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AVERAGING METHODS OF ARBITRARY ORDER, PERIODIC SOLUTIONS AND INTEGRABILITY

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Abstract. In this paper we provide an arbitrary order averaging theory for higher dimensional periodic analytic differential systems. This result extends and improves results on averaging theory of periodic analytic differential systems, and it unifies many different kinds of averaging methods.

Applying our theory to autonomous analytic differential systems, we obtain some conditions on the existence of limit cycles and integrability.

For polynomial differential systems with a singularity at the origin having a pair of pure imaginary eigenvalues, we prove that there always exists a positive number N such that if its first N averaging functions vanish, then all averaging functions vanish, and consequently there exists a neighborhood of the origin filled with periodic orbits. Consequently if all averaging functions vanish, the origin is a center for $n = 2$.

Furthermore, in a punctured neighborhood of the origin, the system is C^∞ completely integrable for $n > 2$ provided that each periodic orbit has a trivial holonomy.

Finally we develop an averaging theory for studying limit cycle bifurcations and the integrability of planar polynomial differential systems near a nilpotent monodromic singularity and some degenerate monodromic singularities.

1. Introduction and statement of the main results

To know when a differential system has or not periodic solutions is very important for understanding its dynamics. Averaging theory is a good theory for studying the periodic solutions. Of course, the averaging theory is a classical tool for studying the behaviour of nonlinear differential systems. This theory has a long history that starts with the works of Lagrange and Laplace, who work with it in an intuitive way. One of first formalizations of the averaging theory was done by Fatou in 1928 [15]. Later on Bogoliubov and Krylov [4] in the 1930s

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and Bogoliubov [3] in 1945 did very important practical and theoretical contributions to the averaging theory. The ideas of averaging theory has been improved in several directions for the finite and infinite dimensional differentiable systems. More recently Hale also did good contributions to the averaging theory, see the books [20, 21]. For modern expositions and results on the averaging theory see also the books of Sanders, Verhulst and Murdock [37] and Verhulst [38].

Consider periodic analytic differential systems

$$\dot{x} = F_0(t,x) + \sum_{i=1}^{\infty} \varepsilon^i F_i(t,x), \qquad (t,x) \in \mathbb{R} \times \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is an open subset, ε is a parameter with $|\varepsilon|$ sufficiently small, and the $F_i(t,x)$'s are n–dimensional vector valued analytic functions in their variables in $\mathbb{R} \times \Omega$ and periodic of period T in the variable t. For $z \in \Omega$, let $x(t,z,\varepsilon)$ be the solution of (1) satisfying $x(0,z,\varepsilon) = z$. One of the important problems in the study of the dynamics of a differential system (1) is to know when $x(t,z,\varepsilon)$ is a periodic solution. There are many different methods for studying this problem, and the averaging method is one very useful.

In order to apply the averaging theory for studying periodic orbits of the differential system (1), one of the basic assumptions is that the unperturbed system $\dot{x} = F_0(t,x)$ has an invariant manifold formed by periodic orbits. In this direction there are extensive studies for the first, second and third order averaging theories, see for instance [6, 7, 8, 9, 13, 31, 36, 37] and the references therein. Also the averaging theory has broad applications, see e.g. [1, 2, 13, 16, 17, 19, 23, 28, 29, 35] and the references therein.

Recently the averaging theory was extended to arbitrary order for computing periodic orbits. Giné *et al* [18] provided an arbitrary order averaging formula of system (1) when $n = 1$. In [16] the averaging theory in $\mathbb{R}^n$ up to any order in ε for the particular case $F_0(t,x) \equiv 0$ is described in a recursive way and it is applied to the center problem for planar systems. Llibre *et al* [24] further extended the arbitrary order averaging method to any finite dimensional periodic differential system provided that the manifold formed by periodic solutions of the unperturbed differential system $\dot{x} = F_0(t,x)$ is an open subset of Ω. When the manifold formed by periodic solutions of the unperturbed differential system $\dot{x} = F_0(t,x)$ has dimension less than n, Malkin [31] and Roseau [36] provided the averaging theory of first order. For a different and shorter proof, see [6]. Buică *et al* [7, 8] extended the Malkin and Roseau's first order averaging theory to second order. Here we extend these previous results to arbitrary order for any finite dimensional periodic analytic differential system (1). As a consequence our results also extend and improve the ones of [18, 24].

1.1. **Arbitrary order averaging theory of periodic differential systems.** For stating our results, we first consider the initial value problem of the unperturbed differential system of (1)

$$\dot{x}_0(t,z) = F_0(t, x_0(t,z)), \qquad x_0(0,z) = z. \tag{2}$$

Let $V \subset \mathbb{R}^k$, with $1 \le k \le n$, be an open and bounded subset, and let $\eta : \mathrm{cl}(V) \to \mathbb{R}^{n-k}$ be a C^2 function such that $\mathcal{M}_0 := \{z_w = (w, \eta(w)) |\ w \in \mathrm{cl}(V)\} \subset \Omega$ is strictly contained in the set of all periodic solutions of the unperturbed differential system, where cl denotes the closure of a set. Then $\mathcal{M}_0$ is a k–dimensional smooth submanifold in Ω. Of course, if $k = n$ the map η does not appear, and $\mathcal{M}_0 = \mathrm{cl}(V)$. In this paper we have a basic assumption that the unperturbed system of (1), i.e. the initial value problem (2) with $z = z_w$, has the T periodic solutions

$$x_0(t, z_w) = \varphi(t, z_w) \qquad \text{for all } z_w \in \mathcal{M}_0.$$

Let $\Phi(t, z_w)$ be a fundamental matrix solution of the variational equation of system (2) along the solution $x_0(t, z_w) = \varphi(t, z_w)$

$$\dot{y} = \partial F_0(t, \varphi(t, z_w))y, \tag{3}$$

where $\partial F_0(t, \varphi(t, z_w))$ denotes the Jacobian matrix of $F_0(t, x)$ with respect to x taking values at $x = \varphi(t, z_w)$. We inductively define the functions

$$x_j(t, z_w) = \Phi(t, z_w) \int_0^t \Phi^{-1}(s, z_w)\, \mathcal{K}_j(s, z_w) ds, \qquad j = 1, 2, \ldots, \tag{4}$$

where

(5)

$$\mathcal{K}_j(t, z_w) := \sum_{|\alpha|=2}^{j} \mathcal{F}_{0,\alpha,j}(t, z_w) + F_j(t, \varphi_0(t, z_w)) + \sum_{i=1}^{j-1} \sum_{|\alpha|=1}^{j-i} \mathcal{F}_{i,\alpha,j-i}(t, z_w),$$

for $j = 1, 2, \ldots$, are successively known functions, with

$$\mathcal{F}_{i,\alpha,r}(t, z_w) = \left(\mathcal{F}^{(1)}_{i,\alpha,r}(t, z_w), \ldots, \mathcal{F}^{(n)}_{i,\alpha,r}(t, z_w)\right), \quad r = 0, 1, \ldots \tag{6}$$

and for $l = 1, \ldots, n$

$$\begin{aligned} \mathcal{F}^{(l)}_{i,\alpha,r}(t, z_w) = &\frac{\partial^\alpha F^{(l)}_i(t, \varphi(t, z_w))}{\alpha!} \\ &\times \sum_{s_1+\ldots+s_n=r} \left(\sum_{j_1+\ldots+j_{\alpha_1}=s_1} x^{(1)}_{j_1}(t, z_w) \ldots x^{(1)}_{j_{\alpha_1}}(t, z_w) \right) \\ &\times \ldots \times \left(\sum_{j_1+\ldots+j_{\alpha_n}=s_n} x^{(n)}_{j_1}(t, z_w) \ldots x^{(n)}_{j_{\alpha_n}}(t, z_w) \right), \end{aligned} \tag{7}$$

where we have used the notations $x_j(t, z) = (x^{(1)}_j(t, z), \ldots, x^{(n)}_j(t, z))$ and $F_j(t, x) = (F^{(1)}_j(t, x), \ldots, F^{(n)}_j(t, x))$ for $j = 1, 2, \ldots$ And also the notations

$$\frac{\partial^\alpha F^{(l)}_i(t, x)}{\alpha!} = \frac{1}{\alpha!} \frac{\partial^{\alpha_1} \ldots \partial^{\alpha_n} F^{(l)}_i(t, x)}{\partial x^{\alpha_1}_1 \ldots \partial x^{\alpha_n}_n}, \qquad \alpha! = \alpha_1! \ldots \alpha_n!$$

for $\alpha \in \mathbb{Z}^n_+$ with $\mathbb{Z}_+$ the set of nonnegative integers. In (7) and also in the full paper, for simplifying notations we have used the convention that when $\alpha_p = 0$ for some $p = 1, \ldots, n$ we have

$$\begin{aligned} \sum_{j_1+\ldots+j_{\alpha_p}=0} x^{(p)}_{j_1}(t, z_w) \ldots x^{(p)}_{j_{\alpha_p}}(t, z_w) = 1, \\ \sum_{j_1+\ldots+j_{\alpha_p}=s\in\mathbb{N}} x^{(p)}_{j_1}(t, z_w) \ldots x^{(p)}_{j_{\alpha_p}}(t, z_w) = 0. \end{aligned} \tag{8}$$

In this paper we also use the convention $\sum\limits_{s=i}^{j} c_s = 0$ for any c_s if $j < i$.

We note that the integrals which appear in (4) are nested integrals whose computation usually are not easy.

