} \sin^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
&\quad - c_{01} a_{02} \sin^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{21} a_{00} \cos^2 \theta \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
&\quad - c_{00} a_{21} \cos^2 \theta \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - 2 c_{12} c_{00} \sin^3 \theta \cos^2 \theta (2 \cos^2 \theta + 1) \\
&\quad - 2 c_{30} c_{00} \cos^4 \theta \sin \theta (2 \cos^2 \theta + 1) + 2 a_{12} a_{00} \sin^3 \theta \cos^2 \theta (2 \cos^2 \theta - 1) \\
&\quad + 2 a_{03} a_{00} \sin^4 \theta \cos \theta (2 \cos^2 \theta - 1) + 2 a_{30} a_{00} \cos^4 \theta \sin \theta (2 \cos^2 \theta - 1) \\
&\quad + 2 a_{02} a_{01} \sin^4 \theta \cos \theta (2 \cos^2 \theta - 1) \\
&\quad - 2 c_{10} c_{02} \sin^3 \theta \cos^2 \theta (2 \cos^2 \theta + 1) - 2 c_{20} c_{10} \cos^4 \theta \sin \theta (2 \cos^2 \theta + 1) \\
&\quad + 2 a_{11} a_{01} \sin^3 \theta \cos^2 \theta (2 \cos^2 \theta - 1) - c_{20} a_{01} \cos^2 \theta \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta)
\end{aligned}$$

$$\begin{aligned}
& + (-4 \cos^7 \theta - 2 \cos^3 \theta + 6 \cos^5 \theta) a_{21} a_{00} + (-2 \cos^5 \theta - 2 \cos^3 \theta + 4 \cos^7 \theta) c_{21} c_{00} \\
& + (-2 \cos \theta + 6 \cos^5 \theta - 4 \cos^7 \theta) c_{03} c_{00} + (4 \cos^7 \theta - 6 \cos^5 \theta + \cos^3 \theta + \cos \theta) c_{11} a_{01} \\
& + (4 \cos^7 \theta - 6 \cos^5 \theta + \cos^3 \theta + \cos \theta) c_{10} a_{02} + (4 \cos^7 \theta - 6 \cos^5 \theta + \cos^3 \theta + \cos \theta) c_{02} a_{10} \\
& + (4 \cos^7 \theta - 6 \cos^5 \theta + \cos^3 \theta + \cos \theta) c_{01} a_{11} + (-2 \cos \theta + 6 \cos^5 \theta - 4 \cos^7 \theta) c_{02} c_{01} \\
& + (-4 \cos^7 \theta - 2 \cos^3 \theta + 6 \cos^5 \theta) a_{20} a_{01} + (-4 \cos^7 \theta - 2 \cos^3 \theta + 6 \cos^5 \theta) a_{11} a_{10} \\
& + (-2 \cos^5 \theta - 2 \cos^3 \theta + 4 \cos^7 \theta) c_{20} c_{01} + (-2 \cos^5 \theta - 2 \cos^3 \theta + 4 \cos^7 \theta) c_{11} c_{10}, \\
g_{2,5}(\theta) = & -c_{10} a_{30} \cos^4 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{30} a_{10} \cos^4 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta), \\
& -c_{02}^2 \sin^5 \theta \cos \theta (2 \cos^2 \theta + 1) - c_{11}^2 \sin^3 \theta \cos^3 \theta (2 \cos^2 \theta + 1) \\
& -c_{20}^2 \cos^5 \theta \sin \theta (2 \cos^2 \theta + 1) + a_{02}^2 \sin^5 \theta \cos \theta (2 \cos^2 \theta - 1) \\
& + (-7 \cos^4 \theta + 1 - 4 \cos^8 \theta + 10 \cos^6 \theta) a_{01} c_{03} + (-7 \cos^4 \theta + 1 - 4 \cos^8 \theta + 10 \cos^6 \theta) a_{03} c_{01} \\
& + (-7 \cos^4 \theta + 1 - 4 \cos^8 \theta + 10 \cos^6 \theta) a_{02} c_{02} + d_{10} \sin \theta \cos \theta (2 \cos^2 \theta + 1) \\
& + b_{10} \cos^2 \theta (2 \cos^2 \theta - 1) + d_{01} \sin^2 \theta (2 \cos^2 \theta + 1) + a_{11}^2 \sin^3 \theta \cos^3 \theta (2 \cos^2 \theta - 1) \\
& -c_{01} a_{12} \sin^3 \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{30} a_{01} \sin \theta \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& -c_{21} a_{10} \sin \theta \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{20} a_{11} \sin \theta \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& -c_{11} a_{20} \sin \theta \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + 2 a_{10} a_{03} \sin^4 \theta \cos^2 \theta (2 \cos^2 \theta - 1) \\
& -c_{10} a_{21} \sin \theta \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{01} a_{30} \sin \theta \cos^3 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& -c_{12} a_{01} \sin^3 \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{11} a_{02} \sin^3 \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& -c_{10} a_{03} \sin^3 \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{03} a_{10} \sin^3 \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& -c_{02} a_{11} \sin^3 \theta \cos \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + 2 a_{12} a_{01} \sin^4 \theta \cos^2 \theta (2 \cos^2 \theta - 1) \\
& + 2 a_{11} a_{02} \sin^4 \theta \cos^2 \theta (2 \cos^2 \theta - 1) + a_{20}^2 \cos^5 \theta \sin \theta (2 \cos^2 \theta - 1) \\
& + 2 a_{21} a_{01} \sin^3 \theta \cos^3 \theta (2 \cos^2 \theta - 1) + 2 a_{20} a_{02} \sin^3 \theta \cos^3 \theta (2 \cos^2 \theta - 1) \\
& + 2 a_{30} a_{10} \cos^5 \theta \sin \theta (2 \cos^2 \theta - 1) - 2 c_{30} c_{10} \cos^5 \theta \sin \theta (2 \cos^2 \theta + 1) \\
& -2 c_{03} c_{01} \sin^5 \theta \cos \theta (2 \cos^2 \theta + 1) - c_{20} a_{20} \cos^4 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& -2 c_{21} c_{01} \sin^3 \theta \cos^3 \theta (2 \cos^2 \theta + 1) - 2 c_{20} c_{02} \sin^3 \theta \cos^3 \theta (2 \cos^2 \theta + 1) \\
& -2 c_{12} c_{10} \sin^3 \theta \cos^3 \theta (2 \cos^2 \theta + 1) + 2 a_{12} a_{10} \sin^3 \theta \cos^3 \theta (2 \cos^2 \theta - 1) \\
& + 2 a_{03} a_{01} \sin^5 \theta \cos \theta (2 \cos^2 \theta - 1) + (6 \cos^6 \theta - 4 \cos^8 \theta - 2 \cos^2 \theta) c_{10} c_{03} \\
& + (-4 \cos^8 \theta - 2 \cos^4 \theta + 6 \cos^6 \theta) a_{30} a_{01} + (-4 \cos^8 \theta - 2 \cos^4 \theta + 6 \cos^6 \theta) a_{21} a_{10} \\
& + (-4 \cos^8 \theta - 2 \cos^4 \theta + 6 \cos^6 \theta) a_{20} a_{11} + (\cos^2 \theta - 6 \cos^6 \theta + \cos^4 \theta + 4 \cos^8 \theta) c_{11} a_{11} \\
& + (\cos^2 \theta - 6 \cos^6 \theta + \cos^4 \theta + 4 \cos^8 \theta) c_{01} a_{21} + (-2 \cos^6 \theta - 2 \cos^4 \theta + 4 \cos^8 \theta) c_{30} c_{01} \\
& + (-2 \cos^6 \theta - 2 \cos^4 \theta + 4 \cos^8 \theta) c_{21} c_{10} + (-2 \cos^6 \theta - 2 \cos^4 \theta + 4 \cos^8 \theta) c_{20} c_{11} \\
& + (\cos^2 \theta - 6 \cos^6 \theta + \cos^4 \theta + 4 \cos^8 \theta) c_{20} a_{02} + (\cos^2 \theta - 6 \cos^6 \theta + \cos^4 \theta + 4 \cos^8 \theta) c_{12} a_{10} \\
& + (\cos^2 \theta - 6 \cos^6 \theta + \cos^4 \theta + 4 \cos^8 \theta) c_{10} a_{12} + (\cos^2 \theta - 6 \cos^6 \theta + \cos^4 \theta + 4 \cos^8 \theta) c_{02} a_{20} \\
& + (\cos^2 \theta - 6 \cos^6 \theta + \cos^4 \theta + 4 \cos^8 \theta) c_{21} a_{01} + (6 \cos^6 \theta - 4 \cos^8 \theta - 2 \cos^2 \theta) c_{12} c_{01} \\
& + (6 \cos^6 \theta - 4 \cos^8 \theta - 2 \cos^2 \theta) c_{11} c_{02} + b_{01} \sin \theta \cos \theta (2 \cos^2 \theta - 1), \\
g_{2,6}(\theta) = & -2 c_{21} c_{11} \sin^3 \theta \cos^4 \theta (2 \cos^2 \theta + 1) - 2 c_{20} c_{12} \sin^3 \theta \cos^4 \theta (2 \cos^2 \theta + 1) \\
& -2 c_{30} c_{20} \cos^6 \theta \sin \theta (2 \cos^2 \theta + 1) - c_{20} a_{03} \sin^3 \theta \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& + 2 a_{21} a_{11} \sin^3 \theta \cos^4 \theta (2 \cos^2 \theta - 1) \\
& + 2 a_{20} a_{12} \sin^3 \theta \cos^4 \theta (2 \cos^2 \theta - 1) - c_{30} a_{20} \cos^5 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& -c_{20} a_{30} \cos^5 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + (\cos^3 \theta + \cos \theta - 2 \cos^5 \theta) d_{11} \\
& + (-\cos \theta + 3 \cos^3 \theta - 2 \cos^5 \theta) b_{02} + d_{02} \sin^3 \theta (2 \cos^2 \theta + 1) \\
& -c_{20} a_{21} \cos^4 \theta \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + 2 a_{21} a_{02} \sin^4 \theta \cos^3 \theta (2 \cos^2 \theta - 1) \\
& + 2 a_{20} a_{03} \sin^4 \theta \cos^3 \theta (2 \cos^2 \theta - 1) + 2 a_{12} a_{11} \sin^4 \theta \cos^3 \theta (2 \cos^2 \theta - 1)
\end{aligned}$$

$$\begin{aligned}
& +2 a_{03} a_{02} \sin^6 \theta \cos \theta (2 \cos^2 \theta - 1) - 2 c_{03} c_{02} \sin^6 \theta \cos \theta (2 \cos^2 \theta + 1) \\
& +2 a_{11} a_{03} \sin^5 \theta \cos^2 \theta (2 \cos^2 \theta - 1) - 2 c_{11} c_{03} \sin^5 \theta \cos^2 \theta (2 \cos^2 \theta + 1) \\
& -2 c_{30} c_{02} \sin^3 \theta \cos^4 \theta (2 \cos^2 \theta + 1) + 2 a_{12} a_{02} \sin^5 \theta \cos^2 \theta (2 \cos^2 \theta - 1) \\
& -c_{11} a_{30} \cos^4 \theta \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& -c_{11} a_{12} \sin^3 \theta \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{21} a_{02} \sin^3 \theta \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& +b_{20} \cos^3 \theta (2 \cos^2 \theta - 1) - 2 c_{12} c_{02} \sin^5 \theta \cos^2 \theta (2 \cos^2 \theta + 1) \\
& -c_{12} a_{11} \sin^3 \theta \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{30} a_{11} \cos^4 \theta \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& -c_{21} a_{20} \cos^4 \theta \sin \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + 2 a_{30} a_{20} \cos^6 \theta \sin \theta (2 \cos^2 \theta - 1) \\
& -c_{03} a_{20} \sin^3 \theta \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) - c_{02} a_{21} \sin^3 \theta \cos^2 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& +2 a_{30} a_{02} \sin^3 \theta \cos^4 \theta (2 \cos^2 \theta - 1) - c_{03} a_{02} \sin^5 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) \\
& +d_{20} \cos^2 \theta \sin \theta (2 \cos^2 \theta + 1) + b_{11} \cos^2 \theta \sin \theta (2 \cos^2 \theta - 1) \\
& -c_{02} a_{03} \sin^5 \theta (-2 \cos^2 \theta - 1 + 4 \cos^4 \theta) + (\cos \theta + 10 \cos^7 \theta - 7 \cos^5 \theta - 4 \cos^9 \theta) c_{11} a_{03} \\
& + (\cos \theta + 10 \cos^7 \theta - 7 \cos^5 \theta - 4 \cos^9 \theta) c_{03} a_{11} + (\cos^5 \theta - 6 \cos^7 \theta + 4 \cos^9 \theta + \cos^3 \theta) c_{21} a_{11} \\
& + (\cos^5 \theta - 6 \cos^7 \theta + 4 \cos^9 \theta + \cos^3 \theta) c_{11} a_{21} + (\cos^5 \theta - 6 \cos^7 \theta + 4 \cos^9 \theta + \cos^3 \theta) c_{20} a_{12} \\
& + (\cos^5 \theta - 6 \cos^7 \theta + 4 \cos^9 \theta + \cos^3 \theta) c_{12} a_{20} + (\cos^5 \theta - 6 \cos^7 \theta + 4 \cos^9 \theta + \cos^3 \theta) c_{02} a_{30} \\
& + (\cos \theta + 10 \cos^7 \theta - 7 \cos^5 \theta - 4 \cos^9 \theta) c_{12} a_{02} + (\cos \theta + 10 \cos^7 \theta - 7 \cos^5 \theta - 4 \cos^9 \theta) c_{02} a_{12} \\
& + (\cos^5 \theta - 6 \cos^7 \theta + 4 \cos^9 \theta + \cos^3 \theta) c_{30} a_{02} + (-2 \cos^3 \theta + 6 \cos^7 \theta - 4 \cos^9 \theta) c_{21} c_{02} \\
& + (-2 \cos^3 \theta + 6 \cos^7 \theta - 4 \cos^9 \theta) c_{20} c_{03} + (-2 \cos^3 \theta + 6 \cos^7 \theta - 4 \cos^9 \theta) c_{12} c_{11} \\
& + (-4 \cos^9 \theta + 6 \cos^7 \theta - 2 \cos^5 \theta) a_{30} a_{11} + (-4 \cos^9 \theta + 6 \cos^7 \theta - 2 \cos^5 \theta) a_{21} a_{20} \\
& + (4 \cos^9 \theta - 2 \cos^5 \theta - 2 \cos^7 \theta) c_{30} c_{11} + (4 \cos^9 \theta - 2 \cos^5 \theta - 2 \cos^7 \theta) c_{21} c_{20}, \\
g_{2,7}(\theta) = & -c_{21}^2 \sin^3 \theta \cos^5 \theta (2 \cos^2 \theta + 1) + a_{12}^2 \sin^5 \theta \cos^3 \theta (-1 + 2 \cos^2 \theta) \\
& -c_{03}^2 \sin^7 \theta \cos \theta (2 \cos^2 \theta + 1) + a_{03}^2 \sin^7 \theta \cos \theta (-1 + 2 \cos^2 \theta) \\
& +a_{21}^2 \sin^3 \theta \cos^5 \theta (-1 + 2 \cos^2 \theta) - c_{12}^2 \sin^5 \theta \cos^3 \theta (2 \cos^2 \theta + 1) - c_{30}^2 \\
& \cos^7 \theta \sin \theta (2 \cos^2 \theta + 1) + 2 a_{30} a_{12} \sin^3 \theta \cos^5 \theta (-1 + 2 \cos^2 \theta) \\
& -c_{03} a_{12} \sin^5 \theta \cos \theta (4 \cos^4 \theta - 2 \cos^2 \theta - 1) - c_{30} a_{21} \cos^5 \theta \sin \theta (4 \cos^4 \theta - 2 \cos^2 \theta - 1) \\
& -c_{12} a_{21} \sin^3 \theta \cos^3 \theta (4 \cos^4 \theta - 2 \cos^2 \theta - 1) - c_{30} a_{03} \sin^3 \theta \cos^3 \theta (4 \cos^4 \theta - 2 \cos^2 \theta - 1) \\
& -2 c_{21} c_{03} \sin^5 \theta \cos^3 \theta (2 \cos^2 \theta + 1) - 2 c_{12} c_{03} \sin^6 \theta \cos^2 \theta (2 \cos^2 \theta + 1) \\
& +2 a_{30} a_{03} \sin^4 \theta \cos^4 \theta (-1 + 2 \cos^2 \theta) + 2 a_{12} a_{03} (\sin \theta)^6 \cos^2 \theta (-1 + 2 \cos^2 \theta) \\
& +2 a_{21} a_{12} \sin^4 \theta \cos^4 \theta (-1 + 2 \cos^2 \theta) - c_{21} a_{30} \cos^5 \theta \sin \theta (4 \cos^4 \theta - 2 \cos^2 \theta - 1) \\
& -c_{03} a_{30} \sin^3 \theta \cos^3 \theta (4 \cos^4 \theta - 2 \cos^2 \theta - 1) - c_{21} a_{12} \sin^3 \theta \cos^3 \theta (4 \cos^4 \theta - 2 \cos^2 \theta - 1) \\
& +b_{03} \sin^3 \theta \cos \theta (-1 + 2 \cos^2 \theta) + b_{12} \sin^2 \theta \cos^2 \theta (-1 + 2 \cos^2 \theta) \\
& -c_{30} a_{30} \cos^6 \theta (4 \cos^4 \theta - 2 \cos^2 \theta - 1) - a_{03} c_{03} \sin^6 \theta (4 \cos^4 \theta - 2 \cos^2 \theta - 1) \\
& +d_{12} \sin^3 \theta \cos \theta (2 \cos^2 \theta + 1) + d_{21} \sin^2 \theta \cos^2 \theta (2 \cos^2 \theta + 1) \\
& +d_{30} \sin \theta \cos^3 \theta (2 \cos^2 \theta + 1) - 2 c_{30} c_{12} \sin^3 \theta \cos^5 \theta (2 \cos^2 \theta + 1) \\
& +2 a_{21} a_{03} \sin^5 \theta \cos^3 \theta (-1 + 2 \cos^2 \theta) - c_{12} a_{03} \sin^5 \theta \cos \theta (4 \cos^4 \theta - 2 \cos^2 \theta - 1) \\
& +d_{03} \sin^4 \theta (2 \cos^2 \theta + 1) + b_{30} \cos^4 \theta (-1 + 2 \cos^2 \theta) + a_{30}^2 \cos^7 \theta \sin \theta (-1 + 2 \cos^2 \theta) \\
& + (6 \cos^8 \theta - 4 \cos^{10} \theta - 2 \cos^4 \theta) c_{30} c_{03} + (6 \cos^8 \theta - 4 \cos^{10} \theta - 2 \cos^4 \theta) c_{21} c_{12} \\
& + (4 \cos^{10} \theta - 2 \cos^8 \theta - 2 \cos^6 \theta) c_{30} c_{21} + b_{21} \sin \theta \cos^3 \theta (-1 + 2 \cos^2 \theta) \\
& + (-6 \cos^8 \theta + \cos^4 \theta + \cos^6 \theta + 4 \cos^{10} \theta) c_{21} a_{21} + (-4 \cos^{10} \theta - 2 \cos^6 \theta + 6 \cos^8 \theta) a_{30} a_{21} \\
& + (\cos^2 \theta - 7 \cos^6 \theta + 10 \cos^8 \theta - 4 \cos^{10} \theta) c_{03} a_{21} + (-6 \cos^8 \theta + \cos^4 \theta + \cos^6 \theta + 4 \cos^{10} \theta) c_{12} a_{30} \\
& + (\cos^2 \theta - 7 \cos^6 \theta + 10 \cos^8 \theta - 4 \cos^{10} \theta) c_{12} a_{12} + (-6 \cos^8 \theta + \cos^4 \theta + \cos^6 \theta + 4 \cos^{10} \theta) c_{30} a_{12} \\
& + (\cos^2 \theta - 7 \cos^6 \theta + 10 \cos^8 \theta - 4 \cos^{10} \theta) c_{21} a_{03}.
\end{aligned}$$

6. Appendix B

We present the expressions of $A_4, A_2, A_0, A_5, A_3, A_1, \tilde{A}_1$ that appear in equation (9).

$$\begin{aligned}
A_4(\theta) = & e^{5 \sin^2 \theta} \left[\left[- \frac{(4 a_{01} c_{00} I_2 - 8 I_1 a_{01} c_{00} - 8 I_1 c_{00} c_{10} - 4 a_{00} c_{01} I_3 + 4 c_{00} c_{10} I_2) \cos^4 \theta}{I_2 - 2 I_1} \right. \right. \\
& - \frac{(-4 I_1 c_{00} c_{10} + 2 a_{00} c_{01} I_3 - 2 a_{01} c_{00} I_2 + 2 c_{00} c_{10} I_2 + 4 I_1 a_{01} c_{00}) \cos^2 \theta}{I_2 - 2 I_1} \\
& - \left. \frac{(-a_{01} c_{00} I_2 + 2 I_1 a_{01} c_{00} + 2 I_1 a_{00} c_{01} - a_{00} c_{01} I_2)}{I_2 - 2 I_1} \right] \sin \theta \\
& - \frac{(4 c_{00} c_{01} I_3 + 4 a_{00} c_{10} I_2 - 8 I_1 a_{00} a_{01} - 8 I_1 a_{00} c_{10} + 4 a_{00} a_{01} I_2) (\cos \theta)^5}{I_2 - 2 I_1} \\
& - \frac{(12 I_1 a_{00} a_{01} + 4 I_1 a_{00} c_{10} - 2 c_{00} c_{01} I_3 - 2 a_{00} c_{10} I_2 - 6 a_{00} a_{01} I_2) (\cos \theta)^3}{I_2 - 2 I_1} \\
& - \left. \frac{(-2 c_{00} c_{01} I_1 + c_{00} c_{01} I_2 - 4 I_1 a_{00} a_{01} + 2 a_{00} a_{01} I_2 + 2 I_1 a_{00} c_{10} - c_{00} c_{01} I_3 - a_{00} c_{10} I_2) \cos \theta}{I_2 - 2 I_1} \right], \\
A_2(\theta) = & [A_{2,7} \cos^7 \theta + A_{2,6} \cos^6 \theta \sin \theta + A_{2,5} \cos^5 \theta + A_{2,4} \cos^4 \theta \sin \theta + A_{2,3} \cos^3 \theta + A_{2,2} \cos^2 \theta \sin \theta \\
& + A_{2,1} \cos \theta + A_{2,0} \theta \sin \theta] \exp(3 \sin^2 \theta), \\
A_{2,7} = & 4 a_{03} a_{00} - 4 a_{21} a_{00} - 4 c_{30} a_{00} + 4 c_{12} a_{00} - 4 a_{20} a_{01} + 4 a_{02} a_{01} + 4 c_{11} a_{01} - 4 c_{10} a_{20} \\
& + 4 c_{10} a_{02} + 4 \frac{a_{11} c_{01} I_3}{2 I_1 - I_2} + 4 c_{00} a_{12} + 4 c_{03} c_{00} + 8 c_{21} c_{00} + 4 \frac{c_{01} c_{20} I_3}{2 I_1 - I_2} \\
& - 4 \frac{c_{01} c_{02} I_3}{2 I_1 - I_2} + 4 c_{11} c_{10}, \\
A_{2,6} = & -4 a_{12} a_{00} - 4 c_{03} a_{00} - 8 c_{21} a_{00} - 4 a_{11} a_{01} - 4 c_{20} a_{01} + 4 c_{02} a_{01} - 4 \frac{a_{20} c_{01} I_3}{2 I_1 - I_2} \\
& + 4 \frac{a_{02} c_{01} I_3}{2 I_1 - I_2} - 4 a_{11} c_{10} + 4 a_{03} c_{00} - 4 a_{21} c_{00} - 4 c_{30} c_{00} + 4 c_{12} c_{00} \\
& + 4 \frac{c_{01} c_{11} I_3}{2 I_1 - I_2} - 4 c_{20} c_{10} + 4 c_{10} c_{02}, \\
A_{2,5} = & -10 a_{03} a_{00} + 6 a_{21} a_{00} + 2 c_{30} a_{00} - 6 c_{12} a_{00} + 6 a_{20} a_{01} - 10 a_{02} a_{01} - 6 c_{11} a_{01} + 2 c_{10} a_{20} \\
& - 6 c_{10} a_{02} - 6 \frac{a_{11} c_{01} I_3}{2 I_1 - I_2} - 6 c_{00} a_{12} + 2 c_{03} c_{00} - 4 c_{21} c_{00} - 2 \frac{c_{01} c_{20} I_3}{2 I_1 - I_2} \\
& + 6 \frac{c_{01} c_{02} I_3}{2 I_1 - I_2} - 2 c_{11} c_{10}, \\
A_{2,4} = & 6 a_{00} a_{12} - 2 a_{00} c_{03} + 4 a_{00} c_{21} + 6 a_{01} a_{11} + 2 a_{01} c_{20} - 6 a_{01} c_{02} + 2 \frac{a_{20} c_{01} I_3}{2 I_1 - I_2} \\
& - 6 \frac{a_{02} c_{01} I_3}{2 I_1 - I_2} + 2 a_{11} c_{10} - 6 a_{03} c_{00} + 2 a_{21} c_{00} - 2 c_{00} c_{30} - 2 c_{00} c_{12} - 2 \frac{c_{01} c_{11} I_3}{2 I_1 - I_2} \\
& - 2 c_{10} c_{20} - 2 c_{02} c_{10}, \\
A_{2,3} = & 2 b_{00} + 8 a_{00} a_{03} - 2 a_{00} a_{21} + a_{00} c_{30} + a_{00} c_{12} - 2 a_{01} a_{20} + 8 a_{01} a_{02} + a_{01} c_{11} + a_{20} c_{10} \\
& + a_{02} c_{10} - \frac{(-2 I_3 - I_2 + 2 I_1) c_{01} a_{11}}{2 I_1 - I_2} + a_{12} c_{00} - 2 c_{00} c_{03} - 3 c_{00} c_{21} \\
& - \frac{(-I_2 + 2 I_1 + I_3) c_{01} c_{20}}{2 I_1 - I_2} + \frac{(2 I_1 - I_3 - I_2) c_{02} c_{01}}{-I_2 + 2 I_1} - 2 c_{10} c_{11},
\end{aligned}$$

$$\begin{aligned}
A_{2,2} &= 2d_{00} - 2a_{00}a_{12} + a_{00}c_{03} + a_{00}c_{21} - 2a_{01}a_{11} + a_{01}c_{20} + a_{01}c_{02} + a_{20}c_{01} \\
&\quad - \frac{(-2I_3 - I_2 + 2I_1)c_{01}a_{02}}{2I_1 - I_2} + a_{11}c_{10} + a_{03}c_{00} + a_{21}c_{00} - 2c_{00}c_{12} \\
&\quad - \frac{(-I_2 + 2I_1 + I_3)c_{11}c_{01}}{2I_1 - I_2} - 2c_{02}c_{10}, \\
A_{2,1} &= -b_{00} - 2a_{00}a_{03} + a_{00}c_{12} - 2a_{01}a_{02} + a_{01}c_{11} + a_{02}c_{10} + a_{11}c_{01} + a_{12}c_{00} - 2c_{00}c_{03} \\
&\quad - \frac{(-I_2 + 2I_1 + I_3)c_{01}c_{02}}{2I_1 - I_2}, \\
A_{2,0} &= a_{03}c_{00} + d_{00} + a_{00}c_{03} + a_{01}c_{02} + a_{02}c_{01}.
\end{aligned}$$

Now we have

$$\begin{aligned}
A_0(\theta) &= A_{0,9} \cos^9 \theta + A_{0,8} \cos^8 \theta + A_{0,7} \cos^7 \theta + A_{0,6} \cos^6 \theta + A_{0,5} \cos^5 \theta + A_{0,4} \cos^4 \theta \\
&\quad + A_{0,3} \cos^3 \theta + A_{0,2} \cos^2 \theta + A_{0,1} \cos \theta + A_{0,0},
\end{aligned}$$

