This is a preprint of: "A quasiperiodically forced skew-product on the cyclinder without fixed-curves", Lluís Alsedà, Francesc Mañosas, Leopoldo Morales, *Nonlinear Anal.*, vol. 145, 199–263, 2016.

A QUASIPERIODICALLY FORCED SKEW-PRODUCT ON THE CYLINDER WITHOUT FIXED-CURVES

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ABSTRACT. In [1] the Sharkovskiĭ Theorem was extended to periodic orbits of strips of quasiperiodic skew products in the cylinder.

In this paper we deal with the following natural question that arises in this setting: *Does Sharkovskiĭ Theorem holds when restricted to curves instead of general strips?*

We answer this question in the negative by constructing a counterexample: We construct a map having a periodic orbit of period 2 of curves (which is, in fact, the upper and lower circles of the cylinder) and without any invariant curve.

In particular this shows that there exist quasiperiodic skew products in the cylinder without invariant curves.

1. Introduction

We consider the coexistence and implications between periodic objects of maps on the cylinder $\Omega = \mathbb{S}^1 \times \mathbb{I}$, of the form:

$$F\colon \begin{pmatrix} \theta \\ x \end{pmatrix} \longrightarrow \begin{pmatrix} R_\omega(\theta) \\ \zeta(\theta, x) \end{pmatrix},$$

where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, $\mathbb{I}$ is an interval of the real line, $R_\omega(\theta) = \theta + \omega \pmod 1$ with $\omega \in \mathbb{R}\backslash\mathbb{Q}$ and $\zeta(\theta, x) = \zeta_\theta(x)$ is continuous on both variables. The class of all maps of the above type will be denoted by $\mathcal{S}(\Omega)$.

In this setting a very basic and natural question is the following: *is it true that any map in the class $\mathcal{S}(\Omega)$ has an invariant curve?*

In [1], the authors created an appropriate topological framework that allowed them to obtain the following extension of the Sharkovskiĭ Theorem to the class $\mathcal{S}(\Omega)$[1].

Let X be a compact metric space. We recall that a subset $G \subset X$ is *residual* if it contains the intersection of a countable family of open dense subsets in X.

In what follows, $\pi\colon \Omega \longrightarrow \mathbb{S}^1$ will denote the standard projection from Ω to the circle. Given a set $B \subset \mathbb{S}^1$, for convenience we will use the following notation:

$$\upuparrows B := \pi^{-1}(B) = B \times \mathbb{I} \subset \Omega$$

In the particular case when $B = \{\theta\}$, instead of $\upuparrows\{\theta\}$ we will simply write $\upuparrows\theta$. Also, given $A \subset \Omega$, we will denote by $A^{\upuparrows B}$ the set

$$A \cap \upuparrows B = \{(\theta, x) \in \Omega : \theta \in B \text{ and } (\theta, x) \in A\}.$$

Date: February 29, 2016.

1991 *Mathematics Subject Classification.* Primary: 37C55, 37C70.

Key words and phrases. Quasiperiodically forced systems on the cylinder, invariant strips.

The authors have been partially supported by MINECO grant numbers MTM2008-01486, MTM2011-26995-C02-01 and MTM2014-52209-C2-1-P.

[1]As already remarked in [1], instead of $\mathbb{S}^1$ we could take any compact metric space Θ that admits a minimal homeomorphism $R\colon \Theta \longrightarrow \Theta$ such that R^ℓ is minimal for every $\ell > 1$. However, for simplicity and clarity we will remain in the class $\mathcal{S}(\Omega)$.

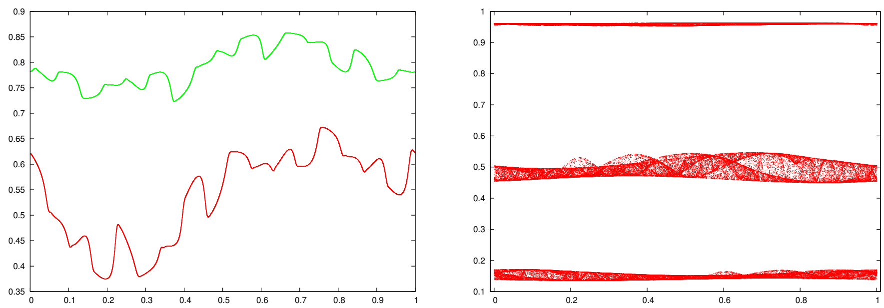
In the particular case when $B = \{\theta\}$, instead of $A^{\upuparrows \theta}$ we will simply write A^{θ}.

Instead of periodic points we use objects that project over the whole $\mathbb{S}^1$, called *strips* in [1, Definition 3.9]. A set $B \subset \Omega$ such that $\pi(B) = \mathbb{S}^1$ (i.e., B projects on the whole $\mathbb{S}^1$) will be called a *circular set.*

Definition 1.1. A *strip in* Ω is a compact circular set $B \subset \Omega$ such that B^{θ} is a closed interval (perhaps degenerate to a point) for every θ in a residual set of $\mathbb{S}^1$. ◘

Given two strips A and B, we will write $A < B$ and $A \leq B$ ([1, Definition 3.13]) if there exists a residual set $G \subset \mathbb{S}^1$, such that for every $(\theta, x) \in A^{\upuparrows G}$ and $(\theta, y) \in B^{\upuparrows G}$ it follows that $x < y$ and, respectively, $x \leq y$. We say that the strips A and B are *ordered* (respectively *weakly ordered*) if either $A < B$ or $A > B$ (respectively $A \leq B$ or $A \geq B$).

Definition 1.2 ([1, Definition 3.15])**.** A strip $B \subset \Omega$ is called *n-periodic* for $F \in \mathcal{S}(\Omega)$ if $F^n(B) = B$ and the image sets $B,\ F(B),\ F^2(B), \dots, F^{n-1}(B)$ are pairwise disjoint and pairwise ordered (see Figure 1 for examples). ◘



(A) $3.28x(1-x) + \frac{4}{100}\cos(2\pi\theta)$ (B) $3.85x(1-x)(1+\frac{111}{10^5}\cos(2\pi\theta))$

FIGURE 1. In the left picture we show an example two periodic orbit of curves, and in the second we show a possible example of a three periodic orbit solid strips.

To state the main theorem of [1] we need to recall the *Sharkovskiĭ Ordering* ([2, 3]). The *Sharkovskiĭ Ordering* is a linear ordering of $\mathbb{N}$ defined as follows:

$$\begin{aligned}
&3 \,{}_{\mathrm{Sh}}\!> 5 \,{}_{\mathrm{Sh}}\!> 7 \,{}_{\mathrm{Sh}}\!> 9 \,{}_{\mathrm{Sh}}\!> \cdots {}_{\mathrm{Sh}}\!> \\
&2 \cdot 3 \,{}_{\mathrm{Sh}}\!> 2 \cdot 5 \,{}_{\mathrm{Sh}}\!> 2 \cdot 7 \,{}_{\mathrm{Sh}}\!> 2 \cdot 9 \,{}_{\mathrm{Sh}}\!> \cdots {}_{\mathrm{Sh}}\!> \\
&4 \cdot 3 \,{}_{\mathrm{Sh}}\!> 4 \cdot 5 \,{}_{\mathrm{Sh}}\!> 4 \cdot 7 \,{}_{\mathrm{Sh}}\!> 4 \cdot 9 \,{}_{\mathrm{Sh}}\!> \cdots {}_{\mathrm{Sh}}\!> \\
&\qquad\qquad\vdots \\
&2^n \cdot 3 \,{}_{\mathrm{Sh}}\!> 2^n \cdot 5 \,{}_{\mathrm{Sh}}\!> 2^n \cdot 7 \,{}_{\mathrm{Sh}}\!> 2^n \cdot 9 \,{}_{\mathrm{Sh}}\!> \cdots {}_{\mathrm{Sh}}\!> \\
&\qquad\qquad\vdots \\
&\cdots {}_{\mathrm{Sh}}\!> 2^n \,{}_{\mathrm{Sh}}\!> \cdots {}_{\mathrm{Sh}}\!> 16 \,{}_{\mathrm{Sh}}\!> 8 \,{}_{\mathrm{Sh}}\!> 4 \,{}_{\mathrm{Sh}}\!> 2 \,{}_{\mathrm{Sh}}\!> 1.
\end{aligned}$$

In the ordering ${}_{\mathrm{Sh}}\!\geq$ the least element is 1 and the largest one is 3. The supremum of the set $\{1, 2, 4, \dots, 2^n, \dots\}$ does not exist.

Sharkovskiĭ Theorem for maps from $\mathcal{S}(\Omega)$ ([1])**.** *Assume that the map $F \in \mathcal{S}(\Omega)$ has a p-periodic strip. Then F has a q-periodic strip for every $q <_{\mathrm{Sh}} p$.*

In view of this result, the new following natural question (that is stronger that the previous one) arises: *Does Theorem 1 holds when restricted to curves?* where a curve is defined as the graph of a continuous map from $\mathbb{S}^1$ to $\mathbb{I}$. More precisely, *is it true that if F has a q-periodic curve and $p \leq_{\text{Sh}} q$ then does there exists a p-periodic curve of F?*

The aim of this paper is to answer both of the above questions in the negative by constructing a counterexample. This is done by the following result which is the main result of the paper.

Theorem A. *There exists a map $T \in \mathcal{S}(\Omega)$ with $f(\theta, \cdot)$ non-increasing for every $\theta \in \mathbb{S}^1$, such that T permutes the upper and lower circles of Ω (thus having a periodic orbit of period two of curves), and T does not have any invariant curve.*

The construction will be done in two steps. First, in Section 3, we construct a strip A which is a pseudo-curve which is not a curve. This strip is obtained as a *limit* of sets defined inductively by using of a collection of *winged boxes* $\mathcal{R}^{\frown}(i^*) \subset \Omega$. Second, we construct a Cauchy sequence $\{T_m\}_{m=0}^{\infty}$ that gives as a limit the function T from Theorem A having A as invariant set. To this end, in Section 4 we define a collection of auxiliary functions G_i defined on the winged boxes $\mathcal{R}^{\frown}(i^*)$. Next, in Section 5 we introduce a notion of *depth* in the set of winged boxes $\mathcal{R}^{\frown}(i^*)$ which defines a convenient stratification in the set of winged boxes $\mathcal{R}^{\frown}(i^*)$. In Section 6 we study the wings of box and its interaction with boxes of higher depth. In Section 7, by using the auxiliary functions from Section 4, the stratification from Section 5 and the technical results from Section 6 we construct the Cauchy sequence $\{T_m\}_{m=0}^{\infty} \subset \mathcal{S}(\Omega)$, we define the map $T = \lim_{m\to\infty} T_m$, and we prove Theorem A.

For clarity, we omit the proofs of all results from Section 7. These proofs will be provided in Sections 8, 9 and 10. Section 2 is devoted to introduce the necessary definitions and, in particular, to introduce the notion of pseudo-curve and some necessary results on the space of pseudo-curves.

2. Definitions and preliminary results

The main aim of this section is to introduce the definition and basic results about pseudo-curves.

Given $G \subset \mathbb{S}^1$ and a map $\varphi\colon G \longrightarrow \mathbb{I}$, $\mathsf{Graph}(\varphi)$ denotes the *graph of* φ. Also, given a set A we will denote the closure of A by $\overline{A}$.

Definition 2.1 (Pseudo-curve)**.** Let G be a residual set of $\mathbb{S}^1$ and let $\varphi\colon G \longrightarrow \mathbb{I}$ be a continuous map from G to $\mathbb{I}$. The set $\overline{\mathsf{Graph}(\varphi)}$, denoted by $\mathsf{A}_{(\varphi,G)}$, will be called a *pseudo-curve.* Notice that every pseudo-curve is a compact circular set.

Also, $\boldsymbol{\mathcal{A}}$ will denote the class of all pseudo-curves. ◘

A set $A \subset \Omega$ is F*-invariant* (respectively *strongly* F*-invariant*) if $F(A) \subset A$ (respectively $F(A) = A$). Observe that if $F \in \mathcal{S}(\Omega)$, every compact F-invariant set is circular. A closed invariant set is called *minimal* if it does not contain any proper closed invariant set.

An *arc of a curve* is the graph of a continuous function from an arc of $\mathbb{S}^1$ to $\mathbb{I}$.

The pseudo-curves have the following properties which are easy to prove:

Lemma 2.2. *Given a pseudo-curve $\mathsf{A}_{(\varphi,G)} \in \boldsymbol{\mathcal{A}}$ the following statements hold.*

(a) $\mathsf{A}^{\theta}_{(\varphi,G)}$ *consists of a single point for every $\theta \in G$. Consequently,*
$$\mathsf{A}^{\upuparrows G}_{(\varphi,G)} = \mathsf{Graph}(\varphi).$$

(b) Every circular compact set contained in a pseudo-curve coincides with the pseudo-curve.

(c) $\mathsf{A}_{(\varphi,G)} = \overline{\mathsf{Graph}(\varphi|_{\widetilde{G}})}$ *for every* $\widetilde{G} \subset G$ *dense in* $\mathbb{S}^1$.
(d) If $\mathsf{A}_{(\varphi,G)}$ *contains a curve then it is a curve.*

Proof. We start by proving (a). By the definition of a pseudo-curve we have $\mathsf{Graph}(\varphi) \subset \mathsf{A}^{\upuparrows G}_{(\varphi,G)}$. To prove the other inclusion fix $\theta \in G$ and $x \in \mathbb{I}$ such that $(\theta, x) \in \mathsf{A}_{(\varphi,G)}$. Then, there exists a sequence $\{(\theta_n, \varphi(\theta_n))\}_{n=1}^{\infty} \subset \mathsf{Graph}(\varphi)$ such that $\lim_{n\to\infty}(\theta_n, \varphi(\theta_n)) = (\theta, x)$. The continuity of φ in G (and hence in θ) implies $x = \varphi(\theta)$ and, therefore, $(\theta, x) \in \mathsf{Graph}(\varphi)$.

Now we prove (b). Assume that $B \subset \mathsf{A}_{(\varphi,G)}$ is a circular compact set. From the assumptions and statement (a) we get $\mathsf{A}^{\upuparrows G}_{(\varphi,G)} = B^{\upuparrows G}$. Hence,

$$\mathsf{A}_{(\varphi,G)} = \overline{\mathsf{Graph}(\varphi)} = \overline{\mathsf{A}^{\upuparrows G}_{(\varphi,G)}} = \overline{B^{\upuparrows G}} \subset B.$$

Now (d) follows directly from (b) and the fact that a curve is compact since it is the graph of a continuous function. Statement (c) also follows from (b) because $\overline{\mathsf{Graph}\left(\varphi|_{\widetilde{G}}\right)} \subset \mathsf{A}_{(\varphi,G)}$ and $\overline{\mathsf{Graph}\left(\varphi|_{\widetilde{G}}\right)}$ is a circular set (since $\widetilde{G}$ is dense in $\mathbb{S}^1$). □

We also will be interested in the pseudo-curves as a possible invariant objects of maps from $\mathcal{S}(\Omega)$. The next lemma studies their properties in this case.

Lemma 2.3. *Let* $F \in \mathcal{S}(\Omega)$ *and assume that* $\mathsf{A}_{(\varphi,G)} \in \boldsymbol{\mathcal{A}}$ *is an* F*-invariant pseudo-curve. Then,*
(a) $\mathsf{A}_{(\varphi,G)}$ *is strongly* F*-invariant and minimal.*
(b) If $\mathsf{A}_{(\varphi,G)}$ *contains an arc of a curve then it is a curve.*

Proof. We start by proving (a). Let $B \subset \mathsf{A}_{(\varphi,G)}$ be a closed invariant set. We have that B is circular and, by Lemma 2.2(b), $B = \mathsf{A}_{(\varphi,G)}$. Hence, $\mathsf{A}_{(\varphi,G)}$ is minimal.

On the other hand, $F(\mathsf{A}_{(\varphi,G)}) \subset \mathsf{A}_{(\varphi,G)}$ implies $F^2(\mathsf{A}_{(\varphi,G)}) \subset F(\mathsf{A}_{(\varphi,G)})$ and, hence, $F(\mathsf{A}_{(\varphi,G)})$ is a compact F-invariant set. Therefore, by the part already proven, $F(\mathsf{A}_{(\varphi,G)}) = \mathsf{A}_{(\varphi,G)}$.

Now we prove (b). Let S be an (open) arc of $\mathbb{S}^1$ and let $\xi \colon S \longrightarrow \mathbb{I}$ be a continuous map such that $\mathsf{Graph}(\xi) \subset \mathsf{A}_{(\varphi,G)}$. Clearly, there exists $m \in \mathbb{N}$ such that $\bigcup_{i=0}^{m} R_\omega^i(S) = \mathbb{S}^1$. Now we set $\xi_0 := \xi$ and, for $i = 1, 2, \ldots, m$, we define $\xi_i \colon R_\omega^i(S) \longrightarrow \mathbb{I}$ by

$$\xi_i(\theta) := f\left(R_\omega^{-1}(\theta), \xi_{i-1}\left(R_\omega^{-1}(\theta)\right)\right).$$

The continuity of f implies that every ξ_i is an arc of a curve and $\mathsf{Graph}(\xi_i) = F(\mathsf{Graph}(\xi_{i-1}))$. Hence,

$$\bigcup_{i=0}^{m} \mathsf{Graph}(\xi_i) = \bigcup_{i=0}^{m} F^i(\mathsf{Graph}(\xi)) \subset \mathsf{A}_{(\varphi,G)}$$

because $\mathsf{A}_{(\varphi,G)}$ is F-invariant.

In view of Lemma 2.2(d) we only have to show that $\bigcup_{i=0}^{m} \mathsf{Graph}(\xi_i)$ is a curve. We will prove prove this by induction.

Assume that $\emptyset \neq M \subsetneq \{0, 1, 2, \ldots, m\}$ verifies that $S_M := \bigcup_{i\in M} R_\omega^i(S)$ is an (open) arc of $\mathbb{S}^1$ and $\bigcup_{i\in M} \mathsf{Graph}(\xi_i)$ is an arc of a curve (initially we can take M to be any unitary subset of $\{0, 1, 2, \ldots, m\}$). Then, there exists a continuous map $\xi_{_M} : S_M \longrightarrow \mathbb{I}$ such that $\mathsf{Graph}(\xi_{_M}) = \bigcup_{i\in M} \mathsf{Graph}(\xi_i)$.

Clearly, there exists $j \in \{0, 1, 2, \ldots, m\}\backslash M$ such that $S_{M,j} := S_M \cap R_\omega^j(S) \neq \emptyset$. The set $S_{M,j}$ is an open arc of $\mathbb{S}^1$ and, by Lemma 2.2(a), $\xi_{_M}\big|_{S_{M,j}\cap G} = \xi_j\big|_{S_{M,j}\cap G}$ because $\mathsf{Graph}(\xi_{_M}), \mathsf{Graph}(\xi_j) \subset \mathsf{A}_{(\varphi,G)}$. Since $S_{M,j} \cap G$ is dense in $S_{M,j}$, given $\theta \in S_{M,j}\backslash G$, there exists a sequence $\{\theta_n\}_{n=0}^{\infty} \subset S_{M,j} \cap G$ converging to θ. The continuity of $\xi_{_M}$ and ξ_j on $S_{M,j}$ implies that $\xi_{_M}(\theta) = \lim_{n\to\infty} \xi_{_M}(\theta_n) = \lim_{n\to\infty} \xi_j(\theta_n) = \xi_j(\theta)$. Consequently, $\xi_{_M}\big|_{S_{M,j}} = \xi_j\big|_{S_{M,j}}$ and $\mathsf{Graph}(\xi_{_M}) \cup \mathsf{Graph}(\xi_j)$ is an arc of a curve (defined on the open arc $S_M \cup R_\omega^j(S)$). By redefining M as $M \cup \{j\}$ and iterating this procedure until $M \cup \{j\} = \{0, 1, 2, \dots, m\}$ we see that the whole $\bigcup_{i=0}^m \mathsf{Graph}(\xi_i)$ is a curve. $\square$

Next we will introduce and study the space of pseudo-curves.

Definition 2.4. We define the *space of pseudo-curve generators* as

$$\mathcal{PCG} := \{(\varphi, G) : G \text{ is a residual set in } \mathbb{S}^1 \text{ and } \varphi \colon G \longrightarrow \mathbb{I} \text{ is a continuous map}\}.$$

On $\mathcal{PCG}$ we also define the *supremum pseudo-metric* $\mathsf{d}_\infty : \mathcal{PCG} \times \mathcal{PCG} \longrightarrow \mathbb{R}^+$ by:

$$\mathsf{d}_\infty\big((\varphi, G), (\varphi', G')\big) := \sup_{\theta \in G \cap G'} |\varphi(\theta) - \varphi'(\theta)|\,.$$

Clearly, $\mathsf{d}_\infty((\varphi, G), (\varphi', G')) = 0$ if and only if $\varphi\big|_{G\cap G'} = \varphi'\big|_{G\cap G'}$ and, hence, d_∞ is a pseudo-metric. ◘

The next lemma will be useful in using the metric d_∞.

Lemma 2.5. *Let $(\varphi, G), (\varphi', G') \in \mathcal{PCG}$. Then,*

$$\mathsf{d}_\infty\big((\varphi, G), (\varphi', G')\big) = \sup_{\theta \in \widetilde{G}} |\varphi(\theta) - \varphi'(\theta)|$$

for every $\widetilde{G} \subset G \cap G'$ dense in $\mathbb{S}^1$.

Proof. Set $\mathsf{d}_{\infty,\widetilde{G}}\big((\varphi, G), (\varphi', G')\big) := \sup_{\theta\in\widetilde{G}} |\varphi(\theta) - \varphi'(\theta)|$. With this notation, we clearly have $\mathsf{d}_{\infty,\widetilde{G}}\big((\varphi, G), (\varphi', G')\big) \le \mathsf{d}_\infty\big((\varphi, G), (\varphi', G')\big)$.

To prove the reverse inequality take $\theta \in (G \cap G')\backslash\widetilde{G}$. Since $\widetilde{G}$ is dense in $\mathbb{S}^1$, there exists a sequence $\{\theta_n\}_{n=0}^\infty \subset \widetilde{G}$ converging to θ. On the other hand, by definition, the maps φ and φ', are continuous in $G \cap G'$ (and, hence, in θ). Consequently, $|\varphi(\theta), \varphi'(\theta)| = \lim_{n\to\infty} |\varphi(\theta_n) - \varphi'(\theta_n)| \le \mathsf{d}_{\infty,\widetilde{G}}\big((\varphi, G), (\varphi', G')\big)$. This ends the proof of the lemma. $\square$

As it is customary we will introduce an equivalent relation in the space of pseudo-curve generators so that the quotient space will be a metric space.

Definition 2.6. Two pseudo-curve generators $(\varphi, G), (\varphi', G') \in \mathcal{PCG}$ are said to be equivalent, denoted by $(\varphi, G) \sim (\varphi', G')$ if and only if $\mathsf{A}_{(\varphi,G)} = \mathsf{A}_{(\varphi',G')}$. Clearly $\sim$ is an equivalence relation in $\mathcal{PCG}$. The $\sim$-equivalence class of $(\varphi, G) \in \mathcal{PCG}$ will be denoted by $[\varphi, G]$. ◘

Remark 2.7. From Lemma 2.2(a,c) it follows that $(\varphi, G) \sim (\varphi', G')$ if and only if $\varphi\big|_{\widetilde{G}} = \varphi'\big|_{\widetilde{G}}$ for every $\widetilde{G} \subset G \cap G'$ dense in $\mathbb{S}^1$. In particular, by taking $\widetilde{G} = G \cap G'$, we get that $\mathsf{d}_\infty((\varphi, G), (\varphi', G')) = 0$ if and only if $(\varphi, G) \sim (\varphi', G')$. ◘

Definition 2.8. The space $\mathcal{PCG}/\!\sim$ will be called the *space of pseudo-curves generator classes* and denoted by $\mathcal{PC}$. Also, on $\mathcal{PC}$ we define the *supremum metric*, also denoted $\mathsf{d}_\infty : \mathcal{PC} \times \mathcal{PC} \longrightarrow \mathbb{R}^+$ by abuse of notation, in the following way. Given $A = [\varphi_A, G_A], B = [\varphi_B, G_B] \in \mathcal{PC}$ we set

$$\mathsf{d}_\infty(A, B) := \mathsf{d}_\infty\big((\varphi_A, G_A), (\varphi_B, G_B)\big).$$

Note that d_∞ is well defined. To see this take $[\varphi_A, G_A] = [\varphi_{'A}, G_{A'}], [\varphi_B, G_B] \in \mathcal{PCG}$. Then, by Lemma 2.5 and Remark 2.7 applied to $\widetilde{G} = G_A \cap G_{A'} \cap G_B$ we get $\mathsf{d}_\infty\big((\varphi_A, G_A), (\varphi_B, G_B)\big) = \mathsf{d}_\infty\big((\varphi_{A'}, G_{A'}), (\varphi_B, G_B)\big)$. ◘

The next result establishes the basic properties of the space of pseudo-curves generator classes $(\mathcal{PC}, \mathsf{d}_\infty)$.

Proposition 2.9. *The space of pseudo-curves generator classes $\mathcal{PC}$ is a complete metric space.*

Proof. The fact that d_∞ is a metric in $\mathcal{PC}$ follows from Remark 2.7.

Now we prove that $\mathcal{PC}$ is complete. Assume that $\{[\varphi_n, G_n]\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{PC}$. We have to see that $\lim_{n\to\infty}[\varphi_n, G_n] \in \mathcal{PC}$.

Set, $G := \cap_{i=1}^\infty G_n$. Since this intersection is countable, G is still a residual set. The definition of d_∞ implies that the sequence $\{\varphi_n(\theta)\}_{n=1}^\infty \subset \mathbb{I}$ is a Cauchy sequence in $\mathbb{I}$ for every $\theta \in G$. So, it is convergent and we can define a map $\varphi\colon G \longrightarrow \mathbb{I}$ by $\varphi(\theta) := \lim_{n\to\infty} \varphi_n(\theta)$.

If $(\varphi, G) \in \mathcal{PCG}$ we have $[\varphi, G] \in \mathcal{PC}$ and, from the definition of φ it follows that

$$\lim_{n\to\infty} \mathsf{d}_\infty([\varphi, G], [\varphi_n, G_n]) = \sup_{\theta\in G\cap G_n} \lim_{n\to\infty} |\varphi(\theta) - \varphi_n(\theta)| = 0.$$

Consequently, $[\varphi, G] = \lim_{n\to\infty}[\varphi_n, G_n]$. Since φ is the uniform limit of a sequence of continuous functions on G, it is continuous on G. That is, $(\varphi, G) \in \mathcal{PCG}$. □

In what follows we want to look at the space $\boldsymbol{\mathcal{A}}$ as a metric space and relate this metric space with $(\mathcal{PC}, \mathsf{d}_\infty)$.

Let ρ denote the euclidean metric in Ω. Then, the space (Ω, ρ) is a compact metric space. We recall that the *Hausdorff metric* is defined in the space of compact subsets of (Ω, ρ), by

$$H_\rho(\mathsf{A}, \mathsf{B}) = \max\left\{\max_{(\theta,x)\in\mathsf{A}} \rho((\theta, x), \mathsf{B}), \max_{(\theta,x)\in\mathsf{B}} \rho((\theta, x), \mathsf{A})\right\}.$$

Then, $(\boldsymbol{\mathcal{A}}, H_\rho)$ is a metric space. To study the relation between $(\mathcal{PC}, \mathsf{d}_\infty)$ and $(\boldsymbol{\mathcal{A}}, H_\rho)$ we need a couple of simple technical results.

Lemma 2.10. *Let $\mathsf{A}, \mathsf{B} \subset \Omega$ be compact circular sets. Then,*

$$H_\rho(\mathsf{A}, \mathsf{B}) \le \max_{\theta\in\mathbb{S}^1} H_\rho\big(\mathsf{A}^\theta, \mathsf{B}^\theta\big).$$

Proof. It follows directly from the definitions:

$$\begin{aligned} H_\rho(\mathsf{A}, \mathsf{B}) &\le \max\left\{\sup_{(\theta,x)\in\mathsf{A}} \rho\Big((\theta, x), \mathsf{B}^\theta\Big), \sup_{(\theta,x)\in\mathsf{B}} \rho\Big((\theta, x), \mathsf{A}^\theta\Big)\right\} \\ &= \max\left\{\sup_{\theta\in\mathbb{S}^1} \max_{\{x\in\mathbb{I}\,:\,(\theta,x)\in\mathsf{A}\}} \rho\Big((\theta, x), \mathsf{B}^\theta\Big),\right. \\ &\qquad\qquad\qquad\left.\sup_{\theta\in\mathbb{S}^1} \max_{\{x\in\mathbb{I}\,:\,(\theta,x)\in\mathsf{B}\}} \rho\Big((\theta, x), \mathsf{A}^\theta\Big)\right\} \\ &= \sup_{\theta\in\mathbb{S}^1} \max\left\{\max_{\{x\in\mathbb{I}\,:\,(\theta,x)\in\mathsf{A}\}} \rho\Big((\theta, x), \mathsf{B}^\theta\Big), \max_{\{x\in\mathbb{I}\,:\,(\theta,x)\in\mathsf{B}\}} \rho\Big((\theta, x), \mathsf{A}^\theta\Big)\right\} \\ &= \sup_{\theta\in\mathbb{S}^1} H_\rho\left(\mathsf{A}^\theta, \mathsf{B}^\theta\right). \end{aligned}$$

□

Proposition 2.11. *Let $(\varphi, G), (\widetilde{\varphi}, \widetilde{G}) \in \mathcal{PCG}$. Then,*

$$H_\rho\left(\mathsf{A}_{(\varphi,G)}, \mathsf{A}_{(\widetilde{\varphi},\widetilde{G})}\right) \le \sup_{\theta\in\mathbb{S}^1} H_\rho\left(\mathsf{A}^\theta_{(\varphi,G)}, \mathsf{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}\right) = \mathsf{d}_\infty\big((\varphi, G), (\widetilde{\varphi}, \widetilde{G})\big).$$

Proof. The first inequality follows from Lemma 2.10.

Now we prove the second equality. By Lemma 2.2(a),

$$\mathsf{d}_\infty\big((\varphi, G), (\widetilde{\varphi}, \widetilde{G})\big) = \sup_{\theta\in G\cap\widetilde{G}} |\varphi(\theta) - \widetilde{\varphi}(\theta)| = \sup_{\theta\in G\cap\widetilde{G}} H_\rho\left(\mathsf{A}^\theta_{(\varphi,G)}, \mathsf{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}\right).$$

So, to end the proof of the lemma, we have to see that

$$H_\rho\left(\mathsf{A}^\theta_{(\varphi,G)}, \mathsf{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}\right) \le \mathsf{d}_\infty\big((\varphi,G),(\widetilde{\varphi},\widetilde{G})\big) \quad \text{for every} \quad \theta \in \mathbb{S}^1 \backslash (G \cap \widetilde{G}).$$

Fix $\theta \in \mathbb{S}^1 \backslash (G \cap \widetilde{G})$. From the definition of the Hausdorff metric it follows that there exist $x, y \in \mathbb{I}$ such that $H_\rho\left(\mathsf{A}^\theta_{(\varphi,G)}, \mathsf{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}\right) = |x-y|$, $(\theta,x) \in \mathsf{A}^\theta_{(\varphi,G)}$, and $(\theta,y) \in \mathsf{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}$.

Since $G \cap \widetilde{G}$ is residual (and thus dense) in $\mathbb{S}^1$, from Lemma 2.2(a,c) it follows that there exists sequences $\{(\theta_n, \varphi(\theta_n))\}_{n=0}^\infty$, $\{(\theta_n, \widetilde{\varphi}(\theta_n))\}_{n=0}^\infty \subset \upuparrows(G \cap \widetilde{G})$ such that $\lim_{n\to\infty}(\theta_n, \varphi(\theta_n)) = (\theta, x)$ and $\lim_{n\to\infty}(\theta_n, \widetilde{\varphi}(\theta_n)) = (\theta, y)$. Hence,

$$H_\rho\left(\mathsf{A}^\theta_{(\varphi,G)}, \mathsf{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}\right) = |x-y| = \lim_{n\to\infty} |\varphi(\theta_n) - \widetilde{\varphi}(\theta_n)| \le \mathsf{d}_\infty\big((\varphi,G),(\widetilde{\varphi},\widetilde{G})\big).$$

□

Proposition 2.11 tells us that that if $\{[\varphi_n, G_n]\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{PC}$ then $\mathsf{A}_{(\varphi_n,G_n)}$ is a Cauchy sequence in $(\boldsymbol{\mathcal{A}}, H_\rho)$, and if $[\varphi, G] = \lim_{n\to\infty}[\varphi_n, G_n]$ then $\mathsf{A}_{(\varphi,G)} = \lim_{n\to\infty} \mathsf{A}_{(\varphi_n,G_n)}$. Unfortunately the space $(\boldsymbol{\mathcal{A}}, H_\rho)$ is not complete as the following simple example shows.

Example 2.12 (The space $(\boldsymbol{\mathcal{A}}, H_\rho)$ is not complete)**.** Consider continuous maps $\xi_n \colon \mathbb{S}^1 \longrightarrow \mathbb{I}$ with $n \in \mathbb{N}$, $n \ge 2$, defined by

$$\xi_n(\theta) = \begin{cases} 2n\theta & \text{if } \theta \in [0, \frac{1}{2n}], \\ 2(1-n\theta) & \text{if } \theta \in [\frac{1}{2n}, \frac{1}{n}], \\ 0 & \text{if } \theta \ge \frac{1}{n}. \end{cases}$$

Clearly, $(\xi_n, \mathbb{S}^1) \in \mathcal{PCG}$ and $H_\rho(\mathsf{A}_{(\xi_n,\mathbb{S}^1)}, \mathsf{A}_{(\xi_m,\mathbb{S}^1)}) \le \frac{1}{\min\{n,m\}}$. Hence, $\{\mathsf{A}_{(\xi_n,\mathbb{S}^1)}\}$ is a Cauchy sequence in $\boldsymbol{\mathcal{A}}$. However, the sequence $\{\mathsf{A}_{(\xi_n,\mathbb{S}^1)}\}$ has no limit in $\boldsymbol{\mathcal{A}}$. Indeed, $\lim_{n\to\infty} \mathsf{A}_{(\xi_n,\mathbb{S}^1)} = \mathsf{L} = (\mathbb{S}^1 \times \{0\}) \cup (\{0\} \times [0,1])$, which is not the closure of the graph of a continuous map on a residual set of $\mathbb{S}^1$ (in other words, $\mathsf{L} \notin \boldsymbol{\mathcal{A}}$). This is consistent with the fact that, clearly, $\{[\xi_n, \mathbb{S}^1]\}$ is not a Cauchy sequence in $(\mathcal{PC}, \mathsf{d}_\infty)$. ◘

3. Construction of a connected pseudo-curve

The aim of this subsection is to construct a strip $\mathsf{A} = \mathsf{A}_{(\gamma,G)}$ as a connected pseudo-curve with certain topological properties that will allow us to define the map $T \in \mathcal{S}(\Omega)$ having this pseudo-curve as the only proper invariant object. The pseudo-curve $\mathsf{A}_{(\gamma,G)}$ will be obtained as a limit in $\mathcal{PC}$ of a sequence of pseudo-curves that will be constructed recursively.

We will start by introducing the necessary notation.

In what follows, for simplicity, we will take the interval $\mathbb{I}$ as the interval $[-2, 2]$. Also, fix $\omega \in [0,1]\backslash\mathbb{Q}$. For any $\ell \in \mathbb{Z}$ set $\ell^* = \ell\omega \pmod 1$ and $O^*(\omega) = \{\ell^* : \ell \in \mathbb{Z}\}$. That is, $O^*(\omega)$ is the orbit of 0 by the rotation of angle ω.

We will denote by $\mathsf{d}_{\mathbb{S}^1}$ the arc distance on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. That is, for $\theta_1, \theta_2 \in \mathbb{S}^1$, we set

$$\mathsf{d}_{\mathbb{S}^1}(\theta_1, \theta_2) := \begin{cases} \theta_2 - \theta_1 & \text{when } \theta_1 \le \theta_2, \text{ and} \\ (\theta_2 + 1) - \theta_1 & \text{when } \theta_1 > \theta_2. \end{cases}$$

The closed arc of $\mathbb{S}^1$ joining θ_1 and θ_2 in the natural direction will be denoted by $[\theta_1, \theta_2]$. That is,

$$[\theta_1, \theta_2] = \begin{cases} \{t \pmod 1 : \theta_1 \le t \le \theta_2\} & \text{when } \theta_1 \le \theta_2, \text{ and} \\ \{t \pmod 1 : \theta_1 \le t \le \theta_2 + 1\} & \text{when } \theta_1 > \theta_2. \end{cases}$$

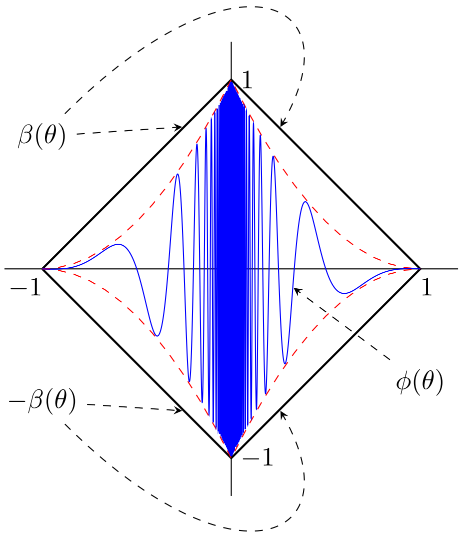


FIGURE 2. The graphs of the functions ϕ (in blue) and $\pm\beta$ in thick black. The red dashed curve is $(1-|x|)^2$.