Now we can state our first main result.

Theorem 1. *For a periodic analytic differential system* (1) *we assume:*

(*i*) *For each $z_w \in \mathcal{M}_0$ the unique solution $x_0(t, z_w)$ of system* (2) *satisfying the condition $x_0(0, z_w) = z_w$ is T periodic.*

(*ii*) *The fundamental matrix solution $\Phi(t, z_w)$ of the variational equation* (3) *satisfies that $\Phi^{-1}(0, z_w) - \Phi^{-1}(T, z_w)$ has in the upper right corner the null $k \times (n-k)$ matrix, while in the lower right corner has the $(n-k) \times (n-k)$ matrix Δ_w, with $\det \Delta_w \neq 0$.*

Then the following statements hold.

(*a*) *Assume that there exists a* $k \in \mathbb{N}$ *such that* $G_k(z_w) \not\equiv 0$ *and* $G_i(z_w) \equiv 0$ *in* $\mathcal{M}_0$ *for* $i < k$, *where*

$$G_j(z_w) = \int_0^T \Phi^{-1}(s, z_w)\, \mathcal{K}_j(s, z_w) ds, \qquad j = 0, 1, \ldots$$

Defining $f_k(w) : cl(V) \to \mathbb{R}^k$ *by* $f_k(w) = \pi G_k(z_w)$, *with* $\pi : \mathbb{R}^n \to \mathbb{R}^k$ *the projection into the first* k *components. If* $w_0 \in V$ *is a zero of* f_k, *i.e.* $f_k(w_0) = 0$, *and* $\det(\partial_w f_k(w_0)) \neq 0$, *then system* (1) *has a* T *periodic solution* $\phi(t, \varepsilon)$ *such that* $\lim_{\varepsilon \to 0} \phi(0, \varepsilon) = z_{w_0}$.

(*b*) *If* $G_k(z_w) \equiv 0$ *for all* $k \in \mathbb{Z}_+$, *then all orbits of system* (1) *with initial points in* $\mathcal{M}_0$ *are periodic of period* T.

Theorem 1 will be proved in section 2. We remark that there are many dynamical systems which satisfy the conditions (*i*) and (*ii*) of Theorem 1, see for instance [7, 26, 27] and the references therein.

Theorem 1 for periodic analytic differential systems extends and improves the main results of [18, 24].

We now turn to autonomous differential systems and study the existence of limit cycles and their local integrability via averaging theory. The problem on limit cycles and integrability of polynomial differential systems (especially for planar polynomial systems) has been extensively studied from different points of view, see for instance [1, 10, 11, 14, 25, 39, 40, 41, 42] and the references therein. But there remain lots of unsolved difficult problems. In what follows we will use our extended arbitrary order averaging theory to investigate the limit cycle bifurcations and the integrability. As far as we know it is the first time that the averaging theory is applied to study the integrability of polynomial differential systems.

1.2. **Averaging method for higher dimensional autonomous differential systems.** For polynomial differential systems near a singularity such that its linear part has a pair of pure imaginary eigenvalues, we can strength Theorem 1 not only on the limit cycle bifurcation but also on the local integrability of the singularity. Consider the following polynomial differential systems of degree m in $\mathbb{R}^n$ with $n \geq 2$

$$
\begin{aligned}
\dot{x} &= -y + f_1(x,y,z) := -y + \sum_{i=2}^{m} f_{1i}(x,y,z),\\
\dot{y} &= x + f_2(x,y,z) := x + \sum_{i=2}^{m} f_{2i}(x,y,z),\\
\dot{z} &= Az + f_3(x,y,z) := Az + \sum_{i=2}^{m} f_{3i}(x,y,z),
\end{aligned}
\tag{9}
$$

where $z = (z^{(3)}, \ldots, z^{(n)}) \in \mathbb{R}^{n-2}$, A is a real square matrix of order $n-2$, f_{1i} and f_{2i} are real homogeneous polynomials of degree i and f_{3i} are $n-2$ dimensional real vector valued homogeneous polynomials of degree i. Of course if $n = 2$ system (9) is a planar polynomial differential system, i.e. the other components of system (9) do not appear.

Taking the change of variables $(x, y, z) \to \varepsilon(x, y, z)$, system (9) becomes the one depending on the parameter ε. Applying the change of variables $x = r\cos\theta, y = r\sin\theta, z = w$ to the resulting system, and choosing θ as the new independent variable, we get the following $n-1$ dimensional analytic periodic differential system

$$
\begin{pmatrix} r'(\theta,\varepsilon) \\ w'(\theta,\varepsilon) \end{pmatrix} = R_0(\theta, r, w) + \sum_{i=1}^{\infty} \varepsilon^i R_i(\theta, r, w),
\tag{10}
$$

where $R_0(\theta, r, w) = \begin{pmatrix} 0 \\ Aw \end{pmatrix}$, and $R_i(\theta, r, w)$ are polynomials in the variables r and w, and trigonometric polynomials in the variables $\cos\theta$ and $\sin\theta$. For more details see the proof of Theorem 2.

Let $z_0 = (r_0, w_0)$ with $r_0 > 0$ be any initial value, and $x(\theta, z_0, \varepsilon) := (r(\theta, z_0, \varepsilon), w(\theta, z_0, \varepsilon))$ be the solution of system (10) satisfying $x(0, z_0, \varepsilon) = z_0$. Since system (10) is analytic in a neighborhood of the origin, we can write the solution $x(\theta, z_0, \varepsilon)$ in the Taylor series

$$
x(\theta, z_0, \varepsilon) = x_0(\theta, z_0) + \sum_{j=1}^{\infty} \varepsilon^j x_j(\theta, z_0).
$$

Let $x_0(\theta, z_0)$ be the solution of system

$$
x'(\theta, z_0) = R_0(\theta, x_0(\theta, z_0)),
\tag{11}
$$

satisfying the initial condition $x(0, z_0) = z_0$, and let $\Phi(\theta, z_0)$ be the fundamental solution matrix of the variational equations of system (11) along the solution $x_0(\theta, z_0)$ such that $\Phi(0, z_0$ is the identity matrix.

Define

$$\mathcal{L}_j(\theta, z_0) := R_j(\theta, x_0(\theta, z_0)) + \sum_{i=1}^{j-1} \sum_{|\alpha|=1}^{j-i} \mathcal{R}_{i,\alpha,j-i}(\theta, z_0), \tag{12}$$

where

$$\mathcal{R}_{i,\alpha,j}(\theta, z_0) = \left(\mathcal{R}_{i,\alpha,j}^{(1)}(\theta, z_0), \dots, \mathcal{R}_{i,\alpha,j}^{(n-1)}(\theta, z_0)\right), \tag{13}$$

and for $l = 1, \dots, n$

$$\begin{aligned} \mathcal{R}_{i,\alpha,j}^{(l)}(\theta, z_0) = & \frac{\partial^\alpha R_i^{(l)}(\theta, x_0(\theta, z_0))}{\alpha!} \\ & \times \sum_{s_1+\dots+s_{n-1}=j} \left(\sum_{j_1+\dots+j_{\alpha_1}=s_1} x_{j_1}^{(1)}(\theta, z_0) \dots x_{j_{\alpha_1}}^{(1)}(\theta, z_0) \right) \\ & \times \dots \times \left(\sum_{j_1+\dots+j_{\alpha_{n-1}}=s_{n-1}} x_{j_1}^{(n-1)}(\theta, z_0) \dots x_{j_{\alpha_{n-1}}}^{(n-1)}(\theta, z_0) \right), \end{aligned} \tag{14}$$

with

$$\begin{aligned} x_j(\theta, z_0) &= \left(x_j^{(1)}(\theta, z_0), \dots, x_j^{(n-1)}(\theta, z_0)\right) \\ &= \left(r_j(\theta, z_0), w_j^{(3)}(\theta, z_0), \dots, w_j^{(n)}(\theta, z_0)\right), \qquad j = 1, 2, \dots \end{aligned}$$

and

$$x_j(\theta, z_0) = \Phi(\theta, z_0) \int_0^\theta \Phi^{-1}(s, z_0) \mathcal{L}_j(s, z_0) ds, \qquad j = 1, 2, \dots,$$

are successively known functions.

Let U be an open subset of $\mathbb{R}^n$. For the differential system (9) a non–locally constant function $H : U \to \mathbb{R}$ such that it is constant on the orbits of (9) contained in U is called a *first integral* of system (9) on U.

Recall that an analytic autonomous differential system in $\mathbb{R}^n$ is *locally C^l completely integrable* in an open subset U of $\mathbb{R}^n$ with $l \in \mathbb{N} \cup \{\infty, \omega\}$, if it has $n-1$ functionally independent C^l first integrals in U.

The notion of holonomy intends to be a replacement of the flow of a vector field in the case when the natural parametrization of the solutions is absent or ignored. Its definition is involved and will not be presented here, for details on it we refer to Section 2.3 of Chapter 1 of [22]. We note that this notion is essential in the statement and proof of our next theorem.

If a singularity in $\mathbb{R}^n$ has a punctured neighborhood filled of periodic orbits it is called a *center*.

Now we can state our next main result.