with

$$\begin{aligned}
A_{0,9}(\theta) &= -(-4a_{21}c_{11} + 4c_{02}c_{03} + 8c_{02}c_{21} - 4c_{03}c_{20} + 4c_{11}c_{12} - 4c_{11}c_{30} - 8c_{20}c_{21} \\
&\quad - 8a_{11}c_{21} + 4a_{12}c_{02} - 4a_{12}c_{20} + 4a_{20}a_{21} - 4a_{20}c_{12} + 4a_{20}c_{30} + 4a_{02}a_{03} - 4a_{02}a_{21} \\
&\quad + 4a_{02}c_{12} - 4a_{02}c_{30} - 4a_{03}a_{20} + 4a_{03}c_{11} - 4a_{11}a_{12} - 4a_{11}c_{03}) e^{\sin^2 \theta}, \\
A_{0,8}(\theta) &= -(-4a_{03}c_{20} + 4a_{11}a_{21} - 4a_{11}c_{12} + 4a_{12}a_{20} - 4a_{02}c_{03} - 8a_{02}c_{21} - 4a_{03}a_{11} \\
&\quad + 4a_{03}c_{02} - 8c_{11}c_{21} - 4c_{12}c_{20} + 4c_{20}c_{30} + 4a_{11}c_{30} - 4a_{02}a_{12} - 4a_{12}c_{11} \\
&\quad + 4a_{20}c_{03} + 8a_{20}c_{21} - 4a_{21}c_{02} + 4a_{21}c_{20} + 4c_{02}c_{12} - 4c_{02}c_{30} - 4c_{03}c_{11}) e^{\sin^2 \theta} \sin \theta, \\
A_{0,7}(\theta) &= -(2c_{11}c_{30} + 4c_{20}c_{21} - 14a_{02}a_{03} + 10a_{02}a_{21} - 10a_{02}c_{12} + 6a_{02}c_{30} + 10a_{03}a_{20} \\
&\quad - 10a_{03}c_{11} + 10a_{11}a_{12} + 2a_{11}c_{03} + 12a_{11}c_{21} - 10a_{12}c_{02} + 6a_{12}c_{20} - 6a_{20}a_{21} + 6a_{20}c_{12} \\
&\quad - 2a_{20}c_{30} + 6a_{21}c_{11} - 2c_{02}c_{03} - 12c_{02}c_{21} - 2c_{03}c_{20} - 6c_{11}c_{12}) e^{\sin^2 \theta}, \\
A_{0,6}(\theta) &= -(2c_{20}c_{30} + 2a_{02}c_{03} + 6a_{12}c_{11} + 2a_{20}c_{03} - 4a_{20}c_{21} + 6a_{21}c_{02} - 2a_{21}c_{20} - 6c_{02}c_{12} \\
&\quad + 2c_{02}c_{30} - 2c_{03}c_{11} + 4c_{11}c_{21} + 2c_{12}c_{20} - 6a_{11}a_{21} + 6a_{11}c_{12} - 2a_{11}c_{30} - 6a_{12}a_{20} \\
&\quad + 6a_{03}c_{20} + 12a_{02}c_{21} + 10a_{03}a_{11} + 10a_{02}a_{12} - 10a_{03}c_{02}) e^{\sin^2 \theta} \sin \theta, \\
A_{0,5}(\theta) &= -[-2b_{20} + 18a_{02}a_{03} - 8a_{11}a_{12} + 2b_{02} - a_{20}c_{12} - a_{20}c_{30} - a_{21}c_{11} - 4c_{02}c_{03} + c_{02}c_{21} \\
&\quad + 2c_{03}c_{20} + 2c_{1,1}c_{30} + 3c_{20}c_{21} + 3a_{11}c_{03} - 3a_{11}c_{21} + 7a_{12}c_{02} - a_{12}c_{20} + 2a_{20}a_{21} \\
&\quad + 7a_{02}c_{12} - a_{02}c_{30} + 2d_{11} - 8a_{02}a_{21} - 8a_{03}a_{20} + 7a_{03}c_{11}] e^{\sin^2 \theta}, \\
A_{0,4}(\theta) &= -[7a_{03}c_{02} - a_{03}c_{20} - 8a_{03}a_{11} - a_{11}c_{30} + 2a_{12}a_{20} - a_{12}c_{11} - a_{11}c_{12} - a_{20}c_{21} \\
&\quad - 8a_{02}a_{12} + 3a_{02}c_{0,3} - 3a_{02}c_{21} + 2c_{03}c_{11} + 3c_{11}c_{21} + 2c_{12}c_{20} + 2a_{11}a_{21} - a_{20}c_{03} \\
&\quad - 2b_{11} + 2d_{02} - 2d_{2,0} + 2c_{02}c_{30} - a_{21}c_{02} - a_{21}c_{20}) e^{\sin^2 \theta} \sin \theta, \\
A_{0,3}(\theta) &= -[-a_{11}c_{21} - d_{11} - a_{12}c_{20} - 3b_{02} + b_{20} + 2a_{11}a_{12} + 3c_{02}c_{21} + 2c_{03}c_{20} - a_{20}c_{12} \\
&\quad - a_{21}c_{11} - 10a_{02}a_{03} + 2a_{02}a_{21} + 2c_{11}c_{12} + 2a_{03}a_{20} - a_{02}c_{30}] e^{\sin^2 \theta}, \\
A_{0,2}(\theta) &= -[2a_{03}a_{11} - a_{03}c_{20} - a_{02}c_{21} - a_{20}c_{03} - a_{21}c_{02} - a_{11}c_{12} - a_{12}c_{11} - d_{02} - d_{20} \\
&\quad + 2c_{02}c_{12} + 2c_{03}c_{11} + b_{11} + 2a_{02}a_{12}] e^{\sin^2 \theta} \sin \theta, \\
A_{0,1}(\theta) &= -(2c_{02}c_{03} - a_{02}c_{12} + b_{02} - d_{11} + 2a_{02}a_{03} - a_{03}c_{11} - a_{11}c_{03} - a_{12}c_{02}) e^{\sin^2 \theta}, \\
A_{0,0}(\theta) &= -[-d_{02} - a_{02}c_{03} - a_{03}c_{02}] e^{\sin^2 \theta} \sin \theta.
\end{aligned}$$

Additionally, we have

$$\begin{aligned}
A_5(\theta) &= [\sin \theta \cos \theta (-1 + 2 \cos^2 \theta) a_{00}^2 + (-4 \cos^4 \theta + 2 \cos^2 \theta + 1) c_{00} a_{00} \\
&\quad - \sin \theta \cos \theta (2 \cos^2 \theta + 1) c_{00}^2] e^{6 \sin^2 \theta}, \\
A_3(\theta) &= 2 \cos^3 \theta \sin \theta (-1 + 2 \cos^2 \theta) a_{00} a_{20} - 2 \cos \theta \sin \theta (\cos^2 \theta - 1) (-1 + 2 \cos^2 \theta) a_{00} a_{02} \\
&\quad - 2 \cos^2 \theta (\cos^2 \theta - 1) (-1 + 2 \cos^2 \theta) a_{00} a_{11} - \cos^2 \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{00} c_{20} \\
&\quad + (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{00} c_{02} - \cos \theta \sin \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{11} a_{00} \\
&\quad - \cos \theta \sin \theta (\cos^2 \theta - 1) (-1 + 2 \cos^2 \theta) a_{01}^2 - (\cos^2 \theta - 1) \frac{2 I_1 - 4 \cos^4 \theta I_3 + 2 \cos^2 \theta I_3 - I_2}{2 I_1 - I_2} a_{01} c_{01} \\
&\quad - \cos \theta \sin \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{10} a_{01} - \cos^2 \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{00} a_{20} \\
&\quad + (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{00} a_{02} - \cos \theta \sin \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{00} a_{11} \\
&\quad - 2 \cos^3 \theta \sin \theta (2 \cos^2 \theta + 1) c_{00} c_{20} + 2 \cos \theta \sin \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) c_{00} c_{02} \\
&\quad + 2 \cos^2 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) c_{11} c_{00} - \cos \theta \sin \theta I_3 \frac{2 I_1 - I_2 + \cos^2 \theta I_3 - 2 \cos^4 \theta I_3}{2 I_1 - I_2} c_{01}^2 \\
&\quad - \cos^2 \theta \frac{2 I_1 + I_3 + 2 \cos^2 \theta I_3 - I_2 - 4 \cos^4 \theta I_3}{2 I_1 - I_2} c_{10} c_{01} - \cos^3 \theta \sin \theta (2 \cos^2 \theta + 1) c_{10}^2] \exp(4 \sin^2 \theta), \\
A_1(\theta) &= \left[-\cos^4 \theta \frac{2 I_1 + I_3 + 2 \cos^2 \theta I_3 - I_2 - 4 \cos^4 \theta I_3}{2 I_1 - I_2} c_{01} c_{30} + \cos^4 \theta (8 \cos^4 \theta - 3 - 4 \cos^2 \theta) c_{10} c_{21} \right. \\
&\quad + 2 \cos^2 \theta (2 \cos^2 \theta - 1) (\cos^2 \theta - 1)^2 a_{02} a_{11} + \cos^2 \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{20} c_{02} \\
&\quad + \sin \theta \cos \theta (2 \cos^2 \theta - 1) (\cos^2 \theta - 1)^2 a_{02}^2 - \sin \theta \cos^3 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta - 1) a_{11}^2 \\
&\quad - 2 \cos^4 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta - 1) a_{11} a_{20} \\
&\quad + e^{2 \sin^2 \theta} \cos^2 \theta (\cos^2 \theta - 1) (8 \cos^4 \theta - 4 \cos^2 \theta - 1) a_{01} c_{21} \\
&\quad + 2 \cos^2 \theta (2 \cos^2 \theta - 1) (\cos^2 \theta - 1)^2 a_{01} a_{12} + \cos^2 \theta (2 \cos^2 \theta - 1) b_{10} \\
&\quad + \cos^2 \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{10} a_{12} + 2 \cos^4 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) c_{11} c_{20} \\
&\quad - \sin \theta \cos \theta (2 \cos^2 \theta + 1) (\cos^2 \theta - 1)^2 c_{02}^2 - 2 \cos^2 \theta (2 \cos^2 \theta + 1) (\cos^2 \theta - 1)^2 c_{11} c_{02} \\
&\quad + \sin \theta \cos^3 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) c_{11}^2 + \cos^2 \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{11} a_{11} \\
&\quad + \cos^2 \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{02} c_{20} \\
&\quad - \sin \theta \cos \theta (\cos^2 \theta - 1) \frac{2 I_1 - 4 \cos^4 \theta I_3 + 2 \cos^2 \theta I_3 - I_2}{2 I_1 - I_2} a_{12} c_{01} \\
&\quad - \sin \theta \cos \theta \frac{-4 \cos^6 \theta I_3 - I_2 + 2 I_1 + 2 \cos^2 \theta I_1 + \cos^2 \theta I_3 + I_3 - 2 \cos^4 \theta I_3 - \cos^2 \theta I_2}{2 I_1 - I_2} c_{01} c_{03} \\
&\quad - \sin \theta \cos^5 \theta (2 \cos^2 \theta + 1) c_{20}^2 - \sin \theta \cos^3 \theta \frac{4 \cos^2 \theta I_3 + 4 I_1 - 8 \cos^4 \theta I_3 + I_3 - 2 I_2}{2 I_1 - I_2} c_{01} c_{21} \\
&\quad + 2 \cos^2 \theta (\cos^4 \theta - \cos^2 \theta - 1 + 2 \cos^6 \theta) c_{03} c_{10} + \sin \theta \cos^5 \theta (-1 + 2 \cos^2 \theta) a_{20}^2 \\
&\quad - \cos^4 \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{20} c_{20} + (\cos^2 \theta - 1)^2 \frac{2 I_1 - 4 \cos^4 \theta I_3 + 2 \cos^2 \theta I_3 - I_2}{2 I_1 - I_2} a_{03} c_{01} \\
&\quad - (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) d_{01} + \sin \theta \cos \theta (2 \cos^2 \theta + 1) d_{10} + \sin \theta \cos \theta (-1 + 2 \cos^2 \theta) b_{01} \\
&\quad - \sin \theta \cos^3 \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{11} c_{20} - \sin \theta \cos^3 \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{01} c_{30} \\
&\quad + (\cos^2 \theta - 1) (4 \cos^6 \theta + 2 \cos^4 \theta - (\cos \theta)^2 - 1) a_{01} c_{03} \\
&\quad - \cos^2 \theta (\cos^2 \theta - 1) \frac{2 I_1 - 4 \cos^4 \theta I_3 + 2 \cos^2 \theta I_3 - I_2}{2 I_1 - I_2} a_{21} c_{01} \\
&\quad - (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) (\cos^2 \theta - 1)^2 a_{02} c_{02} - \sin \theta \cos^3 \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{11} a_{20} \\
&\quad + \cos^2 \theta (\cos^2 \theta - 1) \frac{2 I_1 + I_3 + 2 \cos^2 \theta I_3 - I_2 - 4 \cos^4 \theta I_3}{2 I_1 - I_2} c_{12} c_{01} \\
&\quad - \sin \theta \cos^3 \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{10} a_{21} - 2 \sin \theta \cos^5 \theta (2 \cos^2 \theta + 1) c_{10} c_{30} \\
&\quad + \sin \theta \cos \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{12} a_{01} \\
&\quad - 2 \sin \theta \cos^3 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta - 1) a_{01} a_{21} \\
&\quad + 2 \sin \theta \cos \theta (-1 + 2 \cos^2 \theta) (\cos^2 \theta - 1)^2 a_{01} a_{03} - 2 \sin \theta \cos^3 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta - 1) a_{02} a_{20} \\
&\quad + \sin \theta \cos \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{11} a_{02} \\
&\quad + \sin \theta \cos \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{11} c_{02} \\
&\quad + \sin \theta \cos \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) c_{10} a_{03} \\
&\quad + 2 \sin \theta \cos^3 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) c_{12} c_{10} \\
&\quad \left. + 2 \sin \theta \cos^3 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) c_{02} c_{20} \right] e^{2 \sin^2 \theta},
\end{aligned}$$

$$\begin{aligned}
\tilde{A}_1(\theta) = & \cos \theta \sin \theta (\cos^4 \theta - \cos^2 \theta + 2 \cos^8 \theta + 3 \cos^6 \theta - 1) c_{03}^2 \\
& - (4 \cos^6 \theta + 2 \cos^4 \theta - \cos^2 \theta - 1) (\cos^2 \theta - 1)^2 a_{03} c_{03} \\
& - \cos^5 \theta \sin \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{21} c_{30} + \cos^6 \theta (8 \cos^4 \theta - 3 - 4 \cos^2 \theta) c_{21} c_{30} \\
& + 2 \cos^4 \theta (\cos^4 \theta - \cos^2 \theta - 1 + 2 \cos^6 \theta) c_{03} c_{30} \\
& + \cos^3 \theta \sin \theta (2 \cos^2 \theta + 1) d_{30} + \cos^3 \theta \sin \theta (-1 + 2 \cos^2 \theta) b_{21} \\
& + 2 c_{21}^2 \cos^5 \theta \sin \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) + a_{12}^2 \sin^5 \theta \cos^3 \theta (-1 + 2 \cos^2 \theta) \\
& - \cos^7 \theta \sin \theta (2 \cos^2 \theta + 1) c_{30}^2 + (2 \cos^2 \theta + 1) (\cos^2 \theta - 1)^2 d_{03} \\
& + \cos^3 \theta \sin \theta (4 \cos^4 \theta + 8 \cos^6 \theta - 3 - 3 \cos^2 \theta) c_{03} c_{21} - \cos^2 \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) d_{21} \\
& - \cos^2 \theta (\cos^2 \theta - 1) (-1 + 2 \cos^2 \theta) b_{12} + \cos^4 \theta (-1 + 2 \cos^2 \theta) b_{30} \\
& - \cos \theta \sin \theta (-1 + 2 \cos^2 \theta) (\cos^2 \theta - 1)^3 a_{03}^2 + \cos^4 \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{12} c_{30} \\
& - 2 \cos^2 \theta (\cos^2 \theta - 1) (\cos^4 \theta - \cos^2 \theta - 1 + 2 \cos^6 \theta) c_{03} c_{12} \\
& - \cos \theta \sin \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) d_{12} \\
& - \cos \theta \sin \theta (\cos^2 \theta - 1) (-1 + 2 \cos^2 \theta) b_{03} + 2 \cos^4 \theta (-1 + 2 \cos^2 \theta) (\cos^2 \theta - 1)^2 a_{12} a_{21} \\
& + \cos^4 \theta (\cos^2 \theta - 1) (8 \cos^4 \theta - 4 \cos^2 \theta - 1) a_{21} c_{21} \\
& - \cos^4 \theta (\cos^2 \theta - 1) (8 \cos^4 \theta - 3 - 4 \cos^2 \theta) c_{12} c_{21} \\
& - 2 \cos^2 \theta (-1 + 2 \cos^2 \theta) (\cos^2 \theta - 1)^3 a_{03} a_{12} - \cos^2 \theta (8 \cos^4 \theta - 4 \cos^2 \theta - 1) (\cos^2 \theta - 1)^2 a_{03} c_{21} \\
& - \cos^2 \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) (\cos^2 \theta - 1)^2 a_{12} c_{12} \\
& + \cos^2 \theta (\cos^2 \theta - 1) (4 \cos^6 \theta + 2 \cos^4 \theta - \cos^2 \theta - 1) a_{21} c_{03} \\
& - \cos^5 \theta \sin \theta (\cos^2 \theta - 1) (-1 + 2 \cos^2 \theta) a_{21}^2 \\
& - \cos^3 \theta \sin \theta (2 \cos^2 \theta + 1) (\cos^2 \theta - 1)^2 c_{12}^2 + 2 \cos^5 \theta \sin \theta (\cos^2 \theta - 1) (2 \cos^2 \theta + 1) c_{12} c_{30} \\
& + 2 \cos^3 \theta \sin \theta (-1 + 2 \cos^2 \theta) (\cos^2 \theta - 1)^2 a_{03} a_{21} \\
& + \cos^3 \theta \sin \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{03} c_{30} \\
& + \cos^3 \theta \sin \theta (\cos^2 \theta - 1) (8 \cos^4 \theta - 4 \cos^2 \theta - 1) a_{12} c_{21} \\
& + \cos^3 \theta \sin \theta (\cos^2 \theta - 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) a_{21} c_{12} \\
& - \cos \theta \sin \theta (4 \cos^4 \theta - 1 - 2 \cos^2 \theta) (\cos^2 \theta - 1)^2 a_{03} c_{12} \\
& + \cos \theta \sin \theta (\cos^2 \theta - 1) (4 \cos^6 \theta + 2 \cos^4 \theta - \cos^2 \theta - 1) a_{12} c_{03}.
\end{aligned}$$