The open arc of $\mathbb{S}^1$ joining θ_1 and θ_2 will be denoted by $(\theta_1, \theta_2) = [\theta_1, \theta_2] \backslash \{\theta_1, \theta_2\}$, and is defined analogously with strict inequalities Given an arc $B \subset \mathbb{S}^1$, $\mathsf{Bd}(B)$ will denote the set of endpoints of B.

We will denote the open (respectively closed) ball (in $\mathbb{S}^1$) of radius δ centred at $\theta \in \mathbb{S}^1$ by $B_\delta(\theta)$ (respectively $B_\delta[\theta]$):

$$B_\delta(\theta) = \{\widetilde{\theta} \in \mathbb{S}^1 : \mathsf{d}_{\mathbb{S}^1}(\theta, \widetilde{\theta}) < \delta\} = (\theta - \delta \pmod 1, \theta + \delta \pmod 1), \text{ and}$$
$$B_\delta[\theta] = \overline{B_\delta(\theta)} = \{\widetilde{\theta} \in \mathbb{S}^1 : \mathsf{d}_{\mathbb{S}^1}(\theta, \widetilde{\theta}) \le \delta\} = [\theta - \delta \pmod 1, \theta + \delta \pmod 1].$$

We consider the space Ω endowed the metric induced by the maximum of $\mathsf{d}_{\mathbb{S}^1}$ and the absolute value on $\mathbb{I}$. That is, given $(\theta, x), (\nu, y) \in \Omega$ we set

$$\mathsf{d}_\Omega((\theta, x), (\nu, y)) := \max\left\{\mathsf{d}_{\mathbb{S}^1}(\theta, \nu), |x - y|\right\}.$$

Then, given $A \subset \Omega$ we will denote the *interior of* A by $\mathsf{Int}(A)$ and $\mathsf{diam}(A)$ will denote the *diameter of* A whenever A is compact.

To define the sequence of pseudo-curves that will converge to $\mathsf{A}_{(\gamma,G)}$ we first need to construct an auxiliary family $\{\mathcal{R}(\ell^*)\}_{\ell\in\mathbb{Z}}$ of compact regions in Ω and a family of compact sets $\{\Gamma\varphi_{\ell^*}\}_{\ell\in\mathbb{Z}}$ such that, for every $\ell \in \mathbb{Z}$, $\Gamma\varphi_{\ell^*} \subset \mathcal{R}(\ell^*)$ and it is the restriction of a pseudo-curve generator to $\pi(\mathcal{R}(\ell^*))$. To do this we define the auxiliary functions $\beta\colon [-1,1] \longrightarrow [-1,1]$ and $\phi\colon [-1,1]\backslash\{0\} \longrightarrow [-1,1]$ by (see Figure 2):

$$\beta(x) := 1 - |x| \qquad \text{and} \qquad \phi(x) := (1-|x|)^2 \sin\left(\frac{\pi}{x}\right).$$

Note that $-\beta(x) < \phi(x) < \beta(x)$, for all $x \in [-1,1]\backslash\{0\}$ and the graphs of $-\beta$ and β intersect the closure of the graph of ϕ only at the points $(0,-1)$, $(0,1)$, $(-1,0)$ and $(1,0)$.

To define the families $\{\mathcal{R}(\ell^*)\}_{\ell\in\mathbb{Z}}$ and $\{\Gamma\varphi_{\ell^*}\}_{\ell\in\mathbb{Z}}$ we use the following *generic boxes.*

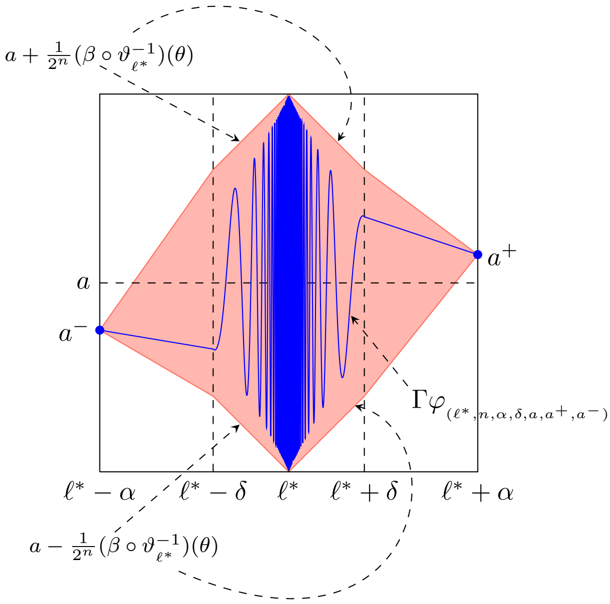


FIGURE 3. The region $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ is the colored filled area, delimited in the rectangle $\upuparrows B_\delta[\ell^*]$ by the graphs of the functions $a \pm \frac{1}{2^n}(\beta \circ \vartheta_{\ell^*}^{-1})(\theta)$. In blue the set $\Gamma\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)}$ inductively defining the pseudo-curve.

For every $\theta \in \mathbb{S}^1$ and $\delta < \frac{1}{2}$, $\vartheta_\theta : [-\delta, \delta] \longrightarrow \mathbb{S}^1$ denotes the map defined by $\vartheta_\theta(x) = x + \theta \pmod 1$. Clearly ϑ_θ is a homeomorphism between $[-\delta, \delta]$ and $B_\delta[\theta]$. Finally $\vartheta_\theta^{-1} : B_\delta[\theta] \longrightarrow [-\delta, \delta]$ denotes the inverse homeomorphism of ϑ_θ.

Definition 3.1 (Generic boxes)**.** Fix $\ell, n \in \mathbb{Z}$, $n \geq |\ell|$, $\alpha \in (0, 2^{-n})$, $\delta \in (0, \alpha)$, $a \in [-1, 1]$ and $a^+, a^- \in B_a(2^{-n}\beta(\delta))$ (see Figure 3). Now we consider the Jordan closed curve in Ω, formed by the graphs of the functions

$$a + 2^{-n}(\beta \circ \vartheta_{\ell^*}^{-1})\big|_{B_\delta[\ell^*]} \quad \text{and} \quad a - 2^{-n}(\beta \circ \vartheta_{\ell^*}^{-1})\big|_{B_\delta[\ell^*]},$$

together with the four segments that join the points:

$$\begin{aligned} &(\ell^* - \alpha, a^-) \text{ with } \big(\ell^* - \delta, a - 2^{-n}\beta(-\delta)\big), \\ &(\ell^* - \alpha, a^-) \text{ with } \big(\ell^* - \delta, a + 2^{-n}\beta(-\delta)\big), \\ &(\ell^* + \alpha, a^+) \text{ with } \big(\ell^* + \delta, a - 2^{-n}\beta(\delta)\big), \text{ and} \\ &(\ell^* + \alpha, a^+) \text{ with } \big(\ell^* + \delta, a + 2^{-n}\beta(\delta)\big). \end{aligned}$$

We denote the closure of the connected component of the complement of the above Jordan curve in Ω that contains the point (ℓ^*, a) by $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ (the coloured region in Figure 3). Observe that $\pi\left(\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)\right)$, the projection of $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ to $\mathbb{S}^1$, is $B_\alpha[\ell^*] = [\ell^* - \alpha, \ell^* + \alpha]$.

We denote by

$$\varphi_{\ell^*} = \varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)} : B_\alpha[\ell^*] \setminus \{\ell^*\} \longrightarrow \mathbb{I}$$

the continuous map defined as follows:

(i) $\varphi_{\ell^*}\big|_{B_\delta[\ell^*]\setminus\{\ell^*\}} = a + (-1)^\ell 2^{-n}(\phi \circ \vartheta_{\ell^*}^{-1})$.
(ii) $\varphi_{\ell^*}(\ell^* - \alpha) = a^-$ and $\varphi_{\ell^*}(\ell^* + \alpha) = a^+$.
(iii) $\varphi_{\ell^*}\big|_{[\ell^*-\alpha,\ell^*-\delta]}$ and $\varphi_{\ell^*}\big|_{[\ell^*+\delta,\ell^*+\alpha]}$ are affine.

We also denote by $\Gamma\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)} \subset \mathcal{R}(\ell^*,n,\alpha,\delta,a,a^+,a^-)$ the closure in Ω of the graph of $\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)}$. ◘

Remark 3.2. The region $\mathcal{R}(\ell^*,n,\alpha,\delta,a,a^+,a^-)$ and the set $\Gamma\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)}$ satisfy the following properties:

(1) $\mathcal{R}(\ell^*,n,\alpha,\delta,a,a^+,a^-) \subset B_\alpha[\ell^*] \times [a-2^{-n}, a+2^{-n}]$.
(2) $\mathsf{diam}(\mathcal{R}(\ell^*,n,\alpha,\delta,a,a^+,a^-)) = \mathsf{diam}(\mathcal{R}(\ell^*,n,\alpha,\delta,a,a^+,a^-)^{\ell^*}) = 2 \cdot 2^{-n}$.
(3) The sets $\Gamma\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)}$ and $\partial\mathcal{R}(\ell^*,n,\alpha,\delta,a,a^+,a^-)$ only intersect at the points $(\ell^*, a-2^{-n}), (\ell^*, a+2^{-n})$, $(\ell^*-\alpha, a^-)$ and $(\ell^*+\alpha, a^+)$.
(4) $\left(\Gamma\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)}\right)^{\ell^*} = \mathcal{R}(\ell^*,n,\alpha,\delta,a,a^+,a^-)^{\ell^*}$ is an interval.
(5) Let $\mathcal{R}(\ell^*,n,\alpha,\delta,a,a^+,a^-)$ and $\mathcal{R}(k^*,\widetilde{n},\widetilde{\alpha},\widetilde{\delta},\widetilde{a},\widetilde{a}^+,\widetilde{a}^-)$ be two regions, then $B_\alpha[\ell^*] \cap B_{\widetilde{\alpha}}[k^*] = \emptyset$ implies
$$\mathcal{R}(\ell^*,n,\alpha,\delta,a,a^+,a^-) \cap \mathcal{R}(k^*,\widetilde{n},\widetilde{\alpha},\widetilde{\delta},\widetilde{a},\widetilde{a}^+,\widetilde{a}^-) = \emptyset.$$

◘

For every $j \in \mathbb{Z}^+$, we set
$$Z_j := \{i \in \mathbb{Z} : |i| \le j\} = \{-j, -j+1, \dots, -1, 0, 1, \dots, j-1, j\} \text{ and}$$
$$Z_j^* := \{i^* : i \in Z_j\}.$$

With the help of the sets $\mathcal{R}(\ell^*,n,\alpha,\delta,a,a^+,a^-)$ and $\Gamma\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)}$, which are the "bricks" of our construction we are ready to define the sequence of pseudo-curve generators $\{(\gamma_j, \mathbb{S}^1 \backslash Z_j^*)\}_{j=0}^\infty$ that we are looking for.

To do this, for every $j \ge 0$ we define

- a strictly increasing sequence $\{n_j\}_{j=0}^\infty \subset \mathbb{N}$,
- a strictly decreasing sequence $\{\alpha_j\}_{j=0}^\infty$ such that $2^{-n_{j+1}} < \alpha_j < 2^{-n_j}$
- and a sequence $\{\delta_j\}_{j=0}^\infty$ with $2^{-n_{j+1}} < \delta_j < \alpha_j$

verifying some technical properties that we will make explicit below, and we define a sequence of boxes $\mathcal{R}(j^*) := \mathcal{R}(j^*, n_j, \alpha_j, \delta_j, a_j, a_j^+, a_j^-)$ and $\mathcal{R}((-j)^*) := \mathcal{R}((-j)^*, n_j, \alpha_j, \delta_j, a_{-j}, a_{-j}^+, a_{-j}^-)$ (for $j=0$ both sets coincide) with projections
$$\pi\left(\mathcal{R}(j^*)\right) = B_{\alpha_j}\left[j^*\right] \quad \text{and} \quad \pi\left(\mathcal{R}((-j)^*)\right) = B_{\alpha_j}\left[(-j)^*\right].$$

Finally, with the use of all these sequences and objects we can define our functions $\gamma_j\big|_{\mathbb{S}^1 \backslash Z_j^*}$.

Observe that we are using the intervals of the form $B_{\alpha_{|\ell|}}[\ell^*]$, $B_{\delta_{|\ell|}}[\ell^*]$ and also $B_{\alpha_{|\ell|-1}}[\ell^*]$ when ℓ is negative. To ease the use of these intervals we introduce the following notation:
$$B_\ell^{\frown}\left[\ell^*\right] := \begin{cases} B_{\alpha_\ell}[\ell^*] & \text{if } \ell \ge 0, \text{ or} \\ B_{\alpha_{|\ell+1|}}[\ell^*] & \text{if } \ell < 0, \end{cases} \quad \text{and} \quad B_\ell^{\frown}\left(\ell^*\right) := \begin{cases} B_{\alpha_\ell}(\ell^*) & \text{if } \ell \ge 0, \text{ or} \\ B_{\alpha_{|\ell+1|}}(\ell^*) & \text{if } \ell < 0. \end{cases}$$

Notice that the ball $B_\ell^{\frown}[\ell^*]$ has diameter α_j for $\ell \in \{j, -(j+1)\}$.

Remark 3.3. With the above notation $B_{\alpha_{|\ell|}}[\ell^*] \subsetneq B_\ell^{\frown}(\ell^*)$ for every $\ell < 0$. Moreover, for $\ell \in \mathbb{Z}$ and $j \in \mathbb{Z}^+$,
$$R_\omega\left(B_{\alpha_j}\left[\ell^*\right]\right) = B_{\alpha_j}\left[(\ell+1)^*\right], \text{ and}$$
$$R_\omega\left(B_\ell^{\frown}\left[\ell^*\right]\right) = \begin{cases} B_{\alpha_\ell}[(\ell+1)^*] & \text{if } \ell \ge 0, \text{ or} \\ B_{\alpha_{|\ell+1|}}[(\ell+1)^*] & \text{if } \ell < 0. \end{cases}$$

Also, the same formulae holds with α replaced by δ and for open balls. ◘

The next crucial definition fixes in detail all quantities and objects mentioned above.

Definition 3.4. We start by defining $\mathcal{R}(0^*) := \mathcal{R}(0^*, n_0, \alpha_0, \delta_0, 0, 0, 0)$ and $\varphi_{0^*} := \varphi_{(0^*, n_0, \alpha_0, \delta_0, 0, 0, 0)}$ by choosing (Definition 3.1) $n_0 = 1$, $\alpha_0 < \frac{1}{2} = 2^{-n_0}$ and $\delta_0 < \alpha_0$ small enough so that the intervals $B^{\frown}_0[0^*] = B_{\alpha_0}[0^*]$, $B_{\alpha_0}[1^*]$ and $B^{\frown}_{-1}[(-1)^*] = B_{\alpha_0}[(-1)^*]$ are pairwise disjoint; and $(-2)^*, 2^* \notin B^{\frown}_{-1}[(-1)^*]$ and, additionally, $\mathsf{Bd}\,(B_{\alpha_0}[0^*]) \cap O^*(\omega) = \emptyset$.

We also set $a_0^+ = a_0^- = a_0 = 0$, and we define the map $\gamma_0 \colon \mathbb{S}^1 \backslash \{0\} \longrightarrow \mathbb{I}$ by

$$\gamma_0(\theta) = \begin{cases} \varphi_{0^*}(\theta) & \text{if } \theta \in B_{\alpha_0}[0^*] \setminus \{0\}, \\ 0 & \text{if } \theta \notin B_{\alpha_0}[0^*]. \end{cases}$$

For consistency with the definition of γ_j in the case $j \geq 1$, we define the map $\gamma_{-1} \colon \mathbb{S}^1 \backslash \{0\} \longrightarrow \mathbb{I}$ by $\gamma_{-1}(\theta) = 0$ for every $\theta \in \mathbb{S}^1$. Then, notice that, $a_0 = \gamma_{-1}(0^*)$, $a_0^{\pm} = \varphi_{0^*}(0^* \pm \alpha_0) = \gamma_{-1}(0^* \pm \alpha_0)$, and $\gamma_0(\theta) = \gamma_{-1}(\theta)$ for every $\theta \notin B_{\alpha_0}[0^*]$.

Next, for every $j \in \mathbb{N}$ we define $\mathcal{R}(j^*)$, $\mathcal{R}((-j)^*)$ and $(\gamma_j, \mathbb{S}^1 \backslash Z_j^*)$ from the corresponding boxes $\mathcal{R}(i^*)$ and $B_{\alpha_{|i|}}[i^*] \subset B^{\frown}_i[i^*]$ for $i \in Z_{j-1}$, and $(\gamma_{j-1}, \mathbb{S}^1 \backslash Z_{j-1}^*)$ as follows. We take n_j, δ_j and α_j such that (see Figure 4 to fix ideas):

(R.1) $n_j > n_{j-1}$, $\delta_j < \alpha_j < 2^{-n_j} < \delta_{j-1} < \alpha_{j-1}$ and

$$\Big(\mathsf{Bd}\,\big(B_{\alpha_j}[(-j)^*]\big) \cup \mathsf{Bd}\,\big(B_{\alpha_j}[j^*]\big)\Big) \cap O^*(\omega) = \emptyset.$$

(R.2) The intervals

$$\begin{aligned} & B^{\frown}_j[j^*] = B_{\alpha_j}[j^*], \\ & R_\omega\big(B_{\alpha_j}[j^*]\big) = B_{\alpha_j}[(j+1)^*], \\ & B^{\frown}_{-j}[(-j)^*] = B_{\alpha_{j-1}}[(-j)^*] \text{ and} \\ & B^{\frown}_{-(j+1)}[(-(j+1))^*] = B_{\alpha_j}[(-(j+1))^*] \end{aligned}$$

are pairwise disjoint,

$$\gamma_{j-1}\big(B_{\alpha_j}[\ell^*]\big) \subset \big[\gamma_{j-1}(\ell^*) - 2^{-n_j}, \gamma_{j-1}(\ell^*) + 2^{-n_j}\big]$$

for every $\ell \in \{j+1, -(j+1)\}$,

$$\begin{aligned} & B^{\frown}_\ell[\ell^*] \cap Z_{j+1}^* = \{\ell^*\} \text{ for } \ell \in \{j, -(j+1)\} \text{ and} \\ & B_{\alpha_j}[(j+1)^*] \cap Z_{j+1}^* = \{(j+1)^*\}, \end{aligned}$$

and $(-(j+2))^*, (j+2)^* \notin B^{\frown}_{-(j+1)}[(-(j+1))^*] = B_{\alpha_j}[(-(j+1))^*]$.

(R.3) $\mathsf{Bd}\,\big(B_{\alpha_{|k|}}[(k+1)^*]\big) \cap \big(B_{\alpha_j}[j^*] \cup B_{\alpha_j}[(-j)^*]\big) = \emptyset$ for every $k \in Z_{j-1}$.

(R.4) Assume that there exists $k \in Z_{j-1}$ such that $B_{\alpha_j}[(j+1)^*] \cap B^{\frown}_k[k^*] \neq \emptyset$ and $|k|$ is maximal verifying these conditions. Then, $B_{\alpha_j}[(j+1)^*]$ is contained in one of the two connected components of $B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ when $B_{\alpha_j}[(j+1)^*] \cap B_{\alpha_{|k|}}[k^*] \neq \emptyset$, and $B_{\alpha_j}[(j+1)^*]$ is contained in one of the two connected components of $B^{\frown}_k(k^*) \setminus B_{\alpha_{|k|}}[k^*]$ if $B_{\alpha_j}[(j+1)^*] \cap B_{\alpha_{|k|}}[k^*] = \emptyset$ (note that, in this case, k must be negative).

(R.5) Let $\ell \in \{j, -(j+1)\}$ (recall that the ball $B^{\frown}_\ell[\ell^*]$ has diameter α_j for these two values of ℓ and only for them).

(R.5.i) If $\ell^* \notin \bigcup_{i \in Z_{j-1}} B^{\frown}_i[i^*]$ then, $B^{\frown}_\ell[\ell^*] \cap B^{\frown}_i[i^*] = \emptyset$ for every $i \in Z_{j-1}$.

(R.5.ii) If $\ell^* \in B^{\frown}_m[m^*]$ for some $m \in Z_{j-1}$ such that $|m|$ is maximal with these properties, then

(R.5.ii.1) $B^{\frown}_\ell[\ell^*] \cap B^{\frown}_i[i^*] = \emptyset$ for every $i \in Z_{j-1}$ such that $|i| \geq |m|$, $i \neq m$, and

(R.5.ii.2) $B^{\frown}_{\ell}[\ell^*]$ is contained in (a connected component of)

$$B^{\frown}_{m}(m^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right) = \left(m^* - \alpha_{|m|-1}, m^* - \alpha_{|m|}\right) \cup \left(m^* - \alpha_{|m|}, m^*\right) \cup \left(m^*, m^* + \alpha_{|m|}\right) \cup \left(m^* + \alpha_{|m|}, m^* + \alpha_{|m|-1}\right)$$

(observe that $B^{\frown}_{\ell}[\ell^*] \subset B^{\frown}_{m}(m^*) \setminus B_{\alpha_{|m|}}[m^*]$ can only happen when $m < 0$ since $B^{\frown}_{m}[m^*] = B_{\alpha_{|m|}}[m^*]$ for $m \geq 0$).

(R.6) Let $\ell \in \{j, -j\}$. If $B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_{m}[m^*] = \emptyset$ for every $m \in Z_j$, $m \neq \ell$ then, to define $\mathcal{R}(\ell^*)$ and the map φ_{ℓ^*}, we set

$$a_\ell = \gamma_{j-1}(\ell^*) = a^{\pm}_{\ell} = \gamma_{j-1}(\ell^* \pm \alpha_j) = 0.$$

Otherwise, there exists $m \in Z_{j-1}$ such that $B^{\frown}_{\ell}[\ell^*]$ is contained in a connected component of $B^{\frown}_{m}(m^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$ and $|m|$ is maximal with these properties. Then, to define $\mathcal{R}(\ell^*)$ and the map φ_{ℓ^*}, we set

(R.6.i) $a_\ell := \gamma_{|m|}(\ell^*)$, $a^{\pm}_{\ell} := \gamma_{|m|}(\ell^* \pm \alpha_j)$ and $\mathsf{Graph}\left(\gamma_{|m|}\big|_{B_{\alpha_j}[\ell^*]}\right) \subset \mathcal{R}(\ell^*)$.

(R.6.ii) Assume that there exists $k \in Z_{|m|} \subset Z_{j-1}$ such that $B^{\frown}_{\ell}[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$. Then, $\mathcal{R}(\ell^*)$ is contained in one of the two connected components of $\mathsf{Int}\left(\mathcal{R}(k^*) \setminus \upuparrows k^*\right)$.

Finally we define $\gamma_j \colon \mathbb{S}^1 \setminus Z^*_j \longrightarrow \mathbb{I}$ by

$$\gamma_j(\theta) = \begin{cases} \varphi_{j^*}(\theta) & \text{if } \theta \in B_{\alpha_j}[j^*] \setminus \{j^*\}, \\ \varphi_{(-j)^*}(\theta) & \text{if } \theta \in B_{\alpha_j}[(-j)^*] \setminus \{(-j)^*\}, \\ \gamma_{j-1}(\theta) & \text{if } \theta \notin \left(B_{\alpha_j}[j^*] \cup B_{\alpha_j}[(-j)^*] \cup Z^*_{j-1}\right). \end{cases}$$

(notice that $Z^*_j = Z^*_{j-1} \cup \{j^*, (-j)^*\}$). ◘

For every $\ell \in \mathbb{Z}$ we define the *winged region associated to* ℓ as

$$\mathcal{R}^{\frown}(\ell^*) := \begin{cases} \mathcal{R}(\ell^*) & \text{if } \ell \geq 0, \text{ or} \\ \mathcal{R}(\ell^*) \cup \mathsf{Graph}\left(\gamma_{|\ell|}\big|_{B^{\frown}_{\ell}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right) & \text{if } \ell < 0. \end{cases}$$

The next technical lemma shows that the objects from Definition 3.4 exist (that is, they are well defined), and studies some of the basic properties of the family of pseudo-curve generators $\{(\gamma_i, \mathbb{S}^1 \setminus Z^*_i)\}_{i=0}^{\infty}$.

Remark 3.5 (Explicit consequences of Definition 3.4)**.** The following statements are easy consequences of Definition 3.4. They are stated explicitly for easiness of usage.

(R.1) $n_j > j$. This follows from Definition 3.4(R.1) and the fact that we have set $n_0 = 1$ and $n_j > n_{j-1}$ for $j \in \mathbb{N}$.

(R.2) For every $j \in \mathbb{N}$,

$$B^{\frown}_{-j}[(-j)^*] \cap Z^*_{j+1} = \{(-j)^*\}.$$

This follows from Definition 3.4(R.2) for $j-1$. We get

$$B^{\frown}_{-j}[(-j)^*] \cap Z^*_j = \{(-j)^*\} \quad \text{and} \quad (-(j+1))^*, (j+1)^* \notin B^{\frown}_{-j}[(-j)^*].$$

which shows the statement.

(R.6) Let $j \in \mathbb{N}$ and $\ell \in \{j, -j\}$, and assume that $B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_{m}[m^*] = \emptyset$ for every $m \in Z_j$, $m \neq \ell$. Then, $\gamma_r\big|_{B^{\frown}_{\ell}[\ell^*]} = \gamma_0\big|_{B^{\frown}_{\ell}[\ell^*]} \equiv 0$ for $r = 1, 2, \ldots, j-1$.

(R.6.i) Assume that here exists $m \in Z_{j-1}$ such that $B^{\frown}_{\ell}[\ell^*]$ is contained in a connected component of $B^{\frown}_{m}(m^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$ and

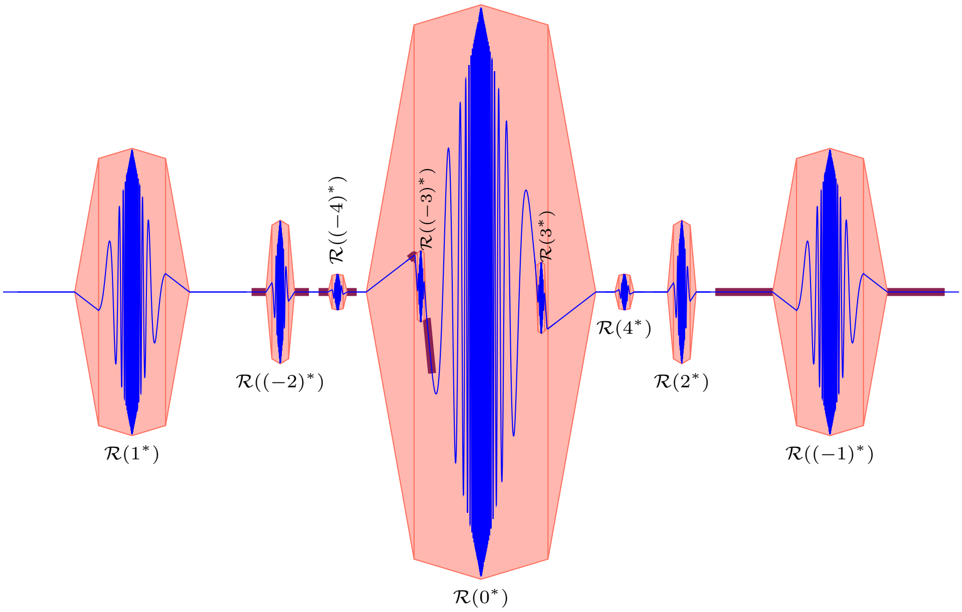


FIGURE 4. The boxes $\mathcal{R}(\ell^*)$ for $\ell \in \{-4,-3,-2,-1,0,1,2,3,4\}$ and the graph of γ_4. The wings are represented as a thick garnet curve surrounding the graph of γ_4. For clarity the scale and separation between boxes is not preserved. The circle $\mathbb{S}^1$ is parametrized as $[-\frac{1}{2}, \frac{1}{2})$.

$|m|$ is maximal with these properties. Then, $\gamma_r\big|_{B^{\frown}_{\ell}[\ell^*]} = \gamma_{|m|}\big|_{B^{\frown}_{\ell}[\ell^*]}$ for $r = |m|+1, |m|+2, \dots, j-1$.

(R.6.ii) Assume that there exists $k \in Z_{|m|} \subset Z_{j-1}$ such that $B^{\frown}_{\ell}[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ and $|k|$ is maximal with these properties. Then, $\gamma_r\big|_{B^{\frown}_{\ell}[\ell^*]} = \gamma_{|k|}\big|_{B^{\frown}_{\ell}[\ell^*]}$ for $r = |k|+1, |k|+2, \dots, |m|$.

To prove (R.6) notice that when $B^{\frown}_{\ell}[\ell^*] \cap B_{\alpha_{|m|}}[m^*] \subset B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_{m}[m^*] = \emptyset$ for every $m \in Z_j$, $m \neq \ell$, from the definition of γ_r for $0 \le r < j$ we get that $\gamma_r\big|_{B^{\frown}_{\ell}[\ell^*]} = \gamma_0\big|_{B^{\frown}_{\ell}[\ell^*]} \equiv 0$ for $r = 1, 2, \dots, j-1$.

(R.6.i) The maximality of $|m|$, together with Definition 3.4(R.2), imply that $B^{\frown}_{\ell}[\ell^*] \cap B_{\alpha_{|i|}}[i^*] \subset B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_{i}[i^*] = \emptyset$ for every $i \in Z_{j-1}$, $|i| \ge |m|$, $i \neq m$. So, by the definition of the functions γ_r,

$$\gamma_r\big|_{B_{\alpha_j}[\ell^*]} = \gamma_{|m|}\big|_{B_{\alpha_j}[\ell^*]} \quad \text{for} \quad r = |m|+1, |m|+2, \dots, j-1.$$

(R.6.ii) When $|k| = |m|$ (R.6.ii) holds trivially. So, assume that $|k| < |m|$. As in the case (R.6.i), the maximality of $|k|$ and Definition 3.4(R.2) imply that $B^{\frown}_{\ell}[\ell^*] \cap B_{\alpha_{|r|}}[r^*] = \emptyset$ for every $r \in Z_{j-1}$, $|r| \ge |k|$, $r \neq k$. So, (R.6.ii) follows from the definition of the functions γ_r. ◘

Lemma 3.6. *For every $j \in \mathbb{Z}^+$ the regions $\mathcal{R}(j^*)$ and $\mathcal{R}((-j)^*)$ (and hence $\mathcal{R}^{\frown}(j^*)$ and $\mathcal{R}^{\frown}((-j)^*)$), and the maps $(\gamma_j, \mathbb{S}^1 \setminus Z^*_j)$ are well defined. Moreover, the following statements hold:*

(a) $(\gamma_j, \mathbb{S}^1 \setminus Z^*_j) \in \mathcal{PCG}$. Furthermore, for every $\ell \in \{j+1, -(j+1)\}$,

$$\gamma_j\left(B_{\alpha_j}[\ell^*]\right) \subset \left[\gamma_j(\ell^*) - 2^{-n_j}, \gamma_j(\ell^*) + 2^{-n_j}\right].$$

(b) $\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}^{\frown}(\ell^*) \subset \mathbb{S}^1 \times [-1,1]$ *and* $\mathsf{Graph}\left(\gamma_j\big|_{\mathbb{S}^1\setminus Z_j^*}\right) \subset \mathbb{S}^1 \times [-1,1]$.

(c) *For* $\ell \in \{j,-j\}$ *we have* $\mathsf{Graph}\left(\gamma_{j-1}\big|_{B_{\alpha_j}[\ell^*]}\right) \subset \mathcal{R}(\ell^*)$, $a_\ell = \gamma_{j-1}(\ell^*)$, *and* $a_\ell^\pm = \varphi_{\ell^*}(\ell^* \pm \alpha_j) = \gamma_{j-1}(\ell^* \pm \alpha_j)$.

(d) $\mathsf{Graph}\left(\gamma_n\big|_{B_{\alpha_j}[\ell^*]\setminus Z_n^*}\right) \subset \mathcal{R}(\ell^*)$ *for every* $n \geq j$ *and* $\ell \in \{j,-j\}$.

(e) *For every* $\ell \in \{j,-j\}$,

$$\gamma_j\big|_{\left(B_\ell^{\frown}[\ell^*]\setminus B_{\alpha_j}(\ell^*)\right)\cup R_\omega\left(B_\ell^{\frown}[\ell^*]\setminus B_{\alpha_j}(\ell^*)\right)} = \gamma_{j-1}\big|_{\left(B_\ell^{\frown}[\ell^*]\setminus B_{\alpha_j}(\ell^*)\right)\cup R_\omega\left(B_\ell^{\frown}[\ell^*]\setminus B_{\alpha_j}(\ell^*)\right)}.$$

Moreover, for every $\theta \in \mathsf{Bd}(B_\ell^{\frown}[\ell^*]\setminus B_{\alpha_j}(\ell^*)) = \mathsf{Bd}(B_{\alpha_j}[\ell^*]) \cup \mathsf{Bd}(B_\ell^{\frown}[\ell^*])$, *we have* $\theta \notin B_n^{\frown}[n^*] \cup B_{-n}^{\frown}[(-n)^*]$ *and* $\gamma_n(\theta) = \gamma_j(\theta) = \gamma_{j-1}(\theta)$ *for every* $n > j$, *and* $R_\omega(\theta) \notin B_{\alpha_n}[n^*] \cup B_{\alpha_n}[(-n)^*]$ *and* $\gamma_n\big(R_\omega(\theta)\big) = \gamma_{j-1}\big(R_\omega(\theta)\big)$ *for every* $n \geq j$.

(f) *For every* $\ell \in \mathbb{Z}$, $\mathcal{R}^{\frown}(\ell^*)$ *is a compact connected set such that* $\pi\left(\mathcal{R}^{\frown}(\ell^*)\right) = B_\ell^{\frown}[\ell^*]$, $\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*]\setminus B_{\alpha_{|\ell|}}(\ell^*)}$ *is continuous and*

$$\mathsf{diam}\left(\mathcal{R}^{\frown}(\ell^*)\right) = \begin{cases} \mathsf{diam}\left(\mathcal{R}(\ell^*)\right) = \mathsf{diam}\left(\mathcal{R}((-\ell)^*)\right) = 2\cdot 2^{-n_\ell} \leq 2^{-\ell} & \text{if } \ell \geq 0,\\ 2\cdot 2^{-n_{|\ell+1|}} \leq 2\cdot 2^{-|\ell|} & \text{if } \ell < 0. \end{cases}$$

(g) *Given* $\ell, m \in \mathbb{Z}$ *such that* $|\ell| \geq |m|$, $\ell \neq m$ *and* $B_\ell^{\frown}[\ell^*] \cap B_m^{\frown}[m^*] \neq \emptyset$, *it follows that* $|\ell| > |m|$, *and either* $B_\ell^{\frown}[\ell^*] \subset B_{\alpha_{|m|}}(m^*)\setminus\{m^*\}$ *and the region* $\mathcal{R}^{\frown}(\ell^*)$ *is contained in one of the two connected components of* $\mathsf{Int}\left(\mathcal{R}(m^*)\setminus \upuparrows m^*\right)$, *or* $m < 0$ *and* $B_\ell^{\frown}[\ell^*]$ *is contained in one of the two connected components of* $B_m^{\frown}(m^*)\setminus B_{\alpha_{|m|}}[m^*]$.

Proof. We start by proving the first statement of the lemma and (a) by induction.

Observe that $n_0 = 1$, α_0, δ_0 and γ_0 are defined so that Definition 3.4(R.1–2) for $j = 0$ and $(\gamma_0, \mathbb{S}^1\setminus Z_0^*) \in \mathcal{PCG}$ are verified except for the obvious fact that $B_{-j}^{\frown}[(-j)^*] = B_j^{\frown}[j^*]$. On the other hand, by construction, $B_{\alpha_0}[0^*]$ is disjoint from $B_{\alpha_0}[1^*]$ and $B_{\alpha_0}[(-1)^*]$. Then, by the definition of γ_0,

$$\gamma_0\left(B_{\alpha_0}[\ell^*]\right) = \{0\} \subset [-\tfrac{1}{2}, \tfrac{1}{2}] = \left[\gamma_0(\ell^*) - 2^{-n_0}, \gamma_0(\ell^*) + 2^{-n_0}\right]$$

for $\ell \in \{1,-1\}$. Hence, (a) holds.

Fix $j > 0$ and assume that we have defined n_ℓ, α_ℓ, δ_ℓ and γ_ℓ such that all Definition 3.4(R.1–6) above and (a) hold for $\ell = 0, 1, \ldots, j-1$.

Since the elements of Z_{j+2}^* are pairwise different, we can choose an integer $n_j > n_{j-1}$ and δ_j and α_j small enough so that

- $0 < \delta_j < \alpha_j < 2^{-n_j} < \delta_{j-1}$,
- $(-(j+2))^*, (j+2)^* \notin B_{-(j+1)}^{\frown}[(-(j+1))^*] = B_{\alpha_j}[(-(j+1))^*]$,
- the three intervals $B_j^{\frown}[j^*] = B_{\alpha_j}[j^*]$, $R_\omega\left(B_{\alpha_j}[j^*]\right) = B_{\alpha_j}[(j+1)^*]$ and $B_{-(j+1)}^{\frown}[(-(j+1))^*]$ are pairwise disjoint,
- $B_\ell^{\frown}[\ell^*] \cap Z_{j+1}^* = \{\ell^*\}$ for $\ell \in \{j, -(j+1)\}$, $B_{\alpha_j}[(j+1)^*] \cap Z_{j+1}^* = \{(j+1)^*\}$ and, additionally,
- $\Big(\mathsf{Bd}\left(B_{\alpha_j}[(-j)^*]\right) \cup \mathsf{Bd}\left(B_{\alpha_j}[j^*]\right)\Big) \cap O^*(\omega) = \emptyset$.

Then, Definition 3.4(R.1) is verified. Moreover, from the above conditions it follows that $B_{\alpha_j}[\ell^*] \cap Z_{j+1}^* = \{\ell^*\}$ for every $\ell \in \{j+1, -(j+1)\}$. Thus, by statement (a) for $j-1$, γ_{j-1} is defined and continuous on $\ell^* \in B_{\alpha_j}[\ell^*]$ because this interval is disjoint from Z_{j-1}^*. Hence, we can decrease the value of α_j (and, accordingly, the value of $0 < \delta_j < \alpha_j$), if necessary, to get

- $$\gamma_{j-1}\left(B_{\alpha_j}[\ell^*]\right) \subset \left[\gamma_{j-1}(\ell^*) - 2^{-n_j}, \gamma_{j-1}(\ell^*) + 2^{-n_j}\right]$$
  for every $\ell \in \{j+1, -(j+1)\}$.

To see that Definition 3.4(R.2) is verified it remains to show that the intervals $B^{\frown}_{j}[j^*]$, $B_{\alpha_j}[(j+1)^*]$ and $B^{\frown}_{-(j+1)}[(-(j+1))^*]$ are disjoint from $B^{\frown}_{-j}[(-j)^*]$. By induction, Definition 3.4(R.2) holds for $j-1$. Thus we see, that $(-(j+1))^*, (j+1)^* \notin B^{\frown}_{-j}[(-j)^*]$, and $R_\omega\left(B_{\alpha_{j-1}}[(j-1)^*]\right) = B_{\alpha_{j-1}}[j^*]$ is disjoint from $B^{\frown}_{-j}[(-j)^*]$. Hence, we can decrease the value of α_j (and, accordingly, the value of $0 < \delta_j < \alpha_j$), if necessary, until $B_{\alpha_j}[(j+1)^*]$ and $B^{\frown}_{-(j+1)}[(-(j+1))^*] = B_{\alpha_j}[(-(j+1))^*]$ are disjoint from $B^{\frown}_{-j}[(-j)^*]$. On the other hand we have that $\alpha_j < 2^{-n_j} < \delta_{j-1} < \alpha_{j-1}$. So, $B^{\frown}_{j}[j^*] = B_{\alpha_j}[j^*] \subset B_{\alpha_{j-1}}[j^*]$ is disjoint from $B^{\frown}_{-j}[(-j)^*]$.

Up to now we have seen that we can choose n_j, δ_j and α_j so that Definition 3.4(R.1–2) hold for j. Let us see that we can choose α_j such that Definition 3.4(R.3) also holds. Observe that for every $\ell, i \in \mathbb{Z}$ and every $m \geq 0$ it follows that $\mathsf{Bd}\left(B_{\alpha_m}[\ell^*]\right) \cap O^*(\omega) \neq \emptyset$ if and only if $\mathsf{Bd}\left(R^i_\omega\left(B_{\alpha_m}[\ell^*]\right)\right) \cap O^*(\omega) = \mathsf{Bd}\left(B_{\alpha_m}[(\ell+i)^*]\right) \cap O^*(\omega) \neq \emptyset$. Therefore, by using Definition 3.4(R.1) inductively, we obtain

$$\bigcup_{k \in Z_{j-1}} \mathsf{Bd}\left(B_{\alpha_{|k|}}[(k+1)^*]\right) \cap \{(-j)^*, j^*\} \subset \bigcup_{k \in Z_{j-1}} \mathsf{Bd}\left(B_{\alpha_{|k|}}[(k+1)^*]\right) \cap O^*(\omega) = \emptyset.$$

Consequently, since $\bigcup_{k \in Z_{j-1}} \mathsf{Bd}\left(B_{\alpha_{|k|}}[(k+1)^*]\right)$ is a finite set, by decreasing again the value of α_j, if necessary, we can achieve that Definition 3.4(R.3) holds for j and Definition 3.4(R.1–2) are still verified.

Next we will take care of Definition 3.4(R.4). If $(j+1)^* \notin \bigcup_{i \in Z_{j-1}} B^{\frown}_{i}[i^*]$, by decreasing again the value of α_j (and δ_j), if necessary, we can achieve that $B_{\alpha_j}[(j+1)^*] \cap \left(\bigcup_{i \in Z_{j-1}} B^{\frown}_{i}[i^*]\right) = \emptyset$ while preserving that Definition 3.4(R.1–3) are verified for j. In this case Definition 3.4(R.4) holds trivially.

Conversely, assume that there exists $k \in Z_{j-1}$ such that $(j+1)^* \in B^{\frown}_{k}[k^*]$ and $|k|$ is maximal verifying these conditions. By Definition 3.4(R.2), k is unique (that is, the condition cannot be verified by k and $-k$ simultaneously). On the other hand, by the Definition 3.4(R.1) for $|k|$ and $|k|-1$ and the comment above, $(j+1)^* \notin \mathsf{Bd}\left(B^{\frown}_{k}[k^*]\right) \cup \mathsf{Bd}\left(B_{\alpha_{|k|}}[k^*]\right)$. Since $k \in Z_{j-1}$, $|k| \leq j-1$ and, hence, $(j+1)^* \notin Z^*_{|k|}$ (in particular $j^* \neq k^*$). Consequently, $(j+1)^*$ is contained in one of the connected components of $B^{\frown}_{k}(k^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup Z^*_{|k|}\right)$. Then, by decreasing again the value of α_j, if necessary, we can get that $B_{\alpha_j}[(j+1)^*]$ is contained in the connected component of $B^{\frown}_{k}(k^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup Z^*_{|k|}\right)$ where $(j+1)^*$ lies, while preserving that Definition 3.4(R.1–3) are verified for j. Consequently, Definition 3.4(R.1–4) hold for j.

Now we will deal with Definition 3.4(R.5). If $\ell^* \notin \bigcup_{i \in Z_{j-1}} B^{\frown}_{i}[i^*]$, by decreasing again the value of α_j, if necessary, we can get Definition 3.4(R.5.i) while preserving that Definition 3.4(R.1–4) are verified for j.

Assume that there exists $m \in Z_{j-1}$ such that $\ell^* \in B^{\frown}_{m}[m^*]$ and $|m|$ is maximal with these properties. As in the above construction, by Definition 3.4(R.1–2), $\ell^* \in B^{\frown}_{m}(m^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$ and m is unique (that is, the condition cannot be verified simultaneously by m and $-m$). Consequently, $\ell^* \notin B^{\frown}_{i}[i^*]$ for every $i \in Z_{j-1}$ such that $|i| \geq |m|$, $i \neq m$. Thus, by decreasing again the value of α_j, if necessary, we can get that Definition 3.4(R.1–4) still hold, Definition 3.4(R.5.ii.1) is verified and the interval $B^{\frown}_{\ell}[\ell^*]$ is contained in the connected component of $B^{\frown}_{m}(m^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$ where ℓ^* lies. So, Definition 3.4(R.5.ii.2) also holds.

We claim that
for every $\ell, m \in \mathbb{Z}$ such that $|m| \le |\ell| \le j$, $\ell \ne m$, either $\widetilde{B_\ell}[\ell^*] \cap \widetilde{B_m}[m^*] = \emptyset$ or $|m| < |\ell|$ and $\widetilde{B_\ell}[\ell^*]$ is contained in a connected component of

$$\widetilde{B_m}(m^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right).$$

We prove the claim by induction. Observe that the claim holds trivially for $|m| \le |\ell| \le 1$ because $\widetilde{B_0}[0^*]$, $\widetilde{B_1}[1^*] = B_{\alpha_1}[1^*] \subset B_{\alpha_0}[1^*]$ and $\widetilde{B_{-1}}[(-1)^*]$ are pairwise disjoint by construction.

Assume that the claim holds for every $|m| \le |\ell| < j$. So, to prove the claim, we may assume that $\ell \in \{j, -j\}$, $m \in Z_{j-1} \cup \{-\ell\}$ and $\widetilde{B_\ell}[\ell^*] \cap \widetilde{B_m}[m^*] \ne \emptyset$. By Definition 3.4($\mathsf{R}$.2), $\widetilde{B_j}[j^*] \cap \widetilde{B_{-j}}[(-j)^*] = \emptyset$. Consequently, $m \ne -\ell$ (that is, $m \in Z_{j-1}$ and $|\ell| = j > |m|$). On the other hand, if $\ell = -j$, Definition 3.4(R.2) for $j-1$ shows that $\widetilde{B_{j-1}}[(j-1)^*]$, $\widetilde{B_{-(j-1)}}[(-(j-1))^*]$ and $\widetilde{B_{-j}}[(-j)^*]$ are pairwise disjoint. Thus, $m \in Z_{j-2}$ in this case.

Hence, by the Definition 3.4(R.5) for j when $\ell = j$ and for $j-1$ when $\ell = -j$, there exists $k \in Z_{j-1}$ (in fact when $\ell = -j$, $k \in Z_{j-2}$) such that $\widetilde{B_\ell}[\ell^*]$ is contained in a connected component of $\widetilde{B_k}(k^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup \{k^*\}\right)$ and $|\ell| = j > |k| \ge |m|$.

If $m = k$ then the claim holds. Otherwise, $m \ne k$ and since $j = |\ell| > |k| \ge |m|$, by the induction hypotheses, $|k| > |m|$, and $\widetilde{B_k}[k^*]$ is contained in a connected component of $\widetilde{B_m}(m^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$. So, the claim holds also in this case. This ends the proof of the claim.

Finally, we consider Definition 3.4(R.6). The fact that either $\widetilde{B_\ell}[\ell^*] \cap \widetilde{B_m}[m^*] = \emptyset$ for every $m \in Z_j$, $m \ne \ell$ or there exists $m \in Z_{j-1}$ such that $\widetilde{B_\ell}[\ell^*]$ is contained in a connected component of $\widetilde{B_m}(m^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$ follows from the claim.

To show that Definition 3.4(R.6.i) can be guaranteed, it is enough to decrease again the value of α_j, if necessary, until $B_{\alpha_j}[\ell^*]$ is disjoint from $Z^*_{|m|}$ and Definition 3.4(R.1–5) are still verified. Thus by (a) for $|m|$, $\gamma_{|m|}$ is well defined and continuous on $B_{\alpha_j}[\ell^*]$. So, we can set $a_\ell := \gamma_{|m|}(\ell^*)$ and, by decreasing again α_j (if necessary), we get $\mathsf{Graph}\left(\gamma_{|m|}\big|_{B_{\alpha_j}[\ell^*]}\right) \subset \mathcal{R}(j^*)$.

To show that Definition 3.4(R.6.ii) can be guaranteed we first assume that $k = m$. As before, if necessary, we can increase the value of n_j and, accordingly, decrease the values of $\alpha_j < 2^{-n_j}$ and $0 < \delta_j < \alpha_j$ so that Definition 3.4(R.1–5) and (R.6.i) are still verified for j and in addition,

$$(\ell^*, a_\ell + 2^{-n_j}), (\ell^*, a_\ell - 2^{-n_j}) \in \mathsf{Int}(\mathcal{R}(k^*))$$

and the region $\mathcal{R}(\ell^*)$ is contained in one of the two connected components of $\mathsf{Int}\left(\mathcal{R}(k^*) \setminus \upuparrows k^*\right)$.

Assume now that $k \ne m$ (recall that $|k| \le |m| < j$). In this case we have $\widetilde{B_\ell}[\ell^*] \subset \widetilde{B_m}(m^*) \cap B_{\alpha_{|k|}}(k^*)$. In particular, $\widetilde{B_m}(m^*) \cap B_{\alpha_{|k|}}(k^*) \ne \emptyset$ and, by the above claim, $|k| < |m|$ and $\widetilde{B_\ell}[\ell^*] \subset \widetilde{B_m}[m^*]$ is contained in a connected component of $\widetilde{B_k}(k^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup \{k^*\}\right)$. The fact that $\widetilde{B_\ell}[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ implies that $\widetilde{B_\ell}[\ell^*] \subset \widetilde{B_m}[m^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$. Then, as above we can increase the value of n_j and, accordingly, decrease the values of $\alpha_j < 2^{-n_j}$ and $0 < \delta_j < \alpha_j$ so that Definition 3.4(R.1–5) and (R.6.i) are still verified,

$$(\ell^*, a_\ell + 2^{-n_j}), (\ell^*, a_\ell - 2^{-n_j}) \in \mathsf{Int}(\mathcal{R}(k^*))$$

and the region $\mathcal{R}(\ell^*)$ is contained in one of the two connected components of $\mathsf{Int}\left(\mathcal{R}(k^*) \setminus \upuparrows k^*\right)$.

Now assume that $|k|$ is not maximal verifying the assumptions. Then, there exists $r \in Z_{|m|} \subset Z_{j-1}$ such that $B^{\frown}_{\ell}[\ell^*] \subset B_{\alpha_{|r|}}(r^*) \setminus \{r^*\}$ and $|r|$ is maximal with these properties.

We have $|k| \le |r| \le |m| < j$ and

$$B^{\frown}_{r}[r^*] \cap B^{\frown}_{k}[k^*] \supset B_{\alpha_{|r|}}(r^*) \cap B_{\alpha_{|k|}}(k^*) \neq \emptyset$$

because $B^{\frown}_{\ell}[\ell^*] \subset B_{\alpha_{|r|}}(r^*) \cap B_{\alpha_{|k|}}(k^*)$. Then, by the claim, $|k| < |r|$ and $B^{\frown}_{r}[r^*]$ is contained in a connected component of $B^{\frown}_{k}(k^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup \{k^*\}\right)$. The fact that $B^{\frown}_{\ell}[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ implies that $B^{\frown}_{r}[r^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$. By the part already proven and Definition 3.4(R.6.ii) for $|r| < j$ we get that $\mathcal{R}(\ell^*)$ is contained in one of the two connected components of $\mathsf{Int}\Big(\mathcal{R}(r^*) \setminus \upuparrows r^*\Big)$ and $\mathcal{R}(r^*)$ is contained in one of the two connected components of $\mathsf{Int}\Big(\mathcal{R}(k^*) \setminus \upuparrows k^*\Big)$. This shows that Definition 3.4(R.6.ii) can be guaranteed.

Let us prove that (a) holds for j. Since the set $\mathbb{S}^1 \setminus Z^*_j$ is residual, to prove that $(\gamma_j, \mathbb{S}^1 \setminus Z^*_j) \in \mathcal{PCG}$ we have to show that $\gamma_j\big|_{\mathbb{S}^1 \setminus Z^*_j}$ is continuous. Note that, from Definition 3.4(R.6.ii), $a^{\pm}_{\ell} = \varphi_{\ell^*}(\ell^* \pm \alpha_j) = \gamma_{j-1}(\ell^* \pm \alpha_j)$. Hence, the continuity of $\gamma_j\big|_{\mathbb{S}^1 \setminus Z^*_j}$ follows from the fact that γ_{j-1} is continuous on $\mathbb{S}^1 \setminus Z^*_{j-1} \supset \mathbb{S}^1 \setminus Z^*_j$ and the continuity of φ_{j^*} and $\varphi_{(-j)^*}$ (Definition 3.1).

This ends the proof of the first statement of the lemma and the first statement of (a). For every $\ell \in \{j+1, -(j+1)\}$, from By Definition 3.4(R.1,2) we get:

$$\begin{aligned} &\gamma_{j-1}\left(B_{\alpha_j}[\ell^*]\right) \subset \left[\gamma_{j-1}(\ell^*) - 2^{-n_j}, \gamma_{j-1}(\ell^*) + 2^{-n_j}\right] \\ &\qquad B_{\alpha_j}[\ell^*] \text{ is disjoint from } B_{\alpha_j}[j^*] \text{ and } B_{\alpha_{j-1}}[(-j)^*] \supset B_{\alpha_j}[(-j))^*], \text{ and} \\ &\qquad\quad \{\ell^*\} \notin B_{\alpha_j}[\ell^*] \cap Z^*_{j-1} \subset B_{\alpha_j}[\ell^*] \cap Z^*_{j+1} = \{\ell^*\}. \end{aligned}$$

So, from the definition of γ_j it follows that

$$\gamma_j\big|_{B_{\alpha_j}[\ell^*]} = \gamma_{j-1}\big|_{B_{\alpha_j}[\ell^*]}$$

and, thus, (a) holds.

Statement (c) follows immediately from Definition 3.4(R.6) and Remark 3.5(R.6).

Next we prove (b,d,e,f,g).

(d) When $n = j$, we get $B_{\alpha_j}[\ell^*] \setminus Z^*_j = B_{\alpha_j}[\ell^*] \setminus \{\ell^*\}$ from Definition 3.4(R.2). Hence, $\mathsf{Graph}\Big(\gamma_j\big|_{B_{\alpha_j}[\ell^*] \setminus Z^*_j}\Big) \subset \mathcal{R}(\ell^*)$ by the definition of γ_j (Definition 3.4) and the definition of φ_{ℓ^*} (Definition 3.1).

Now assume that $n > j$ and fix $\theta \in B_{\alpha_j}[\ell^*] \setminus Z^*_n$. We have to show that the point $(\theta, \gamma_n(\theta)) \in \mathcal{R}(\ell^*)$. If $\theta \notin B_{\alpha_{|m|}}[m^*]$ for every m such that $j < |m| \le n$ then, by the iterative use of the definition of γ_i for $i = j+1, j+2, \ldots, n$ (Definition 3.4) and Definition 3.1,

$$(\theta, \gamma_n(\theta)) = (\theta, \gamma_{n-1}(\theta)) = \cdots = (\theta, \gamma_{j+1}(\theta)) = (\theta, \gamma_j(\theta)) = (\theta, \varphi_{\ell^*}(\theta)) \in \mathcal{R}(\ell^*).$$

Otherwise, by Definition 3.4(R.2), there exists $m \in \mathbb{Z}$ such that $|\ell| < |m| \le n$, $\theta \in B_{\alpha_{|m|}}[m^*] \setminus Z^*_n$, and $\theta \notin B_{\alpha_{|s|}}[s^*]$ for every s such that $|m| < |s| \le n$. This implies that $B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_{m}[m^*] \supset B_{\alpha_j}[\ell^*] \cap B_{\alpha_{|m|}}[m^*] \neq \emptyset$ and $|m|$ is maximal with these properties. So, by the claim for $j = |m|$, $B^{\frown}_{m}[m^*]$ is contained in a connected component of $B^{\frown}_{\ell}(\ell^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|\ell|}}[\ell^*]\right) \cup \{\ell^*\}\right)$. Moreover, since $\theta \in B^{\frown}_{m}(m^*) \cap B_{\alpha_j}[\ell^*] \neq \emptyset$, $B^{\frown}_{m}[m^*] \subset B_{\alpha_{|\ell|}}[\ell^*] \setminus \{\ell^*\}$. Thus, by Definition 3.4(R.6.ii) and Remark 3.5(R.6.ii) for $j = |m|$, ℓ replaced by m and k replaced by ℓ, $\mathcal{R}(m^*) \subset \mathcal{R}(\ell^*)$ and (d) follows from the part already proven by replacing ℓ by m and j by $|m|$.

(g) By the claim we have that for every $\ell, m \in \mathbb{Z}$ such that $|\ell| \geq |m|$, $\ell \neq m$ and $B_\ell^{\frown}[\ell^*] \cap B_m^{\frown}[m^*] \neq \emptyset$, it follows that $|\ell| > |m|$, and $B_\ell^{\frown}[\ell^*]$ is contained in a connected component of $B_m^{\frown}(m^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$. Only it remains to show that if $B_\ell^{\frown}[\ell^*] \subset B_{\alpha_{|m|}}(m^*) \setminus \{m^*\}$, then the region $\mathcal{R}^{\frown}(\ell^*)$ is contained in one of the two connected components of $\mathsf{Int}\left(\mathcal{R}(m^*) \setminus \upuparrows m^*\right)$. By Definition 3.4(R.6.ii) we know that this holds for $\mathcal{R}(\ell^*)$ instead of $\mathcal{R}^{\frown}(\ell^*)$. Hence, if $\ell \geq 0$, (g) holds because $\mathcal{R}^{\frown}(\ell^*) = \mathcal{R}(\ell^*)$. Assume now that $\ell < 0$. Since $\mathcal{R}^{\frown}(\ell^*) = \mathcal{R}(\ell^*) \cup \mathsf{Graph}\left(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right)$ is connected, $\mathcal{R}(\ell^*) \subset \mathcal{R}(m^*)$, and $\mathsf{Int}\left(\mathcal{R}(m^*) \setminus \upuparrows m^*\right)$ has two connected components, it is enough to show that

$$\mathsf{Graph}\left(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right) \subset \mathcal{R}(m^*).$$

Since $B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*) \subset B_\ell^{\frown}[\ell^*] \subset B_{\alpha_{|m|}}(m^*) \setminus \{m^*\}$, statement (g) follows from (d) with ℓ replaced by m, j by $|m|$ and n replaced by $|\ell|$.

(b) With (g) in mind we set

$$\mathsf{D} := \{\ell \in \mathbb{Z} : \mathcal{R}^{\frown}(\ell^*) \not\subset \mathcal{R}(i^*) \text{ for every } i \in \mathbb{Z} \setminus \{\ell\}\}.$$

Clearly,

$$\begin{aligned}\bigcup_{\ell \in \mathbb{Z}} \mathcal{R}^{\frown}(\ell^*) &= \left(\bigcup_{i \in \mathbb{Z} \setminus \mathsf{D}} \mathcal{R}^{\frown}(i^*)\right) \cup \left(\bigcup_{\ell \in \mathsf{D}} \mathcal{R}^{\frown}(\ell^*)\right) \\ &\subset \left(\bigcup_{i \in \mathsf{D}} \mathcal{R}(i^*)\right) \cup \left(\bigcup_{\ell \in \mathsf{D}} \mathcal{R}^{\frown}(\ell^*)\right) = \bigcup_{\ell \in \mathsf{D}} \mathcal{R}^{\frown}(\ell^*)\end{aligned}$$

Claim: *For every* $\ell \in \mathsf{D}$, $\gamma_{|\ell|-1}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)} \equiv 0$.

First we prove statement (b) from the above claim and then we will prove the claim. To this end we start by pointing out few elementary facts.

From the definition of $\mathcal{R}^{\frown}(\ell^*)$ we see that $\mathcal{R}^{\frown}(\ell^*) \setminus \mathcal{R}(\ell^*) = \emptyset$ for every $\ell \geq 0$ and $\mathcal{R}^{\frown}(\ell^*) \setminus \mathcal{R}(\ell^*) \subset \mathsf{Graph}\left(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right)$ for every $\ell < 0$. So, in any case,

$$\mathcal{R}^{\frown}(\ell^*) \setminus \mathcal{R}(\ell^*) \subset \mathsf{Graph}\left(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right) \quad \text{for every} \quad \ell \in \mathbb{Z}.$$

On the other hand, the arc $B_\ell^{\frown}[\ell^*] \supset B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)$ is disjoint from the arc $B_{-\ell}^{\frown}[(-\ell)^*] \supset B_{\alpha_{|\ell|}}[(-\ell)^*]$ by Definition 3.4(R.2). Thus, by Definition 3.4 and (a),

$$\gamma_{|\ell|-1}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)} = \gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}.$$

Furthermore, by the Claim and Definition 3.4(R.6), $a_\ell^+ = a_\ell^- = a_\ell = 0$ for every $\ell \in D$. So, by Remark 3.2(1),

$$\mathcal{R}(\ell^*) \subset B_{\alpha_{|\ell|}}[\ell^*] \times [-2^{-n_{|\ell|}}, 2^{-n_{|\ell|}}] \subset B_{\alpha_{|\ell|}}[\ell^*] \times [-2^{-|\ell|}, 2^{-|\ell|}] \subset B_{\alpha_{|\ell|}}[\ell^*] \times [-1, 1].$$

Therefore, summarizing and using again by the Claim,

$$\begin{aligned}\bigcup_{\ell \in \mathbb{Z}} \mathcal{R}^{\frown}(\ell^*) \subset \bigcup_{\ell \in \mathsf{D}} \mathcal{R}^{\frown}(\ell^*) &\subset \bigcup_{\ell \in \mathsf{D}} \left(\mathcal{R}(\ell^*) \cup \mathsf{Graph}\left(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right)\right) \\ &= \left(\bigcup_{\ell \in \mathsf{D}} \mathcal{R}(\ell^*)\right) \cup \left(\bigcup_{\ell \in \mathsf{D}} \mathsf{Graph}\left(\gamma_{|\ell|-1}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right)\right) \\ &\subset \left(\bigcup_{\ell \in \mathsf{D}} B_{\alpha_{|\ell|}}[\ell^*]\right) \times [-1, 1] \cup \mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times [-1, 1].\end{aligned}$$

So, the first part of (b) is proved, provided that the claim holds. Let us prove the second statement of (b). Observe that, since

$$\left(\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}(\ell^*)\right) \cup \mathbb{S}^1 \times \{0\} \subset \left(\bigcup_{\ell\in\mathbb{Z}} \breve{\mathcal{R}}(\ell^*)\right) \cup \mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times [-1,1],$$

it is enough to show that

$$\mathsf{Graph}\left(\gamma_j\big|_{\mathbb{S}^1\setminus Z_j^*}\right) \subset \left(\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}(\ell^*)\right) \cup \mathbb{S}^1 \times \{0\}$$

for every $j \in \mathbb{Z}^+$. We will prove this statement by induction on j.

By construction we have

$$\mathsf{Graph}\left(\gamma_0\big|_{\mathbb{S}^1\setminus\{0^*\}}\right) \subset \mathcal{R}(0^*) \cup \mathbb{S}^1 \times \{0\} \subset \left(\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}(\ell^*)\right) \cup \mathbb{S}^1 \times \{0\}.$$

So, the statement holds for $j = 0$. Now assume that it holds for some $j \geq 0$, and prove it for $j+1$. By Definition 3.4 and (d),

$$\begin{aligned}\mathsf{Graph}\left(\gamma_{j+1}\big|_{\mathbb{S}^1\setminus Z_{j+1}^*}\right) &\subset \mathcal{R}(j^*) \cup \mathcal{R}((-j)^*) \cup \mathsf{Graph}\left(\gamma_j\big|_{\mathbb{S}^1\setminus Z_j^*}\right)\\ &\subset \mathcal{R}(j^*) \cup \mathcal{R}((-j)^*) \cup \left(\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}(\ell^*)\right) \cup \mathbb{S}^1 \times \{0\}\\ &\subset \left(\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}(\ell^*)\right) \cup \mathbb{S}^1 \times \{0\}.\end{aligned}$$

To end the proof of (b) it remains to show the Claim.

Let $\ell \in \mathsf{D}$ and $m \in Z_{|\ell|}$, $m \neq \ell$. Then, either

$$\text{(1)} \qquad \begin{cases} \breve{B}_{\ell}[\ell^*] \cap \breve{B}_{m}[m^*] = \emptyset \text{ or} \\ |\ell| > |m|\,,\ m < 0 \text{ and } \breve{B}_{\ell}[\ell^*] \subset \breve{B}_{m}(m^*) \setminus B_{\alpha_{|m|}}[m^*]\,. \end{cases}$$

To see this, observe that if $\breve{B}_{\ell}[\ell^*] \cap \breve{B}_{m}[m^*] \neq \emptyset$ then, by (g), $|\ell| > |m|$ and either $\breve{\mathcal{R}}(\ell^*) \subset \mathcal{R}(m^*)$ or $m < 0$ and $\breve{B}_{\ell}[\ell^*] \subset \breve{B}_{m}(m^*) \setminus B_{\alpha_{|m|}}[m^*]$, and the first possibility is ruled out because $\ell \in \mathsf{D}$.

By using iteratively the dichotomy (1) we get that, for every $\ell \in \mathsf{D}$, there exists a sequence $m_0, m_1, \dots, m_k = \ell \in \mathbb{Z}$ with $k \geq 0$ such that $\breve{B}_{m_0}[(m_0)^*] \cap \breve{B}_{q}[q^*] = \emptyset$ for every $q \in Z_{|m_0|}$, $q \neq m_0$ and, in the case $k > 0$, $|m_0| < |m_1| < \dots < |m_k| = |\ell|$ and, for every $p = 0, 1, \dots, k-1$,

- $m_p < 0$,
- $\breve{B}_{m_{p+1}}[(m_{p+1})^*] \subset \breve{B}_{m_p}((m_p)^*) \setminus B_{\alpha_{|m_p|}}[(m_p)^*]$ and
- $\breve{B}_{m_{p+1}}[(m_{p+1})^*] \cap \breve{B}_{q}[q^*] = \emptyset$ for every $q \in Z_{|m_{p+1}|}$, $q \neq m_p, m_{p+1}$ and $|m_p| \leq |q|$.

The condition $\breve{B}_{m_0}[(m_0)^*] \cap \breve{B}_{q}[q^*] = \emptyset$ for every $q \in Z_{|m_0|}$, $q \neq m_0$ implies

$$\gamma_{|m_0|-1}\big|_{\breve{B}_{m_0}[(m_0)^*]} = \gamma_{|m_0|-2}\big|_{\breve{B}_{m_0}[(m_0)^*]} = \dots = \gamma_0\big|_{\breve{B}_{m_0}[(m_0)^*]} \equiv 0$$

by Definition 3.4(R.6) and Remark 3.5(R.6) (with $\ell = m_0$). This ends the proof of the Claim when $k = 0$.

Assume now that $k > 0$. As before we have

$$\gamma_{|m_0|-1}\big|_{\breve{B}_{m_0}[(m_0)^*]\setminus B_{\alpha_{|m_0|}}((m_0)^*)} = \gamma_{|m_0|}\big|_{\breve{B}_{m_0}[(m_0)^*]\setminus B_{\alpha_{|m_0|}}((m_0)^*)}.$$

This, together with the inclusion,

$$\breve{B}_{m_1}[(m_1)^*] \subset \breve{B}_{m_0}((m_0)^*) \setminus B_{\alpha_{|m_0|}}[(m_0)^*]$$

implies that

$$\gamma_{|m_0|}\big|_{B^{\frown}_{m_1}[(m_1)^*]} \equiv 0.$$

Then, by Definition 3.4(R.6.i) and Remark 3.5(R.6.i) with $\ell = m_1$,

$$0 \equiv \gamma_{|m_0|}\big|_{B^{\frown}_{m_1}[(m_1)^*]} = \gamma_{|m_0|+1}\big|_{B^{\frown}_{m_1}[(m_1)^*]} = \cdots = \gamma_{|m_1|-1}\big|_{B^{\frown}_{m_1}[(m_1)^*]}.$$

If $k = 1$ we are done. Otherwise, $k \geq 2$ and, as above,

$$\gamma_{|m_1|}\big|_{B^{\frown}_{m_2}[(m_2)^*]} \equiv 0.$$

By iterating the above arguments at most k times the Claim holds. This ends the proof of (b).

(e) By Definition 3.4(R.2) and Remark 3.5(R.2) it follows that

$$\theta \notin Z^*_{j+1} \cup B_{\alpha_j}(\ell^*) \cup B^{\frown}_{-\ell}((-\ell)^*) \quad \text{for every} \quad \theta \in B^{\frown}_{\ell}[\ell^*] \setminus B_{\alpha_j}(\ell^*).$$

So, by (a), $\gamma_{j-1}(\theta)$ is well defined and γ_{j-1} is continuous at θ. Thus, by the definition of γ_j (Definition 3.4) and the continuity of γ_{j-1} at θ, $\gamma_j(\theta) = \gamma_{j-1}(\theta)$.

Now assume that $\theta \in \mathsf{Bd}(B^{\frown}_{\ell}[\ell^*] \setminus B_{\alpha_j}(\ell^*)) = \mathsf{Bd}(B_{\alpha_j}[\ell^*]) \cup \mathsf{Bd}(B^{\frown}_{\ell}[\ell^*])$. By (g), $\theta \notin B^{\frown}_{n}[n^*] \cup B^{\frown}_{-n}[(-n)^*]$ for every $n > j$. So, by the iterative use of the definition of γ_i for $i = j+1, j+2, \ldots, n$ (Definition 3.4) we get

$$\gamma_j(\theta) = \gamma_{j+1}(\theta) = \cdots = \gamma_{n-1}(\theta) = \gamma_n(\theta).$$

Now we prove the part of (e) concerning $R_\omega(B^{\frown}_{\ell}[\ell^*] \setminus B_{\alpha_j}(\ell^*))$. We first assume that $\ell = j \geq 0$. Then,

$$B^{\frown}_{j}[j^*] = B_{\alpha_j}[j^*], \ \theta \in \mathsf{Bd}(B_{\alpha_j}[j^*]) \quad \text{and} \quad R_\omega(\theta) \in \mathsf{Bd}(B_{\alpha_j}[(j+1)^*]).$$

Again by Definition 3.4(R.2), $R_\omega(\theta) \notin Z^*_{j+1} \cup B_{\alpha_j}[j^*] \cup B^{\frown}_{-j}[(-j)^*]$. So, by (a) and the definition of γ_j (Definition 3.4), $\gamma_{j-1}\big(R_\omega(\theta)\big)$ is well defined and $\gamma_j\big(R_\omega(\theta)\big) = \gamma_{j-1}\big(R_\omega(\theta)\big)$. By Definition 3.4(R.3) (with $j = n$ and $k = \ell = j$), $R_\omega(\theta) \notin B_{\alpha_n}[n^*] \cup B_{\alpha_n}[(-n)^*]$ for every $n > j$. So, $\gamma_n\big(R_\omega(\theta)\big) = \gamma_j\big(R_\omega(\theta)\big)$ as above.

Assume now that $\ell = -j < 0$. In this case we have $B^{\frown}_{\ell}[\ell^*] = B_{\alpha_{|\ell+1|}}[\ell^*]$ and, hence, $R_\omega(\theta) \in B_{\alpha_{|\ell+1|}}[(\ell+1)^*] \setminus B_{\alpha_j}((\ell+1)^*)$. By Definition 3.4(R.1) we have

$$B_{\alpha_j}[(\ell+1)^*] \subset B_{\alpha_{|\ell+1|}}[(\ell+1)^*] \subset B^{\frown}_{\ell+1}[(\ell+1)^*].$$

Thus, $R_\omega(\theta) \in B^{\frown}_{\ell+1}[(\ell+1)^*] \setminus \{(\ell+1)^*\}$. Again by Definition 3.4(R.2) and Remark 3.5(R.2) (with j replaced by $-(\ell+1)$),

$$R_\omega(\theta) \notin Z^*_{\ell} \cup B_{\alpha_{-(\ell+1)}}[(-\ell)^*] \cup B^{\frown}_{\ell}[\ell^*] \supset Z^*_j \cup B_{\alpha_j}[j^*] \cup B^{\frown}_{-j}[(-j)^*].$$

So, by (a) and the definition of γ_j (Definition 3.4), $\gamma_{j-1}\big(R_\omega(\theta)\big)$ is well defined and $\gamma_j\big(R_\omega(\theta)\big) = \gamma_{j-1}\big(R_\omega(\theta)\big)$.

To end the proof of (e), assume as above that $\theta \in \mathsf{Bd}(B_{\alpha_j}[\ell^*]) \cup \mathsf{Bd}(B^{\frown}_{\ell}[\ell^*])$ and, hence, $R_\omega(\theta) \in \mathsf{Bd}(B_{\alpha_j}[(\ell+1)^*]) \cup \mathsf{Bd}\left(B_{\alpha_{|\ell+1|}}[(\ell+1)^*]\right)$. We have to show that, in this case, $R_\omega(\theta) \notin B_{\alpha_n}[n^*] \cup B_{\alpha_n}[(-n)^*]$ for every $n > j$ (the fact that $\gamma_n\big(R_\omega(\theta)\big) = \gamma_j\big(R_\omega(\theta)\big)$ follows as above). When $R_\omega(\theta) \in \mathsf{Bd}(B_{\alpha_j}[(\ell+1)^*])$ this follows from Definition 3.4(R.3) as before. Assume now that $R_\omega(\theta) \in \mathsf{Bd}\left(B_{\alpha_{|\ell+1|}}[(\ell+1)^*]\right)$. Then, by (g), $R_\omega(\theta) \notin B^{\frown}_{n}[n^*] \cup B^{\frown}_{-n}[(-n)^*]$ for every $n > j$.

(f) If $\ell \geq 0$ then the first two statements of (f) follow directly from the definitions. Moreover, by Remarks 3.2(2) and 3.5(R.1),

$$\mathsf{diam}\left(\mathcal{R}^{\frown}(\ell^*)\right) = \mathsf{diam}\left(\mathcal{R}(\ell^*)\right) = \mathsf{diam}\left(\mathcal{R}((-\ell)^*)\right) = 2 \cdot 2^{-n_\ell} \leq 2 \cdot 2^{-(\ell+1)} = 2^{-\ell}.$$

Assume that $\ell < 0$. From Definition 3.4(R.2) and Remark 3.5(R.2) we get $\left(B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)\right) \cap Z^*_{|\ell|} = \emptyset$ and, hence, $\gamma_{|\ell|}$ is continuous in an open neighbourhood of $B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)$ by (a). On the other hand, by (d), $\left(\theta, \gamma_{|\ell|}(\theta)\right) \in \mathcal{R}(\ell^*)$ for every $\theta \in \mathsf{Bd}\left(B_{\alpha_{|\ell|}}[\ell^*]\right) \subset B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)$. Thus,

$$\mathcal{R}^{\frown}(\ell^*) = \mathcal{R}(\ell^*) \cup \mathsf{Graph}\Big(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\Big)$$

is closed, connected and projects onto the whole $B_\ell^{\frown}[\ell^*]$.

On the other hand, by (e) and (a) (since $\ell < 0$, $|\ell + 1| = |\ell| - 1$),

$$\begin{aligned}\gamma_{|\ell|}\left(B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)\right) &= \gamma_{|\ell|-1}\left(B_{\alpha_{|\ell+1|}}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)\right) \\ &\subset \left[\gamma_{|\ell|-1}(\ell^*) - 2^{-n_{|\ell|-1}}, \gamma_{|\ell|-1}(\ell^*) + 2^{-n_{|\ell|-1}}\right].\end{aligned}$$

Thus, by Remark 3.2(1), (c) and Definition 3.4(R.1),

$$\begin{aligned}\mathcal{R}^{\frown}(\ell^*) &= \mathcal{R}(\ell^*) \cup \mathsf{Graph}\Big(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\Big) \\ &\subset B_{\alpha_{|\ell|}}[\ell^*] \times \left[\gamma_{|\ell|-1}(\ell^*) - 2^{-n_{|\ell|}}, \gamma_{|\ell|-1}(\ell^*) + 2^{-n_{|\ell|}}\right] \cup \\ &\qquad \left(B_{\alpha_{|\ell+1|}}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)\right) \times \left[\gamma_{|\ell|-1}(\ell^*) - 2^{-n_{|\ell|-1}}, \gamma_{|\ell|-1}(\ell^*) + 2^{-n_{|\ell|-1}}\right] \\ &\subset B_{\alpha_{|\ell+1|}}[\ell^*] \times \left[\gamma_{|\ell|-1}(\ell^*) - 2^{-n_{|\ell|-1}}, \gamma_{|\ell|-1}(\ell^*) + 2^{-n_{|\ell|-1}}\right].\end{aligned}$$

Hence, by Definition 3.4(R.1) and Remark 3.5(R.1),

$$\mathsf{diam}\left(\mathcal{R}^{\frown}(\ell^*)\right) \leq 2 \cdot \max\{\alpha_{|\ell+1|}, 2^{-n_{|\ell|-1}}\} = 2 \cdot 2^{-n_{|\ell|-1}} \leq 2 \cdot 2^{-|\ell|}.$$

□

The next results allow us to define the limit pseudo-curve generated by the sequence $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$.

Lemma 3.7. *The sequence* $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty \subset \mathcal{PCG}$ *is convergent in* $\mathcal{PCG}$.

Proof. By Proposition 2.9 it suffices to show that $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$ is a Cauchy sequence in $\mathcal{PCG}$. By the definition of γ_i (Definition 3.4) we have

$$\begin{aligned}\mathsf{d}_\infty\left(\gamma_{i-1}, \gamma_i\right) &= \sup_{\theta \in \mathbb{S}^1 \setminus Z_i^*} \left|\gamma_{i-1}(\theta) - \gamma_i(\theta)\right| \\ &= \sup_{\theta \in \left(B_{\alpha_i}[i^*] \setminus \{i^*\}\right) \cup \left(B_{\alpha_i}[(-i)^*] \setminus \{(-i)^*\}\right)} \left|\gamma_{i-1}(\theta) - \gamma_i(\theta)\right|.\end{aligned}$$

By Lemmas 3.6(c,d), and Definition 3.4(R.2) and Remark 3.5(R.2),

$$(\theta, \gamma_{i-1}(\theta)), (\theta, \gamma_i(\theta)) \in \mathcal{R}(\ell^*) \quad \text{for} \quad \theta \in B_{\alpha_i}[\ell^*] \setminus \{\ell^*\} \text{ and } \ell \in \{i, -i\}.$$

Hence, by Lemma 3.6(f),

$$\mathsf{d}_\infty(\gamma_{i-1}, \gamma_i) \leq \mathsf{diam}(\mathcal{R}(i^*)) = \mathsf{diam}(\mathcal{R}((-i)^*)) \leq 2^{-i}.$$

Since n_i is a strictly increasing sequence, for every $m \geq 0$,

$$\mathsf{d}_\infty(\gamma_{i+m}, \gamma_i) \leq \sum_{k=i+1}^{i+m} 2^{-k} < 2^{-(i+1)} \sum_{k=0}^{\infty} \tfrac{1}{2^k} = 2 \cdot 2^{-(i+1)},$$

and consequently $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$ is a Cauchy sequence in $\mathcal{PCG}$. □

Lemma 3.7 allows us to define the following limit pseudo-curve generator of the sequence $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$.

Definition 3.8. There exists $(\gamma, \mathbb{S}^1 \setminus O^*(\omega)) \in \mathcal{PCG}$ such that

$$(\gamma, \mathbb{S}^1 \setminus O^*(\omega)) = \lim_{i\to\infty} (\gamma_i, \mathbb{S}^1 \setminus Z_i^*)$$

(that is, $\gamma(\theta) = \lim_{i\to\infty} \gamma_i(\theta)$ for every $\theta \in \mathbb{S}^1 \setminus O^*(\omega)$). Observe that

$$\mathbb{S}^1 \setminus O^*(\omega) = \bigcap_{i=1}^{\infty} (\mathbb{S}^1 \setminus Z_i^*)$$

is a residual set in $\mathbb{S}^1$. ◘

Now, we are ready to define the sequence of pseudo-curves associated to the sequence $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^{\infty}$, and to the limit pseudo-curve generator $(\gamma, \mathbb{S}^1 \setminus O^*(\omega))$. This will finally define the pseudo-curve A that we want to construct.

Definition 3.9. We denote by

$$\mathsf{A}_j := \mathsf{A}_{(\gamma_j, \mathbb{S}^1 \setminus Z_j^*)} = \overline{\mathsf{Graph}(\gamma_j, \mathbb{S}^1 \setminus Z_j^*)}$$

the pseudo-curve defined by $(\gamma_j, \mathbb{S}^1 \setminus Z_j^*) \in \mathcal{PCG}$, and

$$\mathsf{A} = \mathsf{A}_{(\gamma, \mathbb{S}^1 \setminus O^*(\omega))} := \overline{\mathsf{Graph}(\gamma, \mathbb{S}^1 \setminus O^*(\omega))}.$$

By Definition 3.8 and Proposition 2.11, $\mathsf{A} = \lim_{j\to\infty} \mathsf{A}_{(\gamma_j, \mathbb{S}^1 \setminus Z_j^*)}$. ◘

The next lemmas study the properties the pseudo-curves A_j and A.

Lemma 3.10. *The following statements hold for every $\ell \in \mathbb{Z}$:*
(a) $\mathsf{A}_n^\theta \subset \mathcal{R}(\ell^*)^\theta$ *for every $n \geq |\ell| - 1$ and $\theta \in B_{\alpha_{|\ell|}}[\ell^*]$.*
(b) $\mathsf{A}_n^{\ell^*} = \mathsf{A}_{|\ell|}^{\ell^*} \subset \mathcal{R}(\ell^*)^{\ell^*}$ *for every $n \geq |\ell|$. Moreover, $\mathsf{A}_{|\ell|}^{\ell^*} = \mathcal{R}(\ell^*)^{\ell^*}$ is a non-degenerate interval.*
(c) $\mathsf{A}_\ell^\theta = \{(\theta, \gamma_\ell(\theta)\}$ *for every $\theta \in \mathbb{S}^1 \setminus Z_\ell^*$.*
(d) $\mathsf{A}_{|\ell|} \subset \mathbb{S}^1 \times [-1, 1]$.

Proof. **(a)** By Lemma 3.6(c,d), $\mathsf{Graph}\Big(\gamma_n\big|_{B_{\alpha_{|\ell|}}[\ell^*] \setminus Z_n^*}\Big) \subset \mathcal{R}(\ell^*)$. Then, the statement follows from the compacity of $\mathcal{R}(\ell^*)$.

(b) From the definition of γ_i (Definition 3.4) and Definition 3.4(R.2), for every $n > |\ell|$ there exists an $\varepsilon(n) > 0$ such that $\gamma_n(\theta) = \gamma_{|\ell|}(\theta)$ for every $\theta \in B_{\varepsilon(n)}(\ell^*) \setminus \{\ell^*\}$. Hence $\mathsf{A}_n^{\ell^*} = \mathsf{A}_{|\ell|}^{\ell^*}$. Moreover, $\gamma_{|\ell|}$ coincides with φ_{ℓ^*} in a neighbourhood of ℓ^*. Thus, $\mathsf{A}_{|\ell|}^{\ell^*} = \mathcal{R}(\ell^*)^{\ell^*}$ and it is an interval by Definition 3.1 and Remark 3.2(4).

Finally statement (c) follows from Lemma 2.2(a) and Definition 3.9, and (d) from Lemma 3.6(b). □

Lemma 3.11. *The following statements hold.*
(a) $\mathsf{A}^\theta \subset \mathcal{R}(\ell^*)^\theta$ *for every $\ell \in \mathbb{Z}$ and $\theta \in B_{\alpha_{|\ell|}}[\ell^*]$.*
(b) $\mathsf{A}^{\ell^*} = \mathsf{A}_{|\ell|}^{\ell^*}$ *for every $\ell \in \mathbb{Z}$. In particular A^{ℓ^*} is a non-degenerate interval.*
(c) If $\theta \notin O^*(\omega)$, then $\mathsf{A}^\theta = \{(\theta, \gamma(\theta))\}$.
(d) $\mathsf{A} \subset \mathbb{S}^1 \times [-1, 1]$.

Proof. Statement (c) follows directly from Lemma 2.2(a).

Now we prove (a). From Lemma 3.10(a), $\mathsf{A}_n^\theta \subset \mathcal{R}(\ell^*)$ for every $\ell \in \mathbb{Z}$ and $n \geq |\ell|$. On the other hand, by Definition 3.8 and Proposition 2.11, $\mathsf{A}^\theta = \lim_{n\to\infty} \mathsf{A}_n^\theta$. Hence the result follows from the compacity of $\mathcal{R}(\ell^*)$.

By Lemma 3.10(b) and the part of the lemma already proved we have

$$\mathsf{A}^{\ell^*} = \lim_{n\to\infty} \mathsf{A}_n^{\ell^*} = \mathsf{A}_{|\ell|}^{\ell^*}.$$

Statement (d) follows from Lemma 3.10(d), the compacity of $\mathbb{S}^1 \times [-1,1]$ and the fact that $\mathsf{A} = \lim_{j\to\infty} \mathsf{A}_j$. □

The next proposition, summarizes the main properties of the set A.

Proposition 3.12. *The set A is a connected, does not contain any arc of curve and $\Omega \backslash \mathsf{A}$ has two connected components.*

Proof. From statements (b) and (c) of the previous lemma, we know that A^θ is connected for every $\theta \in \mathbb{S}^1$.

If A is not connected there exist closed (in A) sets U and V such that $U \cap V = \emptyset$ and $U \cup V = \mathsf{A}$. Observe that $\pi(U) \cup \pi(V) = \pi(\mathsf{A}) = \mathbb{S}^1$ because every pseudo-curve is a circular set. Moreover, since A is compact, U and V are also compact sets of Ω. Hence, $\pi(U)$ are $\pi(V)$ compact in $\mathbb{S}^1$. Since $\mathbb{S}^1$ is connected, $\pi(U) \cap \pi(V) \neq \emptyset$. For every $\theta \in \pi(U) \cap \pi(V)$ we have,

$$\mathsf{A}^\theta = (U \cup V)^\theta = U^\theta \cup V^\theta.$$

The sets U^θ and V^θ are closed, non-empty and disjoint. Consequently, A^θ is not connected; a contradiction. This proves that A is connected.

By Lemma 3.11(b), A^{ℓ^*} is a non-degenerate interval for every $\ell \in O^*(\omega)$. Then, since $O^*(\omega)$ is dense in $\mathbb{S}^1$, A does not contain any arc of curve by Lemma 2.3(b).

To prove that $\Omega \backslash \mathsf{A}$ has two connected components we define

$$\Omega_- := \{(\theta, y) \in \Omega : y < \min\{x \in \mathbb{I} : (\theta, x) \in \mathsf{A}\}\}, \text{ and}$$
$$\Omega_+ := \{(\theta, y) \in \Omega : y > \max\{x \in \mathbb{I} : (\theta, x) \in \mathsf{A}\}\}.$$

By Lemma 3.11(d) we know that

$$-1 \leq \min\{x \in \mathbb{I} : (\theta, x) \in \mathsf{A}\} \leq \max\{x \in \mathbb{I} : (\theta, x) \in \mathsf{A}\} \leq 1.$$

Hence, $\Omega \backslash \mathsf{A} = \Omega_- \cup \Omega_+$, Ω_+ and Ω_- are disjoint open circular subsets of Ω and $\Omega_- \supset \mathbb{S}^1 \times [-2,-1]$ and $\Omega_+ \supset \mathbb{S}^1 \times [1,2]$ (in particular, for every $\theta \in \mathbb{S}^1$, Ω_+^θ and Ω_-^θ are non-degenerate intervals). Thus, Ω_+ and Ω_- are arc-wise connected and, hence, connected. □

4. A collection of auxiliary functions G_i defined on the boxes $\mathcal{R}^{\frown}(i^*)$

In this section we define a family of auxiliary functions $G_i \colon \mathcal{R}(i^*) \longrightarrow \Omega$ with $i \in \mathbb{Z}$ and study their properties.

In what follows we consider the supremum metric d_∞ on the class of all functions $F \colon A \longrightarrow \Omega$ with $A \subset \Omega$. That is, given $F, G \colon A \longrightarrow \Omega$ we set

$$\mathsf{d}_\infty(F, G) := \sup_{(\theta,x)\in A} \mathsf{d}_\Omega(F(\theta, x), G(\theta, x)).$$

In the special case when F and G are skew products with the same base, that is when $F(\theta, x) = (R(\theta), f(\theta, x))$ and $G(\theta, x) = (R(\theta), g(\theta, x))$, then

$$\mathsf{d}_\infty(F, G) := \sup_{(\theta,x)\in A} |f(\theta, x) - g(\theta, x)|.$$

Observe that $(\mathcal{S}(\Omega), \mathsf{d}_\infty)$ is a complete metric space.

Before defining the maps G_i we need to introduce the necessary notation, and recall and collect some basic facts that we will use in this definition and to study their properties.

For every $i \in \mathbb{Z}$, we define

$$M_i \colon B_i^{\frown}[i^*] \longrightarrow \mathbb{I} \quad \text{by} \quad M_i(\theta) := \max\{x \in \mathbb{I} : (\theta, x) \in \mathcal{R}^{\frown}(i^*)\}, \text{ and}$$
$$m_i \colon B_i^{\frown}[i^*] \longrightarrow \mathbb{I} \quad \text{by} \quad m_i(\theta) := \min\{x \in \mathbb{I} : (\theta, x) \in \mathcal{R}^{\frown}(i^*)\}.$$

The next simple lemma states the basic properties of the maps m_i and M_i.

Lemma 4.1. *The following statements hold for every $i \in \mathbb{Z}$*
(a) $-1 \le m_i(\theta) \le M_i(\theta) \le 1$ *for every* $\theta \in B_i^{\frown}[i^*]$.
(b) m_i *and* M_i *are continuous.*
(c) $m_i\big|_{B_{\alpha_{|i|}}[i^*]}$ *and* $M_i\big|_{B_{\alpha_{|i|}}[i^*]}$ *are piecewise linear.*
(d) $m_i(\theta) = M_i(\theta) = \gamma_{|i|}(\theta)$ *if and only if* $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$.

Proof. It follows easily from Definition 3.1, the definition of a winged region and Lemma 3.6(b,f). □

Notice that, for every $i \in \mathbb{Z}$,

$$\mathcal{R}^{\frown}(i^*) = \bigcup_{\theta \in B_i^{\frown}[i^*]} \mathcal{R}^{\frown}(i^*)^\theta = \bigcup_{\theta \in B_i^{\frown}[i^*]} \{\theta\} \times [m_i(\theta), M_i(\theta)].$$

In what follows the interval $[m_i(\theta), M_i(\theta)] \subset \mathbb{I}$, defined for every $\theta \in B_i^{\frown}[i^*]$, will be denoted by $\mathbb{I}_{i,\theta}$. Clearly, for every $\theta \in B_i^{\frown}[i^*]$, $\mathcal{R}^{\frown}(i^*)^\theta = \{\theta\} \times \mathbb{I}_{i,\theta}$.

By Definition 3.4(R.2) and Remark 3.5(R.2),

$$B_i^{\frown}[i^*] \setminus \{i^*\} \quad \text{is disjoint from} \quad Z^*_{|i|}.$$

Hence, Lemmas 3.6(a,d) and 3.10(c) can be summarized as:

$$\text{(2)} \qquad \begin{cases} \gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus \{\ell^*\}} & \text{is continuous,} \\ \gamma_{|\ell|}(\theta) \in \mathbb{I}_{\ell,\theta} & \text{for every} \quad \theta \in B_\ell^{\frown}[\ell^*] \setminus \{\ell^*\}, \text{ and} \\ \mathsf{A}^\theta_{|\ell|} = \{(\theta, \gamma_{|\ell|}(\theta)\} & \text{for every} \quad \theta \in B_\ell^{\frown}[\ell^*] \setminus \{\ell^*\} \end{cases}$$

for $\ell \in \{i, i+1\}$.

Now we define a family of continuous maps $G_i \colon \mathcal{R}^{\frown}(i^*) \longrightarrow \Omega$ with $i \in \mathbb{Z}$, by

$$G_i(\theta, x) = \big(R_\omega(\theta), g_i(\theta, x)\big)$$

Also, for every $\theta \in B_i^{\frown}[i^*]$, we will denote the map $g_i(\theta, \cdot) \colon \mathbb{I}_{i,\theta} \longrightarrow \mathbb{I}$ by $g_{i,\theta}$.

To define the functions $g_{i,\theta}$, for clarity, we will consider separately two different situations:

- $i \ge 0$, when $\mathcal{R}^{\frown}(i^*) = \mathcal{R}(i^*)$, $B_i^{\frown}[i^*] = B_{\alpha_{|i|}}[i^*]$ and $G_i(\mathcal{R}(i^*))$ strictly contains the smaller box $\mathcal{R}((i+1)^*)$, and
- $i \le -1$, when $G_i(\mathcal{R}^{\frown}(i^*))$ is strictly contained in the bigger box $\mathcal{R}((i+1)^*)$.

We start by defining $g_{i,\theta}$ for $i \ge 0$ in three different ways, depending on the base point $\theta \in B_{\alpha_i}[i^*]$. In this definition, for simplicity we will use $\mathcal{R}(i^*)$ instead of $\mathcal{R}^{\frown}(i^*)$ and $B_{\alpha_{|i|}}[i^*]$ instead of $B_i^{\frown}[i^*]$.

Notice that, by Definition 3.4(R.1) and Lemma 3.6(c),

$$\text{(3)} \qquad \begin{aligned} &\text{for every } i \ge 0 \\ &B_{\delta_{i+1}}[i^*] \subset B_{\alpha_{i+1}}(i^*) \quad \text{and} \quad B_{\alpha_{i+1}}[i^*] \subset B_{\delta_i}(i^*) \subset B_{\alpha_i}(i^*), \text{ and} \\ &\gamma_{i-1}(i^*) = a_i \quad \text{and} \quad \gamma_i((i+1)^*) = a_{i+1}. \end{aligned}$$

Definition 4.2 (Definition of g_i for $i \ge 0$).

$\boldsymbol{\theta \in B_{\delta_{i+1}}[i^*]}$: $g_{i,\theta}(x) := \gamma_i((i+1)^*) + \frac{2^{n_i}}{2^{n_{i+1}}}\big(\gamma_{i-1}(i^*) - x\big)$.

$\boldsymbol{\theta \in B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)}$: we define $g_{i,\theta}$ to be the unique piecewise affine map with two affine pieces, defined on $\mathbb{I}_{i,\theta}$, whose graph joins $\big(m_i(\theta), M_{i+1}\big(R_\omega(\theta)\big)\big)$ with $\big(\gamma_i(\theta), \gamma_{i+1}\big(R_\omega(\theta)\big)\big)$, and this with the point $\big(M_i(\theta), m_{i+1}\big(R_\omega(\theta)\big)\big)$ (in particular, $g_{i,\theta}\big(\gamma_i(\theta)\big) = \gamma_{i+1}\big(R_\omega(\theta)\big)$),

$\boldsymbol{\theta \in B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*)}$: $g_{i,\theta}(x) := \gamma_{i+1}\big(R_\omega(\theta)\big)$ (that is, $g_{i,\theta}$ is constant).

■

The next lemma states the basic properties of the functions G_i for $i \geq 0$.

Lemma 4.3. *The following statements hold for every $i \geq 0$:*

(a) The map $g_{i,\theta}$ is well defined and non-increasing for every $\theta \in B_{\alpha_i}[i^*]$. Moreover, $-1 \leq g_{i,\theta}(x) \leq 1$ for every $\theta \in B_{\alpha_i}[i^*]$ and $x \in \mathbb{I}_{i,\theta}$. Furthermore, the function G_i is continuous.

(b) $G_i\big|_{\mathcal{R}(i^*)^\theta}$ is affine and $G_i\big(\mathcal{R}(i^*)^\theta\big) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ for every $\theta \in B_{\delta_{i+1}}[i^*]$; $G_i\big|_{\mathcal{R}(i^*)^\theta}$ is piecewise affine with two pieces and $G_i\big(\mathcal{R}(i^*)^\theta\big) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ for every $\theta \in B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)$; and $G_i\big(\mathcal{R}(i^*)^\theta\big) = \mathsf{A}_{i+1}^{R_\omega(\theta)}$ for every $\theta \in B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*)$.

(c) $G_i(\mathsf{A}_i^\theta) = \mathsf{A}_{i+1}^{R_\omega(\theta)}$ for every $\theta \in B_{\alpha_i}[i^*]$.

Proof. We will prove all statements of the lemma simultaneously and according to the regions in the definition of the map g_i.

- We start with the region $\mathcal{R}(i^*)^{\upuparrows B_{\delta_{i+1}}[i^*]}$.

Let $z \in [-\delta_i, \delta_i] \subset \mathbb{R}$ and let $\theta = i^* + z \in B_{\delta_i}[i^*]$. From Definition 3.1 and (3) we get

$$\begin{aligned} m_i(\theta) &= a_i - 2^{-n_i}(1-z) = \gamma_{i-1}(i^*) - 2^{-n_i}(1-z), \text{ and} \\ M_i(\theta) &= a_i + 2^{-n_i}(1-z) = \gamma_{i-1}(i^*) + 2^{-n_i}(1-z). \end{aligned} \tag{4}$$

In a similar way, for every $\theta \in B_{\delta_{i+1}}[i^*]$ (that is, $z \in [-\delta_{i+1}, \delta_{i+1}]$), we have $R_\omega(\theta) = (i+1)^* + z \in B_{\delta_{i+1}}[(i+1)^*]$, and

$$\begin{aligned} m_{i+1}(R_\omega(\theta)) &= a_{i+1} - 2^{-n_{i+1}}(1-z) = \gamma_i((i+1)^*) - 2^{-n_{i+1}}(1-z), \text{ and} \\ M_{i+1}(R_\omega(\theta)) &= a_{i+1} + 2^{-n_{i+1}}(1-z) = \gamma_i((i+1)^*) + 2^{-n_{i+1}}(1-z). \end{aligned} \tag{5}$$

Hence, for every $\theta \in B_{\delta_{i+1}}[i^*]$,

$$\begin{aligned} g_{i,\theta}(m_i(\theta)) &= \gamma_i((i+1)^*) + \tfrac{2^{n_i}}{2^{n_{i+1}}} 2^{-n_i}(1-z) = \gamma_i((i+1)^*) + 2^{-n_{i+1}}(1-z) \\ &= M_{i+1}(R_\omega(\theta)), \\ g_{i,\theta}(M_i(\theta)) &= \gamma_i((i+1)^*) - \tfrac{2^{n_i}}{2^{n_{i+1}}} 2^{-n_i}(1-z) = \gamma_i((i+1)^*) - 2^{-n_{i+1}}(1-z) \\ &= m_{i+1}(R_\omega(\theta)). \end{aligned} \tag{6}$$

So, $g_{i,\theta}\big|_{\mathbb{I}_{i,\theta}}$ is the affine map whose graph joins the point $\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big)$ with $\big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big)$. In particular, $g_{i,\theta}$ sends the interval $\mathbb{I}_{i,\theta}$ affinely onto $\mathbb{I}_{i+1,R_\omega(\theta)}$ or, equivalently, G_i sends the interval $\mathcal{R}(i^*)^\theta$ affinely onto $\mathcal{R}((i+1)^*)^{R_\omega(\theta)}$. Then, by Lemma 3.6(b), this implies that $-1 \leq g_{i,\theta}(x) \leq 1$ for every $x \in \mathbb{I}_{i,\theta}$. Moreover, the continuity of the maps m_i, M_i, $m_{i+1} \circ R_\omega$ and $M_{i+1} \circ R_\omega$ imply that g_i is well defined and continuous on $\mathcal{R}(i^*)^{\upuparrows B_{\delta_{i+1}}[i^*]}$

Next we will prove that $G_i\big(\mathsf{A}_i^\theta\big) = \mathsf{A}_{i+1}^{R_\omega(\theta)}$ for every $\theta \in B_{\delta_{i+1}}[i^*]$. We take $\theta = i^* + z \in B_{\delta_{i+1}}[i^*] \setminus \{i^*\}$. Then, clearly, $z \in [-\delta_{i+1}, \delta_{i+1}] \setminus \{0\} \subset \mathbb{R}$. By Definitions 3.4 and 3.1 and statement (3),

$$\gamma_i(\theta) = \varphi_{i^*}(\theta) = a_i + 2^{-n_i} d = \gamma_{i-1}(i^*) + 2^{-n_i} d \in \mathbb{I}_{i,\theta}, \text{ and}$$
$$\gamma_{i+1}(R_\omega(\theta)) = \varphi_{(i+1)^*}(\theta) = a_{i+1} - 2^{-n_{i+1}} d = \gamma_{i-1}(i^*) - 2^{-n_{i+1}} d \in \mathbb{I}_{i+1,R_\omega(\theta)},$$

where $d = (-1)^i \phi(z)$. So, for every $\theta \in B_{\delta_{i+1}}[i^*] \setminus \{i^*\}$,

$$g_{i,\theta}(\gamma_i(\theta)) = \gamma_i((i+1)^*) - \frac{2^{n_i}}{2^{n_{i+1}}} 2^{-n_i} d = \gamma_{i+1}(R_\omega(\theta)). \tag{7}$$

Thus, from (3) and (2) we get

$$\begin{aligned} G_i\big(\mathsf{A}_i^{\theta}\big) &= G_i\big(\{(\theta, \gamma_i(\theta))\}\big) = \{(R_\omega(\theta), g_{i,\theta}(\gamma_i(\theta)))\} \\ &= \{(R_\omega(\theta), \gamma_{i+1}(R_\omega(\theta)))\} = \mathsf{A}_{i+1}^{R_\omega(\theta)} \end{aligned}$$

for every $\theta \in B_{\delta_{i+1}}[i^*] \setminus \{i^*\}$. On the other hand, by the part already proven, g_{i,i^*} sends the interval $\mathbb{I}_{i,i^*}$ affinely to $\mathbb{I}_{i+1,(i+1)^*}$ or, equivalently, G_i sends the interval $\mathcal{R}(i^*)^{i^*} = \{i^*\} \times \mathbb{I}_{i,i^*}$ affinely onto $\mathcal{R}((i+1)^*)^{(i+1)^*} = \{(i+1)^*\} \times \mathbb{I}_{i,(i+1)^*}$. This implies that $G_i\big(\mathsf{A}_i^{i^*}\big) = \mathsf{A}_{i+1}^{(i+1)^*}$ by Lemma 3.10(b). Hence, $G_i\big(\mathsf{A}_i^{\theta}\big) = \mathsf{A}_{i+1}^{R_\omega(\theta)}$ for every $\theta \in B_{\delta_{i+1}}[i^*]$.

- Now we study $\mathcal{R}(i^*)^{\upuparrows\left(B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)\right)}$.

Observe that $R_\omega(B_\alpha[i^*] \setminus \{i^*\}) = B_\alpha[(i+1)^*] \setminus \{(i+1)^*\}$ for $\alpha \in \{\alpha_i, \alpha_{i+1}\}$. Then, by (2)

$$(8) \qquad \begin{aligned} &\gamma_{i+1} \circ R_\omega\big|_{B_{\alpha_i}[i^*] \setminus \{i^*\}} \quad \text{is continuous, and} \\ &\gamma_{i+1}(R_\omega(\theta)) \in \mathbb{I}_{i+1,R_\omega(\theta)} \quad \text{for every} \quad \theta \in B_{\alpha_{i+1}}[i^*] \setminus \{i^*\}. \end{aligned}$$

So, the continuity of the maps $m_i,\ M_i,\ m_{i+1} \circ R_\omega$ and $M_{i+1} \circ R_\omega$ imply that g_i is well defined and continuous on $\mathcal{R}(i^*)^{\upuparrows\left(B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)\right)}$, and

$$\big(\gamma_i(\theta), \gamma_{i+1}(R_\omega(\theta))\big) \in \mathbb{I}_{i,\theta} \times \mathbb{I}_{i+1,R_\omega(\theta)}$$

for every $\theta \in B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)$. Consequently, $g_{i,\theta}$ maps $\mathbb{I}_{i,\theta}$ piecewise affinely with two pieces onto $\mathbb{I}_{i+1,R_\omega(\theta)}$ or, equivalently, G_i sends the interval $\mathcal{R}(i^*)^{\theta}$ piecewise affinely with two pieces onto $\mathcal{R}((i+1)^*)^{R_\omega(\theta)}$. Again, by Lemma 3.6(b), this implies that $-1 \le g_{i,\theta}(x) \le 1$ for every $x \in \mathbb{I}_{i,\theta}$. On the other hand, from (3) and (2) we have

$$\begin{aligned} G_i\big(\mathsf{A}_i^{\theta}\big) &= G_i\big(\{(\theta, \gamma_i(\theta))\}\big) = \{(R_\omega(\theta), g_{i,\theta}(\gamma_i(\theta)))\} \\ &= \{(R_\omega(\theta), \gamma_{i+1}(R_\omega(\theta)))\} = \mathsf{A}_{i+1}^{R_\omega(\theta)} \end{aligned}$$

for every $\theta \in B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)$.

- Finally, we study the region $\mathcal{R}(i^*)^{\upuparrows\left(B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*)\right)}$.

In this case, by definition and Lemma 3.6(b) we have $-1 \le g_{i,\theta}(x) \le 1$ for every $x \in \mathbb{I}_{i,\theta}$. By (8), $g_i(\cdot, x) = \gamma_{i+1} \circ R_\omega$ is well defined and continuous in both variables on $\mathcal{R}(i^*)^{\upuparrows\left(B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*)\right)}$ because m_i and M_i are continuous. Moreover, for every $\theta \in B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*)$ and x such that $(\theta, x) \in \mathcal{R}(i^*)^{\theta}$, we have

$$\{G_i(\theta, x)\} = \{(R_\omega(\theta), g_i(\theta, x))\} = \{(R_\omega(\theta), \gamma_{i+1}(R_\omega(\theta))\} = \mathsf{A}_{i+1}^{R_\omega(\theta)}$$

by Definition 3.9 and Lemma 2.2(a). Thus, by Lemma 3.10(a),

$$G_i\big(\mathsf{A}_i^{\theta}\big) = G_i\big(\mathcal{R}(i^*)^{\theta}\big) = \mathsf{A}_{i+1}^{R_\omega(\theta)}.$$

From all the previous arguments (b) and (c) follow. To end the proof of (a) we have to see that G_i is well defined and globally continuous. This amounts to show that it is well defined on the fibres

$$\begin{aligned} \mathcal{R}(i^*)^{(i^* \pm \delta_{i+1})} &= \{i^* \pm \delta_{i+1}\} \times \mathbb{I}_{i,i^* \pm \delta_{i+1}} \text{ and} \\ \mathcal{R}(i^*)^{(i^* \pm \alpha_{i+1})} &= \{i^* \pm \alpha_{i+1}\} \times \mathbb{I}_{i,i^* \pm \alpha_{i+1}}. \end{aligned}$$

We will only show that the two definitions of g_i coincide on $\{\theta\} \times \mathbb{I}_{i,\theta}$ with $\theta \in \{i^* + \delta_{i+1}, i^* + \alpha_{i+1}\}$. The case $\theta \in \{i^* - \delta_{i+1}, i^* - \alpha_{i+1}\}$ follows analogously.

We start with $\theta = i^* + \alpha_{i+1} \in B_{\delta_i}(i^*)$. In this case, $R_\omega(\theta) = (i+1)^* + \alpha_{i+1} \in \mathsf{Bd}(B_{\alpha_{i+1}}[(i+1)^*])$ and, by Definition 3.1 and Lemma 3.6(c),

$$M_{i+1}(R_\omega(\theta)) = m_{i+1}(R_\omega(\theta)) = a^+_{i+1} = \gamma_{i+1}(R_\omega(\theta)).$$

Thus, the piecewise affine map whose graph joins the points $\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big)$, $\big(\gamma_i(\theta), \gamma_{i+1}(R_\omega(\theta))\big)$, and $\big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big)$ is the constant map $\gamma_{i+1}(R_\omega(\theta))$. Hence, $g_{i,\theta}$ is well defined for $\theta = i^* + \alpha_{i+1}$.

Now we deal with the case $\theta = i^* + \delta_{i+1} \in B_{\delta_i}[i^*]$. By (6) and (7) we know that the points $\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big)$, $\big(\gamma_i(\theta), \gamma_{i+1}(R_\omega(\theta))\big)$ and $\big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big)$ belong to $\mathsf{Graph}\,\big(x \mapsto \gamma_i((i+1)^*) + \frac{2^{n_i}}{2^{n_{i+1}}}\big(\gamma_{i-1}(i^*) - x\big)\big)$. Consequently, the map $\gamma_i((i+1)^*) + \frac{2^{n_i}}{2^{n_{i+1}}}\big(\gamma_{i-1}(i^*) - x\big)$ coincides with the piecewise affine map whose graph joins $\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big)$, $\big(\gamma_i(\theta), \gamma_{i+1}(R_\omega(\theta))\big)$ and $\big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big)$. This ends the proof of (a). □

Now we define $g_{i,\theta}$ for $i < 0$. In this case, since we are going from a smaller box $\mathcal{R}^{\frown}(i^*)$ to a bigger one, we only need to define $g_{i,\theta}$ in two different ways, depending on the base point $\theta \in B_i^{\frown}[i^*]$.

As in the previous case we need to fix some facts about the elements that we will use in the definition.

By Definition 3.4(R.1) and Lemma 3.6(c),

$$\text{(9)}\quad \begin{aligned} &\text{for every } i < 0\\ &B_{\delta_{|i|}}[(i+1)^*] \subset B_{\alpha_{|i|}}[(i+1)^*] \subset B_{\delta_{|i+1|}}((i+1)^*) \subset B_{\alpha_{|i+1|}}((i+1)^*),\\ &R_\omega\left(B_i^{\frown}[i^*]\right) = B_{\alpha_{|i+1|}}[(i+1)^*], \quad B_{\delta_{|i|}}[i^*] \subset B_{\alpha_{|i|}}(i^*), \text{ and}\\ &\gamma_{|i+1|}(i^*) = a_i \quad\text{and}\quad \gamma_{|i+2|}((i+1)^*) = a_{i+1}. \end{aligned}$$

Consequently, from (2) and Definitions 3.1 and 3.4 we get

$$\begin{aligned} &m_i(\theta) < \gamma_{|i|}(\theta) < M_i(\theta) \text{ and}\\ &m_{i+1}(R_\omega(\theta)) < \gamma_{|i+1|}(R_\omega(\theta)) < M_{i+1}(R_\omega(\theta)) \end{aligned}$$

for every $\theta \in B_{\alpha_{|i|}}(i^*) \setminus \{i^*\}$ (and $R_\omega(\theta) \in B_{\alpha_{|i|}}((i+1)^*) \setminus \{(i+1)^*\}$). Then,

$$\widetilde{\kappa}_i(\theta) = \min\left\{1, \frac{m_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}(\gamma_{|i|}(\theta) - M_i(\theta))}, \frac{M_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}(\gamma_{|i|}(\theta) - m_i(\theta))}\right\} > 0$$

defines a continuous function $\widetilde{\kappa}_i \colon B_{\alpha_{|i|}}(i^*) \setminus B_{\delta_{|i|}}(i^*) \longrightarrow (0, 1]$. To define the map g_i we need an auxiliary function

$$\kappa_i \colon B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*) \longrightarrow [0, 1]$$

such that κ_i is non-decreasing and continuous, $\kappa_i(i^* \pm \delta_{|i|}) = \widetilde{\kappa}_i(i^* \pm \delta_{|i|})$, and $\kappa_i(\theta) \le \widetilde{\kappa}_i(\theta)$ for every $\theta \in B_{\alpha_{|i|}}(i^*) \setminus B_{\delta_{|i|}}(i^*)$. In principle any such function would do, but for definiteness, and to show that such function exists, we note that we can take, for instance,

$$\kappa_i(\theta) = \begin{cases} \inf_{t \in [\theta, i^* - \delta_{|i|}] \cap B_{\alpha_{|i|}}(i^*)} \widetilde{\kappa}_i(t) & \text{if } \theta \le i^* - \delta_{|i|},\\ \inf_{t \in [i^* + \delta_{|i|}, \theta] \cap B_{\alpha_{|i|}}(i^*)} \widetilde{\kappa}_i(t) & \text{if } \theta \ge i^* + \delta_{|i|}. \end{cases}$$

It is easy to check that this map verifies the desired properties.

Definition 4.4 (Definition of g_i for $i < 0$). For every $(\theta, x) \in \mathcal{R}^{\frown}(i^*)$ we set

$$g_{i,\theta}(x) := \begin{cases} \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\left(\gamma_{|i+1|}(i^*) - x\right) + \gamma_{|i+2|}((i+1)^*) & \text{if } \theta \in B_{\delta_{|i|}}[i^*],\\ \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\kappa_i(\theta)\left(\gamma_{|i|}(\theta) - x\right) + \gamma_{|i+1|}(R_\omega(\theta)) & \text{if } \theta \in B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*)\\ \gamma_{|i+1|}(R_\omega(\theta)) & \text{if } \theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*). \end{cases}$$

□

The next lemma states the basic properties of the functions G_i for $i < 0$.

Lemma 4.5. *The following statements hold for every $i < 0$:*

(a) The map $g_{i,\theta}$ is well defined and non-increasing for every $\theta \in B_{\alpha_i}[i^*]$. Moreover, $-1 \le g_{i,\theta}(x) \le 1$ for every $\theta \in B_{\alpha_i}[i^*]$ and $x \in \mathbb{I}_{i,\theta}$. Furthermore, the function G_i is continuous.

(b) $G_i\big|_{\mathcal{R}^{\frown}(i^*)^\theta}$ is affine, $G_i\big(\mathcal{R}^{\frown}(i^*)^\theta\big) \subset \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ for every $\theta \in B_i^{\frown}[i^*]$ and $G_i\big(\mathcal{R}^{\frown}(i^*)^\theta\big) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ for every $\theta \in B_{\delta_{|i|}}[i^*]$.

(c) $G_i(\mathsf{A}^\theta_{|i|}) = \mathsf{A}^{R_\omega(\theta)}_{|i+1|}$ for every $\theta \in B_i^{\frown}[i^*]$.

Proof. First we will prove that the map G_i is continuous and that $G_i\big|_{\mathcal{R}^{\frown}(i^*)^\theta}$ is affine, according to the three regions in the definition.

- As in the previous lemma we start with $\mathcal{R}^{\frown}(i^*)^{\upuparrows B_{\delta_{|i|}}[i^*]} = \mathcal{R}(i^*)^{\upuparrows B_{\delta_{|i|}}[i^*]}$.

As in the same case of Lemma 4.3, by using (9) instead of (3), it follows that $g_{i,\theta}\big|_{\mathbb{I}_{i,\theta}}$ is the affine map whose graph joins the points $\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big)$ and $\big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big)$, g_i is well defined and continuous on $\mathcal{R}(i^*)^{\upuparrows B_{\delta_{|i|}}[i^*]}$,

$$g_{i,\theta}(\gamma_{|i|}(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)) \text{ for every } \theta \in B_{\delta_{|i|}}[i^*] \setminus \{i^*\},$$
$$G_i \text{ sends the interval } \mathcal{R}(i^*)^\theta \text{ affinely onto } \mathcal{R}((i+1)^*)^\theta, \text{ and}$$
$$G_i\big(\mathsf{A}^\theta_{|i|}\big) = \mathsf{A}^{R_\omega(\theta)}_{|i+1|} \quad \text{for every } \theta \in B_{\delta_{|i|}}[i^*].$$

- $\mathcal{R}^{\frown}(i^*)^{\upuparrows\left(B_{\alpha_{|i|}}[i^*]\setminus B_{\delta_{|i|}}(i^*)\right)} = \mathcal{R}(i^*)^{\upuparrows\left(B_{\alpha_{|i|}}[i^*]\setminus B_{\delta_{|i|}}(i^*)\right)}$.

From (2) we know that the maps $\gamma_{|i|}$ and $\gamma_{|i+1|} \circ R_\omega$ are continuous on the domain $B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*)$. Hence, the continuity of g_i follows from the continuity of the maps κ_i, m_i, M_i, $m_{i+1} \circ R_\omega$ and $M_{i+1} \circ R_\omega$.

Notice that, from the definition of g_i in this region we clearly have that

$$g_{i,\theta}(\gamma_{|i|}(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)), \text{ and}$$

$G_i\big|_{\mathcal{R}^{\frown}(i^*)^\theta} = g_i(\theta, \cdot)$ is affine.

- $\mathcal{R}^{\frown}(i^*)^{\upuparrows\left(B_i^{\frown}[i^*]\setminus B_{\alpha_{|i|}}(i^*)\right)}$.

In this case we have $m_i(\theta) = \gamma_{|i|}(\theta) = M_i(\theta)$ by definition. Then, the map $G_i\big|_{\mathcal{R}^{\frown}(i^*)^\theta} = g_i(\theta, \cdot)$ is affine because it is constant, and g_i is continuous because $\gamma_{|i|}$ and $\gamma_{|i+1|} \circ R_\omega$ are continuous on the domain $B_i^{\frown}[i^*] \setminus \{i^*\}$ by (2).

To end the proof of (a) we have to see that G_i is well defined and globally continuous. This amounts to show that it is well defined on the fibres

$$\mathcal{R}(i^*)^{(i^* \pm \delta_{|i|})} \quad \text{and} \quad \mathcal{R}(i^*)^{(i^* \pm \alpha_{|i|})}$$

We start by showing that the two definitions of g_i coincide on the fibres $\mathcal{R}(i^*)^\theta$ for $\theta \in \{i^* \pm \alpha_{|i|}\}$. In this case we have $m_i(\theta) = \gamma_{|i|}(\theta) = M_i(\theta)$. Consequently, $\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\}$ and

$$\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} \kappa_i(\theta) \left(\gamma_{|i|}(\theta) - x\right) + \gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|i+1|}(R_\omega(\theta))$$

for $x \in \mathbb{I}_{i,\theta}$.

Next we consider $\mathcal{R}(i^*)^\theta = \{\theta\} \times \mathbb{I}_{i,\theta}$ with $\theta = i^* + \delta_{|i|}$. We will show that the two definitions of g_i coincide on this set. The case $\theta = i^* - \delta_{|i|}$ follows analogously.

For simplicity we will denote

$$g^{\delta_{|i|}}_{i,\theta}(x) := \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\big(\gamma_{|i+1|}(i^*) - x\big) + \gamma_{|i+2|}((i+1)^*), \text{ and}$$
$$\xi_{i,\theta}(x) := \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\big(\gamma_{|i|}(\theta) - x\big) + \gamma_{|i+1|}(R_\omega(\theta)).$$

Notice that $g^{\delta_{|i|}}_{i,\theta}$ is the map $g_{i,\theta}$ as defined in the first region while

$$\kappa_i(\theta)\left(\xi_{i,\theta} - \gamma_{|i+1|}(R_\omega(\theta))\right) + \gamma_{|i+1|}(R_\omega(\theta))$$

is the map $g_{i,\theta}$ as defined in the second region. In a similar way to the previous lemma we have that $\big(\gamma_{|i|}(\theta), \gamma_{|i+1|}(R_\omega(\theta))\big) \in \mathsf{Graph}\left(g^{\delta_{|i|}}_{i,\theta}\right)$. Hence, since $g^{\delta_{|i|}}_{i,\theta}$ is affine with slope $-\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}$, it follows that $g^{\delta_{|i|}}_{i,\theta} = \xi_{i,\theta}$. So, to end the proof of the lemma, we only have to see that $\kappa_i(i^* + \delta_{|i|}) = \widetilde{\kappa}_i(i^* + \delta_{|i|}) = 1$.

Since the points $\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big)$ and $\big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big)$ also belong to $\mathsf{Graph}\left(g^{\delta_{|i|}}_{i,\theta}\right) = \mathsf{Graph}\left(\xi_{i,\theta}\right)$, it follows that

$$m_{i+1}(R_\omega(\theta)) = \xi_{i,\theta}(M_i(\theta)) = \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\big(\gamma_{|i|}(\theta) - M_i(\theta)\big) + \gamma_{|i+1|}(R_\omega(\theta)), \text{ and}$$
$$M_{i+1}(R_\omega(\theta)) = \xi_{i,\theta}(m_i(\theta)) = \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\big(\gamma_{|i|}(\theta) - m_i(\theta)\big) + \gamma_{|i+1|}(R_\omega(\theta)).$$

This shows that $\widetilde{\kappa}_i(i^* + \delta_{|i|}) = \widetilde{\kappa}_i(\theta) = 1$ and ends the proof of (a).

Now we prove (b) according to the three regions in the definition. From the part of the lemma already proven we already know that $G_i\big|_{\mathcal{R}^{\frown}(i^*)^\theta}$ is affine, and $G_i\big(\mathcal{R}^{\frown}(i^*)^\theta\big) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ for every $\theta \in B_{\delta_{|i|}}[i^*]$. So, to end the proof of (b) we have to see that

$$g_{i,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{i+1,R_\omega(\theta)} \tag{10}$$

for every $\theta \in B^{\frown}_i[i^*] \setminus B_{\delta_{|i|}}[i^*]$ (by definition, since $i < 0$, $B^{\frown}_i[i^*] = B_{\alpha_{|i+1|}}[i^*]$; therefore, $R_\omega(\theta) \in B_{\alpha_{|i+1|}}[(i+1)^*]$ and $\mathbb{I}_{i+1,R_\omega(\theta)} = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$).

For $\theta \in B^{\frown}_i[i^*] \setminus B_{\alpha_{|i|}}(i^*)$, by (2), we have

$$g_{i,\theta}(\mathbb{I}_{i,\theta}) = \{\gamma_{|i+1|}(R_\omega(\theta))\} \subset \mathbb{I}_{i+1,R_\omega(\theta)}.$$

Now we consider $\theta \in B_{\alpha_{|i|}}(i^*) \setminus B_{\delta_{|i|}}[i^*]$. Since

$$\kappa_i(\theta) \le \widetilde{\kappa}_i(\theta) \le \frac{M_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}(\gamma_{|i|}(\theta) - m_i(\theta))},$$

we have

$$\begin{aligned} g_{i,\theta}(m_i(\theta)) &\le \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\frac{M_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}(\gamma_{|i|}(\theta) - m_i(\theta))}\big(\gamma_{|i|}(\theta) - m_i(\theta)\big) + \gamma_{|i+1|}(R_\omega(\theta)) \\ &= M_{i+1}(R_\omega(\theta)). \end{aligned}$$

An analogous computation shows that $g_{i,\theta}(M_i(\theta)) \ge m_{i+1}(R_\omega(\theta))$. Hence, (10) holds because $g_{i,\theta}$ is affine. This ends the proof of (b).

Then, by Lemma 3.6(b), Statement (b) of the lemma implies that $-1 \le g_{i,\theta}(x) \le 1$ for every $x \in \mathbb{I}_{i,\theta}$.

By the part of the lemma already proved we know that $G_i\big(\mathsf{A}^\theta_{|i|}\big) = \mathsf{A}^{R_\omega(\theta)}_{|i+1|}$ for every $\theta \in B_{\delta_{|i|}}[i^*]$. On the other hand, as in the previous lemma, from (9) and (2) we get

$$G_i\big(\mathsf{A}^{\theta}_{|i|}\big) = G_i\big(\{(\theta, \gamma_{|i|}(\theta))\}\big) = \{(R_\omega(\theta), g_{i,\theta}(\gamma_{|i|}(\theta)))\}$$
$$= \{(R_\omega(\theta), \gamma_{|i+1|}(R_\omega(\theta)))\} = \mathsf{A}^{R_\omega(\theta)}_{|i+1|}$$

for every $\theta \in B^{\smile}_i[i^*] \setminus B_{\delta_{|i|}}[i^*]$. So, (c) holds. □

Up to now we have defined the family of auxiliary functions $G_i \colon \mathcal{R}^{\smile}(i^*) \longrightarrow \Omega$ with $i \in \mathbb{Z}$. The next step before being able to define the family $\{T_m\} \subset \mathcal{S}(\Omega)$ is to fix some stratification in the set of boxes $\mathcal{R}^{\smile}(i^*)$.

5. A stratification in the set of boxes $\mathcal{R}^{\smile}(i^*)$

In this section we introduce a notion of *depth* in the set of arcs $B^{\smile}_i[i^*]$ defined earlier. This notion introduce a stratification in the set of boxes $\mathcal{R}^{\smile}(i^*)$ that we study below.

Definition 5.1. For every $\ell \in \mathbb{Z}$ we define the *depth of* ℓ, which will be denoted by $\mathsf{depth}(\ell)$, as the cardinality of the set (see Lemma 3.6(g))

$$\{i \in \mathbb{Z} : B^{\smile}_\ell[\ell^*] \subsetneq B^{\smile}_i[i^*]\} = \{i \in \mathbb{Z} : B^{\smile}_\ell[\ell^*] \cap B^{\smile}_i[i^*] \neq \emptyset\} =$$
$$\{i \in \mathbb{Z} : \mathcal{R}^{\smile}(\ell^*) \subsetneq \mathcal{R}^{\smile}(i^*)\} = \{i \in \mathbb{Z} : \mathcal{R}^{\smile}(\ell^*) \cap \mathcal{R}^{\smile}(i^*) \neq \emptyset\}.$$

Also, for every $m \in \mathbb{Z}^+$, we denote

$$\mathfrak{D}_m := \{\ell \in \mathbb{Z} : \mathsf{depth}(\ell) = m\},$$
$$\mathfrak{D}^*_m := \{i^* : i \in \mathfrak{D}_m\}, \text{ and}$$
$$\mu_m := \min\{|i| : i \in \mathfrak{D}_m\}.$$

The next lemma studies the stratification on $\mathbb{Z}$ created by the notion of *depth*.

Lemma 5.2. *The following statements hold:*

(a) $\mathfrak{D}_{m+1} \subset \{\ell \in \mathbb{Z} : \exists\, i \in \mathfrak{D}_m$ *such that* $B^{\smile}_\ell[\ell^*] \subsetneq B^{\smile}_i[i^*]\}$.
(b) For every $\ell, k \in \mathfrak{D}_m$ *it follows that* $B^{\smile}_\ell[\ell^*] \cap B^{\smile}_k[k^*] = \emptyset$.

Proof. Observe that if $B^{\smile}_\ell[\ell^*] \subsetneq B^{\smile}_i[i^*]$ then $\mathsf{depth}(\ell) \geq \mathsf{depth}(i) + 1$. Hence, (a) holds.

Statement (b) follows from Lemma 3.6(g). □

In what follows, for every $m \in \mathbb{Z}^+$ we set

$$\mathbb{B}^{\smile}_m := \bigcup_{i \in \mathfrak{D}_m} B^{\smile}_i[i^*] \supset \mathfrak{D}^*_m.$$

Note that, by Lemma 5.2(b), $\mathbb{B}^{\smile}_m$ is a disjoint union of closed arcs. Therefore, for every $\theta \in \mathbb{B}^{\smile}_m$, there exists a unique $i \in \mathfrak{D}_m$ such that $\theta \in B^{\smile}_i[i^*]$. We will denote such integer i by $\mathsf{b}^{\smile}(\theta, m) \in \mathfrak{D}_m$.

The next two lemmas study the properties of the winged boxes $B^{\smile}_i[i^*]$ and $\mathcal{R}^{\smile}(i^*)$ according to the depth stratification. Lemma 5.4 is the real motivation to introduce the winged boxes.

Lemma 5.3. *The following statements hold:*

(a) The sequence $\{\mu_m\}_{m=0}^{\infty}$ *is strictly increasing. In particular* $\lim_{m\to\infty} \mu_m = \infty$.
(b) For every $m \in \mathbb{Z}^+$, $\mathbb{B}^{\smile}_m$ *is dense in* $\mathbb{S}^1$, $\mathbb{B}^{\smile}_{m+1} \subset \mathbb{B}^{\smile}_m$ *and* $\mathfrak{D}^*_m \cap \mathbb{B}^{\smile}_{m+1} = \emptyset$.
(c) $O^*(\omega) \subset \mathbb{B}^{\smile}_0$, *and* $\mathsf{A}^\theta = \{(\theta, 0)\}$ *for every* $\theta \in \mathbb{S}^1 \setminus \mathbb{B}^{\smile}_0$.
(d) Let $i \in \mathbb{Z}$ *and* $\theta \in B^{\smile}_i[i^*] \setminus \mathbb{B}^{\smile}_{\mathsf{depth}(i)+1}$. *Then,* $\theta \notin O^*(\omega)$ *unless* $\theta = i^*$, *and* $\mathsf{A}^\theta_n = \mathsf{A}^\theta_{|i|}$ *for every* $n \geq |i|$. *In particular* $\mathsf{A}^\theta = \mathsf{A}^\theta_{|i|}$.

Proof. By Lemmas 5.2(a) and 3.6(g) it follows that for every $m \in \mathbb{Z}+$ and $\ell \in \mathfrak{D}_{m+1}$ there exists $i \in \mathfrak{D}_m$ such that $B^{\frown}_{\ell}[\ell^*] \subsetneq B^{\frown}_{i}[i^*]$ and $|i| < |\ell|$. Thus, $\mathbb{B}^{\frown}_{m+1} \subset \mathbb{B}^{\frown}_{m}$ and $\mu_m < \mu_{m+1}$. This proves (a) and the second statement of (b).

Next we will show that $i^* \notin \mathbb{B}^{\frown}_{m+1}$ for every $i \in \mathfrak{D}_m$. Assume by way of contradiction that there exists $i \in \mathfrak{D}_m$ such that $i^* \in \mathbb{B}^{\frown}_{m+1}$. Let $k = \mathsf{b}^{\frown}(i^*, m+1) \in \mathfrak{D}_{m+1}$. Clearly, $i \neq k$ and $i^* \in B^{\frown}_{k}[k^*]$. Then, by Lemma 3.6(g), $|k| < |i|$ and $B^{\frown}_{i}[i^*] \subsetneq B^{\frown}_{k}[k^*]$. Thus,

$$m = \mathsf{depth}(i) \geq \mathsf{depth}(k) + 1 = m + 2;$$

a contradiction.

Now we prove the first statement of (c). From the definitions and the part of (b) already proven we have

$$O^*(\omega) \subset \bigcup_{i \in \mathbb{Z}} B^{\frown}_{i}[i^*] \subset \bigcup_{m=0}^{\infty} \mathbb{B}^{\frown}_{m} = \mathbb{B}^{\frown}_{0}.$$

To end the proof of (b) it remains to show the density of $\mathbb{B}^{\frown}_{m}$. We will do it by induction on m. Clearly $\mathbb{B}^{\frown}_{0} \supset O^*(\omega)$ is dense in $\mathbb{S}^1$ because so is $O^*(\omega)$. Suppose that (b) holds for $\mathbb{B}^{\frown}_{m}$. We will show that (b) also holds for $\mathbb{B}^{\frown}_{m+1}$. Choose $\theta \in \mathbb{B}^{\frown}_{m}$ and set $i = \mathsf{b}^{\frown}(\theta, m)$. Since $O^*(\omega)$ is dense in $\mathbb{S}^1$, there exists a sequence $\{s_n\}_{n=0}^{\infty} \subset \mathbb{Z}$ such that $s_n^* \in B^{\frown}_{i}(i^*)$ and $\lim_{n\to\infty} s_n^* = \theta$. As above, we get that $\mathsf{depth}(s_n) \geq \mathsf{depth}(i) + 1 = m + 1$. Moreover, $s_n^* \in \mathbb{B}^{\frown}_{\mathsf{depth}(s_n)} \subset \mathbb{B}^{\frown}_{m+1}$ for every n. Consequently, $\mathbb{B}^{\frown}_{m} \subset \overline{\mathbb{B}^{\frown}_{m+1}}$, and the density of $\mathbb{B}^{\frown}_{m+1}$ follows from the density of $\mathbb{B}^{\frown}_{m}$.

Next we prove the second statement of (c). From above it follows that

$$\bigcup_{i \in \mathbb{Z}} B_{\alpha_{|i|}}[i^*] \subset \bigcup_{i \in \mathbb{Z}} B^{\frown}_{i}[i^*] \subset \mathbb{B}^{\frown}_{0}.$$

Hence, by the definition of the maps γ_m (Definition 3.4) it follows that $\gamma_m(\theta) = \gamma_0(\theta) = 0$ for every $\theta \notin \mathbb{B}^{\frown}_{0}$ and $m \in \mathbb{Z}^+$. So, $\gamma(\theta) = \lim_{m\to\infty} \gamma_m(\theta) = 0$, and $\mathsf{A}^\theta = \{(\theta, \gamma(\theta))\} = \{(\theta, 0)\}$ by Lemma 3.11(c). This ends the proof of (c).

(d) If $\theta = i^*$ then the statement follows from Lemmas 3.10(b) and 3.11(b). So, we assume that $\theta \neq i^*$.

By Definition 3.4(R.2) and Remark 3.5(R.2) we get that $\theta \notin Z^*_{|i|+1}$. Hence, if $\theta \in O^*(\omega)$, it follows that $\theta = k^* \in \mathbb{B}^{\frown}_{\mathsf{depth}(k)}$ with $|k| > |i| + 1$ and $B^{\frown}_{k}[k^*] \cap B^{\frown}_{i}[i^*] \neq \emptyset$. Thus, by Lemma 3.6(g), $\mathsf{depth}(k) \geq \mathsf{depth}(i) + 1$. By (b), this implies that $\theta = k^* \in \mathbb{B}^{\frown}_{\mathsf{depth}(i)+1}$; a contradiction. Therefore, $\theta \notin O^*(\omega)$. On the other hand, $\theta \notin B^{\frown}_{-i}[(-i)^*]$ by Definition 3.4(R.2).

If $\theta \notin B_{\alpha_{|k|}}[k^*]$ for every $k \in \mathbb{Z}$ such that $|k| > |i|$, then $\gamma_n(\theta) = \gamma_{|i|}(\theta)$ and $\mathsf{A}^\theta_n = \mathsf{A}^\theta_{|i|}$ for every $n \geq |i|$, by Definition 3.4 and Lemma 3.10(c).

Now assume that $\theta \in B_{\alpha_{|k|}}[k^*]$ for some $k \in \mathbb{Z}$ such that $|k| > |i|$ and $|k|$ is minimal with these properties. If $\theta \in B^{\frown}_{k}(k^*)$, as above we get that $\mathsf{depth}(k) \geq \mathsf{depth}(i) + 1$ and $\theta \in \mathbb{B}^{\frown}_{\mathsf{depth}(k)} \subset \mathbb{B}^{\frown}_{\mathsf{depth}(i)+1}$. Thus, $\theta \in \mathsf{Bd}(B^{\frown}_{k}[k^*]) = \mathsf{Bd}(B_{\alpha_{|k|}}[k^*])$ and $k \geq 0$. So, by Lemma 3.6(c) and the definition of the maps γ_j (Definition 3.4), $\gamma_{|k|}(\theta) = \gamma_{|k|-1}(\theta)$. Moreover, by Lemma 3.6(e), $\gamma_j(\theta) = \gamma_{|k|}(\theta)$ for every $j > |k|$. On the other hand, the minimality of $|k|$ implies that $\theta \notin B_{\alpha_{|\ell|}}[\ell^*]$ for every $\ell \in \mathbb{Z}$ such that $|k| > |\ell| > |i|$. Hence, by the definition of the maps γ_j (Definition 3.4), $\gamma_j(\theta) = \gamma_{|i|}(\theta)$ for every $|k| > j > |i|$. In short, we have proved that $\gamma_j(\theta) = \gamma_{|i|}(\theta)$ for every $j \geq |i|$. Thus, as above, $\mathsf{A}^\theta_n = \mathsf{A}^\theta_{|i|}$ for every $n \geq |i|$. This ends the proof of the lemma. □

Lemma 5.4. *Assume that* $B_i^{\frown}[i^*] \subset B_k^{\frown}[k^*]$ *for some* $i \in \mathfrak{D}_m$, $k \in \mathfrak{D}_{m-1}$ *and* $m \in \mathbb{N}$. *Then,* $|k| < |i|$ *and* $|k+1| < |i+1|$ *unless* $k \geq 0$ *and* $i = -(k+2)$ *(whence* $|k+1| = |i+1|$*). Moreover, the following statements hold:*

(a) For every $\theta \in B_i^{\frown}[i^*]$,

$$\gamma_{|k|}(\theta) = \gamma_{|k|+1}(\theta) = \cdots = \gamma_{|i|-1}(\theta) \in \mathbb{I}_{i,\theta}$$

and, when $|k+1| < |i+1|$,

$$\gamma_{|k+1|}\left(R_\omega(\theta)\right) = \gamma_{|k+1|+1}\left(R_\omega(\theta)\right) = \cdots = \gamma_{|i+1|-1}\left(R_\omega(\theta)\right)$$

(b) For every $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$,

$$\gamma_{|i|}(\theta) = \gamma_{|i|-1}(\theta) \quad \text{and} \quad \mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}.$$

Proof. The fact that $|k| < |i|$ follows from Lemma 3.6(g). Therefore, either $|k+1| < |i+1|$ or $k \geq 0$, $i = -(k+2)$ and $|k+1| = |i+1|$ or $k \geq 0$, $i = -(k+1)$ and $|k+1| > |i+1|$. In the last case, $B_i^{\frown}[i^*] = B_{-(k+1)}^{\frown}[(-(k+1))^*]$ and $B_k^{\frown}[k^*]$ must be disjoint by Definition 3.4(R.2) (with $j = k$); which is a contradiction. Thus $|k+1| < |i+1|$ unless $k \geq 0$ and $i = -(k+2)$ ($|k+1| = |i+1|$).

By Definition 3.4(R.2) and Remark 3.5(R.2), $B_i^{\frown}[i^*] \cap Z^*_{|i|-1} = \emptyset$. Hence, from the definition of the maps γ_j (Definition 3.4), to prove that

$$\gamma_{|k|}\big|_{B_i^{\frown}[i^*]} = \gamma_{|k|+1}\big|_{B_i^{\frown}[i^*]} = \cdots = \gamma_{|i|-2}\big|_{B_i^{\frown}[i^*]} = \gamma_{|i|-1}\big|_{B_i^{\frown}[i^*]},$$

it is enough to show that $B_{\alpha_{|\ell|}}[\ell^*] \cap B_i^{\frown}[i^*] = \emptyset$ for every ℓ such that $|k| < |\ell| < |i|$. Assume that $B_{\alpha_{|\ell|}}[\ell^*] \cap B_i^{\frown}[i^*] \neq \emptyset$ for some ℓ such that $|k| < |\ell| < |i|$. Then,

$$\emptyset \neq B_{\alpha_{|\ell|}}\left[\ell^*\right] \cap B_i^{\frown}\left[i^*\right] \subset B_\ell^{\frown}\left[\ell^*\right] \cap B_i^{\frown}\left[i^*\right] \subset B_\ell^{\frown}\left[\ell^*\right] \cap B_k^{\frown}\left[k^*\right]$$

and, by Lemma 3.6(g),

$$B_i^{\frown}\left[i^*\right] \subsetneq B_\ell^{\frown}\left[\ell^*\right] \subsetneq B_k^{\frown}\left[k^*\right].$$

So, in a similar way as before,

$$m = \mathsf{depth}(i) \geq \mathsf{depth}(\ell) + 1 \geq \mathsf{depth}(k) + 2 = m + 1;$$

a contradiction. This ends the proof of the first statement of (a).

Now we show that if $|k+1| < |i+1| - 1$, then

$$\gamma_{|k+1|}\left(R_\omega(\theta)\right) = \gamma_{|k+1|+1}\left(R_\omega(\theta)\right) = \cdots = \gamma_{|i+1|-1}\left(R_\omega(\theta)\right),$$

and are well defined.

First we prove that $\gamma_\ell\left(R_\omega(\theta)\right)$ is well defined for every $\ell = 0, 1, \ldots, |i+1| - 1$. For every $\theta \in B_i^{\frown}[i^*]$ we have

$$R_\omega(\theta) \in R_\omega\left(B_i^{\frown}\left[i^*\right]\right) = \begin{cases} B_{\alpha_i}[(i+1)^*] & \text{when } i \geq 0, \text{ and} \\ B_{\alpha_{|i+1|}}[(i+1)^*] \subset B_{i+1}^{\frown}[(i+1)^*] & \text{when } i < 0. \end{cases}$$

In any case, by Definition 3.4(R.2) and Remark 3.5(R.2) with $j = i$ when $i \geq 0$ and $\ell = -(j+1) = i+1$ when $i < 0$, and Lemma 3.6(a),

$$R_\omega(\theta) \notin \begin{cases} Z^*_i & \text{when } i \geq 0, \text{ and} \\ Z^*_{|i+1|-1} & \text{when } i < 0, \end{cases}$$

and $\gamma_\ell\left(R_\omega(\theta)\right)$ is well defined for $\ell = 0, 1, \ldots, |i+1| - 1$ (recall that $Z^*_m \subset Z^*_{m+1}$ for every $m \geq 0$).

Now, assume by way of contradiction that

$$\gamma_\ell\left(R_\omega(\theta)\right) \neq \gamma_{\ell-1}\left(R_\omega(\theta)\right) \text{ for some } \ell \in \{|k+1|+1, |k+1|+2, \ldots, |i+1|-1\},$$

and ℓ is minimal with this property (observe that $\ell \geq 1$). By the definition of the map γ_ℓ (Definition 3.4),

$$R_\omega(\theta) \in B_{\alpha_\ell}\left((q+1)^*\right) \quad \text{with} \quad q \in \{\ell-1, -(\ell+1)\}$$

and, hence, $\theta \in B_{\alpha_\ell}(q^*)$.

Since $|k+1|+1 \leq \ell < |i+1|$, when $q = -(\ell+1) \leq -2$,

$$|k+1|+2 \leq -q \leq |i+1| \text{ and } B_{\alpha_\ell}\left(q^*\right) = B^{\frown}_{-(\ell+1)}\left((-(\ell+1))^*\right) = B^{\frown}_{q}\left(q^*\right).$$

Otherwise, when $q = \ell - 1 \geq 0$, $|k+1| \leq q \leq |i+1| - 2$ and

$$B_{\alpha_\ell}\left(q^*\right) \subset B_{\alpha_{\ell-1}}\left((\ell-1)^*\right) = B^{\frown}_{\ell-1}\left((\ell-1)^*\right) = B^{\frown}_{q}\left(q^*\right),$$

by Definition 3.4(R.1).

Next we want to use Lemma 3.6(g) to show that $B^{\frown}_{i}[i^*] \subsetneq B^{\frown}_{q}[q^*] \subsetneq B^{\frown}_{k}[k^*]$. To this end we have to compare $|q|$ with $|i|$ and $|k|$.

Notice $B^{\frown}_{q}[q^*] \cap B^{\frown}_{k}[k^*] \neq \emptyset$ because

$$\theta \in B^{\frown}_{q}\left(q^*\right) \cap B^{\frown}_{i}\left[i^*\right] \subset B^{\frown}_{q}\left(q^*\right) \cap B^{\frown}_{k}\left[k^*\right].$$

If $k \geq 0$, $|q| \geq |k+1| > |k|$. When $k, q < 0$, $|q| \geq |k+1|+2 = |k|+1 > |k|$. If $k < 0$ and $q \geq 0$, $|q| = q \geq |k+1| = |k|-1$. If $q = |k|-1$ (that is, $k = -(q+1)$), as above, by Definition 3.4(R.2) with $j = q$ we get $B^{\frown}_{k}[k^*] \cap B^{\frown}_{q}[q^*] = \emptyset$; a contradiction. So, $|q| > |k|$ unless $|q| = |k|$ and $k < 0 \leq q$. Summarizing, we have shown that $|q| \geq |k|$ and $q \neq k$. Then, from Lemma 3.6(g) we get that $|q| > |k|$ and $B^{\frown}_{q}[q^*] \subsetneq B^{\frown}_{k}[k^*]$.

Now we will study the relation of $B^{\frown}_{q}[q^*]$ with the box $B^{\frown}_{i}[i^*]$. From above we get that $B^{\frown}_{q}[q^*] \cap B^{\frown}_{i}[i^*] \neq \emptyset$. If $i < 0$, $|q| \leq |i+1| = |i|-1$. When $q, i \geq 0$, we have $|q| = q \leq |i+1| - 2 = |i| - 1$. If $i \geq 0$ and $q < 0$, $|q| \leq |i+1| = |i|+1$.

Assume that $i \geq 0$ and $q = -(i+1) < 0$. In this case, additionally, $q = -(\ell+1)$ and, thus, $i = \ell \geq 1$. Then,

$$R_\omega(\theta) \in R_\omega\left(B^{\frown}_{i}\left[i^*\right]\right) = R_\omega\left(B_{\alpha_i}\left[i^*\right]\right) = B_{\alpha_i}\left[(i+1)^*\right], \text{ and}$$
$$R_\omega(\theta) \in B_{\alpha_\ell}\left((q+1)^*\right) = B_{\alpha_i}\left((-i)^*\right) \subset B^{\frown}_{-i}\left((-i)^*\right),$$

which is a contradiction by Definition 3.4(R.2). Summarizing, $|q| < |i|$ unless $|q| = |i|$ and $q < 0 \leq i$ (that is, $|q| \leq |i|$ and $q \neq i$). Then, again by Lemma 3.6(g), $|q| < |i|$ and $B^{\frown}_{i}[i^*] \subsetneq B^{\frown}_{q}[q^*] \subsetneq B^{\frown}_{k}[k^*]$. So, as before,

$$m = \mathsf{depth}(i) \geq \mathsf{depth}(q) + 1 \geq \mathsf{depth}(k) + 2 = m + 1;$$

a contradiction. This ends the proof of (a).

Now we assume that $\theta \in B^{\frown}_{i}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$. By Lemmas 3.6(e) and 4.1(d),

$$\gamma_{|i|}(\theta) = \gamma_{|i|-1}(\theta) \quad \text{and} \quad \mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|i|-1}(\theta)\} = \{\gamma_{|k|}(\theta)\}.$$

On the other hand, by Lemma 5.3(b), $\mathfrak{D}^*_{m-1} \cap \mathbb{B}^{\frown}_{m} = \emptyset$ which implies that $\theta \neq k^*$ because $k^* \in \mathfrak{D}^*_{m-1}$ and $\theta \in B^{\frown}_{i}[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{B}^{\frown}_{m}$. So, by (2),

$$\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}.$$

Now we prove that $\gamma_{|i|-1}(\theta) \in \mathbb{I}_{i,\theta}$ for every $\theta \in B^{\frown}_{i}[i^*]$. From above, we have $\mathbb{I}_{i,\theta} = \{\gamma_{|i|-1}(\theta)\}$ for every $\theta \in B^{\frown}_{i}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$. Moreover, when $\theta \in B_{\alpha_{|i|}}(i^*)$ the statement follows directly from Lemma 3.6(c). Thus, (b) is proved. □

6. Boxes in the wings

To prove Theorem A we will inductively construct a Cauchy sequence $\{T_m\}_{m=0}^{\infty} \subset \mathcal{S}(\Omega)$ that gives the function T from Theorem A as a limit.

This section is devoted to study the points in the wings of boxes in the circle and its interaction with boxes of higher depth. The resulting technology is necessary to be able to construct the sequence $\{T_m\}_{m=0}^{\infty}$ so that it is Cauchy sequence. Unfortunately this will complicate even more the definition of the functions T_m and the proof of its continuity.

We start by introducing some more notation. For every $m \in \mathbb{Z}^+$ we set

$$\mathbb{B}_m := \bigcup_{i \in \mathfrak{D}_m} B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}_m^{\frown}, \text{ and}$$
$$\mathbb{WDB}_m := \Big\{\theta \in \mathbb{B}_m^{\frown} \setminus \mathbb{B}_m : \theta \in \mathbb{B}_j \text{ for some } j > m\Big\}.$$

On the other hand, the smallest number j from the above definition will be called the *least essential depth of θ below m*, and will be denoted by $\mathsf{led}(\theta, m)$. That is, $\mathsf{led}(\theta, m)$ denotes the positive integer larger than m such that

$$\theta \in \mathbb{B}_j^{\frown} \setminus \mathbb{B}_j \text{ for } j = m, m+1, \dots, \mathsf{led}(\theta, m) - 1 \quad \text{and} \quad \theta \in \mathbb{B}_{\mathsf{led}(\theta,m)}.$$

The following simple lemmas are useful to better understand and use the above definitions. The next lemma establishes the relation between boxes in the wings of increasing depth.

Lemma 6.1. *Assume that $\theta \in \mathbb{WDB}_m$ for some $m \in \mathbb{Z}^+$ and set $\ell = \mathsf{led}(\theta, m)$. Then, the following statements hold.*

(a) For every $j = m, m+1, \dots, \ell$ the numbers $\mathsf{i}_j = \mathsf{b}^{\frown}(\theta, j) \in \mathfrak{D}_j$ are well defined and are all of them negative except, perhaps, $\mathsf{i}_\ell = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m))$.