Theorem 2. *For a polynomial differential system* (9) *with the associated* 2π *periodic differential system* (10)*, we define* $z = (r, w)$ *and*

$$g_j(z) = \int_0^{2\pi} \Phi^{-1}(s, z)\mathcal{L}_j(\theta, z)d\theta, \qquad j = 1, 2, \dots$$

Assume that there exists a $\delta > 0$ *such that for all* $|z_0| = |(r_0, w_0)| \le \delta$*, system* (11) *has a* 2π *periodic solution* $x_0(\theta, z_0)$ *satisfying* $x_0(0, z_0) = z_0$*. Then the following statements hold.*

- (a) *Assume that there exists a* $k \in \mathbb{N}$ *such that* $g_k(z) \not\equiv 0$ *and* $g_i(z) \equiv 0$ *for* $i < k$*. If* $z_0 \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-2}$ *is such that* $g_k(z_0) = 0$*, and* $\det(\partial_z g_k(z_0)) \ne 0$*, then system* (10) *has a* 2π *periodic solution* $\phi(\theta, \varepsilon)$ *such that* $\lim_{\varepsilon \to 0} \phi(0, \varepsilon) = z_0$*. Furthermore system* (9) *has a hyperbolic limit cycle associated to* z_0 *in a neighborhood of the origin.*
- (b) *If* $g_k(z) \equiv 0$ *for all* $k \in \mathbb{N}$*, then all orbits of system* (9) *with initial points in a punctured neighborhood of the origin are periodic, i.e., the origin is a center. Moreover we have the next results.*
  - (b_1) *For* $n = 2$ *the origin of system* (9) *is an analytic integrable center.*
  - (b_2) *For* $n > 2$ *if all the periodic orbits have trivial holonomy, then system* (9) *is locally* C^∞ *completely integrable in a punctured neighborhood of the origin.*
- (c) *There exists an* $N \in \mathbb{N}$ *such that if* $g_k(z) \equiv 0$ *for all* $k = 1, \dots, N$*, then* $g_k(z) \equiv 0$ *for all* $k \in \mathbb{N}$*.*
- (d) *For* n *odd if* $g_k(z) \equiv 0$ *for all* $k \in \mathbb{N}$*, system* (9) *is not locally* C^l *completely integrable in a punctured neighborhood of the origin for all* $l \in \mathbb{N} \cup \{\infty, \omega\}$*. Consequently there exist periodic orbits which have nontrivial holonomy.*

We note that for analytic differential systems in the plane if all the averaged functions are zero, then we have a center, and after using the Poincaré center theorem we obtain the integrability in the plane. Thus we provide a new sufficient condition for integrability in the plane, i.e. all the averaging functions are zero. So, in this sense the averaging theory contributes to the integrability in dimension two. But for higher dimension we must add to the condition of all the averaged functions are zero, the assumption of trivial holonomy of the periodic solutions of the center, then using the integrability result given in reference [34], it follows the complete integrability in a neighborhood of the center in higher dimension.

We must also note that the N which appears in statement (c) of Theorem 2 is a uniform constant which depends only on the dimension of the space where is defined the polynomial differential system and on the degree of the polynomial differential system.

It is known that for any analytic vector field in dimension three having an isolated singular point there exists a trajectory through a point different from the singularity tending to the singularity as time goes to $+\infty$ or $-\infty$, see [5]. So clearly statement (d) is empty when $n = 3$. We do not know if it would be empty or not for $n > 3$.

The next example shows that for n even there are systems, whose associated g_k's all vanish.

Example 1. *Consider the dimensional system*

$$\dot{x} = -y + x^2, \quad \dot{y} = x + xy, \quad \dot{z} = -w, \quad \dot{w} = z, \tag{15}$$

in $\mathbb{R}^4$. Since the system $(\dot{x}, \dot{y})$ is reversible under the symmetry $(x, y, t) \to (-x, y, -t)$ and the system $(\dot{z}, \dot{w})$ is a linear center, it follows that the origin of system (15) *in $\mathbb{R}^4$ is a center. Now we shall see that this system satisfies that all the polynomials g_k's defined in Theorem 2 are zero, and consequently this theorem confirms that system* (15) *has a center at the origin of $\mathbb{R}^4$.*

The proof of Example 1 will be given in section 3.

Remark 1. *In Theorem 2 we need the assumption that system* (10) *has 2π periodic solutions filling a neighborhood of the constant solution* $(r, w) = (0, 0)$. *If every entry of the matrix A of system* (9) *is a parameter or $A = 0$, without this assumption we can also obtain the same result as Theorem 2 working in the following way. Scale the parameters $A \to \varepsilon A$, and using the same techniques than we used before the statement of Theorem 2, we get system* (10) *with* $R_0(\theta, r, w) = 0$ *in the cylindric coordinates. This system for* $\varepsilon = 0$ *has only constant solutions, which of course are all 2π–periodic. The rest of the proof follows using the same arguments as in the proof of Theorem 2.*

1.3. **Averaging theory for planar monodromic nilpotent singularities.**

Finally we restrict to real planar polynomial differential systems

$$\dot{x} = y + P(x,y), \qquad \dot{y} = Q(x,y), \tag{16}$$

where $P(x,y), Q(x,y) \in \mathbb{R}[x,y]$ are polynomials without constant and linear terms. Recall that $\mathbb{R}[x,y]$ is the ring of polynomials in the variables x and y with coefficients in $\mathbb{R}$. We note that the origin of system (16) is a nilpotent singularity. We will develop an averaging theory to study the limit cycle bifurcations and the integrability of system (16). A *monodromic singularity* is the one whose small neighborhood is filled with rotating orbits, i.e., it is a focus or a center.

Under the invertible and analytic change of variables $\xi = x$, $\eta = y + P(x,y)$, system (16) becomes

$$\dot{\xi} = \eta, \qquad \dot{\eta} = a_k\xi^k\,(1 + h(\xi)) + b_n\xi^n\eta\,(1 + g(\xi)) + \eta^2 p(\xi,\eta), \tag{17}$$

where $h(\xi), g(\xi) = O(x)$ and $p(\xi,\eta) = O(1)$ are analytic. Theorems 7.2 and 7.3 of [42] show that the origin of system (17) is a monodromic singularity if and only if $k = 2m+1$ for some $m \in \mathbb{N}$, $a_{2m+1} < 0$ and

- (i_1) either $b_n = 0$;
- (i_2) or $n > m$;
- (i_3) or $n = m$ and $\lambda := b_n^2 + 4(m+1)a_{2m+1} < 0$.

The case (i_1), i.e. $b_n = 0$, can be treated as in the case (i_2) when $n = \infty$. So we will not discuss (i_1). The case (i_3) may be studied in a similar way to the case (i_2) but the expressions are much complicated, and so we omit it. In what follows for simplifying notation we set $a_{2m+1} = -a$ with $a > 0$ and $b_n = b$.

Rescaling the variables and the time of system (17) by $\xi \to \varepsilon\xi$, $\eta \to \varepsilon^{m+1}\eta$, $t \to \varepsilon^{-m}t$, and taking the generalized Lyapunov polar coordinate change of variables

$$x = r(\theta)\operatorname{Cs}(\theta), \qquad y = r^{m+1}(\theta)\operatorname{Sn}(\theta), \tag{18}$$

we can write system (17) as

$$\frac{dr}{d\theta} = K_1(\theta,r)\varepsilon + K_2(\theta,r)\varepsilon^2 + \ldots, \tag{19}$$

where

$$K_1(\theta,r) = \begin{cases} -\dfrac{ab_0}{m+1} r\operatorname{Cs}^{2m+1}\theta\operatorname{Sn}\theta + \dfrac{p_0}{m+1} r^2\operatorname{Sn}^3\theta, & n > m+1, \\ -\dfrac{ab_0}{m+1} r\operatorname{Cs}^{2m+1}\theta\operatorname{Sn}\theta + \dfrac{p_0}{m+1} r^2\operatorname{Sn}^3\theta & \\ \qquad\qquad + \dfrac{b}{m+1} r^2\operatorname{Cs}^2\theta\operatorname{Sn}^2\theta, & n = m+1, \end{cases}$$

and $K_j(\theta, r)$ are polynomials in the variables $r, \mathrm{Cs}\,\theta$ and $\mathrm{Sn}\,\theta$ with coefficients polynomials in the coefficient of system (16), the details are omitted. Recall that $\mathrm{Cs}\,\theta$ and $\mathrm{Sn}\,\theta$ are solutions of a Cauchy problem and they are periodic with period T_m. They will be expressed clearly in the proof of the next Theorem 3.

Let $r_0(\theta, r) = r$ be the solution of the initial value problem $r_0'(\theta) = 0$, $r_0(0) = r$. For $j = 1, 2, \ldots$, define

$$r_j(\theta, r) = \int_0^\theta \mathcal{N}_j(s, r)ds, \tag{20}$$

with

$$\mathcal{N}_j(\theta, r) := K_j(\theta, r) + \sum_{i=1}^{j-1} \sum_{l=1}^{j-i} \frac{\partial^l K_i(\theta, z)}{l!} \sum_{j_1+\ldots+j_l=j-i} r_{j_1}(\theta, r) \ldots r_{j_l}(\theta, r). \tag{21}$$

Clearly all $\mathcal{N}_j(\theta, r)$ and $r_j(\theta, r)$ are functions that can be computed recursively.

Now we can state our last main result.

Theorem 3. *For a planar polynomial differential system* (16) *with the associated T_m periodic differential system* (19), *we define*

$$g_j(r) = \int_0^{T_m} \mathcal{N}_j(\theta, r)d\theta, \qquad j = 1, 2, \ldots$$

Then the following statements hold.

- (a) *Assume that there exists a $k \in \mathbb{N}$ such that $g_k(r) \not\equiv 0$ and $g_i(r) \equiv 0$ for $i < k$. If $r_0 > 0$ is such that $g_k(r_0) = 0$, and $g_k'(r_0) \neq 0$, then system* (19) *has a T_m periodic solution $r(\theta, \varepsilon)$ such that $\lim_{\varepsilon \to 0} r(0, \varepsilon) = r_0$. Furthermore system* (9) *has a hyperbolic limit cycle associated to r_0 for $\varepsilon \neq 0$ sufficiently small.*
- (b) *If $g_k(r) \equiv 0$ for all $k \in \mathbb{N}$, then the origin of system* (16) *is a center and there is a neighborhood of the origin where the system is C^∞ integrable.*
- (c) *There exists an $N \in \mathbb{N}$ such that if $g_k(r) \equiv 0$ for all $k = 1, \ldots, N$, then $g_k(r) \equiv 0$ for all $k \in \mathbb{N}$.*

We remark that Theorem 3 deals with perturbations of individual monodromic nilpotent differential systems in the plane. In fact this perturbation technique is applicable to all monodromic nilpotent differential systems in the plane.

This paper is organized as follows. In the next section we will prove Theorem 1 and Corollaries 4 and 5. The proof of Theorem 2 will be given in section 3. In the last section we will prove Theorem 3.

2. Proof of Theorem 1 and two corollaires

This section is devoted to prove Theorem 1 on arbitrary order averaging theory for the analytic periodic differential system (1), and two corollaries of it.

2.1. **Proof of Theorem 1.** For arbitrary $z \in \Omega$, let $x(t, z, \varepsilon)$ be the solution of system (1) satisfying the initial condition $x(0, z, \varepsilon) = z$. Then we have

$$x(t, z, \varepsilon) = z + \sum_{i=0}^{\infty} \varepsilon^i \int_0^t F_i(s, x(s, z, \varepsilon))ds. \tag{22}$$

To study the existence of periodic solutions is equivalent to find the initial value z such that $x(T, z, \varepsilon) = z$. Since system (1) is analytic, the solution $x(t, z, \varepsilon)$ is analytic in its variables. Expanding this solution in the Taylor series in ε, i.e.,

$$x(t, z, \varepsilon) = x_0(t, z) + \sum_{j=1}^{\infty} \varepsilon^j x_j(t, z). \tag{23}$$

Clearly $x_0(t, z)$ satisfies the unperturbed system (2) of system (1), and $x_j(t, z)$ for $j > 0$ satisfy $x_j(0, z) = 0$.

Our next objective will be to find the differential equations that the functions $x_j(t, z)$ satisfy, later on we find the explicit expressions of these functions, which help us to study the displacement function to time T of the differential system (1), and finally the analysis of the zeros of that function will provide us the periodic solutions of period T which appear in the statement of the theorem.

For any $z_w \in \mathcal{M}_0 \subset \Omega$ by assumption the solution $x_0(t, z_w) = \varphi(t, z_w)$ of system (2) is T periodic, we have that the associated solution $x(t, z_w, \varepsilon)$ of system (1) is of the form

$$x(t, z_w, \varepsilon) = x_0(t, z_w) + \sum_{j=1}^{\infty} \varepsilon^j x_j(t, z_w).$$

Set

$$F_i(t, x(t, z_w, \varepsilon)) = \left(F_i^{(1)}(t, x(t, z_w, \varepsilon)), \dots, F_i^{(n)}(t, x(t, z_w, \varepsilon))\right),$$

with $i = 0, 1, 2, \dots$ Doing the Taylor expansion of functions in several variables to each $F_i^{(l)}$, $l = 1, \dots, n$, together with the expansion (23) of the solution $x(t, z, \varepsilon)$, we get

$$
\begin{aligned}
F_i^{(l)}(t, x(t, z_w, \varepsilon)) &= \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{\partial^\alpha F_i^{(l)}(t, \varphi(t, z_w))}{\alpha!} \left(\sum_{j=1}^{\infty} \varepsilon^j x_j(t, z_w) \right)^\alpha \\
&= \sum_{|\alpha|=0}^{\infty} \frac{\partial^\alpha F_i^{(l)}(t, \varphi(t, z_w))}{\alpha!} \left(\sum_{j=1}^{\infty} \varepsilon^j x_j(t, z_w) \right)^\alpha , \qquad (24)
\end{aligned}
$$

where $\alpha \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and for $|\alpha| \geq 1$

$$
\begin{aligned}
\partial^\alpha F_i^{(l)}(t, x) &= \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n} F_j^{(l)}(t, x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \\
\left(\sum_{j=1}^{\infty} \varepsilon^j x_j(t, z_w) \right)^\alpha &= \left(\sum_{j=1}^{\infty} \varepsilon^j x_j^{(1)}(t, z_w) \right)^{\alpha_1} \dots \left(\sum_{j=1}^{\infty} \varepsilon^j x_j^{(n)}(t, z_w) \right)^{\alpha_n},
\end{aligned}
$$

where we have used the notation $x_j = \left(x_j^{(1)}, \dots, x_j^{(n)} \right)$. Note that

$$
\left(\sum_{j=1}^{\infty} \varepsilon^j x_j^{(p)}(t, z_w) \right)^{\alpha_p} = \sum_{s_p=\alpha_p}^{\infty} \varepsilon^{s_p} \sum_{j_1+\dots+j_{\alpha_p}=s_p} x_{j_1}^{(p)}(t, z_w) \dots x_{j_{\alpha_p}}^{(p)}(t, z_w),
$$

where j_s, $s \in \{1, \dots, \alpha_p\}$, is any positive integer such that their summation is s_p. Recall that we have used the convention (8) for $\alpha_p = 0$, $p = 1, \dots, n$. Then we have

$$
\begin{aligned}
\left(\sum_{j=1}^{\infty} \varepsilon^j x_j(t, z_w) \right)^\alpha = \sum_{r=|\alpha|}^{\infty} \varepsilon^r \sum_{s_1+\dots+s_n=r} &\left(\sum_{j_1+\dots+j_{\alpha_1}=s_1} x_{j_1}^{(1)}(t, z_w) \dots x_{j_{\alpha_1}}^{(1)}(t, z_w) \right) \\
&\times \dots \times \left(\sum_{j_1+\dots+j_{\alpha_n}=s_n} x_{j_1}^{(n)}(t, z_w) \dots x_{j_{\alpha_n}}^{(n)}(t, z_w) \right).
\end{aligned}
$$

So we get from (24) that for $i = 0, 1, \dots,$ and $l = 1, \dots, n$,

$$
\begin{aligned}
F_i^{(l)}(t, x(t, z_w, \varepsilon)) &= F_i^{(l)}(t, \varphi(t, z_w)) + \sum_{|\alpha|=1}^{\infty} \sum_{r=|\alpha|} \varepsilon^r \mathcal{F}_{i,\alpha,r}^{(l)}(t, z_w) \\
&= F_i^{(l)}(t, \varphi(t, z_w)) + \sum_{j=1}^{\infty} \varepsilon^j \sum_{|\alpha|=1}^{j} \mathcal{F}_{i,\alpha,j}^{(l)}(t, z_w), \qquad (25)
\end{aligned}
$$

where $\mathcal{F}^{(l)}_{i,\alpha,j}(t,z_w)$ was defined in (7). Furthermore we have for $l = 1,\ldots,n$

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\varepsilon^i\int_0^t F_i^{(l)}(s,x(s,z_w,\varepsilon))ds \\
(26)\quad &= \sum_{i=1}^{\infty}\varepsilon^i\int_0^t F_i^{(l)}(s,\varphi(s,z_w)ds + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\varepsilon^{i+j}\sum_{|\alpha|=1}^{j}\int_0^t \mathcal{F}^{(l)}_{i,\alpha,j}(s,z_w)ds \\
&= \sum_{j=1}^{\infty}\varepsilon^j\int_0^t F_j^{(l)}(s,\varphi(s,z_w))ds + \sum_{j=2}^{\infty}\varepsilon^j\sum_{i=1}^{j-1}\sum_{|\alpha|=1}^{j-i}\int_0^t \mathcal{F}^{(l)}_{i,\alpha,j-i}(s,z_w)ds.
\end{aligned}
$$

Substituting (23), (25) and (26) into (1), and equating the coefficients of ε^j, we get

$$
\dot{x}_1(t,z_w) = \sum_{|\alpha|=1}\mathcal{F}_{0,\alpha,1}(t,z_w) + F_1(t,\varphi(t,z_w)), \tag{27}
$$

$$
\begin{aligned}
\dot{x}_j(t,z_w) = &\sum_{|\alpha|=1}^{j}\mathcal{F}_{0,\alpha,j}(t,z_w) + F_j(t,\varphi(t,z_w)) \\
&+\sum_{i=1}^{j-1}\sum_{|\alpha|=1}^{j-i}\mathcal{F}_{i,\alpha,j-i}(t,z_w), \quad j=2,3,\ldots
\end{aligned} \tag{28}
$$

where

$$
F_j(t,\varphi(t,z_w)) = \left(F_j^{(1)}(t,\varphi(t,z_w)),\ldots,F_j^{(n)}(t,\varphi(t,z_w))\right), \;\; j=1,2,\ldots
$$

Note that

$$
\sum_{|\alpha|=1}\mathcal{F}_{0,\alpha,j}(t,z_w) = \partial F_0(t,\varphi(t,z_w))x_j(t,z_w).
$$

Recall that $\partial F_0(t,\varphi(t,z_w))$ is the Jacobian matrix of $F_0(t,x)$ with respect to x taking values at $x=\varphi(t,z_w)$.

Since $x(t,z_w,\varepsilon)$ satisfies the initial condition $x(0,z_w,\varepsilon)=z_w$, it follows that

$$
x_j(0,z_w) = 0, \qquad j=1,2,\ldots \tag{29}
$$

Notice that

$$
\mathcal{K}_j(t,z_w) = \sum_{|\alpha|=2}^{j}\mathcal{F}_{0,\alpha,j}(t,z_w) + F_j(t,\varphi_0(t,z_w)) + \sum_{i=1}^{j-1}\sum_{|\alpha|=1}^{j-i}\mathcal{F}_{i,\alpha,j-i}(t,z_w),
$$

and

$$\mathcal{F}_{i,\alpha,j}(t,z_w) = \left(\mathcal{F}^{(1)}_{i,\alpha,j}(t,z_w),\dots,\mathcal{F}^{(n)}_{i,\alpha,j}(t,z_w)\right), \quad j=1,2,\dots$$

are successively known functions. Then systems (27) and (28) satisfying the initial conditions (29) have the solutions given in (4), i.e.

$$x_j(t,z_w) = \Phi(t,z_w)\int_0^t \Phi^{-1}(s,z_w)\,\mathcal{K}_j(s,z_w)ds, \quad j=1,2,\dots,$$

For $z_w \in \mathcal{M}_0$, we define the displacement function

$$d(z_w,\varepsilon) = x(T,z_w,\varepsilon) - z_w. \tag{30}$$

Then $x(t,z_w,\varepsilon)$ is a periodic solution of period T of system (1) if and only if $d(z_w,\varepsilon)=0$. Set

$$G_j(z_w) = \Phi(T,z_w)\int_0^T \Phi^{-1}(s,z_w)\,\mathcal{K}_j(s,z_w)ds, \quad j=2,3,\dots$$

Then

$$d(z_w,\varepsilon) = \sum_{j=1}^{\infty}\varepsilon^j G_j(z_w). \tag{31}$$

Proof of statement (*a*). Since $\Phi(t,z_w)$ is invertible for each $z_w \in \mathcal{M}_0$, the zeros of $d(z_w,\varepsilon)$ are the same as those of $\Phi^{-1}(T,z_w)d(z_w,\varepsilon)$. We consider the function

$$P(z,\varepsilon) = \Phi^{-1}(T,z)d(z,\varepsilon),$$

together with its Taylor expansion in ε, i.e.

$$P(z,\varepsilon) = P_0(z) + \varepsilon P_1(z) + \dots + \varepsilon^k P_k(z) + \dots$$

By assumption we have

$$P(z_w,\varepsilon) = \varepsilon^k P_k(z_w) + \dots, \quad f_k(z_w) = \pi P_k(z_w), \quad z_w \in \mathcal{M}_0. \tag{32}$$

We recall that the functions $f_k(z_w)$ are the averaged functions which appear in the statement of Theorem 1.

We claim that

$$\partial_z P(z_w,0) = \Phi^{-1}(0,z_w) - \Phi^{-1}(T,z_w). \tag{33}$$

Indeed, first we have

$$\partial_z P(z,\varepsilon) = \partial_z\Phi^{-1}(T,z)d(z,\varepsilon) + \Phi^{-1}(T,z)(\partial_z x(T,z,\varepsilon) - E),$$

where E is the $n\times n$ identity matrix. In addition $\partial_z x(t,z_w,\varepsilon)$ is the fundamental matrix solution of the variational equation (3) with the initial condition $\partial_z x(0,z_w,0) = E$, so we have

$$\partial_z x(t,z_w,0) = \Phi(t,z_w)\Phi^{-1}(0,z_w).$$

Then the equality (33) follows from the fact that $d(z_w, 0) = \varphi(T, z_w) - z_w \equiv 0$ for all $z_w \in \mathcal{M}_0$, where we have used the fact that $\varphi(t, z_w)$ is the solution of the initial value problem (2), and the assumption that all orbits of the unperturbed system (2) with the initial values in $\mathcal{M}_0$ are T periodic.

Let $\pi^+ : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be the projection of $\mathbb{R}^n$ to its last $n-k$ components. Since $P(z_w, 0) = 0$, we get from (33) and the assumption $\det \Delta_w \neq 0$ that the equation $\pi^+ P(z, \varepsilon) = 0$, by the Implicit Function Theorem, has a unique analytic solution $v = g(u, \varepsilon)$ in a neighborhood of $(z, \varepsilon) = (z_w, 0)$, where we have used the notation $z = (u, v)$ with $u = (z_1, \ldots, z_k)$ and $v = (z_{k+1}, \ldots, z_n)$. Substituting this solution into $\pi P(z, \varepsilon)$ and using the assumptions of the theorem we get that $\pi P(u, g(u, \varepsilon), \varepsilon) = 0$ has a unique analytic solution $u = u(\varepsilon)$ in a neighborhood of $(u, \varepsilon) = (w, 0)$ with $w \in V$. This proves that near each z_{w_0} with w_0 satisfying $f_k(w_0) = 0$ system (1) has a unique analytic solution $x = \phi(t, \varepsilon)$ such that $\lim_{\varepsilon \to 0} \phi(0, \varepsilon) = z_{w_0}$. Statement (a) follows. □

Proof of statement (b). Since $G_k(z_w) \equiv 0$ for all $k \in \mathbb{N}$ and $z_w \in \mathcal{M}_0$, it follows that $d(z, \varepsilon) \equiv 0$ for all $z = z_w \in \mathcal{M}_0$. This implies that all orbits of system (1) with initial points in $\mathcal{M}_0$ are periodic of period T. We complete the proof of statement (b) and consequently the theorem. □

Two results which follows from Theorem 1 are the following two corollaries.

Corollary 4. *For a periodic analytic differential system* (1) *we assume that* $V \subset \Omega \subset \mathbb{R}^n$ *is an open and bounded subset, such that for each* $z \in V$ *the unique solution* $x_0(t, z)$ *of system* (2) *satisfying the condition* $x_0(0, z) = z$ *is* T *periodic. Then the following statements hold.*

(a) *Assume that there exists a* $k \in \mathbb{N}$ *such that* $g_k(z) \not\equiv 0$ *and* $g_i(z) \equiv 0$ *in* V *for* $i < k$, *where*

$$g_j(z) = \int_0^T \Phi^{-1}(s, z)\, \mathcal{K}_j(s, z) ds, \qquad j = 1, 2, \ldots$$

If $z_0 \in V$ *is such that* $g_k(z_0) = 0$ *and* $\det(\partial_z g_k(z_0)) \neq 0$, *then system* (1) *has a* T *periodic solution* $\phi(t, \varepsilon)$ *such that* $\lim_{\varepsilon \to 0} \phi(0, \varepsilon) = z_0$.

(b) *If* $g_j(z) \equiv 0$ *for all* $j \in \mathbb{N}$, *then all orbits of system* (1) *with initial points in* V *are periodic of period* T.

Proof. Corresponding to Theorem 1 we have $k = n$. Now the projection π is the identity map on $\mathbb{R}^n$, and the minor Δ_n of order $n - n = 0$ of

$\Phi^{-1}(0,z) - \Phi^{-1}(T,z)$ does not appear. So the theorem follows directly from Theorem 1. $\square$

We remark that Corollary 4 is different from Theorem 1 only in the region V. In Corollary 4 V is an open subset of $\mathbb{R}^n$ while in Theorem 1 V is an open subset of $\mathbb{R}^k \subsetneq \mathbb{R}^n$.

Corollary 5. *For a periodic analytic differential system* (1) *which has* $F_0(t,x) \equiv 0$, *we define*

$$g_j(z) = \int_0^T \mathcal{K}_j(s,z)ds, \qquad j = 1,2,\ldots$$

with $\mathcal{K}_j(s,z)$ *defined in* (5), (6) *and* (7) *instead of* z_w *and* $\varphi(t,z_w)$ *by* z. *Then the following statements hold.*

(a) *Assume that there exists a* $k \in \mathbb{N}$ *such that* $g_k(z) \not\equiv 0$ *and* $g_i(z) \equiv 0$ *for* $i < k$. *If* $z_0 \in \Omega$ *is such that* $g_k(z_0) = 0$ *and* $\det(\partial_z g_k(z_0)) \neq 0$, *then system* (1) *has a* T *periodic solution* $\phi(t,\varepsilon)$ *such that* $\lim_{\varepsilon\to 0} \phi(0,\varepsilon) = z_0$.

(b) *If* $g_k(z) \equiv 0$ *for all* $k \in \mathbb{N}$, *then all orbits of system* (1) *with initial points in* Ω *are periodic of period* T.

Proof. Since $F_0(t,x) \equiv 0$ for $(t,x) \in \mathbb{R}\times\Omega$, it follows that

$$\dot{x}_0(t,z) = 0, \qquad x_0(0,z) = z \tag{34}$$

with $z \in \Omega$ any initial value. For any $z \in \Omega$, the unique solution $x_0(t,z) \equiv z$ of (34) is a T periodic one of system (34). We are in the case $k = n$ of Theorem 1. The variational equation of system (34) along the solution $x_0(t,z)$ is $\dot{y} = 0$. It has the fundamental matrix $\Phi(t,z) = E$. Moreover we get from (4) that

$$x_j(t,z) = \int_0^t \mathcal{K}_j(s,z)ds, \qquad j = 1,2,\ldots \tag{35}$$

for $z \in \Omega$. The rest of the proof follows as in the proof of Theorem 1. This completes the proof of the theorem. $\square$

We note that Corollaries 4 and 5 extend the main results of [18] from one dimensional periodic analytic differential systems to any finite dimensional periodic analytic differential systems.

In the case of C^k periodic differential systems Corollaries 4 and 5 were proved in [24]. In any case the formulas for applying these corollaries are easier than the formulas given in [24].

3. Proof of Theorem 2 and Example 1

3.1. **Proof of Theorem 2.** For proving Theorem 2 we first transform system (9) into system (1). Taking the change of variables $(x, y, z) \to \varepsilon(x, y, z)$, system (9) can be written in the form

$$\begin{aligned} \dot{x} &= -y + \sum_{i=2}^{m} \varepsilon^{i-1} f_{1i}(x, y, z), \\ \dot{y} &= \quad x + \sum_{i=2}^{m} \varepsilon^{i-1} f_{2i}(x, y, z), \\ \dot{z} &= Az + \sum_{i=2}^{m} \varepsilon^{i-1} f_{3i}(x, y, z). \end{aligned} \tag{36}$$

Doing the change of variables

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = rw,$$

with $w = \big(w^{(3)}, \dots, w^{(n)}\big)$, system (36) becomes

$$\begin{aligned} \dot{r} &= \sum_{i=2}^{m} \varepsilon^{i-1} r^i g_{1i}(\theta, w), \\ \dot{\theta} &= 1 - \sum_{i=2}^{m} \varepsilon^{i-1} r^{i-1} g_{2i}(\theta, w), \\ \dot{w} &= Aw + \sum_{i=2}^{m} \varepsilon^{i-1} r^{i-1} g_{3i}(\theta, w), \end{aligned} \tag{37}$$

where for $i = 2, \dots, m$

$$\begin{aligned} g_{1i}(\theta, w) &= \cos\theta f_{1i}(\cos\theta, \sin\theta, w) + \sin\theta f_{2i}(\cos\theta, \sin\theta, w), \\ g_{2i}(\theta, w) &= \sin\theta f_{1i}(\cos\theta, \sin\theta, w) - \cos\theta f_{2i}(\cos\theta, \sin\theta, w), \\ g_{3i}(\theta, w) &= f_{3i}(\cos\theta, \sin\theta, w) \\ &\qquad - w(\cos\theta f_{1i}(\cos\theta, \sin\theta, w) + \sin\theta f_{2i}(\cos\theta, \sin\theta, w)). \end{aligned}$$

Treating θ as an independent variable, we rewrite system (37) as an $n-1$ dimensional analytic differential system

$$
\begin{aligned}
r' = \frac{dr}{d\theta} &= \sum_{i=2}^{m} \varepsilon^{i-1} r^{i} g_{1i}(\theta, w) \left(1 + \sum_{j=1}^{\infty} \left(\sum_{i=2}^{m} \varepsilon^{i-1} r^{i-1} g_{2i}(\theta, w) \right)^{j} \right), \\
w' = \frac{dw}{d\theta} &= \left(Aw + \sum_{i=2}^{m} \varepsilon^{i-1} r^{i-1} g_{3i}(\theta, w) \right) \\
&\qquad \times \left(1 + \sum_{j=1}^{\infty} \left(\sum_{i=2}^{m} \varepsilon^{i-1} r^{i-1} g_{2i}(\theta, w) \right)^{j} \right),
\end{aligned}
\tag{38}
$$

which is 2π periodic in θ. Rewriting system (38) in the vector form given in (10), i.e.

$$
\begin{pmatrix} r'(\theta) \\ w'(\theta) \end{pmatrix} = R_0(\theta, r, w) + \sum_{i=1}^{\infty} \varepsilon^{i} R_i(\theta, r, w),
$$

where

$$
\begin{aligned}
R_0(\theta, r, w) &= \begin{pmatrix} 0 \\ Aw \end{pmatrix}, \\
R_1(\theta, r, w) &= \begin{pmatrix} r g_{12}(\theta, w) \\ r g_{32}(\theta, w) \end{pmatrix}, \\
R_2(\theta, r, w) &= \begin{pmatrix} r^2 g_{13}(\theta, w) + r^2 g_{12}(\theta, w) g_{22}(\theta, w) \\ r^2 g_{33}(\theta, w) + r^2 g_{32}(\theta, w) g_{22}(\theta, w) \end{pmatrix},
\end{aligned}
$$

and $R_j(\theta, r, w)$, $j = 3, 4, \ldots$, can be expressed as polynomials in $g_{sl}(\theta, w)$ for $s = 1, 2, 3$ and $l = 2, \ldots, j+1$. The exact expressions will be omitted.

Proof of statement (a). Since for $\varepsilon \neq 0$ the periodic orbits of system (9) are uniquely determined by the periodic orbits of system (36), and consequently of system (10). In addition, since $R_0(\theta, r, w)$ is linear in r and w, all its second order partial derivatives vanish with respect to r and w. So from (5) we get (12). Then the next proof is a direct consequence of Corollary 4. □

Proof of statement (b). From Corollary 5 and the proof of statement (a) of that theorem, it follows that all the orbits of system (10) are periodic of period 2π. Hence all the orbits of system (9) in a punctured neighborhood of the origin are periodic.

(b_1). For $n = 2$ system (9) is a planar analytic differential system. Since there is a neighborhood of the origin filled with periodic orbits, the origin is a nondegenerate center. By the Poincaré center theorem system (9) has an analytic first integral in a neighborhood of the origin. Recall the Poincaré center theorem, i.e. if a real planar analytic differential system has a singularity with a pair of pure imaginary eigenvalues, then the origin is a center if and only if the system has an analytic first integral.

(b_2). Since a punctured neighborhood of the origin of system (9) is filled with periodic orbits, by the assumption that each periodic orbit has trivial holonomy we get from Theorem 1 of [34] that system (9) has $n-1$ functionally independent C^∞ first integrals. □

Proof of statement (c). First we claim that all $g_k(z)$ are polynomials in $z = (r, w)$ with coefficients being polynomials in the coefficients of system (9). Later on using the Hilbert basis theorem to these polynomials we shall prove statement (c).

We now prove this claim. From the construction of system (10) we know that each $R_i(\theta, z)$ with $z = (r, w)$, $i = 1, 2, \ldots$, is a polynomial in $\cos\theta, \sin\theta, z$ with coefficients being polynomials in the coefficients of system (9). Next we use induction to prove that $\mathcal{L}_j(\theta, z)$ defined in (12) and consequently $x_j(\theta, z)$ are polynomials in $\cos\theta, \sin\theta, z$ with coefficients being polynomials in the coefficients of system (9).

For $j = 1$, $\mathcal{L}_1(\theta, z) = R_1(\theta, z)$. So

$$x_1(\theta, z) = \int_0^\theta \mathcal{L}_1(s, z)ds,$$

is a polynomial in $\cos\theta, \sin\theta, z$ with coefficients polynomials in the coefficients of system (9). For $j = 2$ we have

$$\mathcal{L}_2(\theta, z) = R_2(\theta, z) + \sum_{|\alpha|=1} \mathcal{R}_{1,\alpha,1}(\theta, z).$$

Note that $\mathcal{R}_{1,\alpha,1}(\theta, z)$ has its components $\mathcal{R}^{(l)}_{1,\alpha,1}(\theta, z)$, $l = 1, \ldots, n-1$, being of the form

$$\mathcal{R}^{(l)}_{1,\alpha,1}(\theta, z) = \frac{\partial^\alpha R_1^{(l)}(\theta, z)}{\alpha!} \sum_{s_1+\ldots+s_{n-1}=1} \left(\sum_{j_1+\ldots+j_{\alpha_1}=s_1} x^{(1)}_{j_1}(\theta, z) \ldots x^{(1)}_{j_{\alpha_1}}(\theta, z) \right)$$
$$\times \ldots \times \left(\sum_{j_1+\ldots+j_{\alpha_{n-1}}=s_{n-1}} x^{(n-1)}_{j_1}(\theta, z) \ldots x^{(n-1)}_{j_{\alpha_1}}(\theta, z) \right).$$

Observe that in the right hand side of this last expression of $\mathcal{R}^{(l)}_{1,\alpha,1}(\theta, z)$ all components are polynomials in the variables $\cos\theta$, $\sin\theta$ and z with coefficients polynomials in the coefficients of system (9). So $\mathcal{R}_{1,\alpha,1}(\theta, z)$ and consequently

$$x_2(\theta, z) = \int_0^\theta \mathcal{L}_2(s, z)ds,$$

are polynomials in the variables $\cos\theta$, $\sin\theta$ and z with coefficients polynomials in the coefficients of system (9).

We assume that for $k \in \mathbb{N}$ all $x_j(\theta, z)$ are polynomials in the variables $\cos\theta$, $\sin\theta$ and z with coefficients polynomials in the coefficients of system (9), $j = 1, \ldots, k-1$. We prove the same for $x_k(\theta, z)$. Since

$$x_k(\theta, z) = \int_0^\theta \mathcal{L}_k(s, z)ds,$$

we only need to prove that $\mathcal{L}_k(\theta, z)$ is a polynomial in the variables $\cos\theta$, $\sin\theta$ and z with coefficients polynomials in the coefficients of system (9). Since

$$\mathcal{L}_k(\theta, z) = R_k(\theta, z) + \sum_{i=1}^{k-1} \sum_{|\alpha|=1}^{k-i} \mathcal{R}_{i,\alpha,k-i}(\theta, z),$$

it remains to prove that $\mathcal{R}_{i,\alpha,j}(\theta, z)$ with $i, j < k$ are polynomials in the variables $\cos\theta$, $\sin\theta$ and z with coefficients polynomials in the coefficients of system (9). But this follows easily from the expressions

$$\mathcal{R}_{i,\alpha,j}(\theta, z) = \left(\mathcal{R}^{(1)}_{i,\alpha,j}(\theta, z), \ldots, \mathcal{R}^{(n-1)}_{i,\alpha,j}(\theta, z)\right),$$

and

$$\mathcal{R}^{(l)}_{i,\alpha,j}(\theta, z) = \frac{\partial^\alpha R^{(l)}_i(\theta, z)}{\alpha!} \sum_{s_1+\ldots+s_{n-1}=j} \left(\sum_{j_1+\ldots+j_{\alpha_1}=s_1} x^{(1)}_{j_1}(\theta, z) \ldots x^{(1)}_{j_{\alpha_1}}(\theta, z) \right) \times \ldots \times \left(\sum_{j_1+\ldots+j_{\alpha_{n-1}}=s_{n-1}} x^{(n-1)}_{j_1}(\theta, z) \ldots x^{(n-1)}_{j_{\alpha_{n-1}}}(\theta, z) \right).$$

Finally the claim follows from $g_k(z) = \int_0^{2\pi} \mathcal{L}_k(s, z)$.

The assumption $g_k(z) \equiv 0$ for all $k \in \mathbb{N}$, is equivalent to the fact that all the coefficients of $g_k(z)$'s vanish. Ordering the coefficients of system (9) as $c_1, \ldots, c_M$ for some $M \in \mathbb{N}$, we get from the last claim that all coefficients of $g_k(z)$'s are polynomials in the variables $c_1, \ldots, c_M$. These polynomials form a countable subset, namely $\{B_k(c_1, \ldots, c_M)\}$, of $\mathbb{R}[c_1, \ldots, c_M]$. By the Hilbert basis theorem it follows that there exists a $K \in \mathbb{N}$ such that the algebraic variety $\bigcap_{i=1}^{\infty} \{B_j(c_1, \ldots, c_M) = 0\}$ is equal to the algebraic variety $\bigcap_{i=1}^{K} \{B_j(c_1, \ldots, c_M) = 0\}$.

The above fact shows that there exists an $N \in \mathbb{N}$ such that $B_1, \ldots, B_K$ appear in the coefficients of $\{g_1(z), \ldots, g_N(z)\}$. So if $g_k(z) \equiv 0$ for $k = 1, \ldots, N$, then

$$B_j(c_1, \ldots, c_M) = 0, \qquad j = 1, \ldots, K,$$

consequently

$$B_j(c_1, \ldots, c_M) = 0, \qquad j = 1, 2, \ldots$$

This implies that $g_k(z) \equiv 0$ for $k = 1, 2 \ldots$ Statement (c) follows. □

Proof of statement (d). The idea of this proof partially comes from [34]. If all $g_k(z)$ are identically zero, then a punctured neighborhood of the origin of system (9) is filled with periodic orbits. If system (9) is C^l completely integrable in a punctured neighborhood of the origin for some $l \in \mathbb{N} \cup \{\infty, \omega\}$, let $h_1(x, y, z), \ldots, h_{n-1}(x, y, z)$ be the functionally independent first integrals of system (9). Then a punctured neighborhood of the origin is foliated by these $n-1$ first integrals, that is each orbit of system (9) in such neighborhood is given by the intersection of the level hypersurfaces of these first integrals. Without loss of generality, we assume that $h_i(0) = 0$, $i = 1, \ldots, n-1$. Since the origin is an isolated singularity of system (9), it follows that the origin $O = (h_1, \ldots, h_{n-1})^{-1}(0, \ldots, 0)$, and consequently O is an isolated singularity of the function $H(x, y, z) = h_1^2(x, y, z) + \ldots + h_{n-1}^2(x, y, z)$. This implies that each level surface $H(x, y, z) = c$ for $c > 0$ sufficiently small is an $n-1$ dimensional invariant topological hypersurface. Since a punctured neighborhood of the origin of system (9) is filled with periodic orbits, we get that the hypersurface $H(x, y, z) = c$ is foliated by periodic orbits. Since any even dimensional sphere has Euler characteristic 2, it follows from the Poincaré–Hopf index theorem [32, p.35] that the vector field associated to system (9) restricted to $H = c$ must have singularities. We are in contradiction with the fact that the sphere $H = c$ is filled up with periodic orbits. This shows that system (9) cannot be C^l completely integrable for any $l \geq 1$. Statement (d) follows. □

This completes the proof of Theorem 2. □

3.2. **Proof of Example 1.** System (15) under the rescaling $x \to \varepsilon x$, $y \to \varepsilon y$, $z \to \varepsilon z$ and $w \to \varepsilon z$, is transformed into

$$\dot{x} = -y + \varepsilon x^2, \quad \dot{y} = x + \varepsilon xy, \quad \dot{z} = -w, \quad \dot{w} = z. \tag{39}$$

Doing the change of variables $x = r\cos\theta,\ y = r\sin\theta,\ z = ru,\ w = rv$, system (39) becomes

$$\dot r = \varepsilon r^2\cos\theta,\quad \dot\theta = 1,\quad \dot u = -v - \varepsilon ur\cos\theta,\quad \dot v = u - \varepsilon vr\cos\theta. \tag{40}$$

Choosing θ as the new independent variable, we can write this last differential system as

$$\frac{dR}{d\theta} = F_0(\theta,R) + \varepsilon F_1(\theta,R), \tag{41}$$

where $R = (r,u,v)$, and

$$F_0(\theta,R) = \begin{pmatrix} 0 \\ -v \\ u \end{pmatrix}, \qquad F_1(\theta,R) = \begin{pmatrix} r^2\cos\theta \\ -ur\cos\theta \\ -vr\cos\theta \end{pmatrix}.$$

Some easy calculations show that system

$$\frac{dR_0(\theta)}{d\theta} = F_0(\theta,R_0(\theta)), \tag{42}$$

satisfying the initial condition $R(0) = R_0 := (r_0,u_0,v_0)$, has the solution

$$R_0(\theta) = (r_0, u_0\cos\theta + v_0\sin\theta, -u_0\sin\theta + v_0\cos\theta).$$

The variational equation of system (42) along the solution $R_0(\theta)$ has the fundamental solution matrix

$$\Phi(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{pmatrix}$$

Let $R(\theta,R_0,\varepsilon)$ be the solution of system (41) satisfying the initial condition $R(0,R_0,\varepsilon) = R_0$. Since system (41) is analytic, we can expand $R(\theta,R_0,\varepsilon)$ in Taylor series

$$R(\theta,R_0,\varepsilon) = R_0(\theta,R_0) + \varepsilon R_1(\theta,R_0) + \varepsilon^2 R_2(\theta,R_0) + \dots$$

In what follows we denote $R_q(\theta, R_0) = (r_q, u_q, v_q)^T$ for $q = 1, 2, \ldots$. Substituting this expression in system (41) we get

$$\frac{dR_1}{d\theta} = JR_1 + \cos\theta \begin{pmatrix} r_0^2 \\ -v_2 \\ u_2 \end{pmatrix}, \qquad R_1(0) = 0 \tag{43}$$

$$\frac{dR_q}{d\theta} = JR_q + \cos\theta \begin{pmatrix} \sum\limits_{s=0}^{q-1} r_s r_{q-1-s} \\ -\sum\limits_{s=0}^{q-1} r_s u_{q-1-s} \\ -\sum\limits_{s=0}^{q-1} r_s v_{q-1-s} \end{pmatrix}, \quad R_q(0) = 0, \tag{44}$$

for $q = 2, 3, \ldots$, where J is the Jacobian matrix of $F_0(\theta, R_0)$ with respect to R_0.

Clearly system (43) has the solution

$$R_1(\theta, R_0) = \begin{pmatrix} r_0^2 \sin\theta \\ -r_0 \sin\theta(u_0\cos\theta + v_0 \sin\theta) \\ -r_0 \sin\theta(u_0 \sin\theta - v_0\cos\theta) \end{pmatrix}.$$

It follows clearly that $g_1(2\pi) = R_1(2\pi, R_0) = 0$. We claim that system (44) has the solution

$$R_q(\theta, R_0) = \begin{pmatrix} r_0^{q+1} \sin^q\theta \\ 0 \\ 0 \end{pmatrix}, \qquad q = 2, 3, \ldots.$$

Indeed, for $q = 2$ the proof follows from direct calculation. The fact that $g_0(2\pi, R_0) = R_0$, and $g_q(2\pi, R_0) = 0$ for $q = 1, 2, \ldots$ follows easily from some calculations and induction. This completes the proof of the example. $\square$

4. Proof of Theorem 3

As we did before the statement of Theorem 3 system (16) under the change of variables

$$\xi = x, \qquad \eta = y + P(x, y), \tag{45}$$

was transformed to system (17), i.e.

$$\dot{x} = y, \qquad \dot{y} = a_k x^k\,(1 + h(x)) + b_n x^n y\,(1 + g(x)) + y^2 p(x, y),$$

where we still use (x, y) instead of (ξ, η). Clearly a_k, b_n, and $h(x)$, $g(x)$, $p(x, y)$ are uniquely determined by the coefficients of system (16). We mention that the change (45) and its inverse are locally analytic. And we are in the case $k = 2m + 1$ for some $m \in \mathbb{N}$, $a_{2m+1} < 0$ and $n > m$. In order to simplify notation we set $a_{2m+1} = -a$ with $a > 0$ and $b_n = b$.

Rescaling the variables and time by $x \to \varepsilon x$, $y \to \varepsilon^{m+1} y$, $t \to \varepsilon^{-m} t$, system (17) can be written in

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ax^{2m+1}(1 + h(\varepsilon x)) + b\,\varepsilon^{n-m} x^n y(1 + g(\varepsilon x)) + \varepsilon y^2 p\,(\varepsilon x, \varepsilon^{m+1} y)\,. \end{aligned} \tag{46}$$

Let $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$ be the solutions of the Cauchy problem

$$\dot{x} = y, \qquad \dot{y} = -ax^{2m+1}, \tag{47}$$

satisfying the initial conditions $x(0) = ((m+1)/a)^{1/(2m+2)}$, $y(0) = 0$. Then we have

$$\frac{d\,\mathrm{Cs}\,\theta}{d\theta} = \mathrm{Sn}\,\theta, \quad \frac{d\,\mathrm{Sn}\,\theta}{d\theta} = -a\,\mathrm{Cs}^{2m+1}\,\theta. \tag{48}$$

Since

$$H(x, y) = \frac{a}{m+1} x^{2m+2} + y^2,$$

is a first integral of system (47), it follows easily that

$$a\,\mathrm{Cs}^{2m+2}\,\theta + (m+1)\,\mathrm{Sn}^2\,\theta = m + 1. \tag{49}$$

Following Lyapunov [30], see also [12], together with some calculations, we get that $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$ are periodic functions.

Take the generalized Lyapunov polar coordinate change of variables (18), i.e.

$$x = r(\theta)\,\mathrm{Cs}(\theta), \qquad y = r^{m+1}(\theta)\,\mathrm{Sn}(\theta),$$

we can write system (46) via (48) and (49) as

$$\begin{aligned} \dot{r} = &-\frac{a}{m+1} r^{m+1}\,\mathrm{Cs}^{2m+1}\,\theta\,\mathrm{Sn}\,\theta\, h(\varepsilon r\,\mathrm{Cs}\theta) \\ &+ \frac{b}{m+1} \varepsilon^{n-m} r^{n+1}\,\mathrm{Cs}^n\,\theta\,\mathrm{Sn}^2\,\theta\,(1 + g(\varepsilon r\,\mathrm{Cs}\,\theta)) \\ &+ \frac{\varepsilon}{m+1} r^{m+2}\,\mathrm{Sn}^3\,\theta\, p(\varepsilon r\,\mathrm{Cs}\,\theta, \varepsilon^{m+1} r^{m+1}\,\mathrm{Sn}\,\theta), \\ \dot{\theta} = r^m &+ \frac{a}{m+1} r^m\,\mathrm{Cs}^{2m+2}\,\theta\, h(\varepsilon r\,\mathrm{Cs}\theta) \\ &- \frac{b}{m+1} \varepsilon^{n-m} r^n\,\mathrm{Cs}^{n+1}\,\theta\,\mathrm{Sn}\,\theta\,(1 + g(\varepsilon r\,\mathrm{Cs}\,\theta)) \\ &- \frac{\varepsilon}{m+1} r^{m+1}\,\mathrm{Cn}\,\theta\,\mathrm{Sn}^2\,\theta\, p(\varepsilon r\,\mathrm{Cs}\,\theta, \varepsilon^{m+1} r^{m+1}\,\mathrm{Sn}\,\theta). \end{aligned}$$

Choosing θ as the independent variable this last equation can be written as in (19), i.e.

$$\frac{dr}{d\theta} = K_1(\theta, r)\varepsilon + K_2(\theta, r)\varepsilon^2 + \dots .$$

As in the proof of Theorem 1 we write

$$r(\theta, r_0, \varepsilon) = r_0 + \sum_{j=1}^{\infty} \varepsilon^j r_j(\theta, r_0),$$

the analytic solution of equation (19) satisfying the initial condition $r(0, r_0, \varepsilon) = r_0$. Then we have $r_0(\theta, r_0) = r_0$, and $r_j(\theta, r_0)$, $j = 1, 2, \dots$, were defined in (20) and (21).

Proof of statement (a). We note that each T_m periodic solution of equation (19) is uniquely determined by $r(T_m, r_0, \varepsilon) = r_0$, and $r(T_m, r_0, \varepsilon) - r_0 = \sum\limits_{j=1}^{\infty} \varepsilon^j g_k(r_0)$ with g_k defined in Theorem 3. So statement (a) follows from the Implicit Function theorem. □

Proof of statement (b). If all $g_k(r) \equiv 0$, then all orbits of equation (19) are T_m periodic, and consequently all orbits of system (16) in a punctured neighborhood of the origin are periodic. This shows that the origin of system (33) is a center. Since the origin is an isolated singularity of system (16), we get from Theorem 1.3 of [33] that system (16) has a C^∞ first integral defined in a neighborhood of the origin and it takes an isolated minimum at the origin. □

Proof of statement (c). From the construction of $g_k(r)$ and arguing as in the proof of statement (c) of Theorem 2, we get that all $g_k(r)$ are polynomials in r with coefficients being polynomials in the coefficients of system (16). Then working in a similar way as in the proof of statement (c) of Theorem 2, we can prove that there exists an $N \in \mathbb{N}$ such that $g_k(r) \equiv 0$ for all $k \in \mathbb{N}$ is equivalent to $g_k(r) \equiv 0$ for $k \in \{1, \dots, N\}$.

We complete the proof of Theorem 3. □

Remark 2. *Theorem 3 for system* (16) *can be extended to higher dimensional polynomial differential systems of the form*

$$\begin{aligned} \dot{x} &= y + P(x, y, z), \\ \dot{y} &= \qquad Q(x, y, z), \qquad (x, y, z) \in \mathbb{R}^n, \\ \dot{z} &= Az + R(x, y, z), \end{aligned} \tag{50}$$

where $z = (z_3, \dots, z_n)$*, and* P, Q, R *are polynomial functions without constant and linear terms. As in the two dimensional case we rewrite* *system* (50) *in the following form*

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = ax^k(1 + f(x, z)) + bx^l y(1 + g(x, z)) + y^2 h(x, y, z), \ (x, y, z) \in \mathbb{R}^n, \\ &\dot{z} = Az + r_0(x, z) + r_1(x, z)y + r_2(x, y, z)y^2, \end{aligned} \tag{51}$$

with f, g, h, r_0, r_1, r_2 *analytic. Assume that each entry of* A *is a parameter or* $A = 0$*, and that* $a < 0$*,* $k = 2m + 1$*,* $l > m$*,* $r_0(x, z) = O(|(x, z)|^{m+2})$ *and* $r_1(0, 0) = 0$*. After taking the rescaling*

$$x \to \varepsilon x, \ y \to \varepsilon^{m+1} y, \ z \to \varepsilon z, \ t \to \varepsilon^{-m} t, \ A \to \varepsilon^{m+1} A,$$

and using the methods and arguments in the proofs of Theorem 2 and Remark 1 we can obtain a result similar to Theorem 2 except for statement (*b*)*. Instead we have that for* $n \geq 2$ *if all the periodic orbits have trivial holonomy, then system* (51) *is locally* C^∞ *completely integrable in a punctured neighborhood of the origin. The details are omitted.*

Remark 3. *Theorem 3 for system* (16) *can also be extended to degenerate singularities. Consider the next system*

$$\dot{x} = y^{2l+1} + P(x, y), \quad \dot{y} = -ax^{2k+1} + Q(x, y), \tag{52}$$

where $a > 0$ *is a constant,* l *and* k *are positive integers, and* $P(x, y)$ *and* $Q(x, y)$ *are polynomials in* x, y *and do not identically vanish simultaneously. Assume that the lowest order terms of* P *and* Q *are of degree at least* $3\max\{k, l\}$ *if* $\max\{k, l\} \geq 2$ *or of degree at least* 4 *if* $k = l = 1$*. After rescaling*

$$x \to \varepsilon^{l+1} x, \ y \to \varepsilon^{k+1} y, \ t \to \varepsilon^{-k-l-2kl} t,$$

and taking the generalized Lyapunov polar coordinate changes

$$x = r^{l+1} \operatorname{Cs} \varphi, \qquad y = r^{k+1} \operatorname{Sn} \varphi,$$

with $\operatorname{Cs} \varphi$ *and* $\operatorname{Sn} \varphi$ *the solution of the initial value problem*

$$\dot{x} = y^{2l+1}, \quad \dot{y} = -ax^{2k+1}, \quad x(0) = ((k + 1)/a)^{1/(2k+2)}, \quad y(0) = 0,$$

i.e.

$$\frac{d\mathrm{Cs}\varphi}{d\varphi} = \mathrm{Sn}^{2l+1}\varphi, \quad \frac{d\mathrm{Sn}\varphi}{d\varphi} = -a\mathrm{Cs}^{2l+1}\varphi,$$

we can transform system (52) *into a periodic differential equation of the form* (19)*. Then using the methods and the arguments of the proof of Theorem 3 we can obtain a result similar to Theorem 3. This result can also be extended from system* (52) *to higher dimensional differential systems with some additional conditions as we did in Remark 2. The details are omitted.*

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