7. Appendix C

Here we present the explicit expressions of $s_5(\theta)$, $s_4(\theta)$, $s_3(\theta)$, $s_2(\theta)$, $s_1(\theta)$, $\tilde{s}_1(\theta)$ that appear in relation (10). Thus $s_5(\theta) = s_{5,1}(\theta)c_{00}^2 + s_{5,2}(\theta)a_{00}^2 + s_{5,3}(\theta)a_{00}c_{00}$, with

$$\begin{aligned}
s_{5,1}(\theta) = & -2e^3 \sin^2 \theta \left(\int_0^\theta e^{3 \sin^2 w} \sin w dw + 2 \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \right) \\
& (\sin \theta + 2 \sin \theta \cos^2 \theta), \\
s_{5,2}(\theta) = & -2e^3 \sin^2 \theta \left(-\int_0^\theta e^{3 \sin^2 w} \cos w dw + 2 \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw \right) \\
& (-\cos \theta + 2 \cos^3 \theta),
\end{aligned}$$

$$\begin{aligned}
s_{5,3}(\theta) = & -2e^{3\sin^2\theta} \left(\int_0^\theta e^{3\sin^2 w} \sin w dw + 2 \int_0^\theta e^{3\sin^2 w} \sin w \cos^2 w dw \right) \\
& (-\cos\theta + 2\cos^3\theta) \\
& -2e^{3\sin^2\theta} \left(-\int_0^\theta e^{3\sin^2 w} \cos w dw + 2 \int_0^\theta e^{3\sin^2 w} \cos^3 w dw \right) \\
& (\sin\theta + 2\sin\theta\cos^2\theta).
\end{aligned}$$

Additionally, we have

$$s_4(\theta) = s_{4,1}(\theta)a_{00}c_{10} - s_{4,2}(\theta)a_{00}a_{01} - s_{4,3}(\theta)c_{00}c_{01} - s_{4,4}(\theta)c_{00}c_{10} - s_{4,5}(\theta)a_{00}c_{01} - s_{4,6}(\theta)a_{01}c_{00},$$

and $s_{4,i}(\theta)$ for $i = 1 \dots, 6$ are the following:

$$\begin{aligned}
s_{4,1}(\theta) = & - \left(-\sin\theta e^{-\sin^2\theta} \int_0^\theta e^{3\sin^2 w} \cos w dw + 4\cos^2\theta \int_0^\theta e^{2\sin^2 w} \sin w \cos w dw \right. \\
& + 2\sin\theta e^{-\sin^2\theta} \int_0^\theta e^{3\sin^2 w} \cos^3 w dw + 4\cos^2\theta \sin\theta e^{-\sin^2\theta} \int_0^\theta e^{3\sin^2 w} \cos^3 w dw \\
& + 8\cos^2\theta \int_0^\theta e^{2\sin^2 w} \cos^3 w \sin w dw - 2\cos^2\theta \sin\theta e^{-\sin^2\theta} \int_0^\theta e^{3\sin^2 w} \cos w dw \\
& \left. - 4 \int_0^\theta e^{2\sin^2 w} \cos^3 w \sin w dw - 2 \int_0^\theta e^{2\sin^2 w} \sin w \cos w dw \right) e^{3\sin^2\theta} \cos\theta, \\
s_{4,2}(\theta) = & - \left(\sin\theta e^{-\sin^2\theta} \int_0^\theta e^{3\sin^2 w} \cos w dw - 2\sin\theta e^{-\sin^2\theta} \int_0^\theta e^{3\sin^2 w} \cos^3 w dw \right. \\
& - 4\cos^2\theta \int_0^\theta e^{2\sin^2 w} \sin w \cos w dw - 4 \int_0^\theta e^{2\sin^2 w} \cos^3 w \sin w dw \\
& - 2\cos^2\theta \sin\theta e^{-\sin^2\theta} \int_0^\theta e^{3\sin^2 w} \cos w dw \\
& + 4\cos^2\theta \sin\theta e^{-\sin^2\theta} \int_0^\theta e^{3\sin^2 w} \cos^3 w dw + 2 \int_0^\theta e^{2\sin^2 w} \sin w \cos w dw \\
& \left. + 8\cos^2\theta \int_0^\theta e^{2\sin^2 w} \cos^3 w \sin w dw \right) e^{3\sin^2\theta} \cos\theta, \\
s_{4,3}(\theta) = & -e^{2\sin^2\theta} \left(8e^{\sin^2\theta} I_1 \cos^2\theta \sin\theta \int_0^\theta e^{2\sin^2 w} dw \right. \\
& - 2I_2 \int_0^\theta e^{3\sin^2 w} \sin w \cos^2 w dw + 2I_1 \int_0^\theta e^{3\sin^2 w} \sin w dw \\
& - 2e^{\sin^2\theta} I_2 \sin\theta \int_0^\theta e^{2\sin^2 w} dw - 2\cos^4\theta I_3 \int_0^\theta e^{3\sin^2 w} \sin w dw \\
& - 4e^{\sin^2\theta} I_2 \cos^2\theta \sin\theta \int_0^\theta e^{2\sin^2 w} dw + 2e^{\sin^2\theta} I_3 \sin\theta \int_0^\theta e^{2\sin^2 w} \cos^2 w dw \\
& + 4I_1 \int_0^\theta e^{3\sin^2 w} \sin w \cos^2 w dw - 4e^{\sin^2\theta} I_3 \sin\theta \int_0^\theta e^{2\sin^2 w} \cos^4 w dw \\
& \left. + 2\cos^2\theta I_3 \int_0^\theta e^{3\sin^2 w} \sin w \cos^2 w dw + 4e^{\sin^2\theta} I_3 \cos^2\theta \sin\theta \int_0^\theta e^{2\sin^2 w} \cos^2 w dw \right)
\end{aligned}$$

$$\begin{aligned}
& +4 e^{\sin^2 \theta} I_1 \sin \theta \int_0^\theta e^{2 \sin^2 w} dw + \cos^2 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \sin w dw \\
& - I_2 \int_0^\theta e^{3 \sin^2 w} \sin w dw - 4 \cos^4 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \\
& - 8 e^{\sin^2 \theta} I_3 \cos^2 \theta \sin \theta \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \Big) \frac{1}{2 I_1 - I_2}, \\
s_{4,4}(\theta) = & - \left(8 e^{\sin^2 \theta} \cos^2 \theta \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right. \\
& + 4 e^{\sin^2 \theta} \cos^2 \theta \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 2 \left(\int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \right) \cos \theta \\
& + \left(\int_0^\theta e^{3 \sin^2 w} \sin w dw \right) \cos \theta + 2 e^{\sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \\
& + 4 \left(\int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \right) \cos^3 \theta + 4 e^{\sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \\
& \left. + 2 \left(\int_0^\theta e^{3 \sin^2 w} \sin w dw \right) \cos^3 w \right) e^{2 \sin^2 \theta} \sin \theta, \\
s_{4,5}(\theta) = & - e^{2 \sin^2 \theta} \left(-2 I_1 \int_0^\theta e^{3 \sin^2 w} \cos w dw + 2 \cos^4 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \cos w dw \right. \\
& - 2 I_2 \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw + 4 e^{\sin^2 \theta} I_3 \cos \theta \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& - 2 e^{\sin^2 \theta} I_3 \cos \theta \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw - 4 \cos^4 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw \\
& + 2 e^{\sin^2 \theta} I_2 \cos \theta \int_0^\theta e^{2 \sin^2 w} dw + I_2 \int_0^\theta e^{3 \sin^2 w} \cos w dw \\
& - 4 e^{\sin^2 \theta} I_2 \cos^3 w \int_0^\theta e^{2 \sin^2 w} dw - 8 e^{\sin^2 \theta} I_3 \cos^3 w \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& + 4 e^{\sin^2 \theta} I_3 \cos^3 w \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw + 4 I_1 \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw \\
& + 8 e^{\sin^2 \theta} I_1 \cos^3 w \int_0^\theta e^{2 \sin^2 w} dw + 2 \cos^2 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw \\
& \left. - \cos^2 \theta I_3 \int_0^\theta e^{3 \sin^2 w} \cos w dw - 4 e^{\sin^2 \theta} I_1 \cos \theta \int_0^\theta e^{2 \sin^2 w} dw \right) \frac{1}{2 I_1 - I_2}, \\
s_{4,6}(\theta) = & - \left(4 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw + 4 \cos^3 \theta e^{-\sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \right. \\
& + 8 \cos^2 \theta \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw - 4 \cos^2 \theta \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \\
& + 2 \cos^3 \theta e^{-\sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \sin w dw - \cos \theta e^{-\sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \sin w dw \\
& \left. - 2 \cos \theta e^{-\sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw - 2 \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) e^{3 \sin^2 \theta} \sin \theta.
\end{aligned}$$

Now we have

$$\begin{aligned}
s_3(\theta) = & s_{3,1}(\theta)a_{00}c_{02} + s_{3,2}(\theta)a_{02}c_{00} + s_{3,3}(\theta)a_{01}c_{01} + s_{3,4}(\theta)a_{00}a_{11} + s_{3,5}(\theta)a_{00}c_{20} \\
& + s_{3,6}(\theta)c_{00}c_{11} + s_{3,7}(\theta)c_{01}c_{10} + s_{3,8}(\theta)a_{00}a_{02} + s_{3,9}(\theta)a_{00}a_{20} \\
& + s_{3,10}(\theta)a_{00}c_{11} + s_{3,11}(\theta)a_{01}c_{10} + s_{3,12}(\theta)a_{11}c_{00} + s_{3,13}(\theta)c_{00}c_{02} + s_{3,14}(\theta)c_{00}c_{20} \\
& + s_{3,15}(\theta)a_{01}^2 + s_{3,16}(\theta)c_{10}^2 + s_{3,17}(\theta)c_{01}^2 + s_{3,18}(\theta)a_{20}c_{00},
\end{aligned}$$

and $s_{3,i}(\theta)$ for $i = 1 \dots, 18$ satisfying the following expressions:

$$\begin{aligned}
s_{3,1}(\theta) = & -2e^{\sin^2 \theta} \cos \theta \left[\left(4 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \right) \cos^2 \theta \right. \\
& \left. - 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw + 2 \left(\int_0^\theta e^{\sin^2 w} \sin^3 w dw \right) \cos^2 \theta - \int_0^\theta e^{\sin^2 w} \sin^3 w dw \right], \\
s_{3,2}(\theta) = & 2e^{\sin^2 \theta} \sin \theta \left[2 \left(\int_0^\theta e^{\sin^2 w} \cos w dw \right) \cos^2 \theta \right. \\
& + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 4 \left(\int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) \cos^2 \theta + \int_0^\theta e^{\sin^2 w} \cos w dw \\
& \left. - 6 \left(\int_0^\theta e^{\sin^2 w} \cos^3 w dw \right) \cos^2 \theta - 3 \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right], \\
s_{3,3}(\theta) = & -e^{\sin^2 \theta} \left[-2I_1 \cos \theta \sin \theta \int_0^\theta e^{2 \sin^2 w} dw + 4I_1 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right. \\
& - 2I_1 \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 4I_1 \sin \theta \left(\int_0^\theta e^{2 \sin^2 w} dw \right) \cos^3 \theta \\
& + 2I_3 \cos \theta \sin \theta \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& + 2I_3 \sin \theta \left(\int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \right) \cos^3 \theta + 2I_3 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^2 \theta \\
& - 2I_2 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw + I_2 \cos \theta \sin \theta \int_0^\theta e^{2 \sin^2 w} dw \\
& - 4I_3 \sin \theta \left(\int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \right) \cos^3 \theta - I_3 \cos \theta \sin \theta \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& + 2I_3 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^4 \theta - 2I_2 \sin \theta \left(\int_0^\theta e^{2 \sin^2 w} dw \right) \cos^3 \theta \\
& - 4I_3 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^3 w \cos^4 \theta - I_3 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^2 \theta \\
& \left. + I_2 \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right] \frac{1}{2I_1 - I_2},
\end{aligned}$$

$$\begin{aligned}
s_{3,4}(\theta) &= 2e^{3\sin^2\theta}\cos\theta\left[\left(2\int_0^\theta e^{\sin^2 w}\cos^2 w\sin wdw\right)\cos^2\theta\right. \\
&\quad -\int_0^\theta e^{\sin^2 w}\cos^2 w\sin wdw+2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin wdw \\
&\quad \left.-4\left(\int_0^\theta e^{\sin^2 w}\cos^4 w\sin wdw\right)\cos^2\theta\right], \\
s_{3,5}(\theta) &= -2e^{3\sin^2\theta}\cos\theta\left[2\left(\int_0^\theta e^{\sin^2 w}\cos^2 w\sin wdw\right)\cos^2\theta\right. \\
&\quad +4\left(\int_0^\theta e^{\sin^2 w}\cos^4 w\sin wdw\right)\cos^2\theta \\
&\quad \left.-\int_0^\theta e^{\sin^2 w}\cos^2 w\sin wdw-2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin wdw\right], \\
s_{3,6}(\theta) &= -2e^{3\sin^2\theta}\sin\theta\left[\int_0^\theta e^{\sin^2 w}\cos wdw-2\int_0^\theta e^{\sin^2 w}\cos^5 wdw\right. \\
&\quad +\int_0^\theta e^{\sin^2 w}\cos^3 wdw+2\left(\int_0^\theta e^{\sin^2 w}\cos wdw\right)\cos^2\theta \\
&\quad \left.+2\left(\int_0^\theta e^{\sin^2 w}\cos^3 wdw\right)\cos^2\theta-4\left(\int_0^\theta e^{\sin^2 w}\cos^5 wdw\right)\cos^2\theta\right], \\
s_{3,7}(\theta) &= -e^{2\sin^2\theta}\left[2I_1\cos\theta\sin\theta\int_0^\theta e^{2\sin^2 w}dw+4I_1\sin\theta\left(\int_0^\theta e^{2\sin^2 w}dw\right)\cos^3\theta\right. \\
&\quad +2I_1\int_0^\theta e^{2\sin^2 w}\sin w\cos wdw+4I_1\int_0^\theta e^{2\sin^2 w}\cos^3 w\sin wdw \\
&\quad +I_3\cos\theta\sin\theta\int_0^\theta e^{2\sin^2 w}\cos^2 wdw-I_2\cos\theta\sin\theta\int_0^\theta e^{2\sin^2 w}dw \\
&\quad -2I_3\cos\theta\sin\theta\int_0^\theta e^{2\sin^2 w}\cos^4 wdw-I_2\int_0^\theta e^{2\sin^2 w}\sin w\cos wdw \\
&\quad +I_3\left(\int_0^\theta e^{2\sin^2 w}\sin w\cos wdw\right)\cos^2\theta-2I_2\sin\theta\left(\int_0^\theta e^{2\sin^2 w}dw\right)\cos^3\theta \\
&\quad -2I_2\int_0^\theta e^{2\sin^2 w}\cos^3 w\sin wdw-4I_3\sin\theta\left(\int_0^\theta e^{2\sin^2 w}\cos^4 wdw\right)\cos^3\theta \\
&\quad +2I_3\sin\theta\left(\int_0^\theta e^{2\sin^2 w}\cos^2 wdw\right)\cos^3\theta+2I_3\left(\int_0^\theta e^{2\sin^2 w}\cos^3 w\sin wdw\right)\cos^2\theta \\
&\quad \left.-4I_3\left(\int_0^\theta e^{2\sin^2 w}\cos^3 w\sin wdw\right)\cos^4\theta-2I_3\left(\int_0^\theta e^{2\sin^2 w}\sin w\cos wdw\right)\cos^4\theta\right]\frac{1}{2I_1-I_2},
\end{aligned}$$

$$\begin{aligned}
s_{3,8}(\theta) &= -2e^{3\sin^2\theta}\cos\theta\left[-3\int_0^\theta e^{\sin^2 w}\cos^3 w dw - 2\left(\int_0^\theta e^{\sin^2 w}\cos w dw\right)\cos^2\theta\right. \\
&\quad + 6\left(\int_0^\theta e^{\sin^2 w}\cos^2 w dw\right)\cos^2\theta + 2\int_0^\theta e^{\sin^2 w^2}\cos^5 w dw \\
&\quad \left. + \int_0^\theta e^{\sin^2 w}\cos w dw - 4\left(\int_0^\theta e^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta\right], \\
s_{3,9}(\theta) &= 2e^{3\sin^2\theta}\cos\theta\left[2\int_0^\theta e^{\sin^2 w}\cos^5 w dw - 4\left(\int_0^\theta e^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta\right. \\
&\quad \left.- \int_0^\theta e^{\sin^2 w}\cos^3 w dw + 2\left(\int_0^\theta e^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta\right], \\
s_{3,10}(\theta) &= -2e^{3\sin^2\theta}\cos\theta\left[2\left(\int_0^\theta e^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta\right. \\
&\quad + 2\left(\int_0^\theta e^{\sin^2 w}\cos w dw\right)\cos^2\theta - \int_0^\theta e^{\sin^2 w}\cos^3 w dw \\
&\quad \left.- 4\left(\int_0^\theta e^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta - \int_0^\theta e^{\sin^2 w}\cos w dw + 2\int_0^\theta e^{\sin^2 w}\cos^5 w dw\right], \\
s_{3,11}(\theta) &= 2e^{2\sin^2\theta}\sin\theta\cos\theta\left[-4\left(\int_0^\theta e^{2\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta\right. \\
&\quad \left.+ \int_0^\theta e^{2\sin^2 w}\sin w\cos w dw\right], \\
s_{3,12}(\theta) &= 2e^{3\sin^2\theta}\sin\theta\left[2\left(\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\right)\cos^2\theta\right. \\
&\quad + \int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw - 2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw \\
&\quad \left.- 4\left(\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\right)\cos^2\theta\right], \\
s_{3,13}(\theta) &= -2e^{3\sin^2\theta}\sin\theta\left[2\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw\right. \\
&\quad + \int_0^\theta e^{\sin^2 w}\sin^3 w dw + 4\left(\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw\right)\cos^2\theta \\
&\quad \left.+ 2\left(\int_0^\theta e^{\sin^2 w}\sin^3 w dw\right)\cos^2\theta\right],
\end{aligned}$$

$$\begin{aligned}
s_{3,14}(\theta) = & -2e^3 \sin^2 \theta \sin \theta \left[2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right. \\
& + 4 \left(\int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right) \cos^2 \theta + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \\
& \left. + 2 \left(\int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right) \cos^2 \theta \right],
\end{aligned}$$

$$\begin{aligned}
s_{3,15}(\theta) = & e^2 \sin^2 \theta \sin \theta \cos \theta \left[\left(-4 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^2 \theta \right. \\
& - \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 2 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^2 \theta \\
& \left. + 2 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right],
\end{aligned}$$

$$\begin{aligned}
s_{3,16}(\theta) = & -e^2 \sin^2 \theta \sin \theta \cos \theta \left[4 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^2 \theta \right. \\
& + \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 2 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^2 \theta \\
& \left. + 2 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right],
\end{aligned}$$

$$\begin{aligned}
s_{3,17}(\theta) = & -e^2 \sin^2 \theta \left[-4 I_1 I_2 \int_0^\theta e^{2 \sin^2 w} dw - 2 I_3^2 \left(\int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \right) \cos^2 \theta \right. \\
& - I_2 I_3 \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw + I_2^2 \int_0^\theta e^{2 \sin^2 w} dw \\
& + 2 I_1 I_3 \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw + 2 I_2 I_3 \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& + 2 I_1 I_3 \left(\int_0^\theta e^{2 \sin^2 w} dw \right) \cos^2 \theta + I_3^2 \left(\int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \right) \cos^2 \theta \\
& - I_2 I_3 \left(\int_0^\theta e^{2 \sin^2 w} dw \right) \cos^2 \theta + 4 I_1^2 \int_0^\theta e^{2 \sin^2 w} dw - 4 I_1 I_3 \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& - 2 I_3^2 \left(\int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \right) \cos^4 \theta + 2 I_2 I_3 \left(\int_0^\theta e^{2 \sin^2 w} dw \right) \cos^4 \theta \\
& \left. - 4 I_1 I_3 \left(\int_0^\theta e^{2 \sin^2 w} dw \right) \cos^4 \theta + 4 I_3^2 \left(\int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \right) \cos^4 \theta \right] \frac{1}{(-I_2 + 2 I_1)^2},
\end{aligned}$$

$$\begin{aligned}
s_{3,18}(\theta) = & 2e^3 \sin^2 \theta \sin \theta \left[-2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right. \\
& \left. + 2 \left(\int_0^\theta e^{\sin^2 w} \cos^3 w dw \right) \cos^2 \theta - 4 \left(\int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) \cos^2 \theta \right].
\end{aligned}$$

Now we have

$$\begin{aligned}
s_2(\theta) = & s_{2,1}(\theta) a_{00} c_{30} + s_{2,2}(\theta) a_{20} c_{10} + s_{2,3}(\theta) a_{00} c_{12} + s_{2,4}(\theta) a_{02} c_{10} + s_{2,5}(\theta) a_{12} c_{0,0} + s_{2,6}(\theta) a_{00} a_{21} \\
& + s_{2,7}(\theta) a_{01} a_{20} + s_{2,8}(\theta) a_{01} c_{11} + s_{2,9}(\theta) a_{11} c_{01} + s_{2,10}(\theta) c_{00} c_{21} + s_{2,11}(\theta) c_{01} c_{20} + s_{2,12}(\theta) c_{10} c_{11} \\
& + s_{2,13}(\theta) a_{00} a_{03} + s_{2,14}(\theta) a_{01} a_{02} + s_{2,15}(\theta) c_{00} c_{03} + s_{2,16}(\theta) c_{01} c_{02} + s_{2,17}(\theta) a_{00} a_{12} + s_{2,18}(\theta) a_{03} c_{00} \\
& + s_{2,19}(\theta) a_{11} c_{10} + s_{2,20}(\theta) a_{01} c_{02} + s_{2,21}(\theta) a_{01} c_{20} + s_{2,22}(\theta) a_{01} a_{11} + s_{2,23}(\theta) a_{00} c_{03} \\
& + s_{2,24}(\theta) a_{00} c_{21} + s_{2,25}(\theta) a_{02} c_{01} + s_{2,26}(\theta) a_{20} c_{01} + s_{2,27}(\theta) a_{21} c_{00} \\
& + s_{2,28}(\theta) c_{00} c_{12} + s_{2,29}(\theta) c_{00} c_{30} + s_{2,30}(\theta) c_{01} c_{11} + s_{2,31}(\theta) c_{02} c_{10} + s_{2,32}(\theta) c_{10} c_{20},
\end{aligned}$$

and $s_{2,i}(\theta)$ for $i = 1, \dots, 32$ satisfying the following expressions:

$$\begin{aligned}
s_{2,1}(\theta) = & \frac{1}{6} \left(12 \cos^2 \theta \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw - 12 \cos^4 \theta \sin \theta \left(\int_0^\theta e^{3 \sin^2 w} \cos w dw \right) e^{-3 \sin^2 \theta} \right. \\
& + 2 \cos^6 \theta - 14 \cos^2 \theta - 6 \cos^2 \theta \sin \theta \left(\int_0^\theta e^{3 \sin^2 w} \cos w dw \right) e^{-3 \sin^2 \theta} \\
& \left. + 24 \cos^4 \theta \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw - 3 \cos^4 \theta + 7 + 8 \cos^8 \theta \right) e^{3 \sin^2 \theta} \cos \theta, \\
s_{2,2}(\theta) = & \left(-2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right. \\
& \left. + 2 \left(\int_0^\theta e^{\sin^2 w} \cos^3 w dw \right) \cos^2 \theta - 4 \left(\int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) \cos^2 \theta \right) e^{2 \sin^2 \theta} \sin \theta \cos \theta, \\
s_{2,3}(\theta) = & -\frac{1}{6} \left(-12 \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw - 9 \cos^4 \theta \right. \\
& - 12 \cos^4 \theta \sin \theta \left(\int_0^\theta e^{3 \sin^2 w} \cos w dw \right) e^{-3 \sin^2 \theta} + 6 \cos^2 \theta \sin \theta \left(\int_0^\theta e^{3 \sin^2 w} \cos w dw \right) e^{-3 \sin^2 \theta} \\
& - 10 \cos^6 \theta - 12 \cos^2 \theta \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw + 6 \sin \theta \left(\int_0^\theta e^{3 \sin^2 w} \cos w dw \right) e^{-3 \sin^2 \theta} \\
& \left. + 8 \cos^8 \theta + 16 \cos^2 \theta + 24 \cos^4 \theta \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw - 5 \right) e^{3 \sin^2 \theta} \cos \theta, \\
s_{2,4}(\theta) = & - \left(-4 \left(\int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) \cos^2 \theta + 3 \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 6 \left(\int_0^\theta e^{\sin^2 w} \cos^3 w dw \right) \cos^2 \theta \right. \\
& \left. - \int_0^\theta e^{\sin^2 w} \cos w dw - 2 \left(\int_0^\theta e^{\sin^2 w} \cos w dw \right) \cos^2 \theta - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) e^{2 \sin^2 \theta} \sin \theta \cos \theta, \\
s_{2,5}(\theta) = & -\frac{1}{3} \left(6 \cos^4 \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw + 4 e^{3 \sin^2 \theta} \cos^7 \theta - 18 \left(\int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \right) \cos^2 \theta \right. \\
& - 6 e^{3 \sin^2 \theta} \cos^5 \theta - 9 \left(\int_0^\theta e^{3 \sin^2 w} \sin w dw \right) \cos^2 \theta + 2 e^{3 \sin^2 \theta} \cos \theta + 3 \int_0^\theta e^{3 \sin^2 w} \sin w dw \\
& \left. + 12 \cos^4 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw + 6 \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \right) \cos^2 \theta,
\end{aligned}$$

$$\begin{aligned}
s_{2,6}(\theta) &= \frac{1}{6} \left(-12 \cos^2 \theta \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw + 1 - 2 \cos^2 \theta \right. \\
&\quad - 10 \cos^6 \theta - 12 \cos^4 \theta \sin \theta \left(\int_0^\theta e^{3 \sin^2 w} \cos w dw \right) e^{-3 \sin^2 \theta} + 3 \cos^4 \theta \\
&\quad + 24 \cos^4 \theta \sin \theta e^{-3 \sin^2 \theta} \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw \\
&\quad \left. + 6 \cos^2 \theta \sin \theta \left(\int_0^\theta e^{3 \sin^2 w} \cos w dw \right) e^{-3 \sin^2 \theta} + 8 \cos^8 \theta \right) e^{3 \sin^2 \theta} \cos \theta, \\
s_{2,7}(\theta) &= \left(2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 4 \left(\int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) \cos^2 \theta \right. \\
&\quad \left. - \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 2 \left(\int_0^\theta e^{\sin^2 w} \cos^3 w dw \right) \cos^2 \theta \right) e^{2 \sin^2 \theta} \sin \theta \cos \theta, \\
s_{2,8}(\theta) &= - \left(2 \left(\int_0^\theta e^{\sin^2 w} \cos^3 w dw \right) \cos^2 \theta + 2 \left(\int_0^\theta e^{\sin^2 w} \cos w dw \right) \cos^2 \theta - \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right. \\
&\quad \left. - 4 \left(\int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) \cos^2 \theta - \int_0^\theta e^{\sin^2 w} \cos w dw + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) e^{2 \sin^2 \theta} \sin \theta \cos \theta, \\
s_{2,9}(\theta) &= e^{2 \sin^2 \theta} \left(-I_2 \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + 2 I_1 \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad - 4 I_1 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 2 I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad + I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - 2 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \\
&\quad \left. + 4 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw + 2 I_2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right) \frac{1}{2 I_1 - I_2}, \\
s_{2,10}(\theta) &= -\frac{1}{3} \cos^2 \theta \left(-6 \left(\int_0^\theta e^{3 \sin^2 w} \sin w dw \right) \cos^2 \theta - 2 e^{3 \sin^2 \theta} \cos \theta - 3 \int_0^\theta e^{3 \sin^2 w} \sin w dw \right. \\
&\quad - 6 \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw - 12 \left(\int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \right) \cos^2 \theta - 6 e^{3 \sin^2 \theta} \cos^3 \theta \\
&\quad \left. + 8 e^{3 \sin^2 \theta} \cos^7 \theta + 12 \cos^4 \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw + 24 \cos^4 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \right), \\
s_{2,11}(\theta) &= -e^{2 \sin^2 \theta} \left(2 I_1 \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + 2 I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right. \\
&\quad - I_2 \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \\
&\quad - 2 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - 4 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad \left. - 2 I_2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw + 4 I_1 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right) \frac{1}{2 I_1 - I_2},
\end{aligned}$$

$$\begin{aligned}
s_{2,12}(\theta) = & - \left(\int_0^\theta e^{\sin^2 w} \cos w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right. \\
& + 2 \left(\int_0^\theta e^{\sin^2 w} \cos w dw \right) \cos^2 \theta + 2 \left(\int_0^\theta e^{\sin^2 w} \cos^3 w dw \right) \cos^2 \theta \\
& \left. - 4 \left(\int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) \cos^2 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \\
s_{2,13}(\theta) = & -\frac{1}{6} \left(-12 \cos^4 \theta \sin \theta \int_0^\theta e^{3 \sin^2 w} \cos w dw - 6 \sin \theta \int_0^\theta e^{3 \sin^2 w} \cos w dw \right. \\
& + 24 \cos^4 \theta \sin \theta \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw + 12 \sin \theta \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw \\
& - 36 \cos^2 \theta \sin \theta \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw - 22 e^{3 \sin^2 \theta} \cos^6 \theta + e^{3 \sin^2 \theta} \\
& - 8 e^{3 \sin^2 \theta} \cos^2 \theta + 8 e^{3 \sin^2 \theta} \cos^8 \theta + 18 \sin \theta \cos^2 \theta \int_0^\theta e^{3 \sin^2 w} \cos w dw \\
& \left. + 21 e^{3 \sin^2 \theta} \cos^4 \theta \right) \cos \theta, \\
s_{2,14}(\theta) = & - \left(-3 \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 2 \left(\int_0^\theta e^{\sin^2 w} \cos w dw \right) \cos^2 \theta + 6 \left(\int_0^\theta e^{\sin^2 w} \cos^3 w dw \right) \cos^2 \theta \right. \\
& + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + \int_0^\theta e^{\sin^2 w} \cos w dw - 4 \left(\int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) \cos^2 \theta \left. \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \\
s_{2,15}(\theta) = & -2 e^{3 \sin^2 \theta} \cos^7 \theta - \cos^4 \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw - 2 \cos^6 \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw \\
& - 2 e^{3 \sin^2 \theta} \cos^5 \theta - 2 \cos^4 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \\
& + \frac{10}{3} e^{3 \sin^2 \theta} \cos^3 \theta - 4 \cos^6 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw - \frac{4}{3} \cos^9 \theta e^{3 \sin^2 \theta} \\
& + 2 e^{3 \sin^2 \theta} \cos \theta + \int_0^\theta e^{3 \sin^2 w} \sin w dw + 2 \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw, \\
s_{2,16}(\theta) = & -e^{2 \sin^2 \theta} \left(-2 I_2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw + 2 I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \right. \\
& + 2 I_1 \int_0^\theta e^{\sin^2 w} \sin^3 w dw + 4 I_1 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw - 4 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \\
& \left. + I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw - I_2 \int_0^\theta e^{\sin^2 w} \sin^3 w dw - 2 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw \right) \frac{1}{2 I_1 - I_2}, \\
s_{2,17}(\theta) = & \frac{1}{3} \left(-6 \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw + 3 \int_0^\theta e^{3 \sin^2 w} \cos w dw + 2 e^{3 \sin^2 \theta} \cos^2 \theta \sin \theta \right. \\
& + 6 \cos^4 \theta \int_0^\theta e^{3 \sin^2 w} \cos w dw + 4 e^{3 \sin^2 \theta} \cos^6 \theta \sin \theta - 6 e^{3 \sin^2 \theta} \cos^4 \theta \sin \theta \\
& \left. - 9 \cos^2 \theta \int_0^\theta e^{3 \sin^2 w} \cos w dw + (18 \cos^2 \theta - 12 \cos^4 \theta) \int_0^\theta e^{3 \sin^2 w} \cos^3 w dw \right) \cos^2 \theta,
\end{aligned}$$

$$\begin{aligned}
s_{2,18}(\theta) = & -\frac{1}{6} \left(8 e^3 \sin^2 \theta \cos^8 \theta - 14 e^3 \sin^2 \theta \cos^6 \theta - e^3 \sin^2 \theta + 12 \cos^5 \theta \int_0^\theta e^3 \sin^2 w \sin w dw \right. \\
& + 24 \cos^5 \theta \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw + 4 e^3 \sin^2 \theta \cos^2 \theta - 36 \cos^3 \theta \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw \\
& - 18 \cos^3 \theta \int_0^\theta e^3 \sin^2 w \sin w dw + 6 \cos \theta \int_0^\theta e^3 \sin^2 w \sin w dw \\
& \left. + 12 \cos \theta \int_0^\theta e^3 \sin^2 w \sin w \cos^2 w dw + 3 e^3 \sin^2 \theta \cos^4 \theta \right) \sin \theta, \\
s_{2,19}(\theta) = & \left(2 \left(\int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right) \cos^2 \theta + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
& \left. - 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 4 \left(\int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right) \cos^2 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \\
s_{2,20}(\theta) = & - \left(4 \left(\int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \right) \cos^2 \theta - 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \right. \\
& \left. - \int_0^\theta e^{\sin^2 w} \sin^3 w dw + 2 \left(\int_0^\theta e^{\sin^2 w} \sin^3 w dw \right) \cos^2 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \\
s_{2,21}(\theta) = & - \left(2 \left(\int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right) \cos^2 \theta + 4 \left(\int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right) \cos^2 \theta \right. \\
& \left. - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta \\
s_{2,22}(\theta) = & \left(2 \left(\int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right) \cos^2 \theta - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
& \left. + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 4 \left(\int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right) \cos^2 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \\
s_{2,23}(\theta) = & 2 e^3 \sin^2 \theta \cos^6 \theta \sin \theta + \frac{4}{3} e^3 \sin^2 \theta \cos^8 \theta \sin \theta + 2 \int_0^\theta e^3 \sin^2 w \cos^3 w dw + \frac{8}{3} e^3 \sin^2 \theta \cos^4 \theta \sin \theta \\
& - 2 \cos^4 \theta \int_0^\theta e^3 \sin^2 w \cos^3 w dw - 2 e^3 \sin^2 \theta \cos^2 \theta \sin \theta + 2 \cos^6 \theta \int_0^\theta e^3 \sin^2 w \cos w dw \\
& - 4 \cos^6 \theta \int_0^\theta e^3 \sin^2 w \cos^3 w dw - \int_0^\theta e^3 \sin^2 w \cos w dw + \cos^4 \theta \int_0^\theta e^3 \sin^2 w \cos w dw, \\
s_{2,24}(\theta) = & \frac{1}{3} \left(6 \int_0^\theta e^3 \sin^2 w \cos^3 w dw + 12 \cos^4 \theta \int_0^\theta e^3 \sin^2 w \cos w dw + 12 \cos^2 \theta \int_0^\theta e^3 \sin^2 w \cos^3 w dw \right. \\
& - 3 \int_0^\theta e^3 \sin^2 w \cos w dw + 8 e^3 \sin^2 \theta \cos^6 \theta \sin \theta - 24 \cos^4 \theta \int_0^\theta e^3 \sin^2 w \cos^3 w dw \\
& \left. - 6 \cos^2 \theta \int_0^\theta e^3 \sin^2 w \cos w dw - 2 e^3 \sin^2 \theta \cos^2 \theta \sin \theta \right) \cos^2 \theta,
\end{aligned}$$

$$\begin{aligned}
s_{2,25}(\theta) = & -e^{2 \sin^2 \theta} \left(3 I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 2 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right. \\
& + 6 I_1 \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 4 I_1 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 4 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
& + 2 I_2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + I_2 \int_0^\theta e^{\sin^2 w} \cos w dw - 6 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \\
& - 2 I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 I_1 \int_0^\theta e^{\sin^2 w} \cos w dw \\
& \left. - 3 I_2 \int_0^\theta e^{\sin^2 w} \cos^3 w dw - I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right) \frac{1}{2 I_1 - I_2}, \\
s_{2,26}(\theta) = & e^{2 \sin^2 \theta} \left(-2 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 4 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right. \\
& - I_2 \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 2 I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 I_2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
& + 2 I_1 \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 4 I_1 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \left. \right) \frac{1}{2 I_1 - I_2}, \\
s_{2,27}(\theta) = & \frac{1}{6} \left(-6 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw + 12 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw - 3 e^{3 \sin^2 \theta} \cos^4 \theta \right. \\
& - 2 e^{3 \sin^2 \theta} \cos^2 \theta - 2 e^{3 \sin^2 \theta} \cos^6 \theta - e^{3 \sin^2 \theta} + 24 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \\
& \left. - 12 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw + 8 e^{3 \sin^2 \theta} \cos^8 \theta \right) \sin \theta, \\
s_{2,28}(\theta) = & -\frac{1}{6} \left(5 e^{3 \sin^2 \theta} + 24 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \right. \\
& - 12 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw - 12 \cos \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \\
& - 6 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw - 15 e^{3 \sin^2 \theta} \cos^4 \theta - 6 \cos \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw \\
& + 4 e^{3 \sin^2 \theta} \cos^2 \theta + 8 e^{3 \sin^2 \theta} \cos^8 \theta + 12 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw \\
& \left. - 2 e^{3 \sin^2 \theta} \cos^6 \theta \right) \sin \theta, \\
s_{2,29}(\theta) = & \frac{1}{6} \left(-14 e^{3 \sin^2 \theta} \cos^2 \theta + 10 e^{3 \sin^2 \theta} \cos^6 \theta + 12 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw \right. \\
& + 6 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w dw + 12 \cos^3 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw + 8 e^{3 \sin^2 \theta} \cos^8 \theta \\
& \left. + 3 e^{3 \sin^2 \theta} \cos^4 \theta - 7 e^{3 \sin^2 \theta} + 24 \cos^5 \theta \int_0^\theta e^{3 \sin^2 w} \sin w \cos^2 w dw \right) \sin \theta,
\end{aligned}$$

$$\begin{aligned}
s_{2,30}(\theta) = & -e^{2 \sin^2 \theta} \left(-4 I_1 \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right. \\
& - 2 I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 I_2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \\
& + 2 I_1 \int_0^\theta e^{\sin^2 w} \cos w dw - I_2 \int_0^\theta e^{\sin^2 w} \cos^3 w dw + I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \\
& + 2 I_1 \int_0^\theta e^{\sin^2 w} \cos^3 w dw + I_3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \\
& \left. - I_2 \int_0^\theta e^{\sin^2 w} \cos w dw + 4 I_3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right) \frac{1}{2 I_1 - I_2}, \\
s_{2,31}(\theta) = & - \left(2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw + \int_0^\theta e^{\sin^2 w} \sin^3 w dw + 4 \left(\int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \right) \cos^2 \theta \right. \\
& \left. + 2 \left(\int_0^\theta e^{\sin^2 w} \sin^3 w dw \right) \cos^2 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \\
s_{2,32}(\theta) = & - \left(2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw + 4 \left(\int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right) \cos^2 \theta \right. \\
& \left. + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + 2 \left(\int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right) \cos^2 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta.
\end{aligned}$$

We also have

$$\begin{aligned}
s_1(\theta) = & s_{1,1}(\theta) a_{03} c_{10} + s_{1,2}(\theta) a_{01} c_{03} + s_{1,3}(\theta) a_{03} c_{01} + s_{1,4}(\theta) a_{01} c_{30} + s_{1,5}(\theta) c_{10} c_{21} + s_{1,6}(\theta) a_{01} a_{12} \\
& + s_{1,7}(\theta) c_{01} c_{12} + s_{1,8}(\theta) c_{01} a_{11} + s_{1,9}(\theta) c_{03} c_{10} + s_{1,10}(\theta) a_{12} c_{10} + s_{1,11}(\theta) c_{01} c_{21} \\
& + s_{1,12}(\theta) c_{10} c_{12} + s_{1,13}(\theta) a_{21} c_{10} + s_{1,14}(\theta) a_{01} a_{21} + s_{1,15}(\theta) c_{10} c_{30} + s_{1,16}(\theta) a_{01} c_{21} \\
& + s_{1,17}(\theta) a_{01} a_{03} + s_{1,18}(\theta) a_{01} c_{12} + s_{1,19}(\theta) a_{12} c_{01} + s_{1,20}(\theta) c_{01} c_{03} + s_{1,21}(\theta) a_{21} c_{01},
\end{aligned}$$

and $s_{1,i}(\theta)$ for $i = 1 \dots, 21$ are given in the following expressions:

$$\begin{aligned}
s_{1,1}(\theta) = & -\frac{1}{12} \left(24 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^4 \theta - e^{2 \sin^2 \theta} + 12 \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right. \\
& + 4 e^{2 \sin^2 \theta} \cos^2 \theta - 36 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^2 \theta - 72 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^2 \theta \\
& + 24 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw + 8 e^{2 \sin^2 \theta} \cos^8 \theta + 48 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^4 \theta \\
& \left. + 3 e^{2 \sin^2 \theta} \cos^4 \theta - 14 e^{2 \sin^2 \theta} \cos^6 \theta \right) \cos \theta \sin \theta,
\end{aligned}$$

$$\begin{aligned}
s_{1,2}(\theta) = & -\frac{1}{3} e^{2 \sin^2 \theta} \cos^6 \theta + 2 \cos^6 \theta \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw - \frac{2}{3} e^{2 \sin^2 \theta} \cos^{10} \theta \\
& - 2 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^4 \theta - \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \\
& + \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^4 \theta + \frac{7}{3} e^{2 \sin^2 \theta} \cos^4 \theta - 4 \cos^6 \theta \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \\
& - e^{2 \sin^2 \theta} \cos^2 \theta - \frac{1}{3} e^{2 \sin^2 \theta} \cos^8 \theta + 2 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw, \\
s_{1,3}(\theta) = & e^{2 \sin^2 \theta} (\cos \theta - 1) (\cos \theta + 1) (-8 \cos^8 \theta I_3 + 14 \cos^6 \theta I_3 - 7 \cos^4 \theta I_3 + 8 \cos^4 \theta I_1 \\
& - 4 \cos^4 \theta I_2 + 24 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& + 48 I_1 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 48 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& - 24 I_2 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + \cos^2 \theta I_3 - 10 \cos^2 \theta I_1 + 5 \cos^2 \theta I_2 \\
& + 12 I_2 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 12 I_3 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& - 24 I_1 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& + 24 I_3 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw - I_2 + 2 I_1 \Big) \frac{1}{12(I_2 - 2 I_1)}, \\
s_{1,4}(\theta) = & -\frac{1}{12} \left(24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw - 7 + 3 \cos^4 \theta \right. \\
& - 8 \cos^8 \theta + 14 \cos^2 \theta - 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \\
& - 2 \cos^6 \theta + 12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \\
& \left. - 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) e^{2 \sin^2 \theta} \sin \theta \cos \theta, \\
s_{1,5}(\theta) = & -\frac{1}{3} \left[-e^{2 \sin^2 \theta} \cos^2 \theta + 12 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^4 \theta \right. \\
& + 24 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^4 \theta - 3 \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \\
& - 6 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^2 \theta - 6 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \\
& - 12 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^2 \theta + 4 e^{2 \sin^2 \theta} \cos^8 \theta \\
& \left. - 3 e^{2 \sin^2 \theta} \cos^4 \theta \right] \cos^2 \theta,
\end{aligned}$$

$$\begin{aligned}
s_{1,6}(\theta) = & -\frac{1}{3} \left(12 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \cos^4 \theta + 9 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^2 \theta \right. \\
& + 6 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw - 3 \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 4 e^{2 \sin^2 \theta} \cos^4 \theta \\
& - 6 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^4 \theta - 5 e^{2 \sin^2 \theta} \cos^6 \theta + 2 e^{2 \sin^2 \theta} \cos^8 \theta \\
& \left. - 18 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^2 \theta - e^{2 \sin^2 \theta} \cos^2 \theta \right) \cos^2 \theta, \\
s_{1,7}(\theta) = & -\frac{1}{12} e^{2 \sin^2 \theta} \left(12 I_2 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \right. \\
& - 12 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw - 24 I_1 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& + 12 I_2 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + 24 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& + 24 I_3 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw - 12 I_3 \cos \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& - 24 I_1 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + 24 I_3 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} (\cos^4 w)^2 dw \\
& + 48 I_1 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 24 I_2 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& - 48 I_3 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw - 16 \cos^4 \theta I_3 + 9 \cos^6 \theta I_3 + 5 \cos^2 \theta I_3 \\
& - 12 \cos^2 \theta I_1 - 8 I_3 \cos^{10} \theta + 8 \cos^6 \theta I_1 - 4 \cos^6 \theta I_2 - 5 I_2 + 10 I_1 + 6 \cos^2 \theta I_2 \\
& \left. - 6 \cos^4 \theta I_1 + 3 \cos^4 \theta I_2 + 10 \cos^8 \theta I_3 \right) \frac{1}{2 I_1 - I_2}, \\
s_{1,8}(\theta) = & \frac{1}{12} e^{2 \sin^2 \theta} \left(-4 \cos^6 \theta I_2 - 48 I_3 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \right. \\
& - 3 \cos^4 \theta I_2 + 14 \cos^4 \theta I_3 - 24 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& + 3 \cos^6 \theta I_3 - 2 \cos^8 \theta I_3 + 24 I_3 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& + 6 \cos^4 \theta I_1 + 8 \cos^6 \theta I_1 + 7 I_2 - 24 I_2 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& - 12 I_2 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + 48 I_1 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 8 I_3 \cos^{10} \theta \\
& - 7 \cos^2 \theta I_3 + 12 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& \left. + 24 I_1 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 14 I_1 \right) \frac{1}{-I_2 + 2 I_1},
\end{aligned}$$

$$\begin{aligned}
s_{1,9}(\theta) = & - \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^4 \theta + \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \\
& - e^{2 \sin^2 \theta} \cos^8 \theta - 2 \cos^6 \theta \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw - e^{2 \sin^2 \theta} \cos^6 \theta \\
& + 5/3 e^{2 \sin^2 \theta} \cos^4 \theta + 2 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \\
& - 4 \cos^6 \theta \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw - \frac{2}{3} e^{2 \sin^2 \theta} \cos^{10} \theta \\
& + e^{2 \sin^2 \theta} \cos^2 \theta - 2 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^4 \theta, \\
s_{1,10}(\theta) = & -\frac{1}{3} \left(e^{2 \sin^2 \theta} \cos^2 \theta + 12 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^4 \theta - 3 e^{2 \sin^2 \theta} \cos^6 \theta \right. \\
& - 9 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^2 \theta + 6 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^4 \theta \\
& + 6 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw - 18 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^2 \theta \\
& \left. + 3 \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 2 e^{2 \sin^2 \theta} \cos^8 \theta \right) \cos^2 \theta, \\
s_{1,11}(\theta) = & -\frac{1}{3} e^{2 \sin^2 \theta} \cos^2 \theta \left(-6 I_1 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + 4 \cos^7 \theta \sin \theta I_3 \right. \\
& - 4 \cos^3 \theta \sin \theta I_1 + 6 I_2 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& - 12 I_1 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 3 I_3 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& + 2 \cos^3 \theta \sin \theta I_2 + 24 I_1 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + 12 I_3 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& + 12 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw - 2 \cos \theta \sin \theta I_1 \\
& - 12 I_2 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + \cos \theta \sin \theta I_2 \\
& + 6 I_3 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& - 24 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw + 3 I_2 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& \left. - \cos^3 \theta \sin \theta I_3 - 6 I_3 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \right) \frac{1}{2 I_1 - I_2},
\end{aligned}$$

$$\begin{aligned}
s_{1,12}(\theta) = & -\frac{1}{12} \left(-12 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw - 15 \cos^4 \theta - 2 \cos^6 \theta \right. \\
& + 24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw - 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \\
& - 24 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw + 8 \cos^8 \theta + 4 \cos^2 \theta \\
& + 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \\
& \left. - 12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 5 \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \\
s_{1,13}(\theta) = & \frac{1}{12} \left(-12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right. \\
& + 24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 8 \cos^8 \theta - 3 \cos^4 \theta \\
& - 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw + 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \\
& \left. - 1 - 2 \cos^2 \theta - 2 \cos^6 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \\
s_{1,14}(\theta) = & \frac{1}{12} \left(1 + 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw + 8 \cos^8 \theta \right. \\
& - 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw - 10 \cos^6 \theta \\
& - 24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw - 2 \cos^2 \theta \\
& \left. + 12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 3 \cos^4 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \\
s_{1,15}(\theta) = & \frac{1}{12} \left(24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw + 10 \cos^6 \theta \right. \\
& + 24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \\
& + 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \\
& + 3 \cos^4 \theta + 12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 8 \cos^8 \theta \\
& \left. - 14 \cos^2 \theta - 7 \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta,
\end{aligned}$$

$$\begin{aligned}
s_{1,16}(\theta) = & -\frac{1}{3} \left(-6 \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw - 12 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^4 \theta \right. \\
& + e^{2 \sin^2 \theta} \cos^2 \theta + 4 e^{2 \sin^2 \theta} \cos^8 \theta - 12 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^2 \theta \\
& + 24 \left(\int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) \cos^4 \theta - 4 e^{2 \sin^2 \theta} \cos^6 \theta \\
& + 6 \left(\int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^2 \theta - e^{2 \sin^2 \theta} \cos^4 \theta \\
& \left. + 3 \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right) \cos^2 \theta, \\
s_{1,17}(\theta) = & -\frac{1}{12} \left(48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right. \\
& - 12 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw - 24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \\
& + 36 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 8 \cos^8 \theta + 1 + 21 \cos^4 \theta \\
& + 24 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw - 72 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw - 8 \cos^2 \theta \\
& \left. - 22 \cos^6 \theta \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta, \\
s_{1,18}(\theta) = & -\frac{1}{12} \left(-24 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \right. \\
& - 24 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw + 16 \cos^2 \theta - 5 \\
& + 12 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw \\
& - 9 \cos^4 \theta - 24 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw - 10 \cos^6 \theta \\
& + 12 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \sin w \cos w dw + 8 \cos^8 \theta \\
& \left. + 48 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^3 w \sin w dw \right) e^{2 \sin^2 \theta} \cos \theta \sin \theta,
\end{aligned}$$

$$\begin{aligned}
s_{1,19}(\theta) = & -\frac{1}{3} e^{2 \sin^2 \theta} \cos^2 \theta \left(2 \cos \theta \sin \theta I_1 - 18 I_1 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \right. \\
& - \cos \theta \sin \theta I_2 - 3 \cos^5 \theta \sin \theta I_3 + \cos^3 \theta \sin \theta I_2 \\
& + 9 I_2 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + 18 I_3 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& + \cos^3 \theta \sin \theta I_3 - 6 I_2 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& + 6 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw + 6 I_1 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& + 2 \cos \theta^7 \sin \theta I_3 - 3 I_2 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 9 I_3 \cos^2 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& - 12 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw + 12 I_1 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& \left. - 2 \cos^3 \theta \sin \theta I_1 - 6 I_3 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw + 3 I_3 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \right) \frac{1}{2 I_1 - I_2},
\end{aligned}$$

$$\begin{aligned}
s_{1,20}(\theta) = & -\frac{1}{3} e^{2 \sin^2 \theta} \left(-6 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \right. \\
& + 2 \cos^3 \theta \sin \theta I_2 + 12 I_1 \cos^6 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& - 6 I_2 \cos^6 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 3 I_2 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& - 3 \cos^3 \theta \sin \theta I_3 - 12 I_3 \cos^6 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& + 6 I_1 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 6 I_1 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + \cos^5 \theta \sin \theta I_2 \\
& + 4 \cos^5 \theta \sin \theta I_3 + 3 \cos \theta \sin \theta I_2 + 3 \cos^7 \theta \sin \theta I_3 \\
& - 3 I_3 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw + 3 I_3 \cos^4 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& + 3 I_2 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw - 6 \cos \theta \sin \theta I_1 - 2 \cos^5 \theta \sin \theta I_1 \\
& + 6 I_3 e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw - 4 \cos^3 \theta \sin \theta I_1 + 2 I_3 \cos^9 \theta \sin \theta \\
& \left. + 6 I_3 \cos^6 \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \right) \frac{1}{2 I_1 - I_2},
\end{aligned}$$

$$\begin{aligned}
s_{1,21}(\theta) = & \frac{1}{12} e^{2 \sin^2 \theta} \left(2 \cos^4 \theta I_3 - 48 I_3 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \right. \\
& - 3 \cos^6 \theta I_3 - 4 \cos^6 \theta I_2 + I_2 - 12 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& - 6 \cos^4 \theta I_1 - 8 I_3 \cos^{10} \theta - 2 I_1 + 24 I_3 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^4 w dw \\
& - 24 I_2 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + 12 I_2 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \\
& - 24 I_1 \cos^3 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw + 10 \cos^8 \theta I_3 \\
& + 8 \cos^6 \theta I_1 - \cos^2 \theta I_3 + 24 I_3 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} \cos^2 w dw \\
& \left. + 3 \cos^4 \theta I_2 + 48 I_1 \cos^5 \theta \sin \theta e^{-2 \sin^2 \theta} \int_0^\theta e^{2 \sin^2 w} dw \right) \frac{1}{2 I_1 - I_2}.
\end{aligned}$$

Now we present the expression of $s_0(\theta)$.

$$\begin{aligned}
s_0(\theta) = & s_{0,1}(\theta) a_{12} c_{11} + s_{0,2}(\theta) a_{20} c_{03} + s_{0,3}(\theta) a_{20} c_{21} + s_{0,4}(\theta) a_{21} c_{11} + s_{0,5}(\theta) a_{21} c_{02} + s_{0,6}(\theta) a_{21} c_{20} \\
& + s_{0,7}(\theta) c_{02} c_{30} + s_{0,8}(\theta) a_{02} a_{12} + s_{0,9}(\theta) a_{02} c_{03} + s_{0,10}(\theta) a_{02} c_{21} + s_{0,11}(\theta) a_{03} a_{11} + s_{0,12}(\theta) a_{03} c_{02} \\
& + s_{0,13}(\theta) a_{03} c_{20} + s_{0,14}(\theta) a_{11} a_{21} + s_{0,15}(\theta) a_{11} c_{12} + s_{0,16}(\theta) a_{11} c_{30} + s_{0,17}(\theta) a_{12} a_{20} \\
& + s_{0,18}(\theta) a_{11} a_{12} + s_{0,19}(\theta) a_{03} a_{20} + s_{0,20}(\theta) a_{02} a_{21} + s_{0,21}(\theta) a_{11} c_{21} + s_{0,22}(\theta) c_{20} c_{30} \\
& + s_{0,23}(\theta) c_{11} c_{03} + s_{0,24}(\theta) c_{02} c_{03} + s_{0,25}(\theta) c_{11} c_{21} + s_{0,26}(\theta) c_{12} c_{20} + s_{0,27}(\theta) c_{02} c_{12} + s_{0,28}(\theta) a_{02} c_{12} \\
& + s_{0,29} a_{02} c_{30} + s_{0,30}(\theta) a_{02} a_{03} + s_{0,31}(\theta) a_{20} a_{21} + s_{0,32}(\theta) c_{11} c_{30} + s_{0,33}(\theta) c_{20} c_{21} + s_{0,34}(\theta) c_{02} c_{21} \\
& + s_{0,35}(\theta) c_{03} c_{20} + s_{0,36}(\theta) c_{11} c_{12} + s_{0,37}(\theta) a_{20} c_{30} + s_{0,38}(\theta) a_{12} c_{20} + s_{0,39}(\theta) a_{20} c_{12} \\
& + s_{0,40}(\theta) a_{11} c_{03} + s_{0,41}(\theta) a_{12} c_{02} + s_{0,42}(\theta) a_{03} c_{11}.
\end{aligned}$$

and $s_{0,i}(\theta)$ for $i = 1 \dots, 42$ are the following:

$$\begin{aligned}
s_{0,1}(\theta) = & -\cos^2 \theta \left(2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw + \int_0^\theta e^{\sin^2 w} \cos w dw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right. \\
& - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
& \left. + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right), \\
s_{0,2}(\theta) = & -4 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
& + 2 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw - \int_0^\theta e^{\sin^2 w} \cos^3 w dw + \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw, \\
s_{0,3}(\theta) = & \cos^2 \theta \left(2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 8 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right. \\
& \left. - \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right),
\end{aligned}$$

$$\begin{aligned}
s_{0,4}(\theta) &= \cos^3 \theta \sin \theta \left(2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw - \int_0^\theta e^{\sin^2 w} \cos w dw - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right. \\
&\quad \left. + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw - \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right), \\
s_{0,5}(\theta) &= \cos^3 \theta \sin \theta \left(2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw - \int_0^\theta e^{\sin^2 w} \sin^3 w dw \right. \\
&\quad \left. + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw - 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \right), \\
s_{0,6}(\theta) &= \cos^3 \theta \sin \theta \left(2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad \left. + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right), \\
s_{0,7}(\theta) &= \cos^3 \theta \sin \theta \left(\int_0^\theta e^{\sin^2 w} \sin^3 w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw \right. \\
&\quad \left. + 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \right), \\
s_{0,8}(\theta) &= -\cos^2 \theta \left(6 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 3 \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 9 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right. \\
&\quad - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
&\quad \left. - 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw - \int_0^\theta e^{\sin^2 w} \cos w dw + 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right), \\
s_{0,9}(\theta) &= -6 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 3 \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 3 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \\
&\quad + 4 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
&\quad + 2 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos w dw - \int_0^\theta e^{\sin^2 w} \cos w dw + \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw, \\
s_{0,10}(\theta) &= -\cos^2 \theta \left(-3 \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 12 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right. \\
&\quad + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 8 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
&\quad \left. + \int_0^\theta e^{\sin^2 w} \cos w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right), \\
s_{0,11}(\theta) &= \cos \theta \sin \theta \left(2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad \left. - 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right),
\end{aligned}$$

$$\begin{aligned}
s_{0,12}(\theta) &= -\cos \theta \sin \theta \left(2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw + \int_0^\theta e^{\sin^2 w} \sin^3 w dw \right. \\
&\quad - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \\
&\quad \left. + 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw - 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \right), \\
s_{0,13}(\theta) &= -\cos \theta \sin \theta \left(2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad \left. + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right), \\
s_{0,14}(\theta) &= -\cos^3 \theta \sin \theta \left(2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad \left. - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right), \\
s_{0,15}(\theta) &= \cos \theta \sin \theta \left(-\cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad \left. + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right), \\
s_{0,16}(\theta) &= -\cos^3 \theta \sin \theta \left(\int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad \left. - 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right), \\
s_{0,17}(\theta) &= \cos^2 \theta \left(-4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right. \\
&\quad + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + \int_0^\theta e^{\sin^2 w} \cos^3 w dw \\
&\quad \left. - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right), \\
s_{0,18}(\theta) &= \cos^2 \theta \left(2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw, \right. \\
&\quad + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \\
&\quad - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad \left. + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right),
\end{aligned}$$

$$\begin{aligned}
s_{0,19}(\theta) &= \cos \theta \sin \theta \left(-4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right. \\
&\quad + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + \int_0^\theta e^{\sin^2 w} \cos^3 w dw \\
&\quad \left. - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right), \\
s_{0,20}(\theta) &= \cos^3 \theta \sin \theta \left(6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right. \\
&\quad - 3 \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
&\quad \left. + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw + \int_0^\theta e^{\sin^2 w} \cos w dw \right), \\
s_{0,21}(\theta) &= \cos^2 \theta \left(- \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad \left. + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 8 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right), \\
s_{0,22}(\theta) &= \cos^3 \theta \sin \theta \left(\int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad \left. + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right), \\
s_{0,23}(\theta) &= -2 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos w dw + \int_0^\theta e^{\sin^2 w} \cos w dw - \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \\
&\quad + 4 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
&\quad - 2 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + \int_0^\theta e^{\sin^2 w} \cos^3 w dw - \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw, \\
s_{0,24}(\theta) &= -2 \left(\int_0^\theta e^{\sin^2 w} \sin^3 w dw \right) \cos^6 \theta + \int_0^\theta e^{\sin^2 w} \sin^3 w dw - \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw \\
&\quad - 4 \left(\int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \right) \cos^6 \theta + 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \\
&\quad - 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw, \\
s_{0,25}(\theta) &= -\cos^2 \theta \left(- \int_0^\theta e^{\sin^2 w} \cos w dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right. \\
&\quad + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 8 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
&\quad \left. - \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right),
\end{aligned}$$

$$\begin{aligned}
s_{0,26}(\theta) &= -\cos \theta \sin \theta \left(-\cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad \left. + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right), \\
s_{0,27}(\theta) &= -\cos \theta \sin \theta \left(-\cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw \right) - \int_0^\theta e^{\sin^2 w} \sin^3 w dw \\
&\quad + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw - 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \\
&\quad + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \Big), \\
s_{0,28}(\theta) &= -\cos \theta \sin \theta \left(-3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 3 \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right. \\
&\quad + 6 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
&\quad + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \\
&\quad \left. + \int_0^\theta e^{\sin^2 w} \cos w dw - 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right), \\
s_{0,29}(\theta) &= \cos^3 \theta \sin \theta \left(3 \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right. \\
&\quad \left. - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - \int_0^\theta e^{\sin^2 w} \cos w dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right), \\
s_{0,30}(\theta) &= -\cos \theta \sin \theta \left(6 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 3 \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right. \\
&\quad - 9 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
&\quad + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw - \int_0^\theta e^{\sin^2 w} \cos w dw \\
&\quad \left. + 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right), \\
s_{0,31}(\theta) &= -\cos^3 \theta \sin \theta \left(-4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right. \\
&\quad \left. + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw - \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right), \\
s_{0,32}(\theta) &= \cos^3 \theta \sin \theta \left(\int_0^\theta e^{\sin^2 w} \cos w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right. \\
&\quad \left. - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right),
\end{aligned}$$

$$\begin{aligned}
s_{0,33}(\theta) &= -\cos^2 \theta \left(-\int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad \left. - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw + 8 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right), \\
s_{0,34}(\theta) &= -\cos^2 \theta \left(-\int_0^\theta e^{\sin^2 w} \sin^3 w dw - 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw \right. \\
&\quad + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw - 2 \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \\
&\quad \left. - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw + 8 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw \right), \\
s_{0,35}(\theta) &= -2 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \\
&\quad - \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - 4 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw, \\
s_{0,36}(\theta) &= -\cos \theta \sin \theta \left(-\cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw - \int_0^\theta e^{\sin w^2} \cos w dw \right. \\
&\quad + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
&\quad - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw - \int_0^\theta e^{\sin w^2} \cos^3 w dw \\
&\quad \left. + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right), \\
s_{0,37}(\theta) &= -\cos^3 \theta \sin \theta \left(-2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 4 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \right. \\
&\quad \left. + \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right), \\
s_{0,38}(\theta) &= -\cos^2 \theta \left(2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \right. \\
&\quad - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \\
&\quad \left. + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw \right), \\
s_{0,39}(\theta) &= \cos \theta \sin \theta \left(2 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 4 \cos^4 \theta \int_0^\theta e^{\sin w^2} \cos^5 w dw \right. \\
&\quad \left. - \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw - \int_0^\theta e^{\sin^2 w} \cos^3 w dw + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right),
\end{aligned}$$

$$\begin{aligned}
s_{0,40}(\theta) &= 2 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw - \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw + \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^2 w \sin w dw \\
&\quad - 4 \cos^6 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw + 2 \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw - 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^4 w \sin w dw, \\
s_{0,41}(\theta) &= -\cos^2 \theta \left(2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw + \int_0^\theta e^{\sin^2 w} \sin^3 w dw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \sin^3 w dw \right. \\
&\quad \left. + (4 \cos^4 \theta + 2 - 6 \cos^2 \theta) \int_0^\theta e^{\sin^2 w} \sin^3 w \cos^2 w dw, \right. \\
s_{0,42}(\theta) &= -\cos \theta \sin \theta \left(2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos w dw + \int_0^\theta e^{\sin^2 w} \cos w dw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos w dw \right. \\
&\quad - 4 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw - 2 \int_0^\theta e^{\sin^2 w} \cos^5 w dw + 6 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^5 w dw \\
&\quad \left. + 2 \cos^4 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw + \int_0^\theta e^{\sin^2 w} \cos^3 w dw - 3 \cos^2 \theta \int_0^\theta e^{\sin^2 w} \cos^3 w dw \right).
\end{aligned}$$

Now we have

$$\begin{aligned}
\tilde{s}_1(\theta) &= \tilde{s}_{1,1}(\theta)c_{03}c_{12} + \tilde{s}_{1,2}(\theta)c_{03}c_{30} + \tilde{s}_{1,3}(\theta)a_{03}c_{03} + \tilde{s}_{1,4}(\theta)a_{21}c_{03} + \tilde{s}_{1,5}(\theta)c_{30}c_{12} + \tilde{s}_{1,6}(\theta)c_{12}a_{21} \\
&\quad + \tilde{s}_{1,7}(\theta)c_{30}a_{21} + \tilde{s}_{1,8}(\theta)a_{03}a_{21} + \tilde{s}_{1,9}(\theta)a_{03}c_{30} + \tilde{s}_{1,10}(\theta)a_{03}c_{12} + \tilde{s}_{1,11}(\theta)a_{12}c_{03} + \tilde{s}_{1,12}(\theta)a_{12}c_{21} \\
&\quad + \tilde{s}_{1,13}(\theta)c_{03}c_{21} + \tilde{s}_{1,14}(\theta)a_{21}c_{21} + \tilde{s}_{1,15}(\theta)a_{21}a_{12} + \tilde{s}_{1,16}(\theta)c_{12}a_{12} + \tilde{s}_{1,17}(\theta)a_{03}c_{21} \\
&\quad + \tilde{s}_{1,18}(\theta)a_{03}a_{12} + \tilde{s}_{1,19}(\theta)c_{30}c_{21} + \tilde{s}_{1,20}(\theta)c_{30}a_{12} + \tilde{s}_{1,21}(\theta)c_{12}c_{21} + \tilde{s}_{1,22}(\theta)a_{12}^2 \\
&\quad + \tilde{s}_{1,23}(\theta)c_{21}^2 + \tilde{s}_{1,24}(\theta)c_{30}^2 + \tilde{s}_{1,25}(\theta)a_{03}^2 + \tilde{s}_{1,26}(\theta)c_{03}^2 + \tilde{s}_{1,27}(\theta)a_{21}^2 + \tilde{s}_{1,28}(\theta)c_{12}^2,
\end{aligned}$$

and $\tilde{s}_{1,i}(\theta)$ for $i = 1 \dots, 28$ satisfying the following expressions:

$$\begin{aligned}
\tilde{s}_{1,1}(\theta) &= -\frac{1}{12} (16 \cos^8 \theta + 34 \cos^6 \theta + 37 \cos^4 \theta + 8 \cos^2 \theta - 5) (\cos^2 \theta - 1)^2, \\
\tilde{s}_{1,2}(\theta) &= \frac{1}{12} (\cos^2 \theta - 1) (16 \cos^{10} \theta + 38 \cos^8 \theta + 53 \cos^6 \theta + 15 \cos^4 \theta - 7 \cos^2 \theta - 7), \\
\tilde{s}_{1,3}(\theta) &= -\frac{1}{12} (16 \cos^8 \theta + 14 \cos^6 \theta + 15 \cos^4 \theta - 16 \cos^2 \theta + 1) (\cos^2 \theta - 1)^2, \\
\tilde{s}_{1,4}(\theta) &= \frac{1}{12} (\cos^2 \theta - 1) (16 \cos^{10} \theta + 18 \cos^8 \theta + 19 \cos^6 \theta - 15 \cos^4 \theta - \cos^2 \theta - 1), \\
\tilde{s}_{1,5}(\theta) &= \frac{1}{12} \cos \theta \sin^5 \theta (2 \cos^2 \theta + 1) (8 \cos^4 \theta + 12 \cos^2 \theta + 7), \\
\tilde{s}_{1,6}(\theta) &= \frac{1}{12} \cos \theta \sin^5 \theta (16 \cos^6 \theta + 12 \cos^4 \theta - 2 \cos^2 \theta + 1), \\
\tilde{s}_{1,7}(\theta) &= \frac{1}{6} \cos^3 \theta \sin^3 \theta (8 \cos^6 \theta + 8 \cos^4 \theta + 5 \cos^2 \theta - 3), \\
\tilde{s}_{1,8}(\theta) &= \frac{1}{12} \cos \theta \sin^5 \theta (-1 + 2 \cos^2 \theta) (8 \cos^4 \theta + 1), \\
\tilde{s}_{1,9}(\theta) &= \frac{1}{12} \cos \theta \sin^5 \theta (16 \cos^6 \theta + 12 \cos^4 \theta + 6 \cos^2 \theta - 7), \\
\tilde{s}_{1,10}(\theta) &= \frac{1}{6} \cos \theta \sin^7 \theta (8 \cos^4 \theta + 4 \cos^2 \theta - 3), \\
\tilde{s}_{1,11}(\theta) &= -\frac{4}{3} \cos^3 \theta \sin^3 \theta (\cos^6 \theta + \cos^4 \theta + \cos^2 \theta - 1), \\
\tilde{s}_{1,12}(\theta) &= -\frac{2}{3} \cos^5 \theta \sin^3 \theta (4 \cos^4 \theta - \cos^2 \theta - 1), \\
\tilde{s}_{1,13}(\theta) &= \frac{2}{3} \cos^3 \theta \sin \theta (-5 \cos^2 \theta - 2 + 5 \cos^6 \theta + 4 \cos^4 \theta + 4 \cos^8 \theta),
\end{aligned}$$

$$\begin{aligned}
\tilde{s}_{1,14}(\theta) &= -\frac{1}{12} \sin^2 \theta \cos^2 \theta (32 \cos^8 \theta - 4 \cos^6 \theta - 6 \cos^4 \theta - 3 \cos^2 \theta - 1), \\
\tilde{s}_{1,15}(\theta) &= \frac{1}{12} \sin^4 \theta \cos^2 \theta (-1 + 2 \cos^2 \theta) (8 \cos^4 \theta + \cos^2 \theta + 1), \\
\tilde{s}_{1,16}(\theta) &= \frac{1}{12} \sin^6 \theta \cos^2 \theta (16 \cos^4 \theta + 10 \cos^2 \theta - 5), \\
\tilde{s}_{1,17}(\theta) &= -\frac{1}{12} \sin^4 \theta \cos^2 \theta (32 \cos^6 \theta - 12 \cos^4 \theta - 6 \cos^2 \theta + 1), \\
\tilde{s}_{1,18}(\theta) &= \frac{1}{12} \sin^6 \theta \cos^2 \theta (8 \cos^2 \theta - 1) (-1 + 2 \cos^2 \theta), \\
\tilde{s}_{1,19}(\theta) &= -\frac{1}{12} \sin^2 \theta \cos^2 \theta (32 \cos^8 \theta + 36 \cos^6 \theta + 14 \cos^4 \theta - 21 \cos^2 \theta - 7), \\
\tilde{s}_{1,20}(\theta) &= \frac{1}{12} \sin^4 \theta \cos^2 \theta (16 \cos^6 \theta + 14 \cos^4 \theta + 7 \cos^2 \theta - 7), \\
\tilde{s}_{1,21}(\theta) &= -\frac{1}{12} \sin^4 \theta \cos^2 \theta (32 \cos^6 \theta + 28 \cos^4 \theta - 10 \cos^2 \theta - 5), \\
\tilde{s}_{1,22}(\theta) &= \frac{1}{3} \cos^5 \theta \sin^5 \theta (-1 + 2 \cos^2 \theta), \\
\tilde{s}_{1,23}(\theta) &= \frac{1}{3} \cos^5 \theta \sin \theta (2 \cos^2 \theta + 1) (4 \cos^4 \theta - 1 - 2 \cos^2 \theta), \\
\tilde{s}_{1,24}(\theta) &= \frac{1}{12} \cos^3 \theta \sin^3 \theta (2 \cos^2 \theta + 1) (4 \cos^4 \theta + 7 \cos^2 \theta + 7), \\
\tilde{s}_{1,25}(\theta) &= \frac{1}{12} \cos \theta \sin^7 \theta (2 \cos \theta + 1) (2 \cos \theta - 1) (-1 + 2 \cos^2 \theta), \\
\tilde{s}_{1,26}(\theta) &= \frac{1}{3} \cos \theta \sin \theta (\cos^4 \theta + 2 \cos^2 \theta + 3) (2 \cos^6 \theta - 1 + \cos^4 \theta), \\
\tilde{s}_{1,27}(\theta) &= \frac{1}{12} \cos^3 \theta \sin^3 \theta (-1 + 2 \cos^2 \theta) (4 \cos^4 \theta + \cos^2 \theta + 1), \\
\tilde{s}_{1,28}(\theta) &= \frac{1}{12} \cos \theta \sin^7 \theta (4 \cos^2 \theta + 5) (2 \cos^2 \theta + 1).
\end{aligned}$$

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