(b)
$$|\mathsf{i}_m| < |\mathsf{i}_{m+1}| < \dots < |\mathsf{i}_{\ell-1}| < |\mathsf{i}_\ell|, \textit{ and}$$
$$\theta \in B_{\alpha_{|\mathsf{i}_\ell|}}[(\mathsf{i}_\ell)^*] \subset B_{\mathsf{i}_{\ell-1}}^{\frown}((\mathsf{i}_{\ell-1})^*) \setminus B_{\alpha_{|\mathsf{i}_{\ell-1}|}}[(\mathsf{i}_{\ell-1})^*]$$
$$\subset B_{\mathsf{i}_{\ell-2}}^{\frown}((\mathsf{i}_{\ell-2})^*) \setminus B_{\alpha_{|\mathsf{i}_{\ell-2}|}}[(\mathsf{i}_{\ell-2})^*] \subset \dots \subset B_{\mathsf{i}_m}^{\frown}((\mathsf{i}_m)^*) \setminus B_{\alpha_{|\mathsf{i}_m|}}[(\mathsf{i}_m)^*].$$

(c) For every $j = m, m+1, \dots, \ell-1$, $B_{\alpha_{|\mathsf{i}_\ell|}}[(\mathsf{i}_\ell)^*] \subset \mathbb{WDB}_j$, $\mathsf{led}(\nu, j) = \mathsf{led}(\theta, m)$ and $\mathsf{b}^{\frown}(\nu, \mathsf{led}(\nu, j)) = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m)) = \mathsf{i}_\ell$ for every $\nu \in B_{\alpha_{|\mathsf{i}_\ell|}}[(\mathsf{i}_\ell)^*]$.

(d) $\mathbb{I}_{\mathsf{i}_m,\nu} = \{\gamma_{|\mathsf{i}_m|}(\nu)\} \subset \mathbb{I}_{\mathsf{i}_\ell,\nu}$ for every $\nu \in B_{\alpha_{|\mathsf{i}_\ell|}}((\mathsf{i}_\ell)^*)$ and

$$\mathbb{I}_{\mathsf{i}_m,\nu} = \{\gamma_{|\mathsf{i}_m|}(\nu)\} = \{m_{\mathsf{i}_\ell}(\nu)\} = \{M_{\mathsf{i}_\ell}(\nu)\} = \{\gamma_{|\mathsf{i}_\ell|}(\nu)\} = \mathbb{I}_{\mathsf{i}_\ell,\nu}$$

for every $\nu \in \mathsf{Bd}\left(B_{\alpha_{|\mathsf{i}_\ell|}}[(\mathsf{i}_\ell)^*]\right)$.

Proof. Since $B_i^{\frown}[i^*] = B_{\alpha_i}[i^*]$ for every $i \geq 0$,

$$\mathbb{B}_m^{\frown} \setminus \mathbb{B}_m = \bigcup_{\substack{i \in \mathfrak{D}_m \\ i<0}} \left(B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}[i^*]\right) \tag{11}$$

for every $m \in \mathbb{Z}^+$.

Statement (a) follows from Lemma 5.3(b) and (11). Then, (b) follows from Lemma 3.6(g). Statement (c) is an easy consequence of (b) and the definitions.

Now we prove (d) iteratively. Fix $\nu \in B_{\alpha_{|\mathsf{i}_\ell|}}((\mathsf{i}_\ell)^*)$. By (b)

$$\nu \in B_{\mathsf{i}_{m+1}}^{\frown}((\mathsf{i}_{m+1})^*) \setminus B_{\alpha_{|\mathsf{i}_{m+1}|}}[(\mathsf{i}_{m+1})^*] \subset B_{\mathsf{i}_m}^{\frown}((\mathsf{i}_m)^*) \setminus B_{\alpha_{|\mathsf{i}_m|}}[(\mathsf{i}_m)^*]$$

provided that $\ell = \mathsf{led}(\theta, m) > m+1$. Hence, by Lemmas 4.1(d) and 5.4,

$$\gamma_{|\mathsf{i}_m|}(\nu) = \gamma_{|\mathsf{i}_m|+1}(\nu) = \cdots = \gamma_{|\mathsf{i}_{m+1}|}(\nu), \text{ and}$$
$$\mathbb{I}_{\mathsf{i}_m,\nu} = \{\gamma_{|\mathsf{i}_m|}(\nu)\} = \{\gamma_{|\mathsf{i}_{m+1}|}(\nu)\} = \mathbb{I}_{\mathsf{i}_{m+1},\nu}.$$

By iterating this argument we get,

$$\gamma_{|\mathsf{i}_m|}(\nu) = \gamma_{|\mathsf{i}_m|+1}(\nu) = \cdots = \gamma_{|\mathsf{i}_{\ell-1}|}(\nu) \quad \text{and} \quad \mathbb{I}_{\mathsf{i}_m,\nu} = \mathbb{I}_{\mathsf{i}_{\ell-1},\nu}.$$

Again by (b) and Lemmas 4.1(d) and 5.4,

$$\gamma_{|\mathsf{i}_m|}(\nu) = \gamma_{|\mathsf{i}_m|+1}(\nu) = \cdots = \gamma_{|\mathsf{i}_\ell|}(\nu) \quad \text{and} \quad \mathbb{I}_{\mathsf{i}_m,\nu} = \mathbb{I}_{\mathsf{i}_\ell,\nu}$$

when $\nu \in \mathsf{Bd}\left(B_{\alpha_{|\mathsf{i}_\ell|}}[(\mathsf{i}_\ell)^*]\right)$ and, otherwise,

$$\gamma_{|\mathsf{i}_m|}(\nu) = \gamma_{|\mathsf{i}_m|+1}(\nu) = \cdots = \gamma_{|\mathsf{i}_{\ell-1}|}(\nu) \quad \text{and} \quad \mathbb{I}_{\mathsf{i}_m,\nu} \subset \mathbb{I}_{\mathsf{i}_\ell,\nu}.$$

□

Equipped with above results and definition we are going to define two maps, analogous to the maps m_i and M_i, on the wings of the negative boxes.

Definition 6.2. For every $m \in \mathbb{Z}^+$ we define

$$\mathfrak{WFD}_m := \{\mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m)) \,:\, \theta \in \mathbb{WDB}_m\} \subset \mathbb{Z},$$
$$\mathbb{WIB}_m := \mathsf{Int}(\mathbb{WDB}_m) = \bigcup_{i \in \mathfrak{WFD}_m} B_{\alpha_{|i|}}(i^*),$$
$$\mathbb{WB}^{\frown}_m := \bigcup_{\substack{i \in \mathfrak{D}_m \\ i<0}} \left(B^{\frown}_i[i^*] \setminus B_{\alpha_{|i|}}(i^*)\right), \text{ and}$$
$$\mathbb{EB}^{\frown}_m := \bigcup_{i \in \mathfrak{D}_m} \mathsf{Bd}\left(B^{\frown}_i[i^*]\right) \subset \mathbb{B}^{\frown}_m.$$

By Lemma 6.1(a,c), $\mathfrak{WFD}_m$ is well defined and

$$\mathbb{WIB}_m \subset \mathbb{WDB}_m \subset \mathbb{B}^{\frown}_m \setminus \mathbb{B}_m \subset \mathbb{WB}^{\frown}_m.$$

Consequently,

$$\mathbb{B}^{\frown}_m = \mathbb{B}_m \cup \mathbb{WB}^{\frown}_m.$$

Then, we can define functions $\tau_m\colon \mathbb{WB}^{\frown}_m \longrightarrow \mathbb{I}$ and $\lambda_m\colon \mathbb{WB}^{\frown}_m \longrightarrow \mathbb{I}$ as follows:

$$\tau_m(\theta) := \begin{cases} M_{\mathsf{b}^{\frown}(\theta,\mathsf{led}(\theta,m))}(\theta) & \text{if } \theta \in \mathbb{WIB}_m, \\ \gamma_{|\mathsf{b}^{\frown}(\theta,m)|}(\theta) & \text{otherwise,} \end{cases}$$

$$\lambda_m(\theta) := \begin{cases} m_{\mathsf{b}^{\frown}(\theta,\mathsf{led}(\theta,m))}(\theta) & \text{if } \theta \in \mathbb{WIB}_m, \\ \gamma_{|\mathsf{b}^{\frown}(\theta,m)|}(\theta) & \text{otherwise.} \end{cases}$$

Clearly, by Lemmas 4.1(a) and 3.6(b),

$$-1 \le \lambda_m(\theta) \le \tau_m(\theta) \le 1$$

for every $\theta \in \mathbb{WB}^{\frown}_m$. So, we can define

$$\mathbb{IW}_{m,\theta} := [\lambda_m(\theta), \tau_m(\theta)] \subset [0,1].$$

◘

The next lemmas will help us in the definition and study of the maps T_m.

Lemma 6.3. *The following statements hold for every $m \in \mathbb{Z}^+$.*
(a) $\mathbb{WIB}_m \cap \mathbb{B}_m = \mathbb{WIB}_m \cap \mathbb{EB}^{\frown}_m = \emptyset$.

(b) Let $\theta \in \mathbb{WB}_m^{\frown}$. Then, $\mathbb{I}_{\mathsf{b}^{\frown}(\theta,m),\theta} = \left\{ \gamma_{\left|\mathsf{b}^{\frown}(\theta,m)\right|}(\theta) \right\}$,

$$\begin{aligned} \mathbb{I}_{\mathsf{b}^{\frown}(\theta,m),\theta} &= \mathbb{IW}_{m,\theta} && \text{when } \theta \notin \mathbb{WIB}_m, \text{ and} \\ \mathbb{I}_{\mathsf{b}^{\frown}(\theta,m),\theta} &\subset \mathbb{IW}_{m,\theta} && \text{when } \theta \in \mathbb{WIB}_m. \end{aligned}$$

(c) Assume that $m \in \mathbb{N}$ and let U be a connected component of $\mathbb{WB}_m^{\frown}$ such that $U \subset \mathbb{WB}_{m-1}^{\frown}$. Then, $\mathbb{WDB}_m \cap U \subset \mathbb{WDB}_{m-1}$, $\mathbb{WIB}_m \cap U = \mathbb{WIB}_{m-1} \cap U$ and $\mathbb{IW}_{m,\theta} = \mathbb{IW}_{m-1,\theta}$ for every $\theta \in U$.

Proof. **(a)** By Lemma 6.1(b),

$$\theta \in B^{\frown}_{\mathsf{b}^{\frown}(\theta,m)}\left((\mathsf{b}^{\frown}(\theta,m))^*\right) \setminus B_{\alpha_{\left|\mathsf{b}^{\frown}(\theta,m)\right|}}\left[(\mathsf{b}^{\frown}(\theta,m))^*\right]$$

and $\mathsf{b}^{\frown}(\theta,m) < 0$ for every $\theta \in \mathbb{WIB}_m \subset \mathbb{WDB}_m$. So, by Lemma 5.2(b), we get $\theta \notin \mathbb{B}_m \cup \mathbb{EB}_m^{\frown}$.

(b) The fact that $\mathbb{I}_{\mathsf{b}^{\frown}(\theta,m),\theta} = \left\{ \gamma_{\left|\mathsf{b}^{\frown}(\theta,m)\right|}(\theta) \right\}$ follows from Lemma 4.1(d). The other two statements follow from Definition 6.2 and Lemma 6.1(d).

(c) The assumption that U is a connected component of $\mathbb{WB}_m^{\frown}$ and $U \subset \mathbb{WB}_{m-1}^{\frown}$ implies by Lemmas 5.2(b) and 3.6(g) that there exist $i \in \mathfrak{D}_m$ and $k \in \mathfrak{D}_{m-1}$, $i, k < 0$, such that U is a connected component of

$$B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset B_k^{\frown}(k^*) \setminus B_{\alpha_{|k|}}[k^*] \subset \mathbb{WB}_{m-1}^{\frown}.$$

Again by Lemma 5.2(b) this implies that $U \subset \mathbb{B}_{m-1}^{\frown} \setminus \mathbb{B}_{m-1}$. Moreover, by definition, $\mathbb{WDB}_m \subset \mathbb{B}_m^{\frown} \setminus \mathbb{B}_m$. Consequently, $\mathbb{WDB}_m \cap U \subset \mathbb{WDB}_{m-1}$.

Let $\theta \in \mathbb{WIB}_m \cap U \subset \mathbb{WDB}_m \cap U \subset \mathbb{WDB}_{m-1} \cap U$. By Definition 6.2 and Lemma 6.1(a,b), $i = \mathsf{b}^{\frown}(\theta,m)$ and there exists $\ell = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta,m)) \in \mathfrak{WFD}_m$ such that

$$\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset B_k^{\frown}(k^*) \setminus B_{\alpha_{|k|}}[k^*].$$

Therefore, again by Lemma 6.1(a–c) and Definition 6.2, $\mathsf{led}(\theta, m-1) = \mathsf{led}(\theta, m)$,

$$\ell = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta,m)) = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta,m-1)) \in \mathfrak{WFD}_{m-1}$$

and $\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset \mathbb{WIB}_{m-1}$. Hence, $\mathbb{WIB}_m \cap U \subset \mathbb{WIB}_{m-1}$.

Now assume that $\theta \in \mathbb{WIB}_{m-1} \cap U$. As above, there exist $r = \mathsf{b}^{\frown}(\theta,m) \in \mathfrak{D}_m$ and $\ell = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m-1)) \in \mathfrak{WFD}_{m-1}$ such that

$$\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset B_r^{\frown}(r^*) \setminus B_{\alpha_{|r|}}[r^*] \subset B_k^{\frown}(k^*) \setminus B_{\alpha_{|k|}}[k^*].$$

Since $\theta \in U \subset B_i^{\frown}[i^*]$, Lemma 5.2(b) gives $i = r$ and $\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset U$. Moreover, by Lemma 6.1(c), $\ell = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m-1)) = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta,m)) \in \mathfrak{WFD}_m$ and, so, $\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset \mathbb{WIB}_m$. Thus, $\mathbb{WIB}_m \cap U = \mathbb{WIB}_{m-1} \cap U$.

To end the proof of the lemma we have to show that $\mathbb{IW}_{m,\theta} = \mathbb{IW}_{m-1,\theta}$ for every $\theta \in U$. Assume first that $\theta \in U \setminus \mathbb{WIB}_m \subset \mathbb{WB}_m^{\frown} \setminus \mathbb{WIB}_m$. Then,

$$\theta \in U \setminus \mathbb{WIB}_m = U \setminus \mathbb{WIB}_{m-1} \subset \mathbb{WB}_{m-1}^{\frown} \setminus \mathbb{WIB}_{m-1}$$

and, by (b) and Lemmas 4.1(d) and 5.4,

$$\mathbb{IW}_{m,\theta} = \mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} = \mathbb{I}_{k,\theta} = \mathbb{IW}_{m-1,\theta}.$$

If $\theta \in U \cap \mathbb{WIB}_m = U \cap \mathbb{WIB}_{m-1}$ then we get

$$\begin{aligned} \mathbb{IW}_{m,\theta} &= \left[m_{\mathsf{b}^{\frown}(\theta,\mathsf{led}(\theta,m))}(\theta), M_{\mathsf{b}^{\frown}(\theta,\mathsf{led}(\theta,m))}(\theta)\right] \\ &= \left[m_{\mathsf{b}^{\frown}(\theta,\mathsf{led}(\theta,m-1))}(\theta), M_{\mathsf{b}^{\frown}(\theta,\mathsf{led}(\theta,m-1))}(\theta)\right] = \mathbb{IW}_{m-1,\theta} \end{aligned}$$

from Definition 6.2 and Lemma 6.1(c). □

Lemma 6.4. *Let $m \in \mathbb{Z}^+$ and let U be a connected component of $\mathbb{WB}_m^{\smile}$. Then, the functions $\lambda_m\big|_U$ and $\tau_m\big|_U$ are continuous.*

Proof. We will prove only the continuity of $\lambda_m\big|_U$. The proof of the continuity of $\tau_m\big|_U$ is analogous.

By Lemmas 6.1(c) and 4.1(b) we get

(12) for every $\ell \in \mathfrak{WFD}_m$, $\ell = \mathsf{b}^{\smile}(\nu, \mathsf{led}(\nu, m))$ for every $\nu \in B_{\alpha_{|\ell|}}[\ell^*]$, and the function m_ℓ is continuous on $B_{\alpha_{|\ell|}}[\ell^*]$.

Let $\ell \in \mathfrak{WFD}_m$ be such that $B_{\alpha_{|\ell|}}(\ell^*) \subset \mathbb{WIB}_m \cap U$. Thus, by (12), the function $\lambda_m = m_\ell$ is continuous on $B_{\alpha_{|\ell|}}(\ell^*)$.

So, we have to show that λ_m is continuous at every $\theta \in U \setminus \mathbb{WIB}_m$. To show this we will use a simple usual ε–δ game. Fix $\varepsilon > 0$.

By Lemma 5.2(b) it follows that U is a connected component of $B_i^{\smile}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$ for some $i \in \mathfrak{D}_m$, $i < 0$, and

$$\mathsf{b}^{\smile}(\nu, m) = i \quad \text{for every} \quad \nu \in U. \tag{13}$$

By Lemma 3.6(a) and Definition 3.4(R.2) and Remark 3.5(R.2), the function $\gamma_{|i|}\big|_U$ is continuous. So,

(14) there exists $\overline{\delta}_{|i|} = \overline{\delta}_{|i|}(\theta) > 0$ such that $\left|\gamma_{|i|}(\theta), \gamma_{|i|}(\nu)\right| < \varepsilon/2$ provided that $\mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \overline{\delta}_{|i|}$.

On the other hand, by (12),

(15) for every $\ell \in \mathfrak{WFD}_m$, there exists $\delta_\ell > 0$ such that $\left|m_\ell(\widetilde{\theta}), m_\ell(\nu)\right| < \varepsilon/2$ for every $\widetilde{\theta} \in \mathsf{Bd}\left(B_{\alpha_{|\ell|}}[\ell^*]\right)$ and $\nu \in \mathsf{Bd}\, B_{\alpha_{|\ell|}}[\ell^*]$ such that $\mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta_\ell$.

Now we will define δ. Note that there exists $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon/2$. Then we set:

$$\delta = \delta(\theta) := \min\left\{\overline{\delta}_{|i|}(\theta), \min\{\delta_\ell : \ell \in \mathfrak{WFD}_m \text{ and } |\ell| < N\}\right\}.$$

Clearly, $\delta > 0$ because the set $\{\ell \in \mathfrak{WFD}_m : |\ell| < N\}$ is finite.

To end the proof of the lemma we have to show that

$$|\lambda_m(\theta) - \lambda_m(\nu)| < \varepsilon$$

whenever $\nu \in U$ and $\mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta$.

Assume that $\nu \in U$ and $\mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta$ (recall that we have the assumption that $\theta \notin \mathbb{WIB}_m$). If $\nu \notin \mathbb{WIB}_m$, then $\mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta \le \overline{\delta}_{|i|}(\theta)$ and, by (13) and (14),

$$|\lambda_m(\theta) - \lambda_m(\nu)| = \left|\gamma_{|i|}(\theta) - \gamma_{|i|}(\nu)\right| < \varepsilon/2 < \varepsilon.$$

Now assume that there exists $\ell \in \mathfrak{WFD}_m$ such that $\nu \in B_{\alpha_{|\ell|}}(\ell^*) \subset \mathbb{WIB}_m$. Clearly, there exists $\widetilde{\theta} \in \mathsf{Bd}\left(B_{\alpha_{|\ell|}}[\ell^*]\right)$ such that

$$\mathsf{d}_{\mathbb{S}^1}(\theta, \widetilde{\theta}) < \mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta \le \overline{\delta}_{|i|}(\theta) \text{ and}$$
$$\mathsf{d}_{\mathbb{S}^1}(\widetilde{\theta}, \nu) < \mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta.$$

Observe that, by Lemma 5.2(b), $\widetilde{\theta} \notin \mathbb{WIB}_m$. Hence, by (13) and Lemma 6.1(c,d),

$$\lambda_m(\widetilde{\theta}) = \gamma_{|i|}(\widetilde{\theta}) = m_\ell(\widetilde{\theta}).$$

If $|\ell| < N$, then $\mathsf{d}_{\mathbb{S}^1}(\widetilde{\theta}, \nu) < \delta \le \delta_\ell$ and, by (15), $\left|m_\ell(\widetilde{\theta}) - m_\ell(\nu)\right| < \varepsilon/2$. Otherwise, by Lemma 3.6(f),

$$\left|m_\ell(\widetilde{\theta}) - m_\ell(\nu)\right| < \mathsf{diam}\left(\mathcal{R}(\ell^*)\right) \le 2^{-|\ell|} \le 2^{-N} < \varepsilon/2.$$

In any case, $\left|m_\ell(\widetilde{\theta}) - m_\ell(\nu)\right| < \varepsilon/2$. Thus, again by (13) and (14),

$$\begin{aligned}|\lambda_m(\theta) - \lambda_m(\nu)| &\leq \left|\lambda_m(\theta) - \lambda_m(\widetilde{\theta})\right| + \left|\lambda_m(\widetilde{\theta}) - \lambda_m(\nu)\right| \\ &= \left|\gamma_{|i|}(\theta) - \gamma_{|i|}(\widetilde{\theta})\right| + \left|m_\ell(\widetilde{\theta}) - m_\ell(\nu)\right| < \varepsilon.\end{aligned}$$

□

7. A Cauchy sequence of skew products. Proof of Theorem A

In this section prove Theorem A. To do this we inductively construct a Cauchy sequence $\{T_m\}_{m=0}^\infty \subset \mathcal{S}(\Omega)$ that gives the function T from Theorem A as a limit.

The sequence $\{T_m\}_{m=0}^\infty \subset \mathcal{S}(\Omega)$ is defined so that

$$T_m(\theta, x) = (R_\omega(\theta), f_m(\theta, x))$$

and $f_m \colon \Omega \longrightarrow \mathbb{I}$ is continuous in both variables. To build these functions we will use the auxiliary functions $G_i \colon \mathcal{R}(i^*) \longrightarrow \Omega$ with $i \in \mathbb{Z}$ from Section 4. The maps $f_m(\theta, \cdot)$ will also be denoted as $f_{m,\theta}$, and will be defined non-increasing, and such that $f_{m,\theta}(2) = -2$ and $f_{m,\theta}(-2) = 2$ for every $\theta \in \mathbb{S}^1$.

To make more evident the strategy of the construction of this sequence of maps we will separate several cases, and we will state without proofs the results that study these maps. After establishing all the definitions and results related to the construction of the sequence $\{T_m\}_{m=0}^\infty$ without having been distracted by the technicalities involving the proofs, we will proceed to provide the missing proofs. More precisely, we will start by defining the map T_0 and stating without proof the proposition that summarizes the necessary properties of this map. Next we will inductively define the maps $\{T_m\}_{m=1}^\infty \subset \mathcal{S}(\Omega)$ and state without proof the proposition that establishes the properties of the whole sequence $\{T_m\}_{m=0}^\infty$.

Then, as we have said, we prove Theorem A and in the next three sections we will provide all pending proofs.

In what follows $\mathcal{C}(\mathbb{I}, \mathbb{I})$ will denote the class of all continuous maps from $\mathbb{I}$ to itself. We endow $\mathcal{C}(\mathbb{I}, \mathbb{I})$ with the supremum metric denoted by $\|\cdot\|$ so that $(\mathcal{C}(\mathbb{I}, \mathbb{I}), \|\cdot\|)$ is a complete metric space.

Next we define the map T_0.

Definition 7.1 (The map T_0)**.** Assume first that $\theta \in \mathbb{B}_0^{\frown}$ and let $\mathsf{i} = \mathsf{b}^{\frown}(\theta, 0)$ (that is $\theta \in B_{\mathsf{i}}^{\frown}[\mathsf{i}^*]$). In this case we set:

$$f_{0,\theta}(x) = \begin{cases} g_{\mathsf{i},\theta}(x) & \text{if } x \in \mathbb{I}_{\mathsf{i},\theta}, \\ \frac{g_{\mathsf{i},\theta}(m_{\mathsf{i}}(\theta)) - 2}{m_{\mathsf{i}}(\theta) + 2}(x+2) + 2 & \text{if } x \in [-2, m_{\mathsf{i}}(\theta)], \\ \frac{g_{\mathsf{i},\theta}(M_{\mathsf{i}}(\theta)) + 2}{M_{\mathsf{i}}(\theta) - 2}(x-2) - 2 & \text{if } x \in [M_{\mathsf{i}}(\theta), 2]. \end{cases}$$

If $\theta \in \mathbb{S}^1 \backslash \mathbb{B}_0^{\frown}$ then we define $f_{0,\theta}$ to be the unique piecewise affine map with two affine pieces whose graph joins the point $(-2, 2)$ with $(0, \gamma(R_\omega(\theta)))$, and this with the point $(2, -2)$. ◘

Next we introduce some more notation to be able to define the maps $\{T_m\}_{m=1}^\infty$. For every $k \in \mathbb{Z}$ we set

$$\mathsf{V}_{k^*}^{\frown} := \upuparrows B_k^{\frown}[k^*] = B_k^{\frown}[k^*] \times \mathbb{I}$$

and, for every $m \in \mathbb{Z}^+$,

$$\mathbb{V}_m^{\frown} := \upuparrows \mathbb{B}_m^{\frown} = \mathbb{B}_m^{\frown} \times \mathbb{I} = \bigcup_{i \in \mathfrak{D}_m} \mathsf{V}_{i^*}^{\frown}.$$

Definition 7.2 (The maps T_m with $m > 0$)**.** Now we assume that we have defined the function T_{m-1} for some $m \geq 1$ and we define

$$T_m(\theta, x) = (R_\omega(\theta), f_m(\theta, x))$$

as follows. By Lemma 5.2(b), for every $(\theta, x) \in \mathbb{V}^{\frown}_{m}$, we have

$$\theta \in B^{\frown}_{\mathsf{i}}[\mathsf{i}^*] \subset \mathbb{B}^{\frown}_{m} \quad \text{with} \quad \mathsf{i} = \mathsf{b}^{\frown}(\theta, m) \in \mathfrak{D}_{m}$$

(and, of course, $x \in \mathbb{I}$). Then we define:

$$f_{m,\theta}(x) = \begin{cases} f_{m-1,\theta}(x) & \text{if } \theta \in \mathbb{S}^1 \backslash \mathbb{B}^{\frown}_{m};\ x \in \mathbb{I}, \\ g_{\mathsf{i},\theta}(x) & \text{if } \theta \in \mathbb{B}_{m};\ x \in \mathbb{I}_{\mathsf{i},\theta}, \\ \frac{2-g_{\mathsf{i},\theta}(m_{\mathsf{i}}(\theta))}{2-f_{m-1,\theta}(m_{\mathsf{i}}(\theta))}(f_{m-1,\theta}(x)-2)+2 & \text{if } \theta \in \mathbb{B}_{m};\ x \in [-2, m_{\mathsf{i}}(\theta)], \\ \frac{2+g_{\mathsf{i},\theta}(M_{\mathsf{i}}(\theta))}{2+f_{m-1,\theta}(M_{\mathsf{i}}(\theta))}(f_{m-1,\theta}(x)+2)-2 & \text{if } \theta \in \mathbb{B}_{m};\ x \in [M_{\mathsf{i}}(\theta), 2], \\ \gamma_{|\mathsf{i}+1|}(R_\omega(\theta)) & \text{if } \theta \in \mathbb{WB}^{\frown}_{m};\ x \in \mathbb{IW}_{m,\theta}, \\ \frac{2-\gamma_{|\mathsf{i}+1|}(R_\omega(\theta))}{2-f_{m-1,\theta}(\lambda_m(\theta))}(f_{m-1,\theta}(x)-2)+2 & \text{if } \theta \in \mathbb{WB}^{\frown}_{m};\ x \in [-2, \lambda_m(\theta)], \\ \frac{2+\gamma_{|\mathsf{i}+1|}(R_\omega(\theta))}{2+f_{m-1,\theta}(\tau_m(\theta))}(f_{m-1,\theta}(x)+2)-2 & \text{if } \theta \in \mathbb{WB}^{\frown}_{m};\ x \in [\tau_m(\theta), 2]. \end{cases}$$

Since $\mathbb{V}^{\frown}_{m} \subset \mathbb{V}^{\frown}_{m-1}$, $f_{m-1,\theta}$ is defined on $\mathbb{V}^{\frown}_{m}$. Moreover, the above formula defines $f_{m,\theta}$ for every $\theta \in \mathbb{B}^{\frown}_{m}$ since, by Definition 6.2, $\mathbb{B}^{\frown}_{m} = \mathbb{B}_{m} \cup \mathbb{WB}^{\frown}_{m}$. We also remark that $f_{m,\theta}$ formally is defined in two different ways when $\theta \in \mathbb{WB}^{\frown}_{m} \cap \mathbb{B}_{m}$. Later on we will show that $f_{m,\theta}$ is well defined. ◘

The next proposition studies the maps $\{T_m\}_{m=0}^{\infty}$ and describes their properties.

Proposition 7.3. *The following statements hold for every $m \in \mathbb{Z}^+$.*

(a) The map T_m is well defined, continuous and belongs to $\mathcal{S}(\Omega)$.

(b) For every $\theta \in \mathbb{S}^1$, $f_{m,\theta}$ is non-increasing, and $f_{m,\theta}(2) = -2$, $f_{m,\theta}(-2) = 2$. Moreover, $-1 \leq f_{0,\theta}\left(M_{\mathsf{b}^{\frown}(\theta,m)}(\theta)\right) \leq f_{0,\theta}\left(m_{\mathsf{b}^{\frown}(\theta,m)}(\theta)\right) \leq 1$ for every $\theta \in \mathbb{B}^{\frown}_{m}$.

(c) For every $i \in \mathfrak{D}_{m}$, $T_m\big|_{\mathcal{R}^{\frown}(i^*)} = G_i$, $T_m\left(\mathsf{A}^{i^*}_{|i|}\right) = \mathsf{A}^{(i+1)^*}_{|i+1|}$, and $T_k\big|_{\{i^*\}\times\mathbb{I}} = T_m\big|_{\{i^*\}\times\mathbb{I}}$ (that is, $f_{k,i^*} = f_{m,i^*}$) for every $k > m$.

The next result shows that the sequence $\{T_m\}_{m=0}^{\infty}$ has a limit in $\mathcal{S}(\Omega)$.

Proposition 7.4. *For every $m \geq 2$ and $\theta \in \mathbb{S}^1$,*

$$\|f_{m,\theta} - f_{m-1,\theta}\| \leq 2 \cdot 2^{-\left|\mathsf{b}^{\frown}(\theta,m-1)\right|}. \tag{16}$$

Moreover, the sequence $\{T_m\}_{k=0}^{\infty}$ is a Cauchy sequence.

Finally we are ready to prove the main result of the paper. It follows from the next result which gives a more concrete version of Theorem A.

Theorem 7.5. *There exists a map $T \in \mathcal{S}(\Omega)$ with $f(\theta,\cdot)$ non-increasing for every $\theta \in \mathbb{S}^1$, such that T permutes the upper and lower circles of Ω (thus having a periodic orbit of period two of curves), and there exists a connected pseudo-curve $\mathsf{A} \subset \Omega$ which does not contain any arc of a curve such that $T(\mathsf{A}) = \mathsf{A}$ and there does not exist any T-invariant curve.*

Proof. By Propositions 7.3 and 7.4, there exists a map

$$T(\theta, x) = (R_\omega(\theta), f(\theta, x)) = (R_\omega(\theta), \lim_{m\to\infty} f_m(\theta, x)) \in \mathcal{S}(\Omega)$$

with $f(\theta,\cdot)$ non-increasing for every $\theta \in \mathbb{S}^1$ such that T permutes the upper and lower circles of Ω (that is, $f(\theta, 2) = -2$ and $f(\theta, -2) = 2$). As the connected set A we take the one given by Proposition 3.12 (and Definition 3.9).

To end the proof of the theorem we need to show that $T(\mathsf{A}) = \mathsf{A}$, since this already implies that there does not exist any T-invariant curve. To see it, assume by way of contradiction that there exists an invariant curve and denote its graph by B. Since B is the graph of a (continuous) curve, it is compact and connected. On the other hand, let Ω_+ and Ω_- be the two connected components of $\Omega\backslash\mathsf{A}$ from the proof of Proposition 3.12. The facts that $T(\mathsf{A}) = \mathsf{A}$, $f(\theta, \cdot)$ is decreasing for every $\theta \in \mathbb{S}^1$, and T permutes the upper and lower circles of Ω imply that $T(\Omega_+) = \Omega_-$ and $T(\Omega_-) = \Omega_+$. Hence, by the invariance of B, $B \not\subseteq \Omega_+$ and $B \not\subseteq \Omega_-$. The connectivity of A and B imply that there exists $(\theta, x) \in \mathsf{A} \cap B$. Consequently,

$$B = \overline{\{T^n(\theta, x) : n \in \mathbb{Z}^+\}} \subset \mathsf{A};$$

a contradiction because A does not contain any arc of a curve.

So, only it remains to prove that $T(\mathsf{A}) = \mathsf{A}$. By using Proposition 7.3(c) and Lemma 3.11(b) we get that $T_m\left(\mathsf{A}^{i^*}\right) = \mathsf{A}^{(i+1)^*}$, and $T_k\big|_{\mathsf{A}^{i^*}} = T_m\big|_{\mathsf{A}^{i^*}}$ for every $k, m \in \mathbb{Z}^+,\ k \geq m$ and $i \in \mathfrak{D}_m$. Consequently, by the definition of the map T we have, $T(\mathsf{A}^{i^*}) = \mathsf{A}^{(i+1)^*}$ for every $i \in \mathbb{Z}$ or, equivalently, $T\big(\mathsf{A}^{\upuparrows O^*(\omega)}\big) = \mathsf{A}^{\upuparrows O^*(\omega)}$.

Now we consider A^θ with $\theta \in \mathbb{S}^1\backslash O^*(\omega)$. Since $O^*(\omega)$ is dense in $\mathbb{S}^1$, there exists a sequence $\{(\theta_n, x_n)\}_{n=0}^\infty \subset \mathsf{A}^{\upuparrows O^*(\omega)}$ such that $\lim_{n\to\infty} \theta_n = \theta$. By the compacity of A we can assume (by taking a convergent subsequence, if necessary) that $\{(\theta_n, x_n)\}_{n=0}^\infty$ is convergent to a point $(\theta, x) \in \mathsf{A}$. By Lemma 3.11(c), $\mathsf{A}^\theta = (\theta, x)$ (and $x = \gamma(\theta)$). On the other hand, by the part of the statement already proven, $T(\theta_n, x_n) \in \mathsf{A}$ for every n. Hence, by the continuity of T and the compacity of A,

$$T(\theta, x) = (R_\omega(\theta), f(\theta, x)) = \lim_{n\to\infty} T(\theta_n, x_n) \in \mathsf{A}^{R_\omega(\theta)}.$$

Since $\theta \notin O^*(\omega)$ we have that $R_\omega(\theta) \notin O^*(\omega)$ and, again by Lemma 3.11(c), $\mathsf{A}^{R_\omega(\theta)}$ consists of a unique point. Hence, $T(\mathsf{A}^\theta) = \mathsf{A}^{R_\omega(\theta)}$ for every $\theta \in \mathbb{S}^1\backslash O^*(\omega)$. Equivalently, $T\Big(\mathsf{A}^{\upuparrows\left(\mathbb{S}^1\backslash O^*(\omega)\right)}\Big) = \mathsf{A}^{\upuparrows\left(\mathbb{S}^1\backslash O^*(\omega)\right)}$. This ends the proof of the theorem. □

8. Proof of Proposition 7.3 in the case $m = 0$

This section is devoted to prove Proposition 7.3 for $m = 0$; that is, to study the map T_0. It is the first technical counterpart of Section 7.

To prove Proposition 7.3 for T_0 we will need some more notation and a technical lemma.

Given a skew product $F(\theta, x) = (R_\omega(\theta), \zeta(\theta, x)$ from $\Omega = \mathbb{S}^1 \times \mathbb{I}$ to itself we define the *fibre map function of* F, $\mathsf{fib}(F)\colon \mathbb{S}^1 \longrightarrow \mathcal{C}(\mathbb{I}, \mathbb{I})$ by $\mathsf{fib}(F)(\theta) := \zeta(\theta, \cdot)$. A simple exercise shows that F is continuous if and only if $\zeta(\theta, \cdot)$ is continuous for every $\theta \in \mathbb{S}^1$, and $\mathsf{fib}(F)$ is continuous.

Lemma 8.1. *Let $\theta \in \mathsf{Bd}\left(\widetilde{B_i}[i^*]\right)$ for some $i \in \mathfrak{D}_0$. Then, $m_i(\theta) = M_i(\theta) = 0$, $g_i(\theta, m_i(\theta)) = \gamma(R_\omega(\theta))$, and $f_{0,\theta}$ is the unique piecewise affine map with two affine pieces whose graph joins the point $(-2, 2)$ with $(0, \gamma(R_\omega(\theta)))$, and this with the point $(2, -2)$.*

Proof. By Lemma 4.1(d) and Definition 7.1, we have $m_i(\theta) = M_i(\theta)$. Hence, $f_{0,\theta}$ is the piecewise affine map with two affine pieces whose graph joins the point $(-2, 2)$ with $\left(m_i(\theta), g_{i,\theta}\left(m_i(\theta)\right)\right)$, and this with the point $(2, -2)$. So, we need to show that $m_i(\theta) = 0$, and $g_{i,\theta}\left(m_i(\theta)\right) = \gamma(R_\omega(\theta))$.

