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Continua of periodic points for planar integrable rational maps

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Abstract

We present three alternative methodologies to find continua of periodic points with a prescribed period for rational maps having rational first integrals. The first two have been already used for other authors and apply when the maps are birational and the generic level sets of the corresponding first integrals have either genus 0 or 1. As far as we know, the third one is new and it works for rational maps without imposing topological properties to the invariant level sets. It is based on a computational point of view, and relies on the use of resultants in a suitable setting. We apply them to several examples, including the 2-periodic Lyness composition maps and some of the celebrated McMillan-Gumowski-Mira maps.

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1 Introduction

A planar *rational* map $F : \mathcal{U} \to \mathbb{K}^2$, where $\mathcal{U} \subseteq \mathbb{K}^2$ is an open set and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, is called *birational* if it has a rational inverse F^{-1}. Such a map is *integrable* if there exists a non-constant function $V : \mathcal{U} \to \mathbb{K}$ such that $V(F(x, y)) = V(x, y)$, which is called a *first integral* of F. If a map F possesses a first integral V, then the level sets of V are invariant under F.

We consider integrable rational (or birational) maps that have a *rational first integral* $V(x, y) = V_1(x, y)/V_2(x, y)$, thus preserving the fibration given by the pencil of algebraic curves

$$C_h := \{V_1(x, y) - h\, V_2(x, y) = 0\}, \text{ for } h \in \mathrm{Im}(V). \tag{1.1}$$

When each curve C_h is invariant under the iterates of F it is said that the map *preserves the fibration* given by the curves $\{C_h\}$.

Integrable planar birational maps typically exhibit continua of periodic orbits lying in some of the level curves of their first integrals. In fact, a part of the very few globally periodic cases([12]), a typical phase portrait includes curves densely filling an open set of the phase space and each one of them full of periodic points with the same period. Moreover, in this case the set of periods of the map is infinite and it contains, among others, all the periods greater than a particular one. Nevertheless, it is usually very difficult to find any of these continua of periodic points by using either the direct approach of solving the system $F^p(x, y) = (x, y)$, for a given $p \in \mathbb{N}$, or any of their equivalent expressions

$$F^m(x, y) = F^{m-p}(x, y),$$

for some $1 \leq m \leq p$, where $F^0 = \mathrm{Id}$, $F^1 = F$ and $F^k = F \circ F^{k-1}, k \geq 2$. Similarly, F^{-1} is the inverse of F and $F^{-k} = (F^{-1})^k$.

In this paper we present three alternative effective methodologies for finding continua of periodic points with a prescribed period for rational maps having rational first integrals. The first two have been already used for other authors and only apply when the maps are birational and the generic level sets of the corresponding first integrals have either genus 0 or 1. To the best of our knowledge, the third one is new and it works for rational maps, non-necessarily birational, without imposing any restriction to the invariant level sets. It is based on a computational point of view, and relies on the use of resultants in a suitable setting.

Concerning the first approach, in case that the curves C_h in the fibration (1.1) have generically genus 0, it is well-known that each map $F_{|C_h}$ is conjugate to a Möbius transformation ([30]). This fact permits, when the maps $F_{|\{|V|=|h|\}}$ are conjugate to rotations, to give an explicit expression of rotation number $\rho(h)$, and to determine the curves in (1.1) filled with p-periodic points by solving the equations

$$\rho(h) = \frac{q}{p} \text{ for all } 1 \leq q \leq p-1, \quad \text{coprime with } p. \tag{1.2}$$

See the proof of Theorem 2.1 in Section 2 for more details.

The second approach deals with the most studied case which corresponds to the situation where the invariant curves in the preserved fibration (1.1) have generic genus 1

(also called elliptic fibrations). This case is important because birational maps preserving elliptic fibrations appear in many physical applications([28]). Some classical maps like the Lyness and the integrable McMillan-Gumowski-Mira ones, which are included in a large family known as QRT maps preserve elliptic fibrations, see [27] and the references therein.

Although in the elliptic case there is not an explicit expression of the rotation number function it is possible identify the energy levels where the periodic orbits are located by using the group estructure of the elliptic curves in the fibration (1.1), see for instance [3, 15, 33]. In Section 3 we give an example of this approach.

As we have already said, the last method presented is purely computational, it works for rational maps that are not necessarily birational (a trivial example of rational map, not birational, and with a rational first integral $V(x, y) = y$, is $F(x, y) = (x^2, y)$), and it is only based on the assumption that the continua of periodic orbits lie on the level sets of the rational first integral. This new method, together with some examples of application, is developed in Section 4.

Another advantage of our approach is that it can be extended to integrable rational maps in higher dimensions. In this paper we are not concerned with this question but we believe that, using our point of view, we could also reproduce the results of [29] on this subject, which were obtained using more algebraic tools.

Observe, however, that removing the hypotheses on F to be rational, one should not expect that the continua of periodic orbits of integrable maps in the plane coincide with the level sets of their first integrals, as discussed in Section 5.

Throughout the paper we will assume that F is a rational map with a rational first integral $V = V_1/V_2$, where both F and V are defined in an open and dense set of the domain of definition of the dynamical system, the *good set* $\mathcal{G}(F) \subset \mathbb{K}^2$. In the real case we also will assume that there exists a set $\mathcal{H} \subset \mathrm{Im}(V) \subset \mathbb{R}$ such that the set $\mathcal{U} := \{(x, y) \in \mathbb{R}^2 \colon V(x, y) \in \mathcal{H}\} \cap \mathcal{G}(F)$ is a non empty open set of $\mathbb{R}^2$.

Sections 2 and 3 deal with the genus 0 and genus 1 situations, respectively. Both have the same schemes, first we recall the methods and then we apply them to some interesting maps. Finally, in Section 4, we introduce and apply to several examples our computational method to find continua of periodic points. As an illustration of the applicability of our results we dedicate Section 4.4.2 to describe the continua of real 5-periodic points for a McMillan-Gumovski-Mira map.

2 The genus 0 case

2.1 Explicit expression of the rotation number

Suppose that each generic curve $C_h := \{V_1(x, y) - hV_2(x, y) = 0\}$ for h in an open set $\mathcal{H} \subset \mathbb{K}$ is irreducible in $\mathbb{C}$ and has genus 0. Then the Cayley-Riemann Theorem ensures that it can be *rationally parameterized*, that is, that there exist rational functions $P_{1,h}(t), P_{2,h}(t) \in \mathbb{C}(t)$ such that for almost all of the values $t \in \mathbb{K}$ we have $(P_{1,h}(t), P_{2,h}(t)) \in C_h$; and reciprocally, for almost all point $(x, y) \in C_h$, there exists $t \in \mathbb{K}$ such that $(x, y) = P_h(t) := (P_{1,h}(t), P_{2,h}(t))$. A rational parametrization of C_h is not unique but it is always possible to obtain a *proper* one, that is, a parametrization P such that P^{-1} is also rational, which is unique modulus Möbius transformations. The reader is referred to [30] for a further insight about these facts.

Given a birational map F preserving each curve C_h of the fibration (1.1) defined in $\mathbb{K}^2$, a simple computation shows that if we take a proper parametrization of each curve $P_h(t)$, the action of $F_{|C_h}$ is conjugate to the Möbius transformation defined in $\widehat{\mathbb{K}} := \mathbb{K} \cup \{\infty\}$, given by $M_h = P_h^{-1} \circ F_h \circ P_h$, see for instance [22, Proposition 1].

The dynamics of Möbius maps is well-known. With respect the searching of periodic orbits, when M_h is conjugate to a rotation, the rotation number $\rho(h)$ of M_h, which is the one of $F_{|C_h}$, can be explicitly obtained. Indeed, consider the map $M(t) = (at + b)/(ct + d)$ where $a, b, c, d \in \mathbb{K}$ with $c \neq 0$, and defined for $t \in \widehat{\mathbb{K}}$. If

$$\Delta = (d - a)^2 + 4bc \neq 0 \text{ and } \xi = \frac{a + d + \sqrt{\Delta}}{a + d - \sqrt{\Delta}} \text{ is such that } |\xi| = 1,$$

then M is conjugated to a rotation in $\widehat{\mathbb{K}}$ with rotation number $\rho := \arg(\xi) \pmod{2\pi}$. In particular, M is periodic with minimal period p if and only if ξ is a p-primitive root of the unity. See for instance [7, Section 2.2].

Once we have the expression of the rotation number function $\theta(h)$ associated to each map $F_{|C_h}$, we can characterize the continua of periodic points by solving Equation (1.2).

2.2 The Bastien and Rogalski maps

As an example of this specific methodology we consider the one-parameter family of maps

$$F_a(x, y) = \left(y, \frac{a - y + y^2}{x}\right), \tag{2.1}$$

defined in $(\mathbb{R}^+)^2$ with $a > 1/4$. This family has been introduced and studied by G. Bastien and M. Rogalski in [4]. Each map F_a has the first integral

$$V_a(x, y) = \frac{x^2 + y^2 - x - y + a}{xy}, \tag{2.2}$$

so it preserves the fibration of $(\mathbb{R}^+)^2$ given by the conics:

$$C_h = \left\{x^2 + y^2 - x - y + a - hxy = 0, \text{ for } h > 2 - 1/a\right\}. \tag{2.3}$$

A similar study could be done extending this map to $\mathbb{C}^2$ and taking $a \in \mathbb{C}$.

By using the *parametrization by lines method* ([30]) it is not difficult to find the following proper rational parametrization of the curves C_h in the fibration (2.3), see [22] for more details:

$$P_{1,h}(t) = \frac{2\delta t - ah^2 - (1+\delta)h - 2 + 4a}{2(-t^2 + ht - 1)} + a,$$

$$P_{2,h}(t) = \frac{(-ah + \delta - 1)\,t^2 + (4\,a - 2)\,t - ah - \delta - 1}{2(-t^2 + ht - 1)},$$

whose inverse is given by

$$P_h^{-1}(x, y) = \frac{-2\,\delta\,x + (ah^2 + (\delta + 1)h - 4\,a + 2)\,y - ah + 2\,a + \delta - 1}{(ah^2 + (1-\delta)h - 4\,a + 2)\,x + 2\,\delta\,y + ah - 2\,a - \delta + 1}.$$

For a fixed value of $a > 1/4$ and for $2 - 1/a < h < 2$, the level curves C_h are ellipses. A computation gives

$$M_h(t) = P_h^{-1} \circ F_a \circ P_h(t) = \frac{(h+1)t - 1}{t+1}, \tag{2.4}$$

so if $2 - 1/a < h < 2$ then $F_{a|C_h}$ is conjugate to $M_{h|\widehat{\mathbb{R}}}$, which is, in turn, conjugate to a rotation with rotation number

$$\rho(h) = \arg\left(\frac{h - i\sqrt{4 - h^2}}{2}\right) \mod 2\pi, \tag{2.5}$$

see [4] and also [22]. By solving the Equation (1.2) for $p \le 25$, we will obtain:

Theorem 2.1. *The curves C_h of the fibration (2.3) are filled of periodic points with minimal period p for $p \le 25$, if and only if $2 - 1/a < h < 2$, and it is satisfied $c_p(h) = 0$ for:*

$$\begin{aligned}
c_3(h) &= h + 1, & c_4(h) &= h,\\
c_5(h) &= h^2 + h - 1, & c_6(h) &= h - 1,\\
c_7(h) &= h^3 + h^2 - 2h - 1, & c_8(h) &= h^2 - 2,\\
c_9(h) &= h^3 - 3h + 1, & c_{10}(h) &= h^2 - h - 1,\\
c_{11}(h) &= h^5 + h^4 - 4h^3 - 3h^2 + 3h + 1, & c_{12}(h) &= h^2 - 3,\\
c_{13}(h) &= h^6 + h^5 - 5h^4 - 4h^3 + 6h^2 + 3h - 1, & c_{14}(h) &= h^3 - h^2 - 2h + 1,\\
c_{15}(h) &= h^4 - h^3 - 4h^2 + 4h + 1, & c_{16}(h) &= h^4 - 4h^2 + 2,
\end{aligned}$$

$$
\begin{aligned}
c_{17}(h) &= h^8 + h^7 - 7h^6 - 6h^5 + 15h^4 + 10h^3 - 10h^2 - 4h + 1,\\
c_{18}(h) &= h^3 - 3h - 1,\\
c_{19}(h) &= h^9 + h^8 - 8h^7 - 7h^6 + 21h^5 + 15h^4 - 20h^3 - 10h^2 + 5h + 1,\\
c_{20}(h) &= h^4 - 5h^2 + 5,\\
c_{21}(h) &= h^6 - h^5 - 6h^4 + 6h^3 + 8h^2 - 8h + 1,\\
c_{22}(h) &= h^5 - h^4 - 4h^3 + 3h^2 + 3h - 1,\\
c_{23}(h) &= h^{11} + h^{10} - 10h^9 - 9h^8 + 36h^7 + 28h^6 - 56h^5 - 35h^4\\
&\quad + 35h^3 + 15h^2 - 6h - 1,\\
c_{24}(h) &= h^4 - 4h^2 + 1,\\
c_{25}(h) &= h^{10} - 10h^8 + 35h^6 + h^5 - 50h^4 - 5h^3 + 25h^2 + 5h - 1.
\end{aligned}
$$

Observe that the above conditions characterizing the existence of continua of periodic points in the curves C_h do not depend on the parameter a. This is due to the appearance of a simply as an additive constant in the expression of the curves. Of course, this fact is also reflected on the rotation number function (2.5) which is independent of a. For the sake of brevity we only display the values for $p \leq 25$ but the method allows to go much further.

Proof of Theorem 2.1. Observe that since the rotation number function is given by (2.5), then Equation (1.2) is satisfied, and the level h is filled by points of *minimal* period p, if and only if

$$\xi_p := \frac{h - i\sqrt{4-h^2}}{2}, \tag{2.6}$$

is a *primitive* p-root of the unity. These conditions are contained in the set of solutions of the system

$$\left\{\begin{array}{l} \xi^2 - h\xi + 1 = 0,\\ \xi^p - 1 = 0. \end{array}\right.$$

Therefore, we can obtain the values of h for which (2.6) holds by looking for those factors of

$$\phi_p(h) := \mathrm{Res}(\xi^2 - h\xi + 1, \xi^p - 1; \xi)$$

that do not appear in $\phi_k(h)$ for any $k < p$ that is a divisor of p. Here, as usual, $\mathrm{Res}(\cdot,\cdot;\xi)$ denotes the resultant of both polynomials with respect to ξ. Recall that this resultant only vanishes when these two polynomials have a common (real or complex) zero, see for instance [20]. We only detail the computations when $p = 25$:

$$
\begin{aligned}
\phi_{25}(h) = {}& \mathrm{Res}(\xi^2 - h\xi + 1, \xi^{25} - 1; \xi)\\
= {}& (2-h)\left(h^2 + h - 1\right)^2 \left(h^{10} - 10h^8 + 35h^6 + h^5 - 50h^4 - 5h^3\right.\\
& \left. + 25h^2 + 5h - 1\right)^2.
\end{aligned}
$$

We can exclude the factor $2-h$ because when $h = 2$, $F_{a|C_2}$ is not conjugate to a rotation. The vanishing of the factor h^2+h-1 corresponds to periodic points of minimal period 5. Thus the condition characterizing periodic points of minimal period 25 is $c_{25}(h) = 0$. ∎

3 The elliptic case

3.1 Periodic orbits seen as finite order points of a group

If a curve $C_h := \{V_1(x,y) - hV_2(x,y) = 0\}$ for $h \in \mathcal{H}$ has genus 1 (i.e. is an *elliptic curve*) then it has an associated group structure([31]). It is known that this fact allows to characterize the dynamics of any birational map preserving it. As a consequence, we will locate the level sets where the periodic orbits lay.

We start recalling the *chord-tangent group* law associated with an elliptic curve $C \in \mathbb{C}\mathbf{P}^2$, where as usual, we consider the extension of the algebraic curve in $\mathbb{K}^2$ to $\mathbb{C}\mathbf{P}^2$ given by the *homogenization process*, [31, Section I.2]. Given two points P and Q in C we define the addition $P+Q$ as in Figure 3.1: (i) Select a point $\mathcal{O} \in C$ to be the neutral element of the inner addition; (ii) Take the chord passing through P and Q (the tangent line if $P = Q$). It will always intersect C at a unique third point denoted by $P * Q$; (iii) The point $P + Q$ is then defined as $\mathcal{O} * (P * Q)$. The curve endowed with this inner addition $(C, +, \mathcal{O})$ is an *abelian group*.

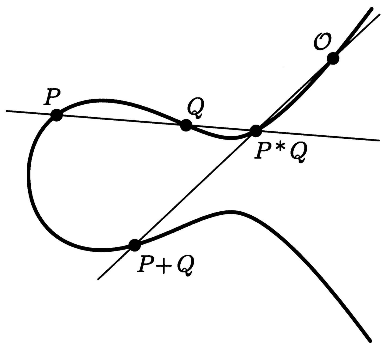


Figure 3.1: Group law with an affine neutral element $\mathcal{O}$.

The relation between the dynamics of a birational map preserving an elliptic curve and the group law of the curve is given by a result of Jogia, Roberts and Vivaldi [18, Theorem 3], JRV Theorem from now on, which states that *if F is a birational map over a field $\mathbb{K}$ not of characteristic 2 or 3, that preserves an elliptic curve C defined over $\mathbb{K}$. Then there exist points $\mathcal{O}, Q \in C$ such that the map can be expressed in terms of the group law $+$ on C as either (a) $F_{|C} : P \mapsto P + Q$, or (b) $F_{|C} : P \mapsto i(P) + Q$, where i is a periodic automorphism whose period is either 2, 3, 4 or 6, and in this case F has the same period as i.*

When F is a birational map preserving an elliptic curve $(C, +, \mathcal{O})$ whose dynamics corresponds to case (a) of the JRV Theorem, then $F^n_{|C}(P) = P + n\,Q$. In this case notice that P is a p-periodic point if and only if

$$p\,Q = \overbrace{Q + \cdots + Q}^{p} = \mathcal{O}, \tag{3.1}$$

that is, if and only if Q is a *finite order* point of the group $(C,+,\mathcal{O})$.

Equation (3.1) can be used to determine the invariant curves where are located the periodic points. See for instance [3, 15, 33] to see how to use this point of view in the case of the paradigmatic example given by the Lyness maps. In the next section we give an example of how to take advantage of equation (3.1) to characterize the invariant curves C_h where lie the periodic orbits for a particular family of maps, the 2-periodic Lyness composition maps.

3.2 The 2-periodic Lyness composition maps

We consider the 2-parametric family of maps given by

$$F_{b,a}(x,y) := \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right), \tag{3.2}$$

with $a,b\in\mathbb{C}$, with $ab\neq 0$, that we will call 2-*periodic Lyness composition maps* because they appear as *composition maps*, to describe the sequences generated by the 2-periodic difference equation

$$u_{n+2} = \frac{a_n+u_{n+1}}{u_n}, \text{ where } a_n = \begin{cases} a & \text{for} \quad n=2\ell+1, \\ b & \text{for} \quad n=2\ell. \end{cases} \tag{3.3}$$

More concretely, $F_{b,a}(x,y) := (F_b\circ F_a)(x,y)$ where F_a and F_b are the Lyness maps $F_\alpha(x,y) = (y,(\alpha+y)/x)$, associated to the autonomous Lyness recurrence $u_{n+2} = (\alpha+u_{n+1})/u_n$. These maps have been introduced and studied in [6, 10, 19].

Each map $F_{b,a}$ has the first integral

$$V_{b,a}(x,y) = \frac{(bx+a)\,(ay+b)\,(ab+ax+by)}{xy}, \tag{3.4}$$

so it preserves each curve of the fibration given by the curves

$$C_h = \{(bx+a)(ay+b)(ax+by+ab)-hxy=0\}\,,\; h\in\mathbb{C}. \tag{3.5}$$

By taking homogeneous coordinates $[x:y:z]\in\mathbb{C}\mathbf{P}^2$ these curves extend to

$$\widetilde{C_h} := \{(bx+az)(ay+bz)(ax+by+abz)-hxyz=0,\; x,y,z\in\mathbb{C}\},$$

and the map $F_{b,a}$ also extends to a polynomial one in $\mathbb{C}\mathbf{P}^2$

$$\widetilde{F}_{b,a}([x:y:z]) = [ayz+y^2 : az^2+bxz+yz : xy], \tag{3.6}$$

except at the points $[x:0:0]$, $[-a:0:b]$ and $[0:-a:1]$.

Some computations (see the Appendix for a brief comment about them) show that, when $ab \neq 0$, the curves $\widetilde{C_h}$ are elliptic except when $h = 0$ or h is a solution of

$$h^4 + p_3(a,b)h^3 + p_2(a,b)h^2 + p_1(a,b)h + p_0(a,b) = 0, \tag{3.7}$$

for some polynomials $p_j(a,b)$ given in this appendix. In particular we get that equation (3.5) gives an invariant fibration with generic fibers of genus 1.

Observe that for all the values of h, the curves $\widetilde{C_h}$ contain the three points at infinity $[1:0:0]$, $[0:1:0]$, $[b:-a:0]$. It is not difficult to prove that taking $\mathcal{O} := [0:1:0]$, for any elliptic level h, the map (3.6) can be re-written as

$$\widetilde{F}_{b,a|\widetilde{C_h}}([x:y:z]) = [x:y:z] + [1:0:0], \tag{3.8}$$

see [6].

Now, we can use Equation (3.1), which in our case writes as:

$$p \cdot [1:0:0] \;=\; [0:1:0], \tag{3.9}$$

in order to obtain the curves where are located the continua of periodic orbits with a prescribed period p. The main result is the following:

Theorem 3.1. *Consider the family of complex maps $F_{a,b}$ given in* (3.2)*, with $ab \neq 0$. The following statements hold:*

(i) The map $F_{1,1}$ is globally 5-periodic.

(ii) There are no elliptic curves C_h filled with 2 or 3 periodic points of these maps.

(iii) The elliptic curves C_h filled with p-periodic points of these maps with period $p \in \{4,\ldots,10\}$, are given by those values a, b, and $h \neq 0$ such that h is not a solution of Equation (3.7) and $c_p(a,b,h_p) = 0$, where:

$$\begin{aligned}
c_4(a,b,h) =\;& h - BA,\\
c_5(a,b,h) =\;& (ab-1)\,h + BA,\\
c_6(a,b,h) =\;& h^2 + \left(a^2b^2 - 2a^3 - 2b^3 + 3ab\right) h + BA\left(-a^2b^2 + a^3 + b^3 - 2ab\right),\\
c_7(a,b,h) =\;& -h^3 + \left(a^3b^3 - 4a^2b^2 + 3a^3 + 3b^3 - 3ab\right) h^2\\
& -BA\left(-4a^2b^2 + 3a^3 + 3b^3 - 3ab\right) h + B^3A^3,\\
c_8(a,b,h) =\;& h^3 + \left(-2a^3 - 2b^3 + 5ab - 1\right) h^2\\
& +BA\left(-a^2b^2 + a^3 + b^3 - 4ab + 2\right) h - B^2A^2,\\
c_9(a,b,h) =\;& -h^5 + \left(-4a^2b^2 + 5a^3 + 5b^3 - 6ab\right) h^4\\
& + \left(a^5b^5 - 10a^4b^4 + 17a^5b^2 + 17a^2b^5 - 10a^6 - 38a^3b^3\right.\\
& \left. -10b^6 + 24a^4b + 24ab^4 - 15a^2b^2\right) h^3 + BA\left(10a^4b^4 - 17a^5b^2\right.\\
& \left. -17a^2b^5 + 10a^6 + 37a^3b^3 + 10b^6 - 26a^4b - 26ab^4 + 19a^2b^2\right) h^2\\
& - \left(4a^4b^4 - 9a^5b^2 - 9a^2b^5 + 5a^6 + 20a^3b^3 + 5b^6 - 14a^4b\right.\\
& \left. -14ab^4 + 12a^2b^2\right) B^2A^2h + \left(a^4b^4 - 2a^5b^2 - 2a^2b^5 + a^6\right.\\
& \left. +5a^3b^3 + b^6 - 3a^4b - 3ab^4 + 3a^2b^2\right) B^3A^3,
\end{aligned}$$

$$
\begin{aligned}
c_{10}(a,b,h) = \; & (-ab-1)\,h^5 + \big(-a^3b^3 + 4a^4b + 4ab^4 - 12a^2b^2 + 5a^3 + 5b^3 \\
& -5ab\big)\,h^4 + \big(-a^5b^5 + 6a^6b^3 + 6a^3b^6 - 6a^7b - 35a^4b^4 - 6ab^7 \\
& +41a^5b^2 + 41a^2b^5 - 10a^6 - 56a^3b^3 - 10b^6 + 20a^4b + 20ab^4 \\
& -10a^2b^2\big)\,h^3 + BA\,\big(a^5b^5 - 5a^6b^3 - 5a^3b^6 + 4a^7b + 30a^4b^4 + 4ab^7 \\
& -37a^5b^2 - 37a^2b^5 + 10a^6 + 55a^3b^3 + 10b^6 - 20a^4b - 20ab^4 \\
& +10a^2b^2\big)\,h^2 - \big(a^5b^5 - 2a^6b^3 - 2a^3b^6 + a^7b + 12a^4b^4 + ab^7 \\
& -15a^5b^2 - 15a^2b^5 + 5a^6 + 25a^3b^3 + 5b^6 - 10a^4b - 10ab^4 \\
& +5a^2b^2\big)\,B^2A^2h + B^5A^5,
\end{aligned}
$$

with $A = a^2 - b$ and $B = a - b^2$.

Remark 3.2. (a) With respect statement (ii), it is easy to verify that indeed the maps $F_{b,a}$ have not 2-periodic orbits. Moreover, when $ab \neq 0$, they have not 3-periodic orbits [6, Proposition 20]. In fact, $F_{0,0}$ is globally 3-periodic.

(b) The computations of the expressions $c_p(a,b,h)$ are formal, so they only characterize the curves filled with p-periodic points, provided that they are elliptic ones. For instance, when $AB = 0$, the condition $c_4(a,b,h) = 0$ is $h = 0$, and one obtains a value that does not correspond neither to an elliptic curve nor to a continua of 4-periodic points. That is the reason why we need to include the characterization of the elliptic curves as a hypothesis of statement (iii).

(c) If we restrict our attention to $a, b \in \mathbb{R}$ and to continua of real p-periodic points it may happen some of the curves C_{h_p} have no real points.

(d) We have also obtained the conditions for $p \in \{11, 12, 13, 14\}$. They are polynomials in h, with polynomials coefficients in a and b, and respective degrees $8, 7, 11$ and 10. We omit their explicit expressions for the sake of shortness. They will appear in [21].

To prove the statements (ii) and (iii) of the above result we need to compute the elements of the form $P + nQ$, so we briefly recall the computation procedure. Consider the points $Q = [1 : 0 : 0]$ and $\mathcal{O} = [0 : 1 : 0]$ which belong to the elliptic curves $(C_h, +, \mathcal{O})$. For any point $P \in C_h$, the point $P + Q$ is given by $P + Q = (P * Q) * \mathcal{O}$. So, if we consider a point $[x_0 : y_0 : 1]$, to compute $[x_1 : y_1 : 1] = [x_0 : y_0 : 1] + Q$, we first calculate $[x_0 : y_0 : 1] * Q$ by taking the horizontal line $y = y_0$ passing through $[x_0 : y_0 : 1]$ and Q, which cuts C_h in a second point $[x_1 : y_0 : 1] = [x_0 : y_0 : 1] * Q$. Finally, to get $[x_1 : y_0 : 1] * \mathcal{O}$, we consider the vertical line $x = x_1$ passing through $[x_1 : y_0 : 1]$ which cuts C_h at the new point $[x_1 : y_1 : 1] = [x_0 : y_0 : 1] + Q$.

Observe that since Q is tangent to the asymptote $y = -b/a$ of C_h, the point $Q * Q$, which is necessary to get $2Q = (Q*Q)*\mathcal{O}$, is obtained by computing the intersection of the line $y = -b/a$ with C_h which is the affine point $[x_1 : -b/a : 1]$. The expression of $2Q$ is finally obtained by computing the intersection of the line $x = x_1$, passing through $[x_1 : -b/a : 1]$, with C_h.

Proof of Theorem 3.1. (i) It is a simple computation. Moreover it is a consequence of the well-known fact that the Lyness map F_1 is globally 5-periodic.

(ii-iii) Applying the formal rules explained in the preamble to the proof we obtain, with the assistance of a computer algebra system, the points $\pm nQ$, for $n \leq 5$. We get

$$
\begin{aligned}
Q &= [1:0:0],\\
2Q &= [0:-a:1],\\
3Q &= \left[\frac{h-aA}{bA} : \frac{-h+AB}{abA} : 1\right],\\
4Q &= \left[\frac{h-A\left(a^2b-b^2+a\right)}{a\left(-h+aA\right)} : -\frac{b\left(\left(ab-1\right)Ah+BA^2\right)}{h^2-\left(-b^2+2a\right)Ah+aBA^2} : 1\right],\\
5Q &= \left[\frac{-a\left(ah^2-A\left(2a^2-b\right)h+BA^3\right)}{\left(-h+A\left(a^2b-b^2+a\right)\right)\left(-h+BA\right)} :\right.\\
&\qquad \left.\frac{\left(-h+aA\right)p(a,b,h)}{b\left(-h+A\left(a^2b-b^2+a\right)\right)\left(\left(ab-1\right)h+BA\right)} : 1\right],
\end{aligned}
$$

with $p(a,b,h) = h^2 + \left(a^2b^2 - 2a^3 - 2b^3 + 3ab\right)h + BA\left(-a^2b^2 + a^3 + b^3 - 2ab\right)$. To compute the points $-nQ$, we use that $-P = P * (\mathcal{O} * \mathcal{O})$ (see [32, page 18]). In our case $\mathcal{O} * \mathcal{O} = [-a/b : 0 : 1]$. We arrive to:

$$
\begin{aligned}
-Q =&[-b:0:1],\\
-2Q =&\left[\frac{h-BA}{abB} : -\frac{h+bB}{aB} : 1\right],\\
-3Q =&\left[-\frac{aB\left(\left(ab-1\right)h+BA\right)}{\left(h+bB\right)\left(-h+BA\right)} : \frac{-h+B\left(-ab^2+a^2-b\right)}{b\left(h+bB\right)} : 1\right],\\
-4Q =&\left[\frac{-\left(h+bB\right)q_0(a,b,h)}{a\left(-h+B\left(-ab^2+a^2-b\right)\right)\left(\left(ab-1\right)h+\left(-b^2+a\right)A\right)} :\right.\\
&\qquad \left.\frac{b\left(bh^2-B\left(-2b^2+a\right)h+AB^3\right)}{\left(-h+BA\right)\left(-h+B\left(-ab^2+a^2-b\right)\right)} : 1\right],\\
-5Q =&\left[\frac{\left(h-B\left(-ab^2+a^2-b\right)\right)q_1(a,b,h)}{b\left(bh^2-B\left(-2b^2+a\right)h+AB^3\right)q_2(a,b,h)} :\right.\\
&\qquad \left.\frac{\left(-h+BA\right)q_3(a,b,h)}{ab\left(\left(ab-1\right)h+BA\right)q_4(a,b,h)} : 1\right], \quad \text{where}
\end{aligned}
$$

$$\begin{aligned}
q_0(a,b,h) =&h^2+\left(a^2b^2-2a^3-2b^3+3ab\right)h+BA\left(-a^2b^2+a^3+b^3-2ab\right),\\
q_1(a,b,h) =&-h^3+\left(a^3b^3-4a^2b^2+3a^3+3b^3-3ab\right)h^2\\
&-BA\left(-4a^2b^2+3a^3+3b^3-3ab\right)h+B^3A^3,\\
q_2(a,b,h) =&h^2+\left(a^2b^2-2a^3-2b^3+3ab\right)h+BA\left(-a^2b^2+a^3+b^3-2ab\right),\\
q_3(a,b,h) =&h^3-B\left(2a^2-3b\right)h^2+\left(-a^3b^2+a^4-3a^2b+3b^2\right)B^2h-b^2AB^3,\\
q_4(a,b,h) =&bh^2-B\left(-2b^2+a\right)h+AB^3.
\end{aligned}$$

The expressions of the points $\pm nQ$ are formal. They give affine points when all the denominators do not vanish, and they give infinite points otherwise. For instance, observe that $3Q$ is an infinite point if and only $A=0$.

Let us prove our results. Observe first that always $2Q=[0:-a:1]\neq\mathcal{O}$. The condition $3Q=\mathcal{O}$ reads as

$$3Q=[a(h-aA):-h+AB:abA]=[0:1:0].$$

Since $ab\neq 0$, we get that $A=0$, but then $3Q=[ah:-h:0]\neq\mathcal{O}$. Hence item (ii) follows.

The p-periodicity conditions are obtained imposing that

$$E\Big(\frac{p+1}{2}\Big)Q=-E\Big(\frac{p}{2}\Big)Q,\quad p=4,5,\ldots,10,$$

where $E(x)$ denotes the integer part of x.

For instance, when $p=4$, the condition $2Q=-2Q$ is

$$[0:-a:1]=[h-BA:-(h+bB)b:abB],$$

which holds if and only if $h-BA=0$, giving rise to condition $c_4(a,b,h)=0$.

Similarly, to characterize the elliptic curves containing 6-periodic points we get that $3Q=-3Q$ is equivalent to:

$$\begin{cases}\dfrac{h-aA}{bA}=-\dfrac{aB\left((ab-1)\,h+AB\right)}{(h+bB)\,(-h+AB)},\\[2ex] \dfrac{-h+AB}{abA}=\dfrac{-h+B\left(-ab^2+a^2-b\right)}{b\,(h+bB)}.\end{cases}$$

From the above equations we arrive to $c_6(a,b,h)=0$. The other conditions can be deduced similarly. ∎

4 A computational approach

4.1 Resultant periodicity conditions (RPC)

Let $\Lambda\in\mathbb{K}^n$ be a set of parameters and let $F_\Lambda:\mathcal{U}\subset\mathbb{K}^2\mapsto\mathbb{K}^2$ be a family of planar integrable rational maps with rational first integrals V_Λ. We present a method, alternative to the one of solving any of the system ${F_\Lambda}^p(x,y) = (x,y)$, for finding continua of p-periodic points lying on the level sets of V_Λ.

The method consists to find the values of h for which the system

$$\left\{\begin{array}{l}(F_\Lambda^p)_1(x,y) = x,\\ V_\Lambda(x,y) = h,\end{array}\right. \tag{4.1}$$

has continua of solutions, where the subscript 1 (resp. 2) indicates the first (resp. second) component of a point. In this way the new unknowns of the problem are simply the values h of the level sets of $V_\Lambda(x,y) = h$. These values of h have to be such that

$$\phi(y,h) := \operatorname{Res}\big(\operatorname{numer}\big((F_\Lambda^p)_1(x,y)\; - \; x\big)\,, \operatorname{numer}\left(V_\Lambda(x,y) - h\right); x\big) \tag{4.2}$$

vanishes identically. In other words we need to collect the factors of the above resultant that only depend on h, if any. Denote by $D_p(\Lambda,h)$ the product of these factors.

We introduce the functions $d_p(\Lambda,h)$ as those factors of $D_p(\Lambda,h)$ that remain after removing from this polynomial all the factors that already appear in some $D_k(\Lambda,h)$ where k is either 1 or a proper divisor of p. We call the conditions $d_p(\Lambda,h) = 0$ the *resultant d-periodicity conditions* associated to the first integral V_Λ (RPC from now on).

We remark that sometimes can be computationally more efficient to take the equation $(F_\Lambda^p)_2(x,y) = y$, instead of the first one in (4.1), or to do in (4.2) the resultant with respect to y instead to the resultant with respect to x.

The main fact is that the energy levels filled with periodic points must satisfy the RPC. Nevertheless, a priori there is no guarantee that all the solutions of them are formed by periodic points because they satisfy only a necessary condition and, moreover, the above resultants could contain some spurious factors. Anyway, in all the examples given in this section we check that the RPC obtained give actual continua of p-periodic points.

In the birational case, the direct method for finding p-periodic points tries to solve some the systems ${F_\Lambda}^m(x,y) = F_\Lambda^{m-p}(x,y)$, for some natural number $m < p$, instead of system when $m = p$. Recall also that in the genus 1 case, it is much more difficult to study the periodicity condition $pQ = \mathcal{O}$ that the equivalent one $mQ = -(p-m)Q$, again for some natural number m, near $p/2$. Similarly, for birational maps, when computing the resultant (4.2) we reach to some computational obstructions that many times can be overcome by using the alternative equivalent formulation

$$\begin{array}{rl}\phi_m(y,h) := & \operatorname{Res}\big(\operatorname{numer}\big((F_\Lambda^m)_1(x,y) - ({F_\Lambda}^{m-p})_1(x,y)\big)\,,\\ & \operatorname{numer}\left(V_\Lambda(x,y) - h\right); x\big)\,,\end{array} \tag{4.3}$$

for some suitable m near $p/2$.

All the RPC computed in these section have been obtained analyzing the factors of all the resultants (4.3), which have been calculated by using the computer algebra software MAPLE v.17 with an Intel Core 2Duo P8700 processor, and by exhausting the limits that this technology allows. The main computational obstruction comes from the fact that the size of the expressions of the resultants requires large resources to guarantee the allocation of enough memory for the computation.

4.2 The feasibility region

If we set $\mathbb{K} = \mathbb{C}$ the RPC are always well defined, but when $\mathbb{K} = \mathbb{R}$, then some RPC $d_p(\Lambda, h) = 0$ can have no real points. So we are only interesed on the cases where there exists a pair $(\Lambda, h) \in \mathbb{R}^n \times \mathbb{R}$ satisfying some RPC, $d_p(\Lambda, h) = 0$, at some real points. We will call *the feasibility region*, to the subset

$$\mathcal{R} = \{(\Lambda, h) \in \mathbb{R}^n \times \mathbb{R} \, : \, \{V_\Lambda(x, y) = h\} \cap \mathbb{R}^2 \neq \emptyset\}.$$

4.3 Auxiliary results.

Of course, we can apply the method blindly, but we insist that it will only work in the case that the continua of periodic points coincide with the level sets of the first integral. In the case of maps defined in $\mathbb{R}^2$, we can guarantee this fact if for any fixed Λ there exists a Lie symmetry X_Λ of the map F_Λ with the same first integral V_Λ (a Lie Symmetry is a vector field which is related with the dynamics of the map since F_Λ maps any orbit of the differential system determined by the vector field, to another orbit of this system, see [9, 17]). From the results in [9] the existence of such a vector field guarantees that the action of the map on any connected components of the level sets of the first integral is conjugated to a linear transformation. Hence if on a closed invariant connected component of an energy level there is no singular point of the associated Lie symmetry X_Λ and there is a p-periodic point of F_Λ then the whole level set is filled with p-periodic points.

It can be seen that if a map F has a first integral $V \in \mathcal{C}^{m+1}(\mathcal{U})$ on $\mathcal{U} \subset \mathbb{R}^2$ and it preserves a measure absolutely continuous with respect the Lebesgue measure with non-vanishing density $\nu \in \mathcal{C}^m$ in $\mathcal{U}$, it holds that the vector field $X = \nu^{-1}(-V_y, V_x)$ is always a Lie Symmetry of F in $\mathcal{U}$. As a consequence of this fact, and of the results of [5, 8], we have the following proposition that characterizes the existence of a continua of periodic points over the level sets of the first integral and proves some regularity properties of the rotation number function.

Proposition 4.1. *Let F be a $\mathcal{C}^2$-measure preserving map with an invariant measure with non-vanishing density $\nu \in \mathcal{C}^1(\mathcal{U})$ and with a first integral $V \in \mathcal{C}^2(\mathcal{U})$. If V has a connected component γ_h of a level set $\{V(x) = h\}$, which is invariant by F and diffeomorphic to $\mathbb{S}^1$, then $F|_{\gamma_h}$ is conjugate to a rotation. Moreover, if ρ_h denotes the rotation number of this map, then the function $\rho(h) : h \to \rho_h$ is continuous and tends to $\theta/(2\pi)$ when γ_h approaches to an non-degenerated fixed point with eigenvalues $e^{\pm i\theta}$, where $0 \neq \theta \in \mathbb{R}$.*

Finally, if $F|_{\gamma_h}$ has rotation number $\rho_h = q/p \in \mathbb{Q}$, with $\gcd(p, q) = 1$, then F has a continuum of p-periodic points in $\gamma_h \subset \mathcal{U}$.

4.4 The McMillan-Gumovski-Mira maps

In this section we consider a one-parameter family of maps defined in $\mathbb{R}^2$, considered by E. M. McMillan in [24] and by I. Gumovski and C. Mira in [16],

$$F_a(x,y) = \left(y, -x + \frac{y}{a+y^2}\right) \tag{4.4}$$

for $a > 0$, that we will call *McMillan-Gumovski-Mira maps*. For any $a > 0$ this is a birational map with rational inverse $F_a{}^{-1}(x,y) = ((x - ya - yx^2)/(a + x^2), x)$. Both maps F_a and F_a^{-1} share the polynomial first integral

$$V_a(x,y) := a(x^2 + y^2) + x^2y^2 - xy. \tag{4.5}$$

A computation shows that $\det(DF_a(x,y)) \equiv 1$, so F_a is an integrable map preserving the Lebesgue measure and it has the Lie Symmetry

$$X_a(x,y) = (x - 2ay - 2x^2y, 2ax - y + 2xy^2), \tag{4.6}$$

see [9, Example 2]. By Proposition 4.1, on any simple closed invariant connected component of the level curves, the map is conjugated to a rotation (such connected components exist, for instance, as a consequence of Lemma 4.3 below). Therefore, those levels for which the associated rotation number is rational are filled with periodic points.

4.4.1 Computation of the RPC for the McMillan-Gumovski-Mira maps

The approach developed in the previous section allow us to prove the following result.

Theorem 4.2. *The RPC of the McMillan-Gumovski-Mira maps associated to the first integral (4.5) for* $p \leq 10$ *are given by* $d_p(a,h) = 0$*, where:*

$$\begin{aligned}
d_3(a,h) =& h + a(a+1),\\
d_4(a,h) =& h + a^2,\\
d_5(a,h) =& h^3 + a\,(3a-1)\,h^2 + a\left(3a^3 - 2a^2 - a - 1\right)h + a^4\left(a^2 - a - 1\right),\\
d_6(a,h) =& h^2 + \left(2a^2+1\right)h + a^4,\\
d_7(a,h) =& h^6 + 2a\,(3a+1)\,h^5 + a\left(15a^3 + 10a^2 - a + 3\right)h^4 + a\left(20a^5 + 20a^4\right.\\
& \left.-4a^3 + 8a^2 + a + 1\right)h^3 + a^4\left(15a^4 + 20a^3 - 6a^2 + 6a + 2\right)h^2\\
& + a^4\left(6a^6 + 10a^5 - 4a^4 + a^2 - a - 1\right)h + a^9\left(a^3 + 2a^2 - a - 1\right),\\
d_8(a,h) =& (2h^2 + \left(4a^2+1\right)h + a^2\left(2a^2-1\right))(h^4 + 4h^3a^2 + 6h^2a^4\\
& + a^2\left(2a^2 - 2a + 1\right)\left(2a^2 + 2a + 1\right)h + a^8),
\end{aligned}$$

$$
\begin{aligned}
d_9(a,h) =& h^9+3a\,(3a-1)\,h^8+a\left(36a^3-24a^2-7\right)h^7+a\left(84a^5-84a^4-41a^2\right.\\
&\left.+6a-5\right)h^6+a\left(126a^7-168a^6-99a^4+30a^3-19a^2+4a-1\right)h^5\\
&+a^2\left(126a^8-210a^7-125a^5+60a^4-26a^3+8a^2-3a+1\right)h^4\\
&+a^3\left(84a^9-168a^8-85a^6+60a^5-14a^4-3a^2-1\right)h^3\\
&+a^7\left(36a^7-84a^6-27a^4+30a^3-a^2-8a-1\right)h^2\\
&+a^8\left(9a^8-24a^7-a^5+6a^4+a^3-4a^2+1\right)h+a^{15}\left(a^3-3a^2+1\right),\\
d_{10}(a,h) =& h^6+\left(6a^2+3\right)h^5+\left(15a^4+12a^2+1\right)h^4+2a^4\left(10a^2+9\right)h^3\\
&+3a^4\left(5a^2-1\right)\left(a^2+1\right)h^2+a^4\left(a^2+1\right)\left(6a^4-3a^2+1\right)h+a^{12}.
\end{aligned}
$$

Proof. For $p=2$ we have that $\mathrm{numer}(({F_a}^2)_1(x,y)-x)=-xa-y\,(xy-1)$. Then

$$
\begin{aligned}
&\mathrm{Res}(\mathrm{numer}(({F_a}^2)_1(x,y)-x),V_a(x,y)-h,x)\\
&\qquad=\left(a+y^2\right)\left(4\,ay^4+4\,a^2y^2\;-4\,hy^2-4\,ah-y^2\right).
\end{aligned}
$$

Notice that in this case there is no factor without dependence on the variable y, so we can ensure that there are not energy levels formed by continua of 2-periodic points.

We continue with $p=3$. In this case

$$
\begin{aligned}
\mathrm{numer}(({F_a}^3)_1(x,y)-x) =& -(x+y)\,a^3+\left(-x^3-x^2y-2\,xy^2-2\,y^3-x\right)a^2\\
&-y\left(2\,yx^3+2\,y^2x^2+xy^3+y^4-2\,x^2-1\right)a-xy^2\left(xy+y^2-1\right)(xy-1),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Res}(\mathrm{numer}(({F_a}^3)_1(x,y)-x),V_a(x,y)-h,x)\\
&\qquad=(a(a+1)+h)^2\left(y^2+a\right)^4\left(y^4+2\,y^2a-y^2-h\right).
\end{aligned}
$$

The only factor which is independent of y is $a(a+1)+h$. Therefore the expression of $d_3(a,h)$ follows.

To compute the RPC corresponding to $p=5$ we consider the equation ${F_a}^3(x,y)={F_a}^{-2}(x,y)$. Then

$$
\begin{aligned}
&\mathrm{numer}(({F_a}^3)_1(x,y)-({F_a}^{-2})_1(x,y))=(x-y)\,a^6+(x-y)\left(3x^2+3y^2-1\right)a^5\\
&+(x-y)\left(3x^4+9x^2y^2+3y^4-2x^2-3xy-2y^2-1\right)a^4+(x-y)\\
&\times\left(x^6+9x^4y^2+9x^2y^4+y^6-x^4-6x^3y-4x^2y^2-6xy^3-y^4-x^2+2xy-y^2\right)a^3\\
&+(x-y)\left(3y^2x^6+9x^4y^4+3x^2y^6-3x^5y-2x^4y^2-12x^3y^3-2x^2y^4-3xy^5\right.\\
&\left.+2x^3y+2x^2y^2+2xy^3-x^2+xy-y^2\right)a^2+xy\,(x-y)\,(xy-1)\left(3x^4y^2+3x^2y^4\right.\\
&\left.-3x^3y-x^2y^2-3xy^3+xy-1\right)a+x^3y^3\,(x-y)\,(xy-1)^3
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Res}(\mathrm{numer}(({F_a}^3)_1(x,y)-({F_a}^{-2})_1(x,y)),V_a(x,y)-h,x)=\\
&\qquad d_5^2(a,h)\left(y^2+a\right)^6\left(y^4+2\,ay^2-y^2-h\right).
\end{aligned}
$$

Hence $d_5(a,h) = 0$ is the searched condition. Proceeding similarly for all the other values of $p \leq 10$ we prove the theorem. ∎

4.4.2 The feasibility region for the McMillan-Gumovski-Mira map

Now we determine the feasibility region of a map F_a, that is, those pairs $(a,h) \in \mathbb{R}^2$ such that

$$\{V_a(x,y) = h\} \cap \mathbb{R}^2 = \{a(x^2+y^2) + x^2y^2 - xy = h\} \cap \mathbb{R}^2 \neq \emptyset.$$

Observe that the fixed points of a McMillan-Gumovski-Mira map F_a are $(0,0)$ for any $a > 0$; and $(\pm x_0(a), \pm x_0(a))$, where $x_0(a) = \sqrt{1/2 - a}$, when $0 < a < 1/2$. These fixed points are critical points of the first integral (4.5), and also the unique singular points of the Lie Symmetry (4.6). A straightforward analysis gives:

Lemma 4.3. *The fixed points of the map* (4.4) *are critical points of the first integral* (4.5). *Furthermore*

(i) If $0 < a < 1/2$, the origin is a saddle of $V_a(x,y)$ and the points $(\pm x_0(a), \pm x_0(a))$ are absolute minima.

(ii) If $a \geq 1/2$ the origin is an absolute minimum of $V_a(x,y)$.

The value of the first integral V_a at the absolute minima is given by

$$h_c(a) := \begin{cases} -a^2 + a - 1/4 & \textit{if} \quad 0 < a < 1/2, \\ 0 & \textit{if} \quad a \geq 1/2. \end{cases}$$

(iii) When $0 < a < 1/2$ and $-a^2 + a - 1/4 < h < 0$ the level sets of $V_a(x,y) = h$ are formed by two disjoint closed curves, each one of them diffeomorphic to $\mathbb{S}^1$.

(iv) When $h > 0$ the level sets of $V_a(x,y) = h$ are closed curves, diffeomorphic to $\mathbb{S}^1$.

(v) When $0 < a < 1/2$ the level set $V_a(x,y) = 0$ is formed by the point $(0,0)$ and two closed loops starting and ending at this point. When $a \geq 1/2$ the level set is the point $(0,0)$.

As a consequence of the above result the solutions of the RPC, $d_p(a,h) = 0$, correspond with nonempty curves $\{(x,y) \in \mathbb{R}^2,\, V_a(x,y) = h\}$ for all $h \geq h_c(a)$ and therefore, the region $\mathcal{R} = \{(a,h),\, a > 0 \text{ and } h \geq h_c(a)\} \subset \mathbb{R}^+ \times \mathbb{R}$, is the *feasibility region* for the map F_a. It is displayed in the blue-shaded region of Figure 4.1. Hence,

Corollary 4.4. *If there exist level curves $\{V_a(x,y) = h\} \subset \mathbb{R}^2$ filled with p-periodic points, then these curves correspond with some pairs $(a,h) \in \{d_p(a,h) = 0\} \cap \mathcal{R}$.*

Moreover, as a consequence of the existence of the Lie symmetry (4.6), whose singular points are the fixed points of F_a, by applying Corollary 4.5 we have:

Corollary 4.5. *If $(a,h) \in \{d_p(a,h) = 0\} \cap \mathcal{R}$, and there exists a p-periodic point of F_a on the curve $\{V_a(x,y) = h\}$, then all the curve is filled with p-periodic points.*

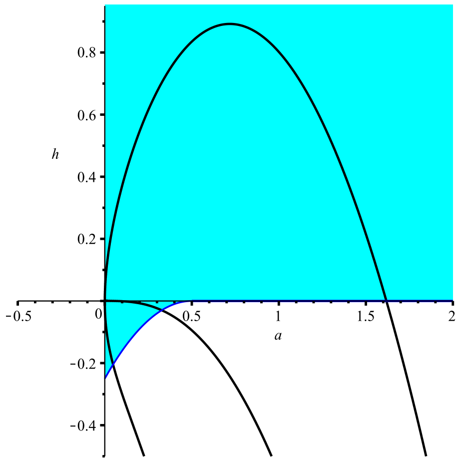


Figure 4.1: Intersection of the curve $d_5(a,h) = 0$ with the feasibility region of the McMillan-Gumovski-Mira map F_a (shaded in blue).

4.4.3 Analysis of the 5-periodic RPC.

In this section we show how to determine the number and shape of the level curves associated to a particular RPC. Our approach, together with some well-known tricks (see Lemma 4.7 below), reduce this problem to a routine task that can be done for each period. In this section, as an illustration, we study with detail the case $p = 5$. The results are summarized as follows:

Theorem 4.6. *Consider the McMillan-Gumovski-Mira map* (4.4) *with* $a > 0$ *and define*

$$a_1 = \frac{3-\sqrt{5}}{16} \approx 0.048 < a_2 = \frac{3+\sqrt{5}}{16} \approx 0.328 < a_3 = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

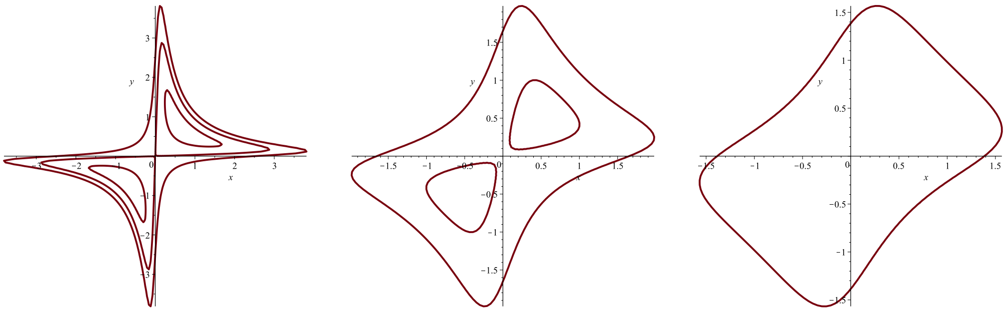
Then the set of real 5-periodic points is given by the smooth non-empty level sets $V_a(x,y) = h$ *for the values of* h *satisfying* $d_5(a,h) = 0$, *with* d_5 *given in Theorem 4.2. It is formed by:*

(i) Five disjoint closed curves diffeomorphic to $\mathbb{S}^1$ *when* $a \in (0, a_1)$. *They correspond to two negative and one positive values of* h.

(ii) Three disjoint closed curves diffeomorphic to $\mathbb{S}^1$ *when* $a \in [a_1, a_2)$. *They correspond to one negative and one positive values of* h.

(iii) One closed curve diffeomorphic to $\mathbb{S}^1$ *when* $a \in [a_2, a_3)$. *It corresponds one positive value of* h.

(iv) The empty set when $a \geq a_3$.



$a \in (0, a_1)$ $a \in [a_1, a_2)$ $a \in [a_2, a_3)$

Figure 4.2: Level sets full of 5-periodic points according different values of a. In the left hand-side figure there are 5 disjoint simple closed curves, but two of them seem to collide at the origin because they pass very near it.

Moreover, the clockwise rotation number on the level sets corresponding to the smallest value of h given in item (i) is $2/5$ and on all the other level sets is $1/5$. The overall situation is displayed and illustrated in Figures 4.1, 4.2 and 4.3.

To prove Theorem 4.6 we will use the following result which is a combination of the results obtained in [13] and [14]. Denote $\Delta_h(P)$ the discriminant of the polynomial $P(h) = a_n h^n + \cdots + a_1 h + a_0$, that is

$$\Delta_h(P) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n} \mathrm{Res}(P(h), P'(h); h).$$

Lemma 4.7. *Let*

$$G_a(h) = g_n(a)h^n + g_{n-1}(a)h^{n-1} + \cdots + g_1(a)h + g_0(a),$$

be a a family of real polynomials depending also polynomially on a real parameter a. Set $I_a = (\phi(a), +\infty)$ where $\phi(a)$ is a continuous function. Suppose that there exists an open interval $\Lambda \subset \mathbb{R}$ such that:

(i) There exists $a_0 \in \Lambda$, such that $G_{a_0}(h)$ has exactly $r \geq 0$ simple roots in I_{a_0}.

(ii) For all $a \in \Lambda$, $G_a(\phi(a)) \cdot g_n(a) \neq 0$.

(iii) For all $a \in \Lambda$, $\Delta_h(G_a) \neq 0$.

Then, for all $a \in \Lambda$, $G_a(h)$ has exactly r simple roots in I_a.

Proof. Observe that from the hypothesis (ii) the roots (real or complex) of $G_a(h)$ depend continuously on a since $g_n(a) \neq 0$. The same hypothesis ensures that there are no real roots trespassing the border of I_a when varying the parameter a. Then, the variation of the number of real roots must be caused by the appearance of some multiple real root, due either to the collision of some real roots or to the collision of some couple of conjugated complex roots. In any case, since by (iii), $\Delta_h(G_a) \neq 0$, multiple roots never appear. Hence the number of real roots of $G_a(h)$ remains constant for all $a \in \Lambda$. ∎

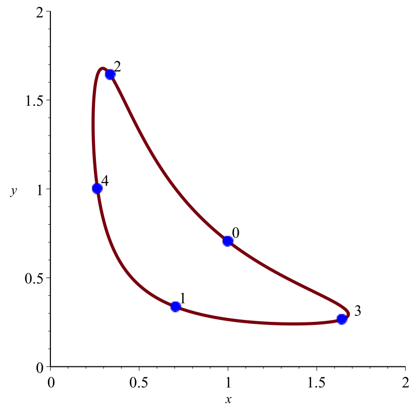


Figure 4.3: Clockwise rotation number $2/5$ of a 5-periodic orbit $\{(x_n, y_n)\}_n$ on a connected component of a level set with $h < 0$. The dots are the points of the orbit and the numbers near them indicate the subscript $n \mod 5$.

Proof of Theorem 4.6. Observe that $d_5(a,h)$ can be rewritten as the one-parametric family of polynomials in h:

$$G_a(h) := d_5(a,h) = g_3(a)h^3 + g_2(a)h^2 + g_1(a)h + g_0(a),$$

where $g_3(a) = 1, g_2(a) = a(3a-1), g_1(a) = a(3a^3-2a^2-a-1), g_0(a) = a^4(a^2-a-1)$.

Remember that $(a,h) \in \mathcal{R}$ if and only if $h \in [h_c(a), +\infty)$. We have to study the number of real zeros of $G_a(h) = 0$ in the feasibility region. We will use Lemma 4.7. First, observe that for all $a > 0$,

$$\Delta_h(G_a(h)) = a^3(32a^2 + 13a + 4) > 0.$$

Therefore hypothesis (iii) is always satisfied.

We split the proof in two cases: $a \in (0, 1/2)$ and $a \geq 1/2$.

Case $a \in (0, 1/2)$**.** In this situation $I_a = (-a^2 + a - 1/4, +\infty)$. Since $g_3(a) \neq 0$, by the lemma, the number of real simple roots in $G_a(h)$ is constant on any open interval I where $G_a(\phi(a)) \neq 0$. A computation shows that

$$G_a(\phi(a)) = G_a\Big(-a^2 + a - \frac{1}{4}\Big) = \frac{3}{8}a - a^2 - \frac{1}{64}$$

vanishes at a_1 and a_2. Hence we know that G_a has a constant number of real roots in I_a for a in each of the intervals $(0,a_1)$, (a_1,a_2) and $(a_2,1/2)$. Moreover, observe that since $g_0(a)\neq 0$ for all $a\in(0,1/2)$, the number of positive and negative solutions of $G_a(h)=0$ neither changes for a in any of the three open intervals. Hence to know the number of level sets candidate to be full of 5-periodic points, and the sign of the corresponding values of h, it suffices to study one concrete value of a in each of the intervals $(0,a_1)$, (a_1,a_2) and $(a_2,1/2)$ together with the values $a=a_1$ and $a=a_2$.

We only detail the computations taking $a=1/50\in(0,a_1)$. For this value,

$$G_{1/50}(h)=h^3-\frac{47}{2500}\,h^2-\frac{127597}{6250000}\,h-\frac{2549}{15625000000}.$$

It is easy to prove that this polynomial has three simple roots, two negative and one positive, all located in $I_{1/50}$. Therefore the same holds for all $a\in(0,a_1)$. By items (iii) and (iv) of Lemma 4.3, these values give rise to five disjoint closed curves, diffeomorphic to $\mathbb{S}^1$, candidates to be full of 5-periodic points.

To ensure that the non-empty level curves corresponding to $d_5(a,h)=0$, are full of 5-periodic points (recall that conditions $d_k(a,h)=0$ are only necessary conditions of periodicity) we need to continue our analysis.

The rotation numbers of F_a on each of the found level sets can be obtained either following the orbit of a given 5-periodic point, see Figure 4.3, or using the properties of the rotation number function introduced in Proposition 4.1. In particular we need to control the limits of the rotation numbers when we level sets tend to infinity, or to the level sets containing the fixed points.

By using the same tools that in [11, Proposition 4] we can prove that the rotation number function $\rho(h)$ tends to $1/4$ when h tends to infinity, tends to 0 when h tends to 0 (this level set corresponds to two homoclinic orbits tending to the origin) and tends to $\arccos(4a-1)/(2\pi)$ when the the ovals of the level sets degenerate to the fixed points $(\pm x_0(a),\pm x_0(a))$. From these limiting behavior of $\rho(h)$ and because it is continuous, we obtain that for $a\in(0,a_1)$ there are at least two values of $h<0$ such that ρ_h is $1/5$ and $2/5$, since these two values belong to the interval $(0,\arccos(4a-1)/(2\pi))$, and there is one positive value of h such that $\rho_h=1/5$, because this value belongs to the interval $(0,1/4)$. Similarly, when $a\in[a_1,a_2)$ there are at least one positive and one negative value of h both corresponding to the rotation number $1/5$ and when $a\in[a_2,1/2)$ there is at least one positive value of h also corresponding to the rotation number $1/5$.

In short, when $a\in(0,1/2)$, we have proved that the found real level sets given by $d_5(a,h)=0$ are actually full of 5-periodic points and have the rotation numbers described in the statement of the theorem.

Case $a\geq 1/2$. Here $I_a=(0,+\infty)$. Note that

$$G_a(0)=g_0(a)=a^4(a^2-a-1),$$

has only one positive root, $a=a_3$. So we have to determine the number of roots of G_a in $(0,\infty)$ when either $a\in(1/2,a_3)$ or when $a\in(a_3,\infty)$. Hence, we can reduce the problem to study the values $a = 1/2$, $a = a_3$ and two (rational) numbers, one in each of the intervals $(1/2, a_3)$ and (a_3, ∞). The rest of the study follows as in the previous case and we omit the details. ∎

4.5 The Bastien-Rogalski and the Lyness 2-periodic maps revisited

We consider again the one-parameter family of maps (2.1) and the family of maps (3.2) studied in Sections 2.2 and 3.2, respectively. Applying the method described in this section we obtain:

Theorem 4.8. *(i) The RPC associated to the first integral (2.2) for the family of Bastien and Rogalski maps (2.1), $d_p(a, h) = 0$, coincide for each $p \leq 15$ with the corresponding expression $c_p(a, h) = 0$ given in Theorem 2.1.*

(ii) The RPC associated to the first integral (3.4) for the family of Lyness 2-periodic maps (3.2) coincide, for each $p \leq 10$, with the corresponding expression $c_p(a, h) = 0$ given in Theorem 3.1.

On one hand the methods used to prove Theorems 2.1 and 3.1 allow to obtain conditions for the existence of continua of periodic points for bigger values of p that the ones obtained with the method introduced in this section. Nevertheless, on the other hand, this new approach is merely computational and it does not require the knowledge neither of the proper rational parametrization in the first case, nor of the expression of the map in terms of the group operation on the elliptic curve (3.8) in the second situation.

5 A final comment

All the methods presented in this paper strongly rely on the fact that the continua of periodic points lie on the level sets of the first integral. This is a feature that integrable birational maps, among others, accomplish. But this is not the scenario that one should expect when integrable diffeomorphisms are considered.

Indeed, let us consider a *real* planar map F that preserves a fibration of curves given by $\{V = h\}_{\{h\in\mathcal{H}\}}$ where $\mathcal{H} \subset \mathbb{R}$ is an open set and such that for each $h \in \mathcal{H}$ each curve $\{V = h\} \cong \mathbb{S}^1$. We can think that $F_h := F_{|V=h}$ defines an one-parametric family of circle maps. If in addition the maps F_h are orientable diffeomorphisms, then generically (i.e. in an open and dense set in the space of orientable diffeomorphisms with an appropriate topology, [26]) the persistence of the rotation number is known to hold, [2, Theorem A] and [25] (see also [1]), so that generically the function $\rho(h)$ giving the rotation number associated to each map F_h is either constant or a cantor function (very far, thus, from scenario that appear when birational maps with rational first integrals are considered).

In the latter case, if there exists $h_0 \in \mathcal{H}$ such that $\rho(h_0) = q/p$, then there exists an open set $h_0 \in \mathcal{U}$ such that for all $h \in \mathcal{U} \subset \mathcal{H}$ the function $\rho(h) \equiv q/p$. So at each level set there are p-periodic points, [26]. By continuity of the solutions of the system $F_h^p(x,y) = \mathrm{Id}$ with respect the parameter h, we get that the periodic points at each level set form a continuum. The fact that generically these periodic points are hyperbolic implies that this continuum do not coincide with the level sets $\{V = h\}$.

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Appendix

The non-elliptic levels of the curves (3.5) can be found either by looking for the singular points (see [6]), or computing the roots of the denominator of the j-invariant associated to them [31]. Here we present a Magma code that allows to do this computation.

```
K<a,b,h>:=FunctionField(Rationals(),3);
A<x,y>:=AffineSpace(K,2);
C:=Curve(A,(b*x+a)*(a*y+b)*(a*x+b*y+a*b)-h*x*y;
CP:=ProjectiveClosure(C);
P:=CP![0,1,0];
E:=EllipticCurve(CP,P);
Factorization(Denominator(jInvariant(E)));
```

Processing this code in the Magma's online calculator page [23], one obtains that the non-elliptic cases occur either when $ab = 0$ or when $ab \neq 0$ and $h = 0$ or h is a root of the polynomial (3.7),

$$h^4 + p_3(a,b)h^3 + p_2(a,b)h^2 + p_1(a,b)h + p_0(a,b) = 0,$$

where $p_3(a,b)$, $p_2(a,b)$, $p_1(a,b)$ and $p_0(a,b)$ are given by:

$$\begin{aligned}
p_3(a,b) = {} & -4\,a^2b^2 - 4\,a^3 - 4\,b^3 + ab,\\
p_2(a,b) = {} & 6\,a^4b^4 + 4\,a^5b^2 + 4\,a^2b^5 + 6\,a^6 - 21\,a^3b^3 + 6\,b^6 - 3\,a^4b - 3\,ab^4,\\
p_1(a,b) = {} & -4\,a^6b^6 + 4\,a^7b^4 + 4\,a^4b^7 + 4\,a^8b^2 - 25\,a^5b^5 + 4\,a^2b^8 - 4\,a^9\\
& +18\,a^6b^3 + 18\,a^3b^6 - 4\,b^9 + 3\,a^7b - 21\,a^4b^4 + 3\,ab^7,\\
p_0(a,b) = {} & \left(-b^2 + a\right)\left(a^2 - b\right)\left(-a^2b^2 + a^3 + b^3\right)^3.
\end{aligned}$$

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