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Heteroclinic, homoclinic and closed orbits in the Chen system

G. Tigan, J. Llibre

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Abstract Bounded orbits such as closed, homoclinic and heteroclinic orbits are discussed in this work for a Lorenz-like 3D nonlinear system. For a large spectrum of the parameters the system has neither closed nor homoclinic orbits but has exactly two heteroclinic orbits, while under other constraints the system has symmetrical homoclinic orbits.

Keywords ODE systems · bifurcations · homoclinic and heteroclinic orbits

1 Introduction

In this paper we consider a Lorenz-like three-dimensional system [13], namely the Chen system. Interesting results reported recently on this system concerning closed, homoclinic and heteroclinic orbits are found in [7]. More results on the Chen system are given in [3], [8], [9], [10], [11], [14], [17] and in some references therein.

In this paper we first refine some results reported in [7] and present a different proof of these results. Secondly, we consider an important case not treated in [7] and show the system has homoclinic orbits for a large spectrum of the parameters, by transforming the system into a new form and using results reported in [1]. Proving existence of homoclinic or heteroclinic orbits in nonlinear ode systems is in general a difficult task [2], [5], [6], [12], [15], [16].

The Chen system is given by:

$$\dot{x} = a(y-x),\ \dot{y} = (c-a)x - xz + cy,\ \dot{z} = xy - bz, \qquad (1)$$

where $a > 0$, $b > 0$ and $c > 0$ are positive real parameters.

G. Tigan
Department of Mathematics, Politehnica University of Timisoara, Romania, E-mail: gheorghe.tigan@upt.ro,
J. Llibre
Department of Mathematics, University Autonoma of Barcelona, Catalonia, Spain, E-mail: jllibre@mat.uab.cat

Recall that, if x_0 is an equilibrium hyperbolic point such that the stable manifold $W^s(x_0)$ intersected with the unstable manifold $W^u(x_0)$ is not empty, then the orbits belonging to $W^s(x_0) \cap W^u(x_0) \neq \Phi$ are called homoclinic orbits. They are doubly asymptotic to an equilibrium point. Similarly, if x^1, x^2 are two hyperbolic equilibrium points such there exists an orbit $\Gamma \subset W^s(x^1) \cap W^u(x^2)$ or inversely $\Gamma \subset W^u(x^1) \cap W^s(x^2)$, then Γ is called a hyperbolic orbit.

The system (1) has the origin O as an equilibrium point for any $a, b, c > 0$ and it has two more equilibrium points
$S_1 = (\sqrt{b(2c-a)}, \sqrt{b(2c-a)}, 2c-a)$
and
$S_2 = (-\sqrt{b(2c-a)}, -\sqrt{b(2c-a)}, 2c-a)$,
for $2c > a$. Assume also further $a > c$. As the transformation $(x,y,z) \to (-x,-y,z)$ leaves the system unchanged, the orbits of the system are symmetrical with respect to the z-axis.

2 Existence of heteroclinic orbits

The matrix of the linearized system to (1) at the origin has the eigenvalues and the corresponding eigenvectors given by:

$$\begin{aligned}
d_1 &= \tfrac{1}{2}c - \tfrac{1}{2}a + \tfrac{1}{2}\sqrt{6ac-3a^2+c^2},\\
u_1 &= \left(\tfrac{1}{2a-2c}\left(a+c-\sqrt{6ac-3a^2+c^2}\right)\ 1\ 0 \right)^T;\\
d_2 &= \tfrac{1}{2}c - \tfrac{1}{2}a - \tfrac{1}{2}\sqrt{6ac-3a^2+c^2},\\
u_2 &= \left(\tfrac{1}{2a-2c}\left(a+c+\sqrt{6ac-3a^2+c^2}\right)\ 1\ 0 \right)^T;\\
d_3 &= -b,\\
u_3 &= \left(0\ 0\ 1\right)^T.
\end{aligned}$$

Considering $2c > a > c > 0$, it follows that $d_1 > 0$ and $d_{2,3} < 0$, that is, the origin is a saddle point having a one-dimensional unstable manifold W_0^u and a two-dimensional stable manifold W_0^s. The tangent unstable subspace TW_0^u is

given by
$TW_0^u = \left\{2(a-c)y = \left(a+c-\sqrt{6ac-3a^2+c^2}\right)x, z=0\right\}$.
The unstable manifold W_0^u contains $O(0,0,0)$ and is tangent to TW_0^u at the origin.

Using the method of undetermined coefficients in a small neighborhood of the origin we get

$$W_0^u = \left\{(x,y,z) \in R^3 \mid \begin{array}{l} y = \quad a_1 x + O(x^2) \\ z = \frac{a_1}{-2a+b+2aa_1}x^2 + O(x^3) \end{array}\right\},$$

where $|x| \ll 1$ and $a_1 = \frac{1}{2(a-c)}\left(a+c-\sqrt{6ac-3a^2+c^2}\right)$.

Note that W_0^u is indeed tangent to TW_0^u since $z'(0) = 0$, $y'(0) = a_1$ and the vector $(1, y'(0), z'(0))^T$ is collinear to the direction vector u_1 of the line TW_0^u. Note also that the z-axis is included in the stable manifold W_0^s.

Denote in the following by $\phi_t u_0 = (x(t,u_0), y(t,u_0), z(t,u_0))$ a solution of the system (1) through the initial point $u_0 = (x_0, y_0, z_0)$ and by W_+^u $\left(W_-^u\right)$ the positive (negative) branch of the unstable manifold W_0^u corresponding to $x > 0$ ($x < 0$).

Define a Lyapunov-like function
$U(x,y,z) = A(y-x)^2 + B\left(z - x^2/b\right)^2 + C\left(x^2 - b(2c-a)\right)^2$.
Choosing
$A = b - 2a \geq 0$, $B = b > 0$ and $C = \frac{1}{2ab}(b-2a) \geq 0$,
it follows that

$$\frac{dU}{dt} = -2(b-2a)(a-c)(x-y)^2 - 2\left(bz - x^2\right)^2 \leq 0. \quad (2)$$

Remark 1 Different from [7], the Lyapunov function U is defined both for $b > 2a$ and $b = 2a$.

PROPOSITION 1 *If $2c > a > c > 0$, $b \geq 2a$ the following assertions are true:*
a) If there exist t_1 and t_2 such that $t_1 < t_2$ and U satisfies $U(\phi_{t_1} u_0) = U(\phi_{t_2} u_0)$, then either
• *u_0 is an equilibrium point of system (1), or*
• *$b = 2a$ and the orbit $\phi_t u_0$ is contained in the parabolic cylinder $bz = x^2$.*
b) Assume $b > 2a$. If $\phi_t u_0 \to O$ as $t \to -\infty$ and $x(t,u_0) > 0$ for some t, then
$U(O) > U(\phi_t u_0)$ and $x(t,u_0) > 0$, for all $t \in \mathbb{R}$.
Consequently $u_0 \in W_+^u$.

Proof a) From (2) and from the hypothesis of a), one gets $\frac{dU}{dt}(\phi_t u_0) \equiv 0$ for all $t \in (t_1, t_2)$, which implies either

$$\dot{x}(\phi_t u_0) = \dot{y}(\phi_t u_0) = \dot{z}(\phi_t u_0) \equiv 0 \quad (3)$$

for all $t \in (t_1, t_2)$, i.e u_0 is one of the equilibria of (1), or $b > 2a$ and the orbit $\phi_t u_0$ is contained in the intersection of the plane $x = y$ with the parabolic cylinder $bz = x^2$. But this latter case leads again to (3), i.e u_0 is one of the equilibria of system (1), because from $\phi_t u_0 \in \{x = y\} \cap \{bz = x^2\}$ for all t, we get $\dot{x}(t,u_0) = 0$, $\dot{z}(t,u_0) = 0$. Hence $x(t) = x_0$, but $y(t) = x(t)$ for all t, i.e $\dot{y}(t,u_0) = 0$. We notice that the three equilibrium points lie on the non-invariant curve $\{x = y\} \cap \{bz = x^2\}$.

Finally, (2) and the hypothesis of a) lead also to $b = 2a$ and the orbit $\phi_t u_0$ is contained in the parabolic cylinder $bz = x^2$ (for these values of the parameters this cylinder is invariant by the flow of system (1), i.e. if an orbit has a point in it the whole orbit is contained in the cylinder).

b) We prove first $U(O) > U(\phi_t u_0)$ for all $t \in \mathbb{R}$; $U(O) > 0$. To this end, assume by contrary that there exists a $t_0 \in \mathbb{R}$ such that $0 < U(O) \leq U(\phi_{t_0} u_0)$. From $\phi_t u_0 \to O$ as $t \to -\infty$ and U continuous on t, it follows that there exists a sequence $t_n \to -\infty$ and an integer positive number n_1 such that $|U(\phi_{t_n} u_0) - U(O)| < \varepsilon$ for all $\varepsilon > 0$ and $n > n_1$. Since $t_n \to -\infty$ and $t_0 \in \mathbb{R}$, there is an integer positive number n_2 such that $t_n < t_0$ for all $n > n_2$. Denote further by $n_0 = max\{n_1, n_2\}$ and take $\varepsilon = \frac{1}{2}(U(\phi_{t_0} u_0) - U(O))$. It is clear that $\varepsilon \geq 0$. Then $U(\phi_{t_n} u_0) - U(\phi_{t_0} u_0) = U(\phi_{t_n} u_0) - U(O) + U(O) - U(\phi_{t_0} u_0) < \varepsilon + U(O) - U(\phi_{t_0} u_0) = -\varepsilon \leq 0$. On the other hand $U(t)$ is decreasing with respect to t, which, by definition, leads to $U(\phi_{t_n} u_0) \geq U(\phi_{t_0} u_0)$ for all $t_n < t_0$ and $n > n_0$. Therefore, $U(\phi_{t_n} u_0) = U(\phi_{t_0} u_0)$ and by virtue of a) we get that u_0 is an equilibrium point of system (1). Since $\phi_t u_0 \to O$ we get $u_0 \equiv O$ and $x(t,u_0) = 0$ for all t. But this contradicts the hypothesis $x(t,u_0) > 0$ for some t. Hence, $U(O) > U(\phi_t u_0)$ for all $t \in \mathbb{R}$.

Let us prove now that $x(t,u_0) > 0$ for all $t \in \mathbb{R}$. Assuming that there exists a $t' \in \mathbb{R}$ such that $x(t', u_0) \leq 0$ and using $x(t'', u_0) > 0$ for some $t'' \in \mathbb{R}$ from the hypothesis of b), one gets that there exists a $\tau \in \mathbb{R}$ such that $x(\tau, u_0) = 0$. As $U(O) > U(\phi_t u_0)$ for all $t \in \mathbb{R}$, it follows that $\phi_\tau u_0 \in \Omega \cap P$, where $\Omega = \{(x,y,z) : U(O) > U(x,y,z)\}$ and P is the plane $x = 0$. On the other hand, $\Omega \cap P$ is given by
$Ay^2 + Bz^2 + Cb^2(2c-a)^2 < Cb^2(2c-a)^2$,
i.e. $Ay^2 + Bz^2 < 0$ with $A, B \geq 0$. It leads to $\Omega \cap P = \Phi$ which is a contradiction. Therefore $x(t,u_0) > 0$ for all $t \in \mathbb{R}$. This completes the proof of the proposition.

Theorem 1 *Consider $2c > a > c > 0$, $b > 2a$ and the above function U. Then the following assertions are true:*
a) The ω-limit of any trajectory of system (1) is an equilibrium point. In particular system (1) has no closed trajectories.
b) System (1) has no homoclinic trajectories.
c) System (1) has exactly two heteroclinic trajectories.

Proof While the function U is different, the proof is similar to the one presented in [7]. However, we choose to present it here as it may be a useful handy exercise for some readers.
a) If $a > c > 0$, the function U is decreasing along trajectories of the system both for $b > 2a$ and $b = 2a$, except perhaps for the orbits on the cylinder $bz = x^2$ if $b = 2a$, and the orbits contained in $\{x = y\} \cap \{bz = x^2\}$ if $b > 2a$, where the function U is constant and equal to zero. This implies that

for all $t \geq 0$

$$0 \leq U(\phi_t u_0) \leq U(u_0), \tag{4}$$

where $\phi_t u_0$ is a trajectory of the system through the initial point u_0. It implies that, the limit $\lim_{t\to\infty} U(\phi_t u_0)$ exists. Denote it by $U^*(u_0)$. From (4) one gets that $U(\phi_t u_0)$ is bounded for $t \geq 0$, which implies further that $x(t,u_0)$, $y(t,u_0)$, $z(t,u_0)$ are bounded for $t \geq 0$, i.e. $\phi_t u_0$ is bounded for $t \geq 0$. Denote by $\Omega(u_0)$ the ω-limit set of the orbit $\phi_t u_0$. It is known that, if $u \in \Omega(u_0)$, then all points of the orbit through u belong to $\Omega(u_0)$, i.e. $\phi_t u \in \Omega(u_0)$. Therefore, for any point $\phi_t u$, $t \geq 0$, there exists a sequence $t_n \to \infty$ for $n \to \infty$ such that $\lim_{n\to\infty} \phi_{t_n} u_0 = \phi_t u$ which leads to

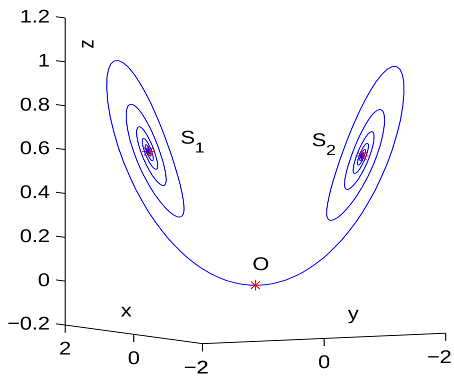
$$U(\phi_t u) = \lim_{n\to\infty} U(\phi_{t_n} u_0) = U^*(u_0) = const,$$

for all $t \geq 0$. So there exists $t_1 < t_2$ such that $U(\phi_{t_1} u) = U(\phi_{t_2} u)$ for all $t \in (t_1,t_2)$, and by Proposition 1 either u is one of the equilibria of the system, or $b = 2a$ and the point u is contained in the invariant cylinder $bz = x^2$, or $b > 2a$ and the point u is contained in the curve $\{x = y\} \cap \{bz = x^2\}$.
Assume $b = 2a$ and the point u is contained in the invariant cylinder $bz = x^2$. Then the function U takes the value zero on this cylinder, and eventually the system can have periodic orbits on the cylinder.
Assume $b > 2a$ and $u \in \{x = y\} \cap \{bz = x^2\}$. Since in the connected curve $\{x=y\}\cap\{bz=x^2\}$ there are the three equilibria of the system, and u is an ω-limit set, u is one of the equilibria of the system by the Bendixson-Poincaré Theorem (see for instance Corollary 1.30 of [4]).
b) Assume that the system has a homoclinic orbit $\gamma(t)$ at one of the equilibria O, S_1 or S_2, that is, $\lim_{t\to\pm\infty} \gamma(t) = q$ where $q \in \{O, S_1, S_2\}$. Since U is decreasing along the trajectories of the system, it follows that

$$U(q) \leq U(\gamma(t)) \leq U(q), \tag{5}$$

c) By statement a) every one-dimensional branch of the unstable manifold W^u has ω-limit an equilibrium point p.
Assume $b > 2a$. Since $U(O) > U(S_i) = 0$ for $i = 1,2$, the equilibrium point p must be either S_1 or S_2, and by the symmetry of the system with respect to the z-axis, one of the two branches of W^u must go to S_1 and the other to S_2, obtaining in this way two heteroclinic orbits. A numerical case with two heteroclinic orbits is illustrated in Fig.1 a). This completes the proof of the theorem.

The case $b = 2a$ has been studied in [7] using a Lyapunov-like function in the form
$V(x,y,z) = a^2(y-x)^2 + \frac{1}{4}\left(x^2 - 2a(2c-a)\right)^2$.
In particular, the following results have been reported:
a) If the negative orbit from a point u_0 is bounded, then the solution $\phi_t u_0$ approaches one of the equilibria of the system as $t \to -\infty$. Consequently, the system has no closed orbits.



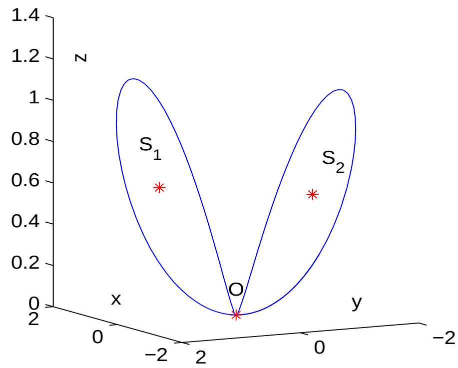


Fig. 1 a) Two symmetrical heteroclinic orbits for $a = 1, b = 2.1, c = 0.8$. (up); b) Two symmetrical homoclinic orbits for $a = 1, b = 1.17, c = 0.8$ (down).

b) The system (1) has no homoclinic orbits.
c) The system (1) has exactly two heteroclinic orbits: one linking O to S_1 and the other O to S_2.

3 Existence of homoclinic orbits

Consider in the following the case $2c > a > c > 0$, $2a > b > 0$ and make the nonsingular transformation:

$$u = \alpha x,\ v = \beta(y-x),\ w = \gamma\left(z - \frac{x^2}{2a}\right),$$

and the rescaling $\tau = rt$. Then the system (1) becomes:

$$\begin{aligned}
\dot{u} &= \frac{a\alpha}{r\beta} v,\\
\dot{v} &= \frac{\beta}{2ar\alpha^3}\left(\frac{2a}{\gamma}\alpha^2 - \frac{2a\alpha^2}{\gamma} w - u^2\right) u + \frac{c-a}{r} v,\\
\dot{w} &= \frac{u^2}{\alpha^2}\left(1 - \frac{b}{2a}\right)\frac{\gamma}{r} - \frac{b}{r} w.
\end{aligned} \tag{6}$$

Choosing

$$\alpha = \sqrt{\frac{1}{2a(2c-a)}} > 0,\ \beta = \frac{\sqrt{2}}{2(2c-a)} > 0,\ \gamma = \frac{1}{2c-a} > 0,$$

and $r = \sqrt{a(2c-a)} > 0$, system (6) reads:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -\left(u^2 + w - 1\right)u - \lambda v, \\ \dot{w} &= -\sigma w + \delta u^2, \end{aligned} \tag{7}$$

where $\lambda = \frac{a-c}{r}$, $\sigma = \frac{b}{\sqrt{a(2c-a)}}$, $\delta = (2a-b)\sqrt{\frac{1}{a(2c-a)}}$. Notice that $\beta = 2ar\alpha^3 = \frac{a}{r}\alpha$, $\gamma = 2a\alpha^2$ and $r^2 = \frac{1}{2\alpha^2}$. But system (7) is treated in [1] and after a laborious study and using a method based on comparison systems, the following result is reported:

Theorem 2 *For each $\sigma > 0$, there is, in the region of positive parameters δ and λ, a bifurcation curve $\{\rho(\sigma,\delta,\lambda) = 0\}$, beginning at $(0,0)$ and going to infinity for $\delta \to \infty$, corresponding to a homoclinic orbit to the saddle point $O(0,0,0)$ of system* (7).

It implies that for any a,b,c with $2c > a > c > 0$ and $2a > b > 0$, system (1) has two symmetrical homoclinic orbits to the equilibrium point O. A particular numerical case in this regard is presented in Fig.1 b) where the homoclinic orbits are depicted.

4 Conclusions

In this paper we have investigated closed, homoclinic and heteroclinic orbits in a three-dimensional autonomous system, known as the Chen system. Using a convenient Lyapunov-like function, we proved that the system under some constraints of its parameters has no homoclinic orbits and no closed orbits but it has exactly two heteroclinic orbits, symmetrically with respect to the z-axis. Moreover, transforming the system to a new form and using some known results from [1], we obtained results on cases not covered in [7] concerning homoclinic orbits. More exactly, we have proved that the system has two symmetrical homoclinic orbits to the equilibrium point O.

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