Lemma 3.6(g) and the fact that $\mathsf{depth}\, i = 0$, $\widetilde{B_i}[i^*] \cap \widetilde{B_\ell}[\ell^*] = \emptyset$ for every $\ell \in Z_{|i|}$, $i \neq \ell$. Consequently, by Definition 3.4(R.6), $m_i(\theta) = M_i(\theta) = a_i^- = 0$.

Now we show that $g_{i,\theta}(m_i(\theta)) = \gamma(R_\omega(\theta))$. From the definition of the map g_i (Definitions 4.2 and 4.4), Lemma 3.6(e) and Definitions 3.8 and 3.4(R.1), we get

$$g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)) = \gamma(R_\omega(\theta)).$$

This ends the proof of the lemma. □

Proof of Proposition 7.3 for $m = 0$. By Lemma 3.6(b),

$$-1 \leq m_{\mathsf{b}^{\frown}(\theta,0)}(\theta) \leq M_{\mathsf{b}^{\frown}(\theta,0)}(\theta) \leq 1$$

for every $\theta \in \mathbb{B}^{\frown}_0$. So, T_0 is well defined.

(b) If $\theta \in \mathbb{S}^1 \setminus \mathbb{B}^{\frown}_0$, then the statement follows directly from Definition 7.1. Now assume that $\theta \in \mathbb{B}^{\frown}_0$ and let $i = \mathsf{b}^{\frown}(\theta, 0)$. From the definition of the maps $g_{i,\theta}$ (Definitions 4.2 and 4.4) and Definition 7.1, it follows that $f_{0,\theta}\big|_{\mathbb{I}_{i,\theta}}$ is piecewise affine and non-increasing. On the other hand, again by Definition 7.1, $f_{0,\theta}\big|_{[-2,m_i(\theta)]}$ and $f_{0,\theta}\big|_{[M_i(\theta),2]}$ are affine with negative slope and $f_{0,\theta}(2) = -2$ and $f_{0,\theta}(-2) = 2$. The fact that

$$-1 \leq f_{0,\theta}\left(M_{\mathsf{b}^{\frown}(\theta,0)}(\theta)\right) \leq f_{0,\theta}\left(m_{\mathsf{b}^{\frown}(\theta,0)}(\theta)\right) \leq 1$$

for every $\theta \in \mathbb{B}^{\frown}_0$ follows from Definition 7.1 and Lemmas 4.3(a) and 4.5(a). This ends the proof of (b).

(c) Recall that

$$\mathcal{R}^{\frown}(i^*) = \bigcup_{\theta \in B^{\frown}_i[i^*]} \{\theta\} \times \mathbb{I}_{i,\theta}.$$

Hence, from Definition 7.1 and the definition of G_i (Definitions 4.2 and 4.4) it follows that

$$T_m(\theta, x) = \big(R_\omega(\theta), f_m(\theta, x)\big) = \big(R_\omega(\theta), g_{i,\theta}(x)\big) = G_i(\theta, x),$$

for every $(\theta, x) \in \mathcal{R}^{\frown}(i^*)$. Thus, $T_0\left(\mathsf{A}^{i^*}_{|i|}\right) = \mathsf{A}^{(i+1)^*}_{|i+1|}$ from Lemmas 3.10(b), 4.3(c) and 4.5(c). On the other hand, Lemma 5.3(b) implies that $i^* \in \mathbb{B}^{\frown}_0$ but $i^* \notin \mathbb{B}^{\frown}_k$ for every $k \in \mathbb{N}$. Then, we get $f_{k,i^*} = f_{0,i^*}$ from Definition 7.2.

(a) Since T_0 is a skew product with base R_ω we only have to prove that f_0 is continuous.

By Definition 7.1, for every $\theta \in \mathbb{S}^1$, the map $f_{0,\theta}$ is continuous. So we have to prove that the map $\mathsf{fib}(T_0)$ (that is, the map $s \mapsto f_{0,s}$) is continuous.

In the rest of the proof we will denote

$$\mathbb{IB}^{\frown}_0 := \bigcup_{i \in \mathfrak{D}_0} B^{\frown}_i(i^*) \subset \mathbb{B}^{\frown}_0.$$

Clearly, since for every $i \in \mathbb{Z}$, the maps m_i and M_i are continuous on $B^{\frown}_i[i^*]$, it follows that the map $s \mapsto f_{0,s}$ is continuous on $\mathbb{IB}^{\frown}_0$. Thus, we have to see that the fibre map function is continuous at every $\theta \in \mathbb{S}^1 \setminus \mathbb{IB}^{\frown}_0$; that is, $\lim_{j\to\infty} f_{0,\theta_j} = f_{0,\theta}$ for every $\{\theta_j\}_{j=1}^\infty \subset \mathbb{S}^1$ converging to θ. Given $\alpha > 0$, we can consider four sets associated to such a sequence:

$$\{j \in \mathbb{N} : \theta_j \in \mathbb{S}^1 \setminus \mathbb{IB}^{\frown}_0\}, \quad \{j \in \mathbb{N} : \theta_j \in \mathbb{IB}^{\frown}_0 \setminus B_\alpha(\theta)\},$$
$$\{j \in \mathbb{N} : \theta_j \in (\theta, \theta + \alpha) \cap \mathbb{IB}^{\frown}_0\} \quad \text{and} \quad \{j \in \mathbb{N} : \theta_j \in (\theta - \alpha, \theta) \cap \mathbb{IB}^{\frown}_0\}.$$

Observe that the second set $\{j \in \mathbb{N} : \theta_j \in \mathbb{IB}^{\frown}_0 \setminus B_\alpha(\theta)\}$ is always finite and that any of the other three sets gives rise to a subsequence of $\{\theta_j\}_{j=1}^\infty$ converging to θ, when it is infinite. Consequently, the continuity of the fibre map function $s \mapsto f_{0,s}$ at θ is equivalent to the fact that $\lim_{j\to\infty} f_{0,\theta_j} = f_{0,\theta}$ for every $\{\theta_j\}_{j=1}^\infty$ converging to θ and such that, for some $\alpha > 0$, $\{\theta_j\}_{j=1}^\infty$ is contained either in $\mathbb{S}^1 \setminus \mathbb{IB}^{\frown}_0$, or

$(\theta, \theta+\alpha) \cap \mathbb{IB}_0^{\frown}$, or $(\theta-\alpha, \theta) \cap \mathbb{IB}_0^{\frown}$. We will only deal with the first two cases since the proof in the last case (for $(\theta-\alpha, \theta)$) can be done symmetrically.

Case 1: $\lim_{j\to\infty} \theta_j = \theta$ *and* $\{\theta_j\}_{j=1}^{\infty} \subset \mathbb{S}^1 \backslash \mathbb{IB}_0^{\frown}$.

By Definition 7.1 and Lemma 8.1, f_{0,θ_j} (respectively $f_{0,\theta}$) is the unique piecewise affine map with two affine pieces whose graph joins the point $(-2, 2)$ with $(0, \gamma(R_\omega(\theta_j)))$ (respectively $(0, \gamma(R_\omega(\theta)))$), and this with the point $(2, -2)$. By Lemma 5.3(c) and Definition 3.8 the function γ is continuous at $R_\omega(\theta) \notin O^*(\omega)$. Hence, $\lim_{j\to\infty} \gamma(R_\omega(\theta_j)) = \gamma(R_\omega(\theta))$ and, thus, $\lim_{j\to\infty} f_{0,\theta_j} = f_{0,\theta}$.

Case 2: $\lim_{j\to\infty} \theta_j = \theta$ *and* $\{\theta_j\}_{j=1}^{\infty} \subset (\theta, \theta+\alpha) \cap \mathbb{IB}_0^{\frown}$.

If there exists $i \in \mathfrak{D}_0$ such that θ is the left endpoint of $B_i^{\frown}[i^*] \subset \mathbb{B}_0^{\frown}$ then the result follows from Definition 7.1, the continuity of the maps m_i and M_i and the continuity of the maps g_i (Lemmas 4.3(a) and 4.5(a)).

Assume now that θ is not the left endpoint of $B_i^{\frown}(i^*)$ for every $i \in \mathfrak{D}_0$. For every $j \in \mathbb{N}$ we set $\mathsf{i}_j := \mathsf{b}^{\frown}(\theta_j, 0) \in \mathfrak{D}_0$ (that is, $\theta_j \in B_{\mathsf{i}_j}^{\frown}((\mathsf{i}_j)^*)$).

We claim that $\lim_{j\to\infty} |\mathsf{i}_j| = \infty$ and consequently, by Definition 3.4(R.1),

$$\lim_{j\to\infty} 2^{-n_{|\mathsf{i}_j+1|}} = \lim_{j\to\infty} 2^{-n_{|\mathsf{i}_j|}} = 0. \tag{17}$$

To prove this claim, assume by way of contradiction that there exists L such that for every $k \in \mathbb{N}$ there exists $j_k \geq k$ such that $|\mathsf{i}_{j_k}| \leq L$. Then,

$$\{\theta_{j_k}\}_{k=1}^{\infty} \subset \bigcup_{k=1}^{\infty} B_{\mathsf{i}_{j_k}}^{\frown}((\mathsf{i}_{j_k})^*)$$

and, since $\{\mathsf{i}_{j_k} : k \in \mathbb{N}\}$ is finite, it follows that there exists $i \in \{\mathsf{i}_{j_k} : k \in \mathbb{N}\} \subset \mathfrak{D}_0$ and a subsequence of $\{\theta_{j_k}\}_{k=1}^{\infty}$, that by abuse of notation will also be called $\{\theta_{j_k}\}$, such that $\{\theta_{j_k}\}_{k=1}^{\infty} \subset B_i^{\frown}(i^*)$. So,

$$\theta = \lim_{k\to\infty} \theta_{j_k} \in B_i^{\frown}[i^*];$$

a contradiction. So, the claim (and hence (17)) holds.

Next we claim that the conditions

(18) $\lim_{j\to\infty} M_{\mathsf{i}_j}(\theta_j) = \lim_{j\to\infty} m_{\mathsf{i}_j}(\theta_j) = 0$, and

(19) there exists a sequence $\{x_j\}_{j=1}^{\infty}$ with $x_j \in \mathbb{I}_{\mathsf{i}_j,\theta_j} = [m_{\mathsf{i}_j}(\theta_j), M_{\mathsf{i}_j}(\theta_j)]$ for every j, such that $\lim_{j\to\infty} f_{0,\theta_j}(x_j) = \gamma(R_\omega(\theta))$

imply

$$\lim_{j\to\infty} f_{0,\theta_j} = f_{0,\theta}.$$

To prove the claim notice that, by Definition 7.1 and Lemma 8.1, $f_{0,\theta}$ is the unique piecewise affine map with two affine pieces whose graph joins the point $(-2, 2)$ with $(0, \gamma(R_\omega(\theta)))$, and this with the point $(2, -2)$. On the other hand, for every j,

- $f_{0,\theta_j}\big|_{[-2, m_{\mathsf{i}_j}(\theta_j)]}$ is the affine map joining the point $(-2, 2)$ with the point $(m_{\mathsf{i}_j}(\theta_j), g_{\mathsf{i}_j}(\theta_j, m_{\mathsf{i}_j}(\theta_j)))$, and
- $f_{0,\theta_j}\big|_{[M_{\mathsf{i}_j}(\theta_j), 2]}$ is the affine map joining the point $(M_{\mathsf{i}_j}(\theta_j), g_{\mathsf{i}_j}(\theta_j, M_{\mathsf{i}_j}(\theta_j)))$ with the point$(2, -2)$

(see Figure 5). Moreover, from the part of the proposition already proven we know that f_{0,θ_j} is non-increasing and continuous. Therefore, the claim holds provided that

$$\lim_{j\to\infty} \mathsf{diam}\left(f_{0,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right)\right) = 0$$

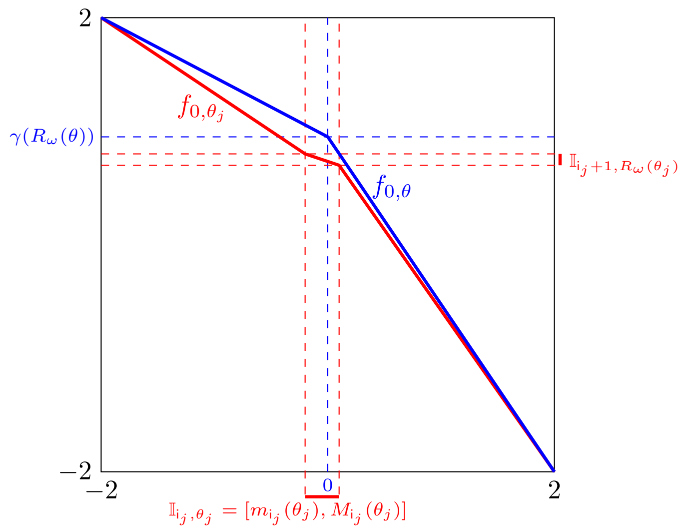


FIGURE 5. A symbolic representation of the maps $f_{0,\theta}$ and f_{0,θ_j} in Case 2 of the proof of Proof of Proposition 7.3 for $m = 0$. The map $f_{0,\theta}$ and the points 0 and $\gamma(R_\omega(\theta))$ are drawn in blue. The map f_{0,θ_j} and the corresponding intervals $\mathbb{I}_{\mathsf{i}_j,\theta_j}$ and $\mathbb{I}_{\mathsf{i}_j+1,R_\omega(\theta_j)}$ are drawn in red.

(see again Figure 5).

When $\theta_j \in B_{\alpha_{\mathsf{i}_j}}[(\mathsf{i}_j)^*] \setminus B_{\alpha_{\mathsf{i}_j+1}}((\mathsf{i}_j)^*)$ and $\mathsf{i}_j \geq 0$, by Definitions 7.1 and 4.2,

$$\mathsf{diam}\left(f_{0,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right)\right) = \mathsf{diam}\left(g_{\mathsf{i}_j,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right)\right) = \mathsf{diam}\left(\{\gamma_{\mathsf{i}_j+1}(R_\omega(\theta_j)\}\right) = 0.$$

Otherwise, by Definition 7.1, and Lemmas 4.3(b) and 4.5(b),

$$\{R_\omega(\theta_j)\} \times f_{0,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right) = \{R_\omega(\theta_j)\} \times g_{\mathsf{i}_j,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right) = G_{\mathsf{i}_j}\left(\mathcal{R}((\mathsf{i}_j)^*)^{\theta_j}\right) \subset \mathcal{R}((\mathsf{i}_j+1)^*)^{R_\omega(\theta_j)}.$$

So, by Remark 3.2(2),

$$\mathsf{diam}\left(f_{0,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right)\right) \leq \mathsf{diam}\left(\mathcal{R}((\mathsf{i}_j+1)^*)\right) \leq 2 \cdot 2^{-n_{|\mathsf{i}_j+1|}}.$$

In any case,

$$0 \leq \mathsf{diam}\left(f_{0,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right)\right) \leq 2 \cdot 2^{-n_{|\mathsf{i}_j+1|}} \quad \text{for every} \quad j \in \mathbb{N}$$

and, by (17), $\lim_{j\to\infty} \mathsf{diam}\left(f_{0,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right)\right) = 0$. This ends the proof of the claim.

By the last claim, to end the proof of the proposition in the case $m = 0$ it is enough to show that (18–19) hold. We start by proving (18). By Lemma 8.1,

$$m_{\mathsf{i}_j}(\mathsf{Bd}(B^{\frown}_{\mathsf{i}_j}[(\mathsf{i}_j)^*])) = M_{\mathsf{i}_j}(\mathsf{Bd}(B^{\frown}_{\mathsf{i}_j}[(\mathsf{i}_j)^*])) = 0,$$

and from the definition of the maps m_{i_j} and M_{i_j}, Definition 3.1 (or Lemma 4.1) and Remark 3.2(2), for every $s \in B^{\frown}_{\mathsf{i}_j}((\mathsf{i}_j)^*)$ we get

$$\begin{aligned} &-1 \leq m_{\mathsf{i}_j}(s) < 0 < M_{\mathsf{i}_j}(s) \leq 1, \text{ and} \\ &M_{\mathsf{i}_j}(s) - m_{\mathsf{i}_j}(s) = \mathsf{diam}(\mathbb{I}_{\mathsf{i}_j,s}) \leq 2 \cdot 2^{-n_{|\mathsf{i}_j|}}. \end{aligned} \tag{20}$$

So, (18) holds by (17). Now we prove (19).

By (2), (3) and (9), it follows that

$$m_{\mathsf{i}_j}(\theta_j) < \gamma_{|\mathsf{i}_j|}(\theta_j) < M_{\mathsf{i}_j}(\theta_j) \qquad \text{if } \theta_j \neq (\mathsf{i}_j)^*, \text{ and}$$
$$m_{\mathsf{i}_j}(\theta_j) < \gamma_{|\mathsf{i}_j|-1}(\theta_j) = 0 < M_{\mathsf{i}_j}(\theta_j) \qquad \text{if } \theta_j = (\mathsf{i}_j)^*.$$

Also, from Definition 7.1, the definitions of G_i and $g_{i,\theta}$ (Definitions 4.2 and 4.4), and Lemmas 4.3(c) and 4.5(c) we get

$$f_{0,\theta_j}(\gamma_{|\mathsf{i}_j|}(\theta_j)) = g_{\mathsf{i}_j,\theta_j}(\gamma_{|\mathsf{i}_j|}(\theta_j)) = \gamma_{|\mathsf{i}_j+1|}(R_\omega(\theta_j)) \qquad \text{if } \theta_j \neq (\mathsf{i}_j)^*,$$
$$f_{0,\theta_j}(\gamma_{\mathsf{i}_j-1}(\theta_j)) = g_{\mathsf{i}_j,\theta_j}(\gamma_{\mathsf{i}_j-1}(\theta_j)) = \gamma_{\mathsf{i}_j}(R_\omega(\theta_j)) \qquad \text{if } \theta_j = (\mathsf{i}_j)^* \text{ and } \mathsf{i}_j \geq 0, \text{ and}$$
$$f_{0,\theta_j}(\gamma_{|\mathsf{i}_j|-1}(\theta_j)) = g_{\mathsf{i}_j,\theta_j}(\gamma_{|\mathsf{i}_j+1|}(\theta_j)) = \gamma_{|\mathsf{i}_j+2|}(R_\omega(\theta_j)) \qquad \text{if } \theta_j = (\mathsf{i}_j)^* \text{ and } \mathsf{i}_j < 0.$$

Thus, to prove (19), we have to show that

$$\begin{cases} \lim_{j\to\infty} \gamma_{|\mathsf{i}_j+1|}(R_\omega(\theta_j)) = \gamma(R_\omega(\theta)) & \text{if } \theta_j \neq (\mathsf{i}_j)^*, \\ \lim_{j\to\infty} \gamma_{\mathsf{i}_j}(R_\omega(\theta_j)) = \gamma(R_\omega(\theta)) & \text{if } \theta_j = (\mathsf{i}_j)^* \text{ and } \mathsf{i}_j \geq 0, \text{ and} \\ \lim_{j\to\infty} \gamma_{|\mathsf{i}_j+2|}(R_\omega(\theta_j)) = \gamma(R_\omega(\theta)) & \text{if } \theta_j = (\mathsf{i}_j)^* \text{ and } \mathsf{i}_j < 0 \end{cases} \tag{21}$$

(that is, we take $x_j := \gamma_{|\mathsf{i}_j|}(\theta_j)$ if $\theta_j \neq (\mathsf{i}_j)^*$, $x_j := \gamma_{\mathsf{i}_j-1}(\theta_j)$ if $\theta_j = (\mathsf{i}_j)^*$ and $\mathsf{i}_j \geq 0$, and $x_j := \gamma_{|\mathsf{i}_j|-1}(\theta_j)$ if $\theta_j = (\mathsf{i}_j)^*$ and $\mathsf{i}_j < 0$).

Let $\varepsilon > 0$. By Lemma 5.3(c) and Definition 3.4(R.1) we have that $\theta \notin O^*(\omega)$ and, hence, $R_\omega(\theta) \notin O^*(\omega)$. By the continuity of γ on $\mathbb{S}^1\backslash O^*(\omega)$ and the fact that $\lim_{i\to\infty} \gamma_i = \gamma$, there exist $\delta > 0$ and $L \in \mathbb{N}$ such that

$$\left|\gamma(R_\omega(\theta)) - \gamma(\widehat{\theta})\right| < \varepsilon/2 \quad \text{for every } \widehat{\theta} \in B_\delta(R_\omega(\theta))\backslash O^*(\omega), \text{ and}$$
$$d_\infty(\gamma, \gamma_i) < \varepsilon/2 \quad \text{for every } i \geq L.$$

Then, since $\lim_{j\to\infty} \theta_j = \theta$ and $\lim_{j\to\infty} |\mathsf{i}_j| = \infty$, there exists $N \in \mathbb{N}$ such that $|\theta - \theta_j| < \delta/2$, and $|\mathsf{i}_j| \geq L + 2$ for every $j \geq N$.

First we will show that

$$\left|\gamma(R_\omega(\theta)) - \gamma_{|\mathsf{i}_j+1|}(R_\omega(\theta_j))\right| \leq \varepsilon$$

for every $j \geq N$ such that $\theta_j \neq (\mathsf{i}_j)^*$. To see it observe that, by Definition 3.4(R.2) and Remark 3.5(R.2), $\theta_j, R_\omega(\theta_j) \notin Z^*_{|\mathsf{i}_j+1|}$ whenever $\theta_j \neq (\mathsf{i}_j)^*$. Thus, $\gamma_{|\mathsf{i}_j+1|}$ is continuous at $R_\omega(\theta_j)$ by Lemma 3.6(a).

Also, there exists a sequence $\{\widehat{\theta}_{j_\ell}\}_{\ell=1}^\infty \subset \left(B_{\delta/2}(\theta_j) \cap B_{\mathsf{i}_j}^{\frown}((\mathsf{i}_j)^*)\right)\backslash O^*(\omega)$ converging to θ_j, because $\mathbb{S}^1\backslash O^*(\omega)$ is dense in $\mathbb{S}^1$. Clearly, for every $j \geq N$, we have $\{R_\omega(\widehat{\theta}_{j_\ell})\}_{\ell=1}^\infty \subset B_\delta(R_\omega(\theta))\backslash O^*(\omega)$ and $\lim_{\ell\to\infty} R_\omega(\widehat{\theta}_{j_\ell}) = R_\omega(\theta_j)$. Moreover, since $\{R_\omega(\widehat{\theta}_{j_\ell})\}_{\ell=1}^\infty \subset \mathbb{S}^1\backslash O^*(\omega) \subset \mathbb{S}^1\backslash Z^*_{|\mathsf{i}_j+1|}$, $\gamma_{|\mathsf{i}_j+1|}$ is defined for every $R_\omega(\widehat{\theta}_{j_\ell})$. Then, for every $j \geq N$ and $\ell \in \mathbb{N}$, we have

$$\begin{aligned}\left|\gamma(R_\omega(\theta)) - \gamma_{|\mathsf{i}_j+1|}(R_\omega(\widehat{\theta}_{j_\ell}))\right| &\leq \left|\gamma(R_\omega(\theta)) - \gamma(R_\omega(\widehat{\theta}_{j_\ell}))\right| + \\ &\qquad \left|\gamma(R_\omega(\widehat{\theta}_{j_\ell})) - \gamma_{|\mathsf{i}_j+1|}(R_\omega(\widehat{\theta}_{j_\ell}))\right| \\ &< \tfrac{\varepsilon}{2} + d_\infty\left(\gamma, \gamma_{|\mathsf{i}_j+1|}\right) < \varepsilon.\end{aligned}$$

Consequently,

$$\left|\gamma(R_\omega(\theta)) - \gamma_{|\mathsf{i}_j+1|}(R_\omega(\theta_j))\right| = \lim_{\ell\to\infty} \left|\gamma(R_\omega(\theta)) - \gamma_{|\mathsf{i}_j+1|}(R_\omega(\widehat{\theta}_{j_\ell}))\right| \leq \varepsilon$$

This ends the proof of the first equality of (21). The second and third equalities of (21) follow as above by replacing $\gamma_{|\mathsf{i}_j+1|}$ by γ_{i_j} (respectively $\gamma_{|\mathsf{i}_j+2|}$), and noting that

$$R_\omega(\theta_j) = R_\omega((\mathsf{i}_j)^*) = \begin{cases} ((\mathsf{i}_j+1))^* \notin Z^*_{\mathsf{i}_j} & \text{if } \mathsf{i}_j \geq 0, \text{ and} \\ ((-(|\mathsf{i}_j|-1)))^* \notin Z^*_{|\mathsf{i}_j|-2} & \text{if } \mathsf{i}_j < 0. \end{cases}$$

This ends the proof of the continuity of T_0, and the proposition for the case $m = 0$. □

9. Proof of Proposition 7.3 for $m > 0$

This section is the second technical counterpart of Section 7 and is devoted to prove Proposition 7.3 for every map T_m with $m > 0$. To do this we will need some more technical results. Also we will use the notion of fibre map function introduced in the previous section.

The next two lemmas establish some basic properties of the maps $T_m\big|_{\mathbb{V}^{\frown}_m}$ and clarify some aspects of Definition 7.2.

Lemma 9.1. *For every $m \in \mathbb{N}$ and for every $\theta \in \mathbb{B}^{\frown}_m$,*

$$f_{m,\theta}\big|_{\mathbb{I}_{\mathsf{i},\theta}} = g_{\mathsf{i},\theta}\big|_{\mathbb{I}_{\mathsf{i},\theta}},$$

where $\mathsf{i} = \mathsf{b}^{\frown}(\theta, m)$. Moreover, assume that $\theta \in \mathbb{WB}^{\frown}_m \setminus \mathbb{WIB}_m$. Then,

$$f_{m,\theta}(x) = \begin{cases} g_{\mathsf{i},\theta}(x) & \text{if } x \in \mathbb{I}_{\mathsf{i},\theta}, \\ \frac{2-g_{\mathsf{i},\theta}(m_\mathsf{i}(\theta))}{2-f_{m-1,\theta}(m_\mathsf{i}(\theta))}(f_{m-1,\theta}(x)-2)+2 & \text{if } x \in [-2, m_\mathsf{i}(\theta)], \\ \frac{2+g_{\mathsf{i},\theta}(M_\mathsf{i}(\theta))}{2+f_{m-1,\theta}(M_\mathsf{i}(\theta))}(f_{m-1,\theta}(x)+2)-2 & \text{if } x \in [M_\mathsf{i}(\theta), 2]. \end{cases}$$

Proof. We start by proving the first statement. When $\theta \in \mathbb{B}_m$ there is nothing to prove. So, assume that $\theta \in \mathbb{B}^{\frown}_m \setminus \mathbb{B}_m$. By Definition 6.2, $\theta \in \mathbb{WB}^{\frown}_m$, $\mathsf{i} < 0$ and $\theta \in B^{\frown}_\mathsf{i}[\mathsf{i}^*] \setminus B_{\alpha_{|\mathsf{i}|}}(\mathsf{i}^*)$. By Lemma 6.3(b),

$$\mathbb{I}_{\mathsf{i},\theta} = \{\gamma_{|\mathsf{i}|}(\theta)\} \subset \mathbb{IW}_{m,\theta}.$$

Consequently, by Definition 7.2 and the definition of the maps $g_{\mathsf{i},\theta}$ for $\mathsf{i} < 0$ (Definition 4.4 — notice that $\mathbb{I}_{\mathsf{i},\theta} \subset \mathcal{R}^{\frown}(\mathsf{i}^*)$ by definition),

$$f_{m,\theta}\left(\gamma_{|\mathsf{i}|}(\theta)\right) = \gamma_{|\mathsf{i}+1|}\left(R_\omega(\theta)\right) = g_{\mathsf{i},\theta}\left(\gamma_{|\mathsf{i}|}(\theta)\right).$$

So, the first statement holds. Now we prove the second one. By Lemma 6.3(b),

$$\mathbb{I}_{\mathsf{i},\theta} = \{m_\mathsf{i}(\theta)\} = \{M_\mathsf{i}(\theta)\} = \{\gamma_{|\mathsf{i}|}(\theta)\} = \{\lambda_m(\theta)\} = \{\tau_m(\theta)\} = \mathbb{IW}_{m,\theta}.$$

Thus, by the part already proven, the formulas

$$\begin{cases} g_{\mathsf{i},\theta}(x) & \text{if } x \in \mathbb{I}_{\mathsf{i},\theta}, \\ \frac{2-g_{\mathsf{i},\theta}(m_\mathsf{i}(\theta))}{2-f_{m-1,\theta}(m_\mathsf{i}(\theta))}(f_{m-1,\theta}(x)-2)+2 & \text{if } x \in [-2, m_\mathsf{i}(\theta)], \\ \frac{2+g_{\mathsf{i},\theta}(M_\mathsf{i}(\theta))}{2+f_{m-1,\theta}(M_\mathsf{i}(\theta))}(f_{m-1,\theta}(x)+2)-2 & \text{if } x \in [M_\mathsf{i}(\theta), 2], \end{cases}$$

and

$$\begin{cases} \gamma_{|\mathsf{i}+1|}\left(R_\omega(\theta)\right) & \text{if } x \in \mathbb{IW}_{m,\theta}, \\ \frac{2-\gamma_{|\mathsf{i}+1|}(R_\omega(\theta))}{2-f_{m-1,\theta}(\lambda_m(\theta))}(f_{m-1,\theta}(x)-2)+2 & \text{if } x \in [-2, \lambda_m(\theta)], \\ \frac{2+\gamma_{|\mathsf{i}+1|}(R_\omega(\theta))}{2+f_{m-1,\theta}(\tau_m(\theta))}(f_{m-1,\theta}(x)+2)-2 & \text{if } x \in [\tau_m(\theta), 2], \end{cases}$$

coincide. □

Lemma 9.2. *The following statements hold for every $m \in \mathbb{N}$ and $i \in \mathfrak{D}_m$:*
(a) The map $T_m\big|_{\mathsf{V}^{\frown}_{i^*}}$ is well defined and continuous.
(b) For every $\theta \in B^{\frown}_i[i^*]$,
 (b.i) $f_{m,\theta}(2) = -2$ and $f_{m,\theta}(-2) = 2$,
 (b.ii) $f_{m,\theta}$ is piecewise affine and non-increasing, and
 (b.iii) $-1 \le f_{m,\theta}(M_i(\theta)) \le f_{m,\theta}(m_i(\theta)) \le 1$.
(c) $T_m\big|_{\mathcal{R}^{\frown}(i^*)} = G_i$ and $T_m\left(\mathsf{A}^{i^*}_{|i|}\right) = \mathsf{A}^{(i+1)^*}_{|i+1|}$.

Proof. Clearly, $T_m\big|_{\mathsf{V}^{\frown}_{i^*}}$ is well defined and continuous if and only if so is $f_m\big|_{\mathsf{V}^{\frown}_{i^*}}$.

We will prove by induction on $m \in \mathbb{Z}^+$ that, (a), (b) and

(b.iv) $f_{m,\theta}\big|_{[-2,-1]}$ and $f_{m,\theta}\big|_{[1,2]}$ are affine, $f_{m,\theta}(-1) < 2$ and $f_{m,\theta}(1) > -2$

hold for every $\theta \in B^{\frown}_i[i^*]$.

First we will show that (a), (b) and (b.iv) hold for $m = 0$ and $i \in \mathfrak{D}_0$ (we are including the map f_0 studied earlier to correctly start the induction process). By Proposition 7.3(a,b) for $m = 0$ we have that $T_0\big|_{\mathsf{V}^{\frown}_{i^*}}$ is well defined and continuous and (b) holds. By Definition 7.1, we also know that $f_{m,\theta}\big|_{[-2,m_i(\theta)]}$ and $f_{m,\theta}\big|_{[M_i(\theta),2]}$ are affine. Then, (b.iv) follows from $-1 \le m_i(\theta) \le M_i(\theta) \le 1$ (see Lemma 4.1(a)) and (b.iii).

Assume now that (a), (b) and (b.iv) hold for some $m - 1 \in \mathbb{Z}+$ and prove it for m and $i \in \mathfrak{D}_m$. By Lemma 5.2(a), $\theta \in B^{\frown}_i[i^*] \subsetneq B^{\frown}_k[k^*]$ for some $k \in \mathfrak{D}_{m-1}$. Consequently, $\mathsf{V}^{\frown}_{i^*} \subset \mathsf{V}^{\frown}_{k^*}$ and $f_{m-1}\big|_{\mathsf{V}^{\frown}_{i^*}}$ is well defined and continuous.

By Lemma 4.1(a) and Definition 6.2,

$$\begin{aligned} &-1 \le m_i(\theta) \le M_i(\theta) \le 1 &&\text{for } \theta \in B^{\frown}_i[i^*], \text{ and} \\ &-1 \le \lambda_m(\theta) \le \tau_m(\theta) \le 1 &&\text{for } \theta \in B^{\frown}_i[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{WB}^{\frown}_m\ (i < 0). \end{aligned} \tag{22}$$

Consequently, by (b.ii) and (b.iv) for $m - 1$,

$$-2 < f_{m-1,\theta}(1) \le f_{m-1,\theta}(M_i(\theta)) \le f_{m-1,\theta}(m_i(\theta)) \le f_{m,\theta}(-1) < 2$$

for every $\theta \in B^{\frown}_i[i^*]$, and

$$-2 < f_{m-1,\theta}(1) \le f_{m-1,\theta}(\tau_m(\theta)) \le f_{m-1,\theta}(\lambda_m(\theta)) \le f_{m,\theta}(-1) < 2$$

for $\theta \in B^{\frown}_i[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{WB}^{\frown}_m$ when $i < 0$.

On the other hand, as it was observed in Definition 7.2, $f_{m,\theta}$ is defined in two different ways when $\theta \in \mathbb{WB}^{\frown}_m \cap \mathbb{B}_m$. In such a case, by Lemmas 6.3(a,b) and 9.1, $\theta \notin \mathbb{WIB}_m$ and both definitions for $f_{m,\theta}$ coincide. Hence, $f_m\big|_{\mathsf{V}^{\frown}_{i^*}}$ is well defined.

Now we prove that $f_m\big|_{\mathsf{V}^{\frown}_{i^*}}$ is continuous by using the continuity of $f_{m-1}\big|_{\mathsf{V}^{\frown}_{i^*}}$. Since $B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}_m$, by Definition 7.2, the continuity of the maps m_i and M_i (see Lemma 4.1(b)), and the continuity of the maps g_i (Lemmas 4.3(a) and 4.5 (a)), $f_m\big|_{\upuparrows B_{\alpha_{|i|}}[i^*]}$ is continuous. Now we assume that $i < 0$ and we study the continuity of $f_m\big|_{\upuparrows U}$ on a connected component U of $B^{\frown}_i[i^*] \setminus B_{\alpha_{|i|}}(i^*)$. Observe that, by Definition 6.2 and Lemma 5.2(b), U is a connected component of $\mathbb{WB}^{\frown}_m$. Then, again by Definition 7.2, the continuity of the maps $\lambda_m\big|_U$ and $\tau_m\big|_U$ (Lemma 6.4), and the continuity of the map $\gamma_{|i|}\big|_U$ (Lemma 3.6(a) and Definition 3.4(R.2) and Remark 3.5(R.2)), $f_m\big|_{\upuparrows U}$ is continuous. Therefore, $f_m\big|_{\mathsf{V}^{\frown}_{i^*}}$ is continuous because it is well defined on $\upuparrows\left(\left(B^{\frown}_i[i^*] \setminus B_{\alpha_{|i|}}(i^*)\right) \cap B_{\alpha_{|i|}}[i^*]\right)$.

Let $\theta \in B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}_m$. By Definition 7.2, and the definition of the maps $g_{i,\theta}$ (Definitions 4.2 and 4.4), $f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}}$ is piecewise affine and non-increasing. So, by Lemma 9.1 for $m - 1$ and Definition 7.2, $f_{m,\theta}(2) = -2$, $f_{m,\theta}(-2) = 2$, and

$f_{m,\theta}\big|_{[-2,m_i(\theta)]}$ and $f_{m,\theta}\big|_{[M_i(\theta),2]}$ are affine transformations of the map $f_{m-1,\theta}$ with positive slope. Hence, (b.i,ii) hold for $f_{m,\theta}$ in this case. Moreover, (b.iv) is verified by (22) and (b.iv) for $m-1$.

Consider $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{WB}_m^{\frown}$. Again by Definition 7.2, $f_{m,\theta}\big|_{\mathbb{IW}_{m,\theta}}$ is constant. Then, (b.i,ii) and (b.iv) hold for $f_{m,\theta}$ as above by replacing $m_i(\theta)$ and $M_i(\theta)$ by $\lambda_m(\theta)$ and $\tau_m(\theta)$, respectively.

By (b.ii) and (22) we have $f_{m,\theta}\left(M_i(\theta)\right) \le f_{m,\theta}\left(m_i(\theta)\right).$ Hence, (b.iii) follows from Lemma 9.1, Definition 7.2, Lemmas 4.3(b) and 3.10(c), Definition 3.4($\mathsf{R}.2$) and Remark 3.5($\mathsf{R}.2$), Lemma 4.5(b) and Lemma 3.6(b).

(c) In a similar way to the proof of Proposition 7.3 for the case $m = 0$,

$$\mathcal{R}^{\frown}(i^*) = \bigcup_{\theta \in B_i^{\frown}[i^*]} \{\theta\} \times \mathbb{I}_{i,\theta} \subset \mathsf{V}_{i^*}^{\frown} \subset \mathbb{V}_m^{\frown}$$

and, by Definition 7.2, Lemma 9.1 and the definition of G_i (Definitions 4.2 and 4.4) it follows that

$$T_m(\theta, x) = \big(R_\omega(\theta), f_m(\theta, x)\big) = \big(R_\omega(\theta), g_{i,\theta}(x)\big) = G_i(\theta, x),$$

for every $(\theta, x) \in \mathcal{R}^{\frown}(i^*)$. Thus, $T_m\left(\mathsf{A}_{|i|}^{i^*}\right) = \mathsf{A}_{|i+1|}^{(i+1)^*}$ from Lemmas 3.10(b), 4.3(c) and 4.5(c). □

The next technical lemma compares the images of $f_{m,\theta}$ and $f_{m-1,\theta}$ on a point. It is an extension of Lemma 5.4.

Lemma 9.3. *Assume that $B_i^{\frown}[i^*] \subset B_k^{\frown}[k^*]$ for some $i \in \mathfrak{D}_m$, $k \in \mathfrak{D}_{m-1}$ and $m \in \mathbb{N}$. Then, for every $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$, $m_i(\theta) = M_i(\theta) = \gamma_i(\theta)$ and*

$$f_{m,\theta}\left(m_i(\theta)\right) = g_{i,\theta}\left(m_i(\theta)\right) = \gamma_{|i+1|}\left(R_\omega(\theta)\right), \text{ and}$$
$$f_{m-1,\theta}\left(m_i(\theta)\right) = g_{k,\theta}\left(m_i(\theta)\right) = \gamma_{|k+1|}\left(R_\omega(\theta)\right).$$

Proof. The fact that $m_i(\theta) = M_i(\theta) = \gamma_i(\theta)$ follows directly from the definitions. The first equation follows from Lemma 9.1, and the definition of the map $g_{i,\theta}$ (Definitions 4.2 and 4.4).

By Lemma 5.4, $\mathbb{I}_{i,\theta} = \{m_i(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}$. Moreover, as in the proof of Lemma 5.4, $\theta \neq k^*$. Consequently, by Definition 7.1, Lemma 9.1, Lemmas 4.3(c) and 4.5(c) and (2) (alternatively, for the last equality check directly the proofs of the Lemmas 4.3(c) and 4.5(c)),

$$f_{m-1,\theta}\left(m_i(\theta)\right) = g_{k,\theta}\left(m_i(\theta)\right) = g_{k,\theta}\left(\gamma_{|k|}(\theta)\right) = \gamma_{|k+1|}\left(R_\omega(\theta)\right).$$

□

The following lemma is the analogue of Lemma 8.1 for $m \ge 1$. To state it we will use the set

$$\upuparrows\mathbb{EB}_m^{\frown} = \mathbb{EB}_m^{\frown} \times \mathbb{I} \subset \mathbb{V}_m^{\frown}.$$

Lemma 9.4. $T_m\big|_{\upuparrows\mathbb{EB}_m^{\frown}} = T_{m-1}\big|_{\upuparrows\mathbb{EB}_m^{\frown}}$ *for every $m \in \mathbb{N}$. Equivalently, $f_{m,\theta} = f_{m-1,\theta}$ for every $m \in \mathbb{N}$ and $\theta \in \mathbb{EB}_m^{\frown}$.*

Proof. Fix $m \in \mathbb{N}$ and $\theta \in \mathbb{EB}_m^{\frown} \subset \mathbb{B}_m^{\frown}$. By Lemma 5.2(a,b), there exist $i \in \mathfrak{D}_m$ and $k \in \mathfrak{D}_{m-1}$ such that $\theta \in \mathsf{Bd}\left(B_i^{\frown}[i^*]\right) \subset B_i^{\frown}[i^*] \subsetneq B_k^{\frown}[k^*]$. So, we are in the assumptions of Lemmas 5.4 and 9.3 and, hence,

$$\mathbb{I}_{i,\theta} = \{m_i(\theta)\} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta},$$
$$f_{m,\theta}\left(m_i(\theta)\right) = g_{i,\theta}\left(m_i(\theta)\right) = \gamma_{|i+1|}\left(R_\omega(\theta)\right), \text{ and}$$
$$f_{m-1,\theta}\left(m_i(\theta)\right) = g_{k,\theta}\left(m_i(\theta)\right) = \gamma_{|k+1|}\left(R_\omega(\theta)\right).$$

Thus, if $i \geq 0$, $\theta \in \mathbb{B}_m$ and, by Definition 7.2 and Lemma 9.2(a), to prove that $f_{m,\theta} = f_{m-1,\theta}$ we only have to show that

$$g_{i,\theta}\left(m_i(\theta)\right) = \gamma_{|i+1|}\left(R_\omega(\theta)\right) = \gamma_{|k+1|}\left(R_\omega(\theta)\right) = f_{m-1,\theta}\left(m_i(\theta)\right).$$

When $i < 0$, $\theta \in \mathbb{WB}^{\frown}_m \cap \mathbb{EB}^{\frown}_m$ and, by Lemma 6.3(a), $\theta \notin \mathbb{WIB}_m$. Then, by Lemma 9.1, we get again that

$$g_{i,\theta}\left(m_i(\theta)\right) = \gamma_{|i+1|}\left(R_\omega(\theta)\right) = \gamma_{|k+1|}\left(R_\omega(\theta)\right) = f_{m-1,\theta}\left(m_i(\theta)\right).$$

implies $f_{m,\theta} = f_{m-1,\theta}$.

If $|k+1| = |i+1|$ there is nothing to prove. So, by Lemma 5.4, we can assume that $|k+1| < |i+1|$ and we have

$$\gamma_{|k+1|}\left(R_\omega(\theta)\right) = \gamma_{|k+1|+1}\left(R_\omega(\theta)\right) = \cdots = \gamma_{|i+1|-1}\left(R_\omega(\theta)\right).$$

Hence, we have to show that $\gamma_{|i+1|}\left(R_\omega(\theta)\right) = \gamma_{|i+1|-1}\left(R_\omega(\theta)\right)$. If $i \geq 0$ we get

$$\gamma_{|i+1|}\left(R_\omega(\theta)\right) = \gamma_{i+1}\left(R_\omega(\theta)\right) = \gamma_i\left(R_\omega(\theta)\right) = \gamma_{|i+1|-1}\left(R_\omega(\theta)\right)$$

by Lemma 3.6(e). Otherwise we have $i < 0$, $\theta \in \mathsf{Bd}\left(B^{\frown}_i[i^*]\right) = \mathsf{Bd}\left(B_{\alpha_{|i+1|}}[i^*]\right)$ and, consequently, $R_\omega(\theta) \in \mathsf{Bd}\left(B_{\alpha_{|i+1|}}[(i+1)^*]\right)$. Again by Lemma 3.6(e) for $j = |i+1|$,

$$\gamma_{|i+1|}\left(R_\omega(\theta)\right) = \gamma_{|i+1|-1}\left(R_\omega(\theta)\right).$$

This ends the proof of the lemma. □

Now we aim at computing two different kind of upper bounds for $\|f_{m,\theta} - f_{m-1,\theta}\|$ (Lemma 9.6 and Proposition 7.4). This will be a key tool in the proof of Propositions 7.3 for $m > 0$ and 7.4. The next two lemmas and remark will be useful to automate and simplify the proofs of these two results.

Lemma 9.5.

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \begin{cases} \left\| f_{m,\theta}\big|_{\mathbb{I}_{\mathsf{b}^{\frown}(\theta,m),\theta}} - f_{m-1,\theta}\big|_{\mathbb{I}_{\mathsf{b}^{\frown}(\theta,m),\theta}} \right\| & \text{when } \theta \in \mathbb{B}^{\frown}_m \setminus \mathbb{WIB}_m, \text{ and} \\ \left\| f_{m,\theta}\big|_{\mathbb{IW}_{m,\theta}} - f_{m-1,\theta}\big|_{\mathbb{IW}_{m,\theta}} \right\| & \text{when } \theta \in \mathbb{WIB}_m, \end{cases}$$

for every $m \geq 2$ and $\theta \in \mathbb{B}^{\frown}_m$.

Proof. Set $i = \mathsf{b}^{\frown}(\theta, m) \in \mathfrak{D}_m$, so that $\theta \in B^{\frown}_i[i^*]$.

When $\theta \in \mathbb{B}^{\frown}_m \setminus \mathbb{WIB}_m = \mathbb{B}_m \cup \mathbb{WB}^{\frown}_m \setminus \mathbb{WIB}_m$, by Definition 7.2 and Lemma 9.1, it is enough to show that

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))|$$

for every $x \in [-2, m_i(\theta)]$, and

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(M_i(\theta)) - f_{m-1,\theta}(M_i(\theta))|$$

for every $x \in [M_i(\theta), 2]$. We will prove the first statement. The second one follows similarly.

Definition 7.2 and Lemma 9.1 give

$$\begin{aligned} f_{m,\theta}(x) - f_{m-1,\theta}(x) &= \frac{2 - g_{i,\theta}\left(m_i(\theta)\right)}{2 - f_{m-1,\theta}\left(m_i(\theta)\right)}\left(f_{m-1,\theta}(x) - 2\right) + 2 - f_{m-1,\theta}(x) \\ &= \frac{2 - f_{m,\theta}\left(m_i(\theta)\right)}{2 - f_{m-1,\theta}\left(m_i(\theta)\right)}\left(f_{m-1,\theta}(x) - 2\right) - \left(f_{m-1,\theta}(x) - 2\right) \\ &= \left(f_{m-1,\theta}(x) - 2\right)\left(\frac{2 - f_{m,\theta}\left(m_i(\theta)\right)}{2 - f_{m-1,\theta}\left(m_i(\theta)\right)} - 1\right) \\ &= \left(2 - f_{m-1,\theta}(x)\right)\frac{f_{m,\theta}\left(m_i(\theta)\right) - f_{m-1,\theta}\left(m_i(\theta)\right)}{2 - f_{m-1,\theta}\left(m_i(\theta)\right)}. \end{aligned}$$

By Lemma 9.2(b), $2 \geq f_{m-1,\theta}(x) \geq f_{m-1,\theta}(m_i(\theta))$ and $1 \geq f_{m-1,\theta}(m_i(\theta))$. Hence,

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| = (2 - f_{m-1,\theta}(x)) \frac{|f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))|}{2 - f_{m-1,\theta}(m_i(\theta))}$$
$$\leq |f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))|.$$

Now assume that $\theta \in \mathbb{WIB}_m \subset \mathbb{WB}_m^{\frown}$. By Definition 7.2 it is enough to show that

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(\lambda_m(\theta)) - f_{m-1,\theta}(\lambda_m(\theta))|$$

for every $x \in [-2, \lambda_m(\theta)]$, and

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(\tau_m(\theta)) - f_{m-1,\theta}(\tau_m(\theta))|$$

for every $x \in [\tau_m(\theta), 2]$. As before, we will prove the first statement. The second one follows similarly. We have

$$f_{m,\theta}(x) - f_{m-1,\theta}(x) = (2 - f_{m-1,\theta}(x)) \frac{f_{m,\theta}(\lambda_m(\theta)) - f_{m-1,\theta}(\lambda_m(\theta))}{2 - f_{m-1,\theta}(\lambda_m(\theta))}.$$

By Lemma 9.2(b), $2 \geq f_{m-1,\theta}(x) \geq f_{m-1,\theta}(\lambda_m(\theta))$ and hence,

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))|$$

provided that $2 - f_{m-1,\theta}(\lambda_m(\theta)) \neq 0$. Assume by way of contradiction that we have $f_{m-1,\theta}(\lambda_m(\theta)) = 2$. Then, by Definition 6.2 and Lemma 9.2(b), $-1 \leq \lambda_m(\theta)$ and

$$2 \geq f_{m-1,\theta}(-1) \geq f_{m-1,\theta}(\lambda_m(\theta)) = 2;$$

which contradicts statement (b.iv) from the proof of Lemma 9.2. □

Next we compute an upper bound for $\|f_{m,\theta} - f_{m-1,\theta}\|$ for every $\theta \in B_i^{\frown}[i^*]$ and $i \in \mathfrak{D}_m$ such that $\mathsf{diam}(B_i^{\frown}[i^*])$ is small enough.

Lemma 9.6. *Assume that T_{m-1} is continuous for some $m \geq 2$ and let ε be positive. Then, there exist $\varrho_m(\varepsilon) \in \mathbb{N}$ such that*

$$\|f_{m,\theta} - f_{m-1,\theta}\| \leq \varepsilon$$

for every $\theta \in B_i^{\frown}[i^*]$ and $i \in \mathfrak{D}_m$ (that is, $B_i^{\frown}[i^*] \subset \mathbb{B}_m^{\frown}$) such that $|i| \geq \varrho_m(\varepsilon)$.

Proof. Since T_{m-1} is uniformly continuous, there exists $\delta_{m-1} = \delta_{m-1}(\varepsilon) > 0$ such that $\mathsf{d}_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) < \varepsilon$ provided that $\mathsf{d}_\Omega((\theta, x), (\nu, y)) < \delta_{m-1}$. We choose $\varrho_m = \varrho_m(\varepsilon) \in \mathbb{N}$ such that

$$3 \cdot 2^{-\varrho_m} < \min\{\delta_{m-1}(\varepsilon/2), \varepsilon/2\}.$$

Assume that $i \in \mathfrak{D}_m$ verifies $|i| \geq \varrho_m(\varepsilon)$ and let $(\theta, x) \in \mathsf{V}_{i^*}^{\frown} = B_i^{\frown}[i^*] \times \mathbb{I}$. When $\theta \in B_i^{\frown}[i^*] \setminus \mathbb{WIB}_m$ we can use Lemma 9.5 with $\mathbb{I}_{i,\theta}$ to compute $\|f_{m,\theta} - f_{m-1,\theta}\|$. We have to show that $|f_{m,\theta}(x) - f_{m-1,\theta}(x)| < \varepsilon$ for every $x \in \mathbb{I}_{i,\theta}$.

Let $\nu \in \mathsf{Bd}(B_i^{\frown}[i^*]) \subset \mathbb{EB}_m^{\frown}$. We have $(\theta, x), (\nu, m_i(\nu)) \in \mathcal{R}^{\frown}(i^*)$ and, by Lemmas 9.2(c) and 3.6(f),

$$\mathsf{d}_\Omega(T_m(\theta, x), T_m(\nu, m_i(\nu))) = \mathsf{d}_\Omega(G_i(\theta, x), G_i(\nu, m_i(\nu)))$$
$$\leq \mathsf{diam}\left(G_i\left(\mathcal{R}^{\frown}(i^*)\right)\right), \text{ and}$$
$$\mathsf{d}_\Omega((\theta, x), (\nu, m_i(\nu))) \leq \mathsf{diam}\left(\mathcal{R}^{\frown}(i^*)\right) \leq 2 \cdot 2^{-|i|} < 3 \cdot 2^{-\varrho_m} < \delta_{m-1}(\varepsilon/2).$$

Thus,

$$\mathsf{d}_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, m_i(\nu))) < \varepsilon/2.$$

Consequently, by Lemma 9.4,

$$\begin{aligned}|f_{m,\theta}(x) - f_{m-1,\theta}(x)| &= \mathsf{d}_\Omega(T_m(\theta,x), T_{m-1}(\theta,x))\\ &\le \mathsf{d}_\Omega(T_m(\theta,x), T_{m-1}(\nu, m_i(\nu))) +\\ &\qquad \mathsf{d}_\Omega(T_{m-1}(\nu, m_i(\nu)), T_{m-1}(\theta,x))\\ &< \mathsf{d}_\Omega(T_m(\theta,x), T_m(\nu, m_i(\nu))) + \varepsilon/2\\ &< \mathsf{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) + \varepsilon/2.\end{aligned}$$

Now we look at the size of $G_i\left(\mathcal{R}^{\smile}(i^*)\right)$. When $i < 0$, from Lemmas 4.5(b) and 3.6(f), we obtain

$$\mathsf{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) \le \mathsf{diam}\left(\mathcal{R}((i+1)^*)\right) \le 2^{-(|i|-1)} < 2\cdot 2^{-|i|}. \tag{23}$$

When $i \ge 0$, from Lemma 4.3(b) we get

$$G_i\left(\mathcal{R}^{\smile}(i^*)\right) = G_i\left(\mathcal{R}(i^*)\right) \subset \mathcal{R}((i+1)^*) \cup \mathsf{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}.$$

Moreover, as in the proof of Lemma 3.6(f) for $\ell < 0$, the set

$$\mathcal{R}((i+1)^*) \cup \mathsf{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}$$

is connected. So, by Lemma 3.6(f),

$$\begin{aligned}\mathsf{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) &\le \mathsf{diam}\left(\mathcal{R}((i+1)^*) \cup \mathsf{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}\right)\\ &\le \mathsf{diam}\left(\mathcal{R}((i+1)^*)\right) + \mathsf{diam}\left(\mathsf{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}\right)\\ &\le 2^{-(i+1)} + \mathsf{diam}\left(\mathsf{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}\right).\end{aligned}$$

As noticed earlier, $B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*)$ is disjoint from

$$B_{\alpha_{i+1}}\left((i+1)^*\right) \cup B^{\smile}_{-(i+1)}\left[(-(i+1))^*\right] \cup Z^*_{i+1}$$

by Definition 3.4(R.2) and Remark 3.5(R.2). So, by Lemma 3.10(c), Definition 3.4 and Lemma 3.6(a),

$$\begin{aligned}\mathsf{A}_{i+1}^{\nu} &= \{(\nu, \gamma_{i+1}(\nu)\} = \{(\nu, \gamma_i(\nu)\}\\ &\in \{\nu\} \times \left[\gamma_i((i+1)^*) - 2^{-n_i}, \gamma_i((i+1)^*) + 2^{-n_i}\right].\end{aligned}$$

for every $\nu \in B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*)$. On the other hand, $\gamma_i((i+1)^*) \in \mathbb{I}_{i+1,(i+1)^*}$ by Lemma 3.6(c). Hence, by Remark 3.2(2), Definition 3.4(R.1) and Remark 3.5(R.1),

$$\begin{aligned}&\mathsf{diam}\left(\mathsf{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}\right)\\ &\quad\le \max\left\{\mathsf{diam}\left(B_{\alpha_i}\left[(i+1)^*\right] \setminus B_{\alpha_{i+1}}\left((i+1)^*\right)\right), 2\cdot(2^{-n_i} + 2^{-n_{i+1}})\right\}\\ &\quad\le 2\cdot\max\left\{\alpha_i, 2^{-n_i} + 2^{-n_{i+1}}\right\} = 2\cdot(2^{-n_i} + 2^{-n_{i+1}})\\ &\quad< 4\cdot 2^{-n_i} \le 2\cdot 2^{-i}.\end{aligned}$$

Summarizing, when $i \ge 0$,

$$\mathsf{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) \le 2^{-(i+1)} + 2\cdot 2^{-i} < 3\cdot 2^{-i}$$

and, from (23),

$$\mathsf{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) < 3\cdot 2^{-|i|} \le 3\cdot 2^{-\varrho_m} < \varepsilon/2$$

for every $i \in \mathbb{Z}^+$. Thus, for every $x \in \mathbb{I}_{i,\theta}$,

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| < \mathsf{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) + \varepsilon/2 < \varepsilon.$$

Now assume that $\theta \in B_i^{\frown}[i^*] \cap \mathbb{WIB}_m$. We can use Lemma 9.5 with $\mathbb{IW}_{m,\theta}$ to compute $\|f_{m,\theta} - f_{m-1,\theta}\|$. We have to show that $|f_{m,\theta}(x) - f_{m-1,\theta}(x)| < \varepsilon$ for every $x \in \mathbb{IW}_{m,\theta}$. Since $\theta \in \mathbb{WIB}_m$, by Definition 6.2 and Lemma 6.3(b), $i < 0$, $\theta \in \mathbb{WB}_m^{\frown}$ and

$$\mathbb{I}_{i,\theta} = \left\{\gamma_{|i|}(\theta)\right\} \subset \mathbb{IW}_{m,\theta} = \mathbb{I}_{\ell,\theta} \ni x$$

with $\ell = \mathsf{b}^{\frown}(\theta, \mathsf{led}\,(\theta, m)) \in \mathfrak{WFD}_m$. In this case we will consider the points $(\theta, x) \in \mathcal{R}(\ell^*)$ and $(\nu, m_i(\nu)), (\theta, \gamma_{|i|}(\theta)) \in \mathcal{R}^{\frown}(i^*)$ with $\nu \in \mathsf{Bd}\,(B_i^{\frown}[i^*]) \subset \mathbb{EB}_m^{\frown}$. By Lemma 6.1(b), Remark 3.2(2) and Lemma 3.6(f), $|i| < |\ell|$ and

$$\begin{aligned}
\mathsf{d}_\Omega((\theta,x),(\nu,m_i(\nu)) &\le \mathsf{d}_\Omega((\theta,x),(\theta,\gamma_{|i|}(\theta)) + \mathsf{d}_\Omega((\theta,\gamma_{|i|}(\theta)),(\nu,m_i(\nu)) \\
&\le \left|x - \gamma_{|i|}(\theta)\right| + \mathsf{diam}\left(\mathcal{R}^{\frown}(i^*)\right) \\
&\le \mathsf{diam}\left(\mathcal{R}(\ell^*)\right) + \mathsf{diam}\left(\mathcal{R}^{\frown}(i^*)\right) \\
&\le 2^{-|\ell|} + 2\cdot 2^{-|i|} < 3\cdot 2^{-|i|} \le 3\cdot 2^{-\varrho_m} < \delta_{m-1}(\varepsilon/2).
\end{aligned}$$

Thus,

$$\mathsf{d}_\Omega(T_{m-1}(\theta,x), T_{m-1}(\nu,m_i(\nu)) < \varepsilon/2.$$

On the other hand, by Lemma 9.2(c), Definition 7.2 and (23),

$$\begin{aligned}
&\mathsf{d}_\Omega(T_m(\theta,x),T_m(\nu,m_i(\nu))) \\
&\quad\le \mathsf{d}_\Omega(T_m(\theta,x),T_m(\theta,\gamma_{|i|}(\theta))) + \mathsf{d}_\Omega(T_m(\theta,\gamma_{|i|}(\theta)),T_m(\nu,m_i(\nu))) \\
&\quad\le \left|f_{m,\theta}(x) - f_{m,\theta}(\gamma_{|i|}(\theta))\right| + \mathsf{d}_\Omega(G_i(\theta,\gamma_{|i|}(\theta)),G_i(\nu,m_i(\nu))) \\
&\quad= \mathsf{d}_\Omega(G_i(\theta,x),G_i(\nu,m_i(\nu))) \le \mathsf{diam}\left(G_i\left(\mathcal{R}^{\frown}(i^*)\right)\right) < 2\cdot 2^{-|i|} \\
&\quad\le 3\cdot 2^{-\varrho_m} < \varepsilon/2.
\end{aligned}$$

So, in a similar way as before, Lemma 9.4 gives

$$\begin{aligned}
|f_{m,\theta}(x) - f_{m-1,\theta}(x)| &= \mathsf{d}_\Omega(T_m(\theta,x),T_{m-1}(\theta,x)) \\
&\le \mathsf{d}_\Omega(T_m(\theta,x),T_{m-1}(\nu,m_i(\nu))) + \\
&\qquad \mathsf{d}_\Omega(T_{m-1}(\nu,m_i(\nu)),T_{m-1}(\theta,x)) \\
&< \varepsilon.
\end{aligned}$$

□

Proof of Proposition 7.3 for $m > 0$. **(a)** We start by proving by induction on m that T_m is continuous for every $m \in \mathbb{Z}^+$.

By Proposition 7.3(a) for $m = 0$, T_0 is continuous. So, we may assume that T_{m-1} is continuous for some $m \in \mathbb{N}$ and prove that T_m is continuous.

Let $\varepsilon > 0$ be fixed but arbitrary, and let $(\theta, x), (\nu, y) \in \Omega$. We have to show that there exists $\delta(\varepsilon) > 0$ such that

$$\mathsf{d}_\Omega(T_m(\theta,x),T_m(\nu,y)) < \varepsilon \quad \text{when} \quad \mathsf{d}_\Omega((\theta,x),(\nu,y)) < \delta.$$

We start by defining $\delta(\varepsilon)$. To this end we need to introduce some more notation and establish some facts about the maps T_m and T_{m-1}.

Since T_{m-1} is uniformly continuous, we know that

(24) there exists $\delta_{m-1} = \delta_{m-1}(\varepsilon) > 0$ such that $\mathsf{d}_\Omega(T_{m-1}(\theta,x),T_{m-1}(\nu,y)) < \varepsilon$ provided that $\mathsf{d}_\Omega((\theta,x),(\nu,y)) < \delta_{m-1}$.

On the other hand, Lemma 9.2(a) tells us that $T_m\big|_{\mathsf{V}_{i^*}^{\frown}}$ is uniformly continuous for every $i \in \mathfrak{D}_m$. So, for every $i \in \mathfrak{D}_m$,

(25) there exists $\delta_{m,i} = \delta_{m,i}(\varepsilon) > 0$ such that $\mathsf{d}_\Omega(T_m(\theta,x),T_m(\nu,y)) < \varepsilon$ for every $(\theta,x),(\nu,y) \in \mathsf{V}_{i^*}^{\frown} \subset \mathbb{V}_m^{\frown}$ verifying $\mathsf{d}_\Omega((\theta,x),(\nu,y)) < \delta_{m,i}(\varepsilon)$.

Then, by using the numbers $\delta_{m-1}(\varepsilon/7)$ given by (24), $\delta_{m,i}(\varepsilon/7)$ given by (25) and $\varrho_m(\varepsilon/7)$ given by Lemma 9.6, we set

$$\delta = \delta(\varepsilon) := \min\left\{\delta_{m-1}(\varepsilon/7), \min\{\delta_{m,i}(\varepsilon/7) : i \in \mathfrak{D}_m \cap Z_{\varrho_m(\varepsilon/7)}\}\right\}.$$

Clearly, $\delta > 0$ because the set $\mathfrak{D}_m \cap Z_{\varrho_m(\varepsilon/7)}$ is finite.

Now we will show that if $\mathsf{d}_\Omega((\theta, x), (\nu, y)) < \delta$, then $\mathsf{d}_\Omega(T_m(\theta, x), T_m(\nu, y)) < \varepsilon$.

Assume first that $(\theta, x), (\nu, y) \in \mathsf{V}^{\frown}_{\ell^*}$ for some $\ell \in \mathfrak{D}_m \cap Z_{\varrho_m(\varepsilon/7)}$. We have

$$\mathsf{d}_\Omega((\theta, x), (\nu, y)) < \delta \le \min\{\delta_{m,i}(\varepsilon/7) : i \in \mathfrak{D}_m \cap Z_{\varrho_m(\varepsilon/7)}\} \le \delta_{m,\ell}(\varepsilon/7).$$

Hence, by (25),

$$\mathsf{d}_\Omega(T_m(\theta, x), T_m(\nu, y)) < \varepsilon/7 < \varepsilon.$$

Next we assume that $(\theta, x), (\nu, y) \in \mathsf{V}^{\frown}_{\ell^*}$ for some $\ell \in \mathfrak{D}_m$ such that $|\ell| > \varrho_m(\varepsilon/7)$ (in particular, $\theta, \nu \in B^{\frown}_\ell[\ell^*]$). In this situation we have

$$\mathsf{d}_\Omega((\theta, x), (\nu, y)) < \delta \le \delta_{m-1}(\varepsilon/7)$$

and, by (24) and Lemma 9.6,

$$\begin{aligned}
\mathsf{d}_\Omega(T_m(\theta, x), T_m(\nu, y)) &\le \mathsf{d}_\Omega(T_m(\theta, x), T_{m-1}(\theta, x)) + \mathsf{d}_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) + \\
&\qquad \mathsf{d}_\Omega(T_{m-1}(\nu, y), T_m(\nu, y)) \\
&= |f_{m,\theta}(x) - f_{m-1,\theta}(x)| + \mathsf{d}_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) + \\
&\qquad |f_{m,\nu}(y) - f_{m-1,\nu}(y)| \\
&\le \|f_{m,\theta} - f_{m-1,\theta}\| + \mathsf{d}_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) + \\
&\qquad \|f_{m,\nu} - f_{m-1,\nu}\| \\
&< \tfrac{3}{7}\varepsilon < \varepsilon.
\end{aligned}$$

In summary, we have proved that

$$\mathsf{d}_\Omega(T_m(\theta, x), T_m(\nu, y)) < \tfrac{3}{7}\varepsilon$$

when $\mathsf{d}_\Omega((\theta, x), (\nu, y)) < \delta$ and $(\theta, x), (\nu, y) \in \mathsf{V}^{\frown}_{\ell^*}$ for some $\ell \in \mathfrak{D}_m$.

Next we assume that $(\theta, x), (\nu, y) \in \mathbb{V}^{\frown}_m$ but $(\theta, x), (\nu, y) \notin \mathsf{V}^{\frown}_{\ell^*}$ for every $\ell \in \mathfrak{D}_m$. By Lemma 5.2(a,b), there exist $i = \mathsf{b}^{\frown}(\theta, m)\,, k = \mathsf{b}^{\frown}(\nu, m) \in \mathfrak{D}_m$, $i \neq k$, such that $\theta \in B^{\frown}_i[i^*]\,$, $(\theta, x) \in \mathsf{V}^{\frown}_{i^*}$, $\nu \in B^{\frown}_k[k^*]$ and $(\nu, y) \in \mathsf{V}^{\frown}_{k^*}$. Then, there exist

$$\widetilde{\theta} \in A \cap \mathsf{Bd}\left(B^{\frown}_i[i^*]\right) \subset \mathbb{EB}^{\frown}_m \quad \text{and} \quad \widetilde{\nu} \in A \cap \mathsf{Bd}\left(B^{\frown}_k[k^*]\right) \subset \mathbb{EB}^{\frown}_m,$$

where A denotes the closed arc of $\mathbb{S}^1$ such that

$$\mathsf{diam}(A) = \mathsf{d}_{\mathbb{S}^1}(\theta, \nu) \quad \text{and} \quad \mathsf{Bd}(A) = \{\theta, \nu\}.$$

Clearly we have, $(\theta, x), \left(\widetilde{\theta}, x\right) \in \mathsf{V}^{\frown}_{i^*}$, $(\nu, y), \left(\widetilde{\nu}, y\right) \in \mathsf{V}^{\frown}_{k^*}$ and, by the previous case,

$$\begin{gathered}
\mathsf{d}_\Omega\left((\theta, x), \left(\widetilde{\theta}, x\right)\right) = \mathsf{d}_{\mathbb{S}^1}\left(\theta, \widetilde{\theta}\right) \le \mathsf{d}_{\mathbb{S}^1}(\theta, \nu) \le \mathsf{d}_\Omega((\theta, x), (\nu, y)) < \delta, \\
\mathsf{d}_\Omega\left(T_m(\theta, x), T_m\left(\widetilde{\theta}, x\right)\right) < \tfrac{3}{7}\varepsilon \\
\mathsf{d}_\Omega\left((\nu, y), \left(\widetilde{\nu}, y\right)\right) = \mathsf{d}_{\mathbb{S}^1}\left(\nu, \widetilde{\nu}\right) \le \mathsf{d}_{\mathbb{S}^1}(\theta, \nu) \le \mathsf{d}_\Omega((\theta, x), (\nu, y)) < \delta, \text{ and} \\
\mathsf{d}_\Omega\left(T_m(\nu, y), T_m\left(\widetilde{\nu}, y\right)\right) < \tfrac{3}{7}\varepsilon.
\end{gathered}$$

On the other hand, $(\widetilde{\theta}, x), (\widetilde{\nu}, y) \in \upuparrows\mathbb{E}\mathbb{B}^{\frown}_{m} \subset \mathbb{V}^{\frown}_{m} \subset \mathbb{V}^{\frown}_{m-1}$ and, by Lemma 9.4 and (24),

$$\begin{aligned}
\mathsf{d}_{\Omega}\left((\widetilde{\theta}, x), (\widetilde{\nu}, y)\right) &= \max\left\{\mathsf{d}_{\mathbb{S}^1}(\widetilde{\theta}, \widetilde{\nu}), |x-y|\right\} \leq \max\left\{\mathsf{d}_{\mathbb{S}^1}(\theta, \nu), |x-y|\right\} \\
&= \mathsf{d}_{\Omega}((\theta, x), (\nu, y)) < \delta \leq \delta_{m,i}(\varepsilon/7), \text{ and} \\
\mathsf{d}_{\Omega}(T_m(\theta, x), T_m(\nu, y)) &\leq \mathsf{d}_{\Omega}\left(T_m(\theta, x), T_m(\widetilde{\theta}, x)\right) + \mathsf{d}_{\Omega}\left(T_m(\widetilde{\theta}, x), T_m(\widetilde{\nu}, y)\right) + \\
&\qquad \mathsf{d}_{\Omega}\left(T_m(\widetilde{\nu}, y), T_m(\nu, y)\right) \\
&< \tfrac{3}{7}\varepsilon + \mathsf{d}_{\Omega}\left(T_{m-1}(\widetilde{\theta}, x), T_{m-1}(\widetilde{\nu}, y)\right) + \tfrac{3}{7}\varepsilon = \varepsilon.
\end{aligned}$$

If $(\theta, x), (\nu, y) \notin \mathbb{V}^{\frown}_{m}$ then, by Definition 7.2 and (24) ,

$$\mathsf{d}_{\Omega}(T_m(\theta, x), T_m(\nu, y)) = \mathsf{d}_{\Omega}(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) < \varepsilon/7 < \varepsilon$$

because $\mathsf{d}_{\Omega}((\theta, x), (\nu, y)) < \delta \leq \delta_{m-1}(\varepsilon/7)$.

Lastly, assume that $(\nu, y) \notin \mathbb{V}^{\frown}_{m}$ but $(\theta, x) \in \mathsf{V}^{\frown}_{i^*} \subset \mathbb{V}^{\frown}_{m}$, for some $i \in \mathfrak{D}_m$ (that is, $\theta \in B^{\frown}_{i}[i^*]$). In this situation, as before, there exists $\widetilde{\theta} \in \mathsf{Bd}\left(B^{\frown}_{i}[i^*]\right) \subset \mathbb{E}\mathbb{B}^{\frown}_{m}$ such that, by Lemma 9.4 and Definition 7.2 ($(\widetilde{\theta}, x) \in \upuparrows\mathbb{E}\mathbb{B}^{\frown}_{m} \subset \mathbb{V}^{\frown}_{m} \subset \mathbb{V}^{\frown}_{m-1}$), and (24),

$$\begin{aligned}
&\mathsf{d}_{\Omega}\left((\theta, x), (\widetilde{\theta}, x)\right) < \delta, \\
&\mathsf{d}_{\Omega}\left((\widetilde{\theta}, x), (\nu, y)\right) < \delta \leq \delta_{m-1}(\varepsilon/7), \\
&\mathsf{d}_{\Omega}\left(T_m(\theta, x), T_m(\widetilde{\theta}, x)\right) < \tfrac{3}{7}\varepsilon, \text{ and} \\
&\mathsf{d}_{\Omega}(T_m(\theta, x), T_m(\nu, y)) \leq \mathsf{d}_{\Omega}\left(T_m(\theta, x), T_m(\widetilde{\theta}, x)\right) + \mathsf{d}_{\Omega}\left(T_m(\widetilde{\theta}, x), T_m(\nu, y)\right) \\
&\qquad < \tfrac{3}{7}\varepsilon + \mathsf{d}_{\Omega}\left(T_{m-1}(\widetilde{\theta}, x), T_{m-1}(\nu, y)\right) < \varepsilon.
\end{aligned}$$

This ends the proof of the continuity of T_m and, hence, of (a).

(b) When $\theta \in \mathbb{B}^{\frown}_{m}$ the statement follows from Lemma 9.2(b). When $\theta \in \mathbb{S}^1 \backslash \mathbb{B}^{\frown}_{m}$, it follows from the part already proven and the continuity of T_m.

(c) The first two statements follow from Lemma 9.2(c) and statement (a). On the other hand, as in the proof of Proposition 7.3(c) for $m = 0$, Lemma 5.3(b) implies that $i^* \in \mathbb{B}^{\frown}_{m}$ but $i^* \notin \mathbb{B}^{\frown}_{k}$ for every $k > m$. Then, we get $f_{k,i^*} = f_{m,i^*}$ from Definition 7.2. □

10. Proof of Proposition 7.4

This section is devoted to prove Proposition 7.4. It is the third technical counterpart of Section 7. In contrast to Lemma 9.6 the bound given by Proposition 7.4. is valid for every $\theta \in \mathbb{B}^{\frown}_{m}$.

Before starting the proof of this proposition we will state and prove a number of very simple lemmas that will help in automating the proof of Proposition 7.4.

Lemma 10.1. *Assume that* $B^{\frown}_{i}[i^*] \subset B^{\frown}_{k}[k^*]$ *for some* $i \in \mathfrak{D}_m$, $k \in \mathfrak{D}_{m-1}$ *and* $m \geq 2$, *and assume that either*

$$i < 0 \text{ and } \theta \in B^{\frown}_{i}[i^*] \backslash \{i^*\} \text{ or } i \geq 0 \text{ and } \theta \in B_{\alpha_i}[i^*] \backslash B_{\alpha_{