A SUFFICIENT CONDITION IN ORDER THAT THE REAL JACOBIAN CONJECTURE IN $\mathbb{R}^2$ HOLDS

FRANCISCO BRAUN[1], JAUME GINÉ[2] AND JAUME LLIBRE[3]

ABSTRACT. Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF(x)$ is different from zero for all $x \in \mathbb{R}^2$ and $F(0, 0) = (0, 0)$. We prove that for the injectivity of F it is sufficient to assume that the higher homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y + gg_y$ do not have real linear factors in common.

1. Introduction and statement of the main result

Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map such that $\det DF$ is nowhere zero. It is clear that F is a local diffeomorphism, but it is not always injective. There are very general well known additional conditions to ensure that F is a global diffeomorfphism, see for instance [8, 11, 13].

If F is a polynomial map, the statement that F is injective is known as the *real Jacobian conjecture*. This conjecture is false, since Pinchuk constructed, in [12], a non injective polynomial map with nonvanishing Jacobian determinant. Thus it is natural to ask for additional conditions in order that this conjecture holds. In [3], for instance, it is showed that it is enough to assume that the degree of f is less than or equal to 3. This result was recently generalized in [2], where it was proved that for the injectivity of F it is enough to assume that the degree of f is less than or equal to 4. If we assume that $\det DF(x) = \text{constant} \neq 0$, then to know if F is injective is an open problem largely known as the *Jacobian conjecture* (see [1] and [10] for details and for surveys on the Jacobian conjecture and related problems).

In the following result we provide a sufficient condition for the validity of the real Jacobian conjecture.

Theorem 1. *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF$ is nowhere zero and $F(0, 0) = (0, 0)$. If the higher homogeneous*

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terms of the polynomials ff_x+gg_x and ff_y+gg_y do not have real linear factors in common, then F is injective.

In the particular case $\deg f = \deg g$, we prove in Lemma 8 bellow that if the homogeneous terms of higher degree of f and g do not have real linear factors in common, then the homogeneous terms of higher degree of ff_x+gg_x and ff_y+gg_y also do not have real linear factors in common. Thus our present result is a generalization of the main result of [4], where besides the assumption $\deg f = \deg g$ it was assumed that the homogeneous terms of higher degree of f and g do not have real linear factors in common (see also [5], for a similar result in $\mathbb{R}^n$). Moreover, the following example shows that Theorem 1 is stronger than the main result of [4].

Example 2. *Let $F = (f, g)$, with $f(x, y) = x - y + x^3$ and $g(x, y) = y + x^3$. Here $\det DF(x, y) = 1 + 6x^2$. The higher homogeneous terms of f and g are both x^3 (which has x as a common factor). Moreover the higher homogeneous terms of*

$$ff_x + gg_x = x - y + 4x^3 + 6x^5, \quad ff_y + gg_y = -x + 2y$$

are $6x^5$ and $-x + 2y$ respectively, which do not have real linear factors in common.

An example satisfying the assumptions of Theorem 1 when $\deg(f) > \deg(g)$ is the following.

Example 3. *Let $F = (f, g)$, with $f(x, y) = x(3 + x^2)/3$ and $g(x, y) = x + y$. Here $\det DF(x, y) = 1 + x^2$ and the higher homogeneous terms of the polynomials*

$$ff_x + gg_x = 2x + y + \frac{4}{3}x^3 + \frac{1}{3}x^5, \quad ff_y + gg_y = x + y.$$

are $x^5/3$ and $x + y$, respectively, which do not have real linear factors in common.

We now recall a result of [7]. Firstly we introduce the notion of quasihomogeneity. Let $w_1, \dots, w_n$ and r be positive integers, and set $w = (w_1, \dots, w_n)$. We say that a polynomial map $h : \mathbb{R}^n \to \mathbb{R}$ is quasihomogeneous of quasidegree r with weight w if $h(\lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n) = \lambda^r h(x_1, \dots, x_n)$ for all $\lambda > 0$ in $\mathbb{R}$ and for all $(x_1, \dots, x_n) \in \mathbb{R}^n$. Given a polynomial map $h : \mathbb{R}^n \to \mathbb{R}$, we denote by h_w its quasihomogeneous term of higher quasidegree. Moreover, for a polynomial map $F = (F^1, \dots, F^n) : \mathbb{R}^n \to \mathbb{R}^n$, we denote $F_w = (F^1_w, \dots, F^n_w)$. The result is the following: Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a a polynomial map such

that $\det DF$ is nowhere zero. If there is a weight w such that the only real solution of $F_w(x) = 0$ is $x = 0$, then F is injective.

The following example shows that our Theorem 1 does not generalize this result.

Example 4. *Let $F = (f, g)$, with $f(x, y) = x^3 + y^3 + x$ and $g(x, y) = y$. Here $\det DF(x, y) = 3x^2 + 1$, and with weight $w = (1, 1)$, we have $f_w(x, y) = x^3 + y^3$ and $g_w(x, y) = y$. Thus $f_w(x, y) = g_w(x, y) = 0$ has only the solution $x = y = 0$. On the other hand,*

$$ff_x + gg_x = (x^3 + y^3 + x)(3x^2 + 1) = 3x^2(x^3 + y^3) + 4x^3 + y^3 + x,$$
$$ff_y + gg_y = (x^3 + y^3 + x)(3y^2) + y = 3y^2(x^3 + y^3) + 3xy^3 + y,$$

whose higher homogeneous parts has the factor $x^3 + y^3$ in common.

In fact in this example, for all weight $w = (w_1, w_2)$, the parts of higher quasidegrees *of $ff_x + gg_x$ and $ff_y + gg_y$ have common real factors. Indeed, independently of w, these higher parts are contained in $3x^5 + 3x^2y^3$ and $3x^3y^2 + 3y^5 + 3xy^3$, respectively. Therefore, if $w_1 = w_2$, then these parts have also the factor $x^3 + y^3$ in common. If $w_1 > w_2$, then it is clear these parts have x in common. Finally, if $w_1 < w_2$, then they have in common the factor y.*

On the other hand, an open problem is to know if Theorem 1 can be attained from the mentioned result of [7]. We discuss the relation between both results in section 4.

We anyway stress that our approach is completely different from theirs. Indeed, our proofs rely only on the qualitative theory of ordinary differential equations, following ideas started by Sabatini in [14], while the proofs in [7] are based in the structure of polynomial maps.

In section 2 we summarize some results that we shall use in the proof of Theorem 1 given in section 3.

2. Preliminary results

A singular point p of a vector field defined in $\mathbb{R}^2$ is a *centre* if it has a neighborhood filled of periodic orbits with the unique exception of p. The *period annulus* of the centre p is the maximal neighborhood $\mathcal{P}$ of p such that all the orbits contained in $\mathcal{P}$ are periodic except of course p. A centre is *global* if its period annulus is the whole $\mathbb{R}^2$.

The next result due to Sabatini, see Theorem 2.3 of [14], will play a main role in the proof of Theorem 1.

Theorem 5. *Let $F = (f, g)$ be a polynomial map with nowhere zero Jacobian determinant such that $F(0, 0) = (0, 0)$. Then the following statements are equivalent.*

(a) *The origin is a global centre for the polynomial vector field* $\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x)$.
(b) F *is a global diffeomorphism of the plane onto itself.*

Let $\mathcal{X}$ be a planar polynomial vector field of degree n and $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2+y_2^2+y_3^2 = 1\}$ (the *Poincaré sphere*). The *Poincaré compactification* of $\mathcal{X}$, denoted by $p(\mathcal{X})$, is an induced vector field on $\mathbb{S}^2$ defined as follows. For more details see Chapter 5 of [9].

Denote by $T_y\mathbb{S}^2$ the tangent space to $\mathbb{S}^2$ at the point y. Assume that $\mathcal{X}$ is defined in the plane $T_{(0,0,1)}\mathbb{S}^2 \equiv \mathbb{R}^2$. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the open northern hemisphere $\mathbb{H}^+$ and other in the open southern hemisphere $\mathbb{H}^-$. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the *infinity* of $\mathbb{R}^2$. In order to extend $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X}$ is a planar polynomial vector field of degree n then $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \setminus \mathbb{S}^1 = \mathbb{H}^+ \cup \mathbb{H}^-$ there are two symmetric copies of $\mathcal{X}$, one in $\mathbb{H}^+$ and other in $\mathbb{H}^-$, and knowing the behaviour of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behaviour of $\mathcal{X}$ at infinity. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

The singular points of $\mathcal{X}$ are called the *finite singular points* of $\mathcal{X}$ or of $p(\mathcal{X})$, while the singular points of $p(\mathcal{X})$ contained in $\mathbb{S}^1$, i.e. at infinity, are called the *infinite singular points* of $\mathcal{X}$ or of $p(\mathcal{X})$. It is known that the infinity singular points appear in pairs diametrically opposed.

Given a polynomial $p : \mathbb{R}^2 \to \mathbb{R}$, we denote by p_k the homogeneous term of degree k of p.

Let q be an infinite singular point and let h be a hyperbolic sector of q. We say that h is *degenerated* if its two separatrices are contained in the equator of $\mathbb{S}^2$ (i.e. in $\mathbb{S}^1$), otherwise h is called *non–degenerated*.

We denote by $\mathcal{G}_{n,m}$ the set of all polynomial vector fields $\mathcal{X} = (P, Q)$ with $\deg(P) = n$ and $\deg(Q) = m$ such that P_n and Q_m have no real linear factors in common.

The next result is due to Cima, Gasull and Mañosas, see Theorem 2.2 of [6].

Theorem 6. *Let q be an infinite singular point of the polynomial Hamiltonian vector field $\mathcal{X} = (-H_y, H_x)$ such that $\deg(H_y) = n$ and $\deg(H_x) = m$. Then the following statements hold.*

(a) *If $n = m$ and $\mathcal{X} \in \mathcal{G}_{n,n}$, then q is a node and hence its topological index is one.*
(b) *If q has some non-degenerated hyperbolic sector h, then the two separatrices of h are tangent to the same direction and in one neighbourhood of q this direction is not between these separatrices. Furthermore, $\mathcal{X} \notin \mathcal{G}_{n,m}$*
(c) *If $n > m$ and $\mathcal{X} \in \mathcal{G}_{n,m}$, then*
  (c.1) *If n is even, then q is a node and hence its topological index is one.*
  (c.2) *If n is odd and m is even, then q has one degenerated hyperbolic sector and one elliptic sector and its topological index is one.*
  (c.3) *If n and m are odd, then either q has two degenerated hyperbolic sectors and topological index zero, or q has two elliptic sectors and topological index two.*

The next result is the Poincaré–Hopf Theorem for the Poincaré compactification of a polynomial vector field. For a proof see Theorem 6.30 of [9].

Theorem 7. *Let $\mathcal{X}$ be a polynomial vector field. If $p(\mathcal{X})$ defined on the Poincaré sphere $\mathbb{S}^2$ has finitely many singular points, then the sum of their topological indices is two.*

We end this section with the lemma mentioned in the introduction section.

Lemma 8. *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF$ is never zero. Suppose $\deg(f) = \deg(g) = k$. If f_k and g_k do not have real linear factors in common, then the higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ also do not have real linear factors in common.*

Proof. We first observe that $(f_k^2 + g_k^2)_x \neq 0$ and $(f_k^2 + g_k^2)_y \neq 0$, because if $(f_k^2+g_k^2)_x = 0$, for instance, then (as we are dealing with polynomials) $f_k = a_{0k}y^k$ and $g_k = b_{0k}y^k$, a contradiction.

Thus the homogeneous parts of higher degree of ff_x+gg_x and ff_y+gg_y are $(f_k^2 + g_k^2)_x/2$ and $(f_k^2 + g_k^2)_y/2$, respectively. If there is a real linear factor $ax+by$ dividing them, then $ax+by$ will be also a factor of $x(f_k^2+g_k^2)_x/2+y(f_k^2+g_k^2)_y/2 = k(f_k^2+g_k^2)$. Hence $ax+by$ is a common factor of f_k and g_k, a contradiction. □

The first example of the introduction section shows that the converse of Lemma 8 is false.

3. Proof of Theorem 1

Assume that we are under the assumptions of Theorem 1.

We consider the function $H : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$H(x,y) = \frac{f(x,y)^2 + g(x,y)^2}{2}$$

and its associated Hamiltonian vector field $\mathcal{X} = (P, Q)$, i.e

$$P = -H_y = -ff_y - gg_y, \qquad Q = H_x = ff_x + gg_x.$$

We claim that each finite singular point of $\mathcal{X}$ is a centre, and thus has index 1. Indeed, $z \in \mathbb{R}^2$ is a singular point of $\mathcal{X}$ if and only if

$$\begin{pmatrix} f_x(z) & g_x(z) \\ f_y(z) & g_y(z) \end{pmatrix} \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives that $f(z) = g(z) = 0$, since $\det DF(z) \neq 0$. Let U be a neighbourhood of z in which F is injective. We have that H is positive in all the points of U different from z, while $H(z) = 0$, which proves that z is an isolated minimum of H. Then all the orbits of $\mathcal{X}$ in a neighbourhood of z (maybe smaller than the neighborhood U) are closed, proving that z is a centre of $\mathcal{X}$.

By Theorem 5, in order to prove Theorem 1 it is enough to prove that $(0,0)$ is a global centre of the vector field $\mathcal{X}$.

From hypothesis, $\mathcal{X} \in \mathcal{G}_{n,m}$. Without loss of generality, we can assume that $\deg(H_y) \geq \deg(H_x)$. Thus from Theorem 6, the index of each infinite singular point of $\mathcal{X}$ is greater than or equal to zero. Moreover, as we saw above, the index of each finite singular of $\mathcal{X}$ is equal to 1. Since the points $(0,0,1)$ and $(0,0,-1)$ of the Poincaré Sphere are finite singular points of $p(\mathcal{X})$ (corresponding to the singular point $(0,0)$ of $\mathcal{X}$), each of them with index 1, it follows from Theorem 7 that $p(\mathcal{X})$ does not have others finite singular points and all the infinite singular points have index 0.

We claim that $p(\mathcal{X})$ either does not have infinite singular points or if it does, they are formed by two degenerated hyperbolic sectors. Indeed, from statement (b) of Theorem 6, if an infinite singular point of $p(\mathcal{X})$ has a non-degenerated hyperbolic sector, then $\mathcal{X} \notin \mathcal{G}_{n,m}$, a contradiction.

Now we shall prove that the boundary of the period annulus $\mathcal{P}$ of the centre of $p(\mathcal{X})$ located at $(0,0,1)$ is the equator $\mathbb{S}^1$ of $\mathbb{H}^+$. This of course will show that the centre $(0,0)$ of $\mathcal{X}$ is global, finishing our proof. Since there are no finite singular points in $\mathbb{H}^+$, except the centre in $(0,0,1)$, and there are either no infinite singular points, or all the infinite singular points are formed by two degenerate hyperbolic sectors,

it follows that the boundary of the period annulus $\mathcal{P}$ is either a finite periodic orbit γ or it is $\mathbb{S}^1$.

If it is $\mathbb{S}^1$, we are done. If not, we consider the Poincaré map π defined in a transversal section S through γ. Since the vector field $p(\mathcal{X})$ is analytic, it follows that π is also analytic. Hence as π is the identity map in $S \cap \mathcal{P}$, it must be the identity in $S \cap (\mathbb{H}^+ \setminus \mathcal{P})$. But then the orbits in $S \cap (\mathbb{H}^+ \setminus \mathcal{P})$ near $\mathcal{P}$ are also periodic, and γ is not the boundary of $\mathcal{P}$, a contradiction.

We finish this section with a characterization for the validity of the real Jacobian conjecture in the plane.

Corollary 9. *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF$ is nowhere zero and $F(0,0) = (0,0)$. Then F is injective if and only if the vector field $\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x)$ has no infinite singular points or each of them is formed by two degenerated hyperbolic sectors.*

Proof. If there exists an infinite singular point of $\mathcal{X}$ having a non-degenerated hyperbolic sector, then it is clear that the centre $(0,0)$ of $\mathcal{X}$ is not global. Hence from Theorem 5, it follows that F is not injective. On the other hand, if there are no infinite singular points or each of them is formed by two degenerated hyperbolic sectors, then it follows from the proof of Theorem 1 that F is injective. □

4. On the equivalence between Theorem 1 and the main result of [7]

We saw in the introduction section that Theorem 1 does not imply the mentioned result of [7]. Up to now we do not know if [7] implies Theorem 1. More precisely, we have the following.

Open question: *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF$ is nowhere zero in $\mathbb{R}^2$ and $F(0,0) = (0,0)$. Set $w_h = (1,1)$. If the system of equations $(ff_x + gg_x)_{w_h}(x,y) = (ff_y + gg_y)_{w_h}(x,y) = 0$ has only the trivial solution, then is there a weight $w = (w_1, w_2)$ such that the system of equations $f_w(x,y) = g_w(x,y) = 0$ has only the trivial solution?*

If this question has positive answer, then [7] implies Theorem 1.

We begin with the following result.

Lemma 10. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial map of degree m such that $\nabla f(x,y) \neq (0,0)$ for all $(x,y) \in \mathbb{R}^2$. Then the equation $f_m(x,y) = 0$ has nontrivial real solutions.*

Proof. We consider the vector field $\mathcal{X} = (-f_y, f_x)$. From hypothesis, $\mathcal{X}$ has no finite singular points. Thus by the Poincaré-Hopf Theorem (Theorem 7), $\mathcal{X}$ must have infinite singular points. Since a polynomial vector field (P, Q) of degree $k = \max\{\deg P, \deg Q\}$ has infinite singular points if and only if there are nontrivial solutions of $-yP_k(x, y) + xQ_k(x, y) = 0$, it follows that $mf_m(x, y) = -y(-f_{my}(x, y)) + xf_{mx}(x, y)$ annihilates for some $(x, y) \neq (0, 0)$. □

The following result shows that the previous open question has affirmative answer if we suppose that the Jacobian determinant of F is a non-zero constant.

Proposition 11. *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF$ is a constant different from zero and $F(0, 0) = (0, 0)$. For $w_h = (1, 1)$, if the only solution of the system of equations $(ff_x + gg_x)_{w_h}(x, y) = (ff_y + gg_y)_{w_h}(x, y) = 0$ is $x = y = 0$, then the same is true for the system $f_{w_h}(x, y) = g_{w_h}(x, y) = 0$.*

Proof. Without loss of generality, we can assume that $m = \deg f \geq \deg g = n$.

We suppose on the contrary that $f_{w_h}(x, y) = g_{w_h}(x, y) = 0$ has nontrivial solutions. We *claim that $f_{mx} = 0$ or $f_{my} = 0$*. Indeed, if both these polynomial are not zero, then

$$(ff_x + gg_x)_{w_h} = f_m f_{mx} + g_m g_{mx}, \quad (ff_y + gg_y)_{w_h} = f_m f_{my} + g_m g_{my}.$$

With our assumptions we cannot have, for instance, $f_m f_{mx} + g_m g_{mx} = 0$, otherwise $(f_m^2 + g_m^2)_x = 0$, and thus, since we are dealing with polynomials, $f_{mx} = g_{mx} = 0$. If $n = m$, then since we are assuming that the system of equations $f_m(x, y) = g_m(x, y) = 0$ has nontrivial solutions, it follows that the system $(f_m f_{mx} + g_m g_{mx})(x, y) = \left(f_m f_{my} + g_m g_{my}\right)(x, y) = 0$ also has nontrivial solutions, a contradiction with the hypothesis. On the other hand, if $n < m$, then $(ff_x + gg_x)_{w_h}$ and $(ff_y + gg_y)_{w_h}$ have the factor f_m in common. Since from hypothesis, f is a submersion, it follows that f_m has real factors as divisors from Lemma 10. Then we reach to a contradiction, concluding the proof of the claim.

By changing x by y and dividing F by a constant, if necessary, we can assume that $f_m = x^m$. If $n = m$, the same arguments show that $g_{mx} = 0$ or $g_{my} = 0$. Since we are supposing that the system $f_m(x, y) = g_m(x, y) = 0$ has nontrivial solutions, it follows that $g_m = b_{m0}x^m$, for some $b_{m0} \neq 0$.

Thus we suppose from now on that $f_m = x^m$ and $g_m = b_{m0}x^m$ ($b_{m0} \neq 0$ or not).

This means that $(ff_x + gg_x)_{w_h} = m(1 + b_{m0}^2)x^{2m-1}$. Therefore, from the hypothesis, we have that x does not divide $(ff_y + gg_y)_{w_h}$. We observe that up to now we have not yet used the hypothesis on the Jacobian determinant.

Now since the Jacobian deteminant of F is constant, it follows that all the non constant homogeneous terms of the polynomial $\det DF(x, y)$ are zero.

The higher possible homogeneous terms of $ff_y + gg_y$ and $\det DF$ are the following, respectively,

$$x^m \left((f_{m-1})_y + b_{m0}g_{m-1_y}\right), \quad mx^{m-1}\left(g_{m-1_y} - b_{m0}f_{m-1_y}\right).$$

Thus $f_{m-1_y} = g_{m-1_y} = 0$. After that, the higher possible homogeneous terms of $ff_y + gg_y$ and $\det DF$ are the following

$$x^m \left((f_{m-2})_y + b_{m0}g_{m-2_y}\right), \quad mx^{m-1}\left(g_{m-2_y} - b_{m0}f_{m-2_y}\right),$$

respectively. Thus $f_{m-2_y} = g_{m-2_y} = 0$. Continuing in this way, it is clear that we reach $f_{m_y} = g_{m_y} = f_{m-1_y} = g_{m-1_y} = \cdots = f_{1_y} = g_{1_y} = 0$, and then $f_y = g_y = 0$, which gives that $\det DF = 0$, a contradiction. □

To show that the above open question has a negative answer, we need examples where for any weight $w = (w_1, w_2)$, $f_w(x, y) = g_w(x, y) = 0$ always have a nontrivial solution, and such that with $w_h = (1, 1)$, $(ff_x + gg_x)_{w_h}(x, y) = (ff_y + gg_y)_{w_h}(x, y) = 0$ has only the solution $x = y = 0$.

By making calculations with Mathematica, we have shown that there are no such examples if $\max\{\deg f, \deg g\} \leq 17$.

Bellow we present a step-by-step idea to construct such an example.

(1) We assume $m = \deg f \geq \deg g = n$.
(2) By Proposition 11, we have to assume that $\det D(f, g)(x, y)$ is not constant, say $\det D(f, g)(x, y) > 0$.
(3) By the first part of the proof of Proposition 11, we can also assume that $f_m = x^m$ and $g_m = b_{m,0}x^m$ ($b_{m,0} \neq 0$ or not).
(4) Thus the homogeneous term of higher degree of $ff_x + gg_x$ is
$$m(1 + b_{m,0}^2)x^{2m-1},$$
and hence we have to look for f and g such that the homogeneous term of higher degree of $ff_y + gg_y$ does not have the factor x.
(5) Analyzing $\det DF(x, y)$, we observe its homogeneous term of higher possible degree is $m(g_{m-1_y} - b_{m,0}f_{m-1_y})x^{m-1}$. This has degree $2m-3$, an odd number. Thus in order to have $\det DF >$

0, we must have $g_{m-1_y} - b_{m,0} f_{m-1_y} = 0$. On the other hand, the homogeneous term of higher possible degree of $ff_y + gg_y$ is $(f_{m-1_y} + b_{m,0} g_{m-1_y})x^m$. From (4) we must have $f_{m-1_y} + b_{m,0} g_{m-1_y} = 0$, and consequently, $f_{m-1_y} = g_{m-1_y} = 0$. This means we have to assume $f_{m-1} = a_{m-1,0} x^{m-1}$ and $g_{m-1} = b_{m-1,0} x^{m-1}$.

(6) Therefore, it follows that the higher homogeneous term that $ff_y + gg_y$ can have is $(f_{m-2_y} + b_{m,0} g_{m-2_y})x^m$. Thus again from (4), we can always suppose $f_{m-2_y} = -b_{m,0} g_{m-2_y}$.

(7) Now the higher homogeneous term of $\det DF(x, y)$ is $m(1 + b_{m,0}^2) g_{m-2_y} x^{m-1}$. Hence in order to have $\det DF > 0$, we have to suppose $g_{m-2_y} x^{m-1} \geq 0$.

(8) Using similar arguments used in steps (3) to (7), we see that if $g_{m-2_y} = 0$, then we will have $f_{m-3_y} = g_{m-3_y} = 0$ and $f_{m-4_y} = -b_{m,0} g_{m-4_y}$. Moreover, in this case, we have to suppose $g_{m-4_y} x^{m-1} \geq 0$.

This gives the idea of an induction procedure to prove that there is not a such example: the idea would be to prove that $g_{m-2_y} = 0$. Then $f_{m-2_y} = 0$ and $f_{m-3_y} = g_{m-3_y} = 0$. Moreover, $f_{m-4_y} = -b_{m,0} g_{m-4_y}$. Then we would prove that $g_{m-4_y} = 0$, and hence $g_{m-6_y} = 0$, and so on.

(9) Consider the weight $w = (m - 2, m)$. We have $f_w(x, y) = x^m - b_{m,0} b_{0,m-2} y^{m-2}$ and $g_w(x, y) = b_{m,0} x^m + b_{0,m-2} y^{m-2}$. Since $f_w(x, y) = g_w(x, y) = 0$ has nontrivial solutions it follows that $b_{0,m-2} = 0$.

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[1] Departamento de Matemática, Universidade Federal de São Carlos Rod. Washington Luís, Km 235, C.P. 676, 13565–905 São Carlos, SP–Brasil

E-mail address: franciscobraun@dm.ufscar.br

[2] Departamento de Matematica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Catalonia, Spain

E-mail address: gine@matematica.udl.cat

[3] Departament de Matemàtiques Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat