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CENTER CYCLICITY OF A FAMILY OF QUARTIC POLYNOMIAL DIFFERENTIAL SYSTEM

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Abstract. In this paper we study the cyclicity of the centers of the quartic polynomial family written in complex notation as

$$\dot{z} = iz + z\bar{z}(Az^2 + Bz\bar{z} + C\bar{z}^2),$$

where $A, B, C \in \mathbb{C}$. We give an upper bound for the cyclicity of any nonlinear center at the origin when we perturb it inside this family. Moreover we prove that this upper bound is sharp.

1. Introduction and statement of the main results

We consider a family of planar polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= \lambda_1 x - y + P(x, y, \lambda), \\ \dot{y} &= x + \lambda_1 y + Q(x, y, \lambda), \end{aligned} \tag{1}$$

where $P, Q \in \mathbb{R}[x, y, \lambda]$ are the polynomial nonlinearities of system (1) and $(\lambda_1, \lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda \subset \mathbb{R}^n$ are the parameters of the family. One of the main problems in the qualitative theory of real planar polynomial systems consists in distinguishing if the singular point located at the origin O of system (1) is either a *center* (i.e. it has a neighborhood U such that $U \setminus \{O\}$ is filled with periodic orbits) or a *focus* (i.e. it has a neighborhood U where all the orbits in $U \setminus \{O\}$ spiral in forward or in backward time to the origin), see [1]. Clearly, the origin of family (1) is a focus when $\lambda_1 \neq 0$.

A characterization of system (1) having a center at the origin is given by the existence of a formal first integral (which in fact it is analytic) $H(x, y) = x^2 + y^2 + \cdots$ (here the dots denote higher order terms) with $\lambda_1 = 0$, see Poincaré [13] and Liapunov [10]. More precisely we seek for a formal series $H(x, y; \lambda) = x^2 + y^2 + \cdots$ in such a way that $\mathcal{X}_\lambda(H) = \sum_{j\geq 1} \eta_j(\lambda)(x^2+y^2)^j$ where $\mathcal{X}_\lambda = (-y+P(x, y, \lambda))\partial_x+(x+Q(x, y, \lambda))\partial_y$ is the associate vector field to family (1) with $\lambda_1 = 0$.

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It is natural to study the *center–problem* in the complex setting. We will associate to system (1) a two–dimensional complex system by using the complex coordinate $z = x + iy \in \mathbb{C}$. Family (1) with $\lambda_1 = 0$ can be written into the form $\dot{z} = iz + F(z, \bar{z}, \lambda)$ where $\bar{z} = x - iy$ and F is given by the polynomial $F(z, \bar{z}, \lambda) = P\left(\frac{1}{2}(z + \bar{z}), \frac{i}{2}(\bar{z} - z), \lambda\right) + iQ\left(\frac{1}{2}(z + \bar{z}), \frac{i}{2}(\bar{z} - z), \lambda\right)$. We can adjoin to this complex polynomial differential equation its complex conjugate forming thus the complex system

$$
\begin{aligned}
\dot{z} &= iz + F(z, \bar{z}, \lambda) = iz + \sum_{j+k=2}^{N} a_{j,k}(\lambda) z^j \bar{z}^k, \\
\dot{\bar{z}} &= -i\bar{z} + \bar{F}(z, \bar{z}, \lambda) = -i\bar{z} + \sum_{j+k=2}^{N} \bar{a}_{j,k}(\lambda) \bar{z}^j z^k.
\end{aligned} \tag{2}
$$

We say that system (2) with $\lambda = \lambda^*$ has a center at the origin $(z, \bar{z}) = (0, 0)$ if and only if it admits a formal first integral $\hat{H}(z, \bar{z}; \lambda^*) = z\bar{z} + \cdots$, see [4, 14]. System (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ has a center at the origin if and only if system (2) has a center at the origin for $\lambda = \lambda^*$.

Denote by $\hat{X} = (iz + \cdots\cdots)\partial_z + (-i\bar{z} + \cdots)\partial_{\bar{z}}$ the family of vector fields in $\mathbb{C}^2$ associated to system (2). We look for a formal series $\hat{H}(z, \bar{z}; \lambda) = z\bar{z} + \cdots \in \mathbb{C}[[z, \bar{z}]]$ such that $\hat{X}(\hat{H}) = \sum_{j \geq 1} f_j(\lambda)(z\bar{z})^{j+1}$. It turns out that $f_j(\lambda) \in \mathbb{R}[x, y]$ are the so-called *focus quantities* for system (1), see [14] and [8]. One can see the nonzero polynomials $f_j(\lambda)$ as the obstruction to the existence of the first integral $\hat{H}$ for system (2) and therefore the obstacles to have a center at the origin in (1). Actually, the origin of system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ is a center if and only if $f_j(\lambda^*) = 0$ for any $j \in \mathbb{N}$.

Let $\mathcal{B}$ and $\mathcal{B}_k$ be the ideals in $\mathbb{R}[\lambda]$ given by $\mathcal{B} = \langle f_j(\lambda) : j \in \mathbb{N}\rangle$ and $\mathcal{B}_k = \langle f_1(\lambda), \ldots, f_k(\lambda)\rangle$, respectively. The ideal $\mathcal{B}$ generated by all the focus quantities is called the *Bautin ideal*. We can also define $\tilde{f}_j \equiv f_j$ mod $\mathcal{B}_{j-1}$, that is, $\tilde{f}_j$ is the remainder of f_j upon division by a Gröbner basis of the ideal $\mathcal{B}_{j-1}$. Clearly $\mathcal{B}_k = \langle f_1(\lambda), \tilde{f}_2(\lambda), \ldots, \tilde{f}_k(\lambda)\rangle$.

The affine variety $\mathbf{V}(\mathcal{B}_k)$ associated to $\mathcal{B}_k$ is the algebraic set $\mathbf{V}(\mathcal{B}_k) = \{\lambda \in \mathbb{R}^{n-1} : f_j(\lambda) = 0 \text{ for } 1 \leq j \leq k\}$, see [3]. To solve the center problem for (1) is to describe the *center variety* $\mathbf{V}(\mathcal{B})$. Clearly $\lambda^* \in \mathbf{V}(\mathcal{B}) \subset \mathbb{R}^{n-1}$ if and only if system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ has a center at the origin.

The polynomial ring $\mathbb{R}[\lambda]$ is Noetherian and then by the Hilbert's basis Theorem any ideal in $\mathbb{R}[\lambda]$ is generated by a finite number of polynomials. We define the *minimal basis* of the finitely generated ideal $\mathcal{B}$ with respect to an ordered basis $B = \{f_1(\lambda), f_2(\lambda), f_3(\lambda), \dots\}$ as the basis $M_{\mathcal{B}}$ defined by the following procedure:

(a) initially set $M_{\mathcal{B}} = \{f_p(\lambda)\}$, where $f_p(\lambda)$ is the first non-zero element of B;
(b) check successive elements $f_j(\lambda)$, starting with $j = p+1$, adjoining $f_j(\lambda)$ to $M_{\mathcal{B}}$ if and only if $f_j(\lambda) \notin \langle M_B \rangle$, the ideal generated by $M_{\mathcal{B}}$.

Let $M_{\mathcal{B}} = \{f_{j_1}, \dots, f_{j_m}\}$ be a minimal base for the Bautin ideal $\mathcal{B}$. The cardinality m of $M_{\mathcal{B}}$ is called the *Bautin depth* of $\mathcal{B}$, see [9].

The *cyclicity* of a center at the origin of system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ is the maximum number of small amplitude limit cycles that can appear bifurcating from that center under arbitrarily small parameter perturbations inside family (1). The concept of cyclicity was introduced by Bautin in the seminal paper [2]. There Bautin showed that the cyclicity problem of a non–degenerate center (center having linear part with nonzero eigenvalues) could be reduced to the problem of finding a minimal base for the Bautin ideal. In reality the cyclicity of any center at the origin of (1) is at most the Bautin depth m of $\mathcal{B}$, see for example [2, 9, 14, 15].

In this work we are interested in the quartic polynomial family written in complex notation as

$$\dot{z} = (i + \lambda_1)z + z\bar{z}(Az^2 + Bz\bar{z} + C\bar{z}^2), \tag{3}$$

with $z = x + iy \in \mathbb{C}$ and parameters $\lambda_1 \in \mathbb{R}$ and $(A, B, C) \in \mathbb{C}^3$. In [11] the authors solve the center problem for such a family and give the cyclicity but only in the simplest case that the origin be a focus, see also [12]. Here we complete the study of family (3) focussing our attention on the harder cyclicity problem of the center at $z = 0$. The main result of this paper is the following.

Theorem 1. *The following statements hold.*

(a) *Any nonlinear center at the origin of the family* (3) *has cyclicity at most* 4 *when we perturb it inside the family.*
(b) *There are perturbations of the linear center* $\dot{z} = iz$ *inside family* (3) *producing* 4 *limit cycles bifurcating from the origin.*

2. Cyclicity and Radicality of The Bautin ideal

The problem of finding the depth of the Bautin ideal is in general a difficult task, this is the reason for which the cyclicity problem of a center is not easy to solve. However, the problem becomes easier when the Bautin ideal is radical.

Recall that the radical $\sqrt{\mathcal{B}}$ of the ideal $\mathcal{B}$ is $\sqrt{\mathcal{B}} = \{p \in \mathbb{R}[\lambda] : p^s \in \mathcal{B}$ for some $s \in \mathbb{N}\}$. If $\mathcal{B} = \sqrt{\mathcal{B}}$ then $\mathcal{B}$ is called a *radical ideal*.

When the Bautin ideal $\mathcal{B}$ is radical we can state the following result stated for the first time in [7], see also [6].

Theorem 2 (Radical Ideal Cyclicity Bound Theorem)**.** *Let m be the cardinality of a minimal basis of the ideal $\mathcal{B}_k$ in the polynomial ring $\mathbb{R}[\lambda]$ with $\lambda \in \mathbb{R}^{n-1}$. Assume that the following two conditions hold:*

(i) $\mathbf{V}(\mathcal{B}_k) = \mathbf{V}(\mathcal{B})$ *holds in* $\mathbb{C}^{n-1}$*;*
(ii) $\mathcal{B}_k$ *is radical.*

Then $\mathcal{B} = \mathcal{B}_k$ and, in particular, the cyclicity of any center at the origin in (1) *is at most m.*

Unfortunately an ideal is not always radical. Suppose now that the center problem has been already solved, that is, we know the center variety $\mathbf{V}(\mathcal{B}_k) = \mathbf{V}(\mathcal{B})$ but $\mathcal{B}_k$ is not radical. In this case we cannot apply Theorem 2 for bounding the cyclicity. The forthcoming Theorem 3 allows to obtain an upper bound on the cyclicity of the center at the origin of family (1) when we have the non-radicality of the Bautin ideal. The idea is to obtain an upper bound of the cyclicity in the varieties associated to the radical components in the primary decomposition of the ideal $\mathcal{B}_k$ following therefore some ideas extracted from [5].

We recall that a polynomial ideal $\mathcal{I}$ in the ring $\mathbb{K}[\mathbf{x}]$ is *primary* if $p\,q \in \mathcal{I}$ implies either $p \in \mathcal{I}$ or a power $q^s \in \mathcal{I}$ for some positive $s \in \mathbb{N}$. By the Lasker-Noether Theorem, any ideal $\mathcal{I}$ can be decomposed as the intersection of a finite number of primary ideals, see [3]. On the other hand, $\mathcal{I}$ is *prime* if whenever $p, q \in \mathbb{K}[\mathbf{x}]$ with $p\,q \in \mathcal{I}$ then either $p \in \mathcal{I}$ or $q \in \mathcal{I}$. Every radical ideal can be written as the intersection of prime ideals. An approach to bound, in some cases, the center ciclycity of system (1) when the Bautin ideal is not radical is the following result, see [7] and also [6].

Theorem 3. *Let s be the cardinality of a minimal basis of the ideal $\mathcal{B}_k$ in the polynomial ring $\mathbb{R}[\lambda]$ with $\lambda \in \mathbb{R}^{n-1}$. Suppose that the center problem at the origin of family* (1) *has been solved and its center variety $\mathbf{V}(\mathcal{B})$ satisfies that $\mathbf{V}(\mathcal{B}_k) = \mathbf{V}(\mathcal{B})$ as varieties in $\mathbb{C}^{n-1}$. Moreover, suppose that a primary decomposition of $\mathcal{B}_k$ can be written as $\mathcal{B}_k = \mathcal{R} \cap \mathcal{N}$ where $\mathcal{R}$ is the intersection of prime ideals in the decomposition with the intersection $\mathcal{N}$ of the remaining ideals in the decomposition.*

Then for any system of family (1) *corresponding to* $\lambda^* \in \mathbf{V}(\mathcal{B}) \setminus \mathbf{V}(\mathcal{N})$, *the Hopf cyclicity of the center at the origin is at most* s.

3. Proof of Theorem 1

In [11] it is proved that system (3) has a center at the origin if and only if one of the following two sets of conditions hold:

(c.1) $\lambda_1 = 2A + \bar{B} = 0$;
(c.2) $\lambda_1 = \mathrm{Im}(AB) = \mathrm{Im}(A^3C) = \mathrm{Im}(\bar{B}^3C) = 0$.

We write $A = a_1 + ia_2$, $B = b_1 + ib_2$, and $C = c_1 + ic_2$ in family (3) so that now $\lambda = (a_1, a_2, b_1, b_2, c_1, c_2) \in \mathbb{R}^6$ and compute the first non-vanishing reduced focal values obtaining

$$\begin{aligned}
f_3(\lambda) &= -2(a_2b_1 - a_1b_2),\\
\tilde{f}_6(\lambda) &= 2(6a_1a_2b_1c_1 + 15a_2b_1^2c_1 + 2a_2^2b_2c_1 - 6b_1^2b_2c_1 - 5a_2b_2^2c_1 + \\
&\quad 2b_2^3c_1 + 2a_1^2b_1c_2 - 6a_2^2b_1c_2 + 5a_1b_1^2c_2 + 2b_1^3c_2 + 15a_2b_1b_2c_2 - \\
&\quad 6b_1b_2^2c_2),\\
\tilde{f}_9(\lambda) &= \frac{1}{32}(-240a_1^4a_2c_1 - 160a_1^2a_2^3c_1 + 80a_2^5c_1 + 5115a_2b_1^4c_1 - \\
&\quad 2550b_1^4b_2c_1 + 3410a_2b_1^2b_2^2c_1 - 1700b_1^2b_2^3c_1 - 1705a_2b_2^4c_1 + \\
&\quad 850b_2^5c_1 + 288a_1^2a_2c_1^3 - 96a_2^3c_1^3 - 1080a_2b_1^2c_1^3 + 504b_1^2b_2c_1^3 + \\
&\quad 360a_2b_2^2c_1^3 - 168b_2^3c_1^3 - 80a_1^5c_2 + 160a_1^3a_2^2c_2 + 240a_1a_2^4c_2 + \\
&\quad 1705a_1b_1^4c_2 + 850b_1^5c_2 + 3410a_2b_1^3b_2c_2 - 1700b_1^3b_2^2c_2 + \\
&\quad 5115a_2b_1b_2^3c_2 - 2550b_1b_2^4c_2 + 96a_1^3c_1^2c_2 - 288a_1a_2^2c_1^2c_2 - \\
&\quad 360a_1b_1^2c_1^2c_2 - 168b_1^3c_1^2c_2 - 1080a_2b_1b_2c_1^2c_2 + 504b_1b_2^2c_1^2c_2 + \\
&\quad 288a_1^2a_2c_1c_2^2 - 96a_2^3c_1c_2^2 - 1080a_2b_1^2c_1c_2^2 + 504b_1^2b_2c_1c_2^2 + \\
&\quad 360a_2b_2^2c_1c_2^2 - 168b_2^3c_1c_2^2 + 96a_1^3c_2^3 - 288a_1a_2^2c_2^3 - \\
&\quad 360a_1b_1^2c_2^3 - 168b_1^3c_2^3 - 1080a_2b_1b_2c_2^3 + 504b_1b_2^2c_2^3),\\
\tilde{f}_{12}(\lambda) &= -\frac{1}{1050}(5628a_1^2a_2c_1 - 1876a_2^3c_1 - 6165a_2b_1^2c_1 + 2379b_1^2b_2c_1 + \\
&\quad 2055a_2b_2^2c_1 - 793b_2^3c_1 + 1876a_1^3c_2 - 5628a_1a_2^2c_2 - \\
&\quad 2055a_1b_1^2c_2 - 6165a_2b_1b_2c_2 - 793b_1^3c_2 + 2379b_1b_2^2c_2)(c_1^2 + c_2^2)^2,
\end{aligned}$$

$$\begin{aligned}\tilde{f}_{15}(\lambda) &= \frac{11}{160}(192a_1^3a_2^3c_1^4 - 64a_1a_2^5c_1^4 + 3b_1^3b_2^3c_1^4 - b_1b_2^5c_1^4 + 64a_1^4a_2^2c_1^3c_2 - \\ &\quad 384a_1^2a_2^4c_1^3c_2 + 64a_2^6c_1^3c_2 - b_1^4b_2^2c_1^3c_2 + 6b_1^2b_2^4c_1^3c_2 - b_2^6c_1^3c_2 + \\ &\quad 128a_1^3a_2^3c_1^2c_2^2 + 128a_1a_2^5c_1^2c_2^2 + 2b_1^3b_2^3c_1^2c_2^2 + 2b_1b_2^5c_1^2c_2^2 + \\ &\quad 64a_1^4a_2^2c_1c_2^3 - 384a_1^2a_2^4c_1c_2^3 + 64a_2^6c_1c_2^3 - b_1^4b_2^2c_1c_2^3 + \\ &\quad 6b_1^2b_2^4c_1c_2^3 - b_2^6c_1c_2^3 - 64a_1^3a_2^3c_2^4 + 192a_1a_2^5c_2^4 - b_1^3b_2^3c_2^4 + \\ &\quad 3b_1b_2^5c_2^4).\end{aligned}$$

We have checked that $\tilde{f}_j(\lambda) \in \mathcal{B}_{15}$ for $j = 18, 19, 20$. It is probable that $\mathcal{B}_{15} = \mathcal{B}$ but we do not have more evidences for making this assertion.

What we will do now is to compute the center variety $\mathbf{V}(\mathcal{B}) \subset \mathbb{R}^6$ associated to the origin of family (3). We claim that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{12})$.

First we have checked that $\mathcal{B}_{15} \subset \sqrt{\mathcal{B}_{12}}$. Since $\mathbf{V}(\mathcal{B}_{12}) = \mathbf{V}(\sqrt{\mathcal{B}_{12}})$, we use the routine `minAssChar` in the `primdec.LIB` library of Singular for finding the prime decomposition of $\sqrt{\mathcal{B}_{12}}$. We get that $\sqrt{\mathcal{B}_{12}} = \cap_{i=1}^4 J_i$ where

$$\begin{aligned} J_1 &= \langle a_2 - 2b_2, a_1 + 2b_1, c_1^2 + c_2^2\rangle, \\ J_2 &= \langle b_1a_2 + a_1b_2, 3b_1^2b_2c_1 - b_2^3c_1 - b_1^3c_2 + 3b_1b_2^2c_2, 3a_1b_1b_2c_1 + \\ &\quad a_2b_2^2c_1 - a_1b_1^2c_2 + 3a_1b_2^2c_2, 3a_1^2b_2c_1 - a_2^2b_2c_1 - a_1^2b_1c_2 - \\ &\quad 3a_1a_2b_2c_2, 3a_1^2a_2c_1 - a_2^3c_1 + a_1^3c_2 - 3a_1a_2^2c_2\rangle, \\ J_3 &= \langle b_1, b_2, a_2c_1 - a_1c_2, c_1^2 + c_2^2, a_1c_1 + a_2c_2, a_1^2 + a_2^2\rangle, \\ J_4 &= \langle 2a_2 - b_2, 2a_1 + b_1\rangle. \end{aligned}$$

Taking into account that the real variety

$$\mathbf{V}(J_3) = \{\lambda \in \mathbb{R}^6 : A = B = C = 0\}$$

corresponds to the linear center $\dot{z} = iz$ and $\mathbf{V}(J_3) \subset \mathbf{V}(J_k)$ for any $k \in \{1, 2, 4\}$ we have that $\mathbf{V}(\mathcal{B}_{12})$ decomposes as the union of irreducible components as

$$\mathbf{V}(\mathcal{B}_{12}) = \mathbf{V}(\sqrt{\mathcal{B}_{12}}) = \mathbf{V}(J_1) \cup \mathbf{V}(J_2) \cup \mathbf{V}(J_4).$$

Center conditions (c.1) and (c.2) written in terms of parameters λ are

(c.1) $\lambda_1 = 2a_1 + b_1 = 2a_2 - b_2 = 0$;

(c.2) $\lambda_1 = a_2b_1 + a_1b_2 = 3a_1^2a_2c_1 - a_2^3c_1 + a_1^3c_2 - 3a_1a_2^2c_2 = -3b_1^2b_2c_1 + b_2^3c_1 + b_1^3c_2 - 3b_1b_2^2c_2 = 0$.

We recall that the origin of system (3) with $(\lambda_1, \lambda) = (0, \lambda^*)$ is a center if all the generators of J_i for any $i \in \{1, 2, 4\}$ vanish at $\lambda = \lambda^*$, hence the claim is proved and the center variety is $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{12}) \subset \mathbb{R}^6$.

Also we want to establish that

$$\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{12}) \text{ holds in } \mathbb{C}^6. \tag{4}$$

For proving that we follow [7] (see also [6]). First we note that the inclusion $\mathbf{V}(\mathcal{B}) \subset \mathbf{V}(\mathcal{B}_{12})$ holds in $\mathbb{C}^6$ since $\mathcal{B}_{12} \subset \mathcal{B}$ by definition. Hence, we only need to check the reverse inclusion. That is that $\mathbf{V}(\mathcal{B}_{12}) \subset \mathbf{V}(\mathcal{B})$ holds in $\mathbb{C}^6$. To prove that we must check whether for any $\lambda^* \in \mathbb{C}^6$ satisfying $f_1(\lambda^*) = \cdots = f_{12}(\lambda^*) = 0$ this implies that $f_j(\lambda^*) = 0$ for all $j \in \mathbb{N}$, or equivalently that there is a formal first integral in $\mathbb{C}[[x, y]]$ of the associated system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ when system (1) is extended to the complex setting with $(x, y, \lambda) \in \mathbb{C}^2 \times \mathbb{C}^6$. Clearly, the former is trivially true if system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ is Hamiltonian. In [6] it is also proved the validity of the above when system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ has a time-reversible center at the origin. We recall that family (3) is *time-reversible* (or reversible with respect to a straight line) if it is invariant under the change of variables $z \mapsto \exp(i\varphi)z$ for some real φ and the reversion of time $t \mapsto -t$.

As can be seen in [11], center condition (c.1) corresponds to an integrable case. That means that, when $\lambda^* \in \mathbf{V}(J_4)$, system (3) with $(\lambda_1, \lambda) = (0, \lambda^*)$ can be written after rescaling by $|z|^2$ as $\dot{z} = i\partial H/\partial \bar{z}$ where $H(z, \bar{z}) = \log |z|^2 - iAz^2\bar{z} + i\bar{A}z\bar{z}^2 + \frac{1}{3}(\bar{C}z^3 - C\bar{z}^3)$. The function $\exp(H)$ is a real analytic first integral in a neighborhood of $(x, y) = (0, 0)$, and it can be clearly extended to a formal first integral in the complex setting.

On other hand, center condition (c.2) corresponds to the time-reversible case. When $\lambda^* \in \mathbf{V}(J_i)$ for any $i \in \{1, 2\}$, it is proved in [11] that (3) with $(\lambda_1, \lambda) = (0, \lambda^*)$ is time-reversible. More precisely, in this case one has $A = -\bar{A}\exp(i\varphi)$, $C = -\bar{C}\exp(-3i\varphi)$, and $B = -\bar{B}\exp(-i\varphi)$ for some real φ. Hence, from Proposition 13 of [6] we deduce the existence of a formal first integral of (3) extended to the complex setting. In summary, we have proved (4).

At this point we have to see wether $\mathcal{B}_{12}$ or $\mathcal{B}_{15}$ are a radical ideals or not. Unfortunately they are not and we cannot apply Theorem 2 for bounding the cyclicity of the center of family (3). We are forced to use Theorem 3 in order to prove Theorem 1.

To find the primary decomposition of $\mathcal{B}_{12}$ we will use either of the routines `primdecGTZ` or `primdecSY` in the `primdec.LIB` library of Singular. We get the primary decomposition $\mathcal{B}_{12} = \bigcap_{j=1}^{6} I_j$ being $I_j = \sqrt{I_j}$

for $j = 1, 2, 3$ and $I_j \neq \sqrt{I_j}$ when $j = 4, 5, 6$. More precisely, we find

$$\begin{aligned}
\sqrt{I_4} &= \langle a_1^2 + a_2^2, -a_2 c_1 + a_1 c_2, c_1 a_1 + c_2 a_2, c_1^2 + c_2^2, b_2, b_1 \rangle, \\
\sqrt{I_5} &= \langle a_1^2 + a_2^2, a_2 c_1 + a_1 c_2, c_1 a_1 - c_2 a_2, c_1^2 + c_2^2, b_2, b_1 \rangle, \\
\sqrt{I_6} &= \langle c_1^2 + c_2^2, b_2, b_1, a_1, a_2 \rangle.
\end{aligned}$$

Setting $\mathcal{N} = \bigcap_{i=4}^{6} I_i$ as it is defined in Theorem 3, now we use the `Intersect` command of MAPLE to get a set of generators of $\sqrt{\mathcal{N}}$, namely

$$\sqrt{\mathcal{N}} = \bigcap_{i=4}^{6} \sqrt{I_i} = \langle b_1, b_2, a_1^2 + a_2^2, c_1^2 + c_2^2 \rangle.$$

Finally, taking into account that $\mathbf{V}(\mathcal{N}) = \mathbf{V}(\sqrt{\mathcal{N}})$ holds in any ground field we obtain

$$\mathbf{V}(\mathcal{N}) = \{\lambda \in \mathbb{R}^6 : A = B = C = 0\} = \{0\}.$$

This means that $\lambda^* \in \mathbf{V}(\mathcal{B}) \setminus \mathbf{V}(\mathcal{N})$ if and only if λ^* corresponds to any nonlinear center of (3).

Finally, since the ideal

$$\mathcal{B}_{12} = \langle f_3(\lambda), \tilde{f}_6(\lambda), \tilde{f}_9(\lambda), \tilde{f}_{12}(\lambda) \rangle$$

has a minimal basis with cardinality four, as a consequence of Theorem 3, we have proved that any nonlinear center at the origin in family (3) has cyclicity at most 4.

Only remains to prove the second claim in Theorem 1. We will consider perturbations of the linear center $\dot{z} = iz$ inside family (3).

First we will see that the point $\lambda^* = 0$ corresponding to the linear center is not isolated from the set of points in the parameter space $\mathbb{R}^6$ corresponding to a system in family (3) possessing a forth order weak focus at the origin. Perturbing from $\lambda(0) = \lambda^* = 0$ to $\lambda(\varepsilon) = (\varepsilon, \varepsilon/\sqrt{5}, 0, 0, 0, \varepsilon) \in \mathbb{R}^6$ with the small perturbation parameter ε we have $f_3 = \tilde{f}_6 = \tilde{f}_9 = 0$ and $\tilde{f}_{12} = -268/375\, \varepsilon^8$. The perturbed system is $\dot{z} = iz + z\bar{z}(A(\varepsilon) z^2 + B(\varepsilon) z\bar{z} + C(\varepsilon)\bar{z}^2)$ with $A(\varepsilon) = \varepsilon(1 - i/\sqrt{5})$, $B(\varepsilon) = 0$ and $C(\varepsilon) = i\varepsilon$. Notice also that $A^2(\varepsilon) = 5|A(\varepsilon)|^2 - 6|C(\varepsilon)|^2 = 0$. Then, the perturbed system has a forth order weak focus at the origin and following Theorem 6 of [11] a further arbitrarily small perturbation can produce four limit cycles bifurcating from the focus at the origin. Therefore the claim is proved finishing the proof of the theorem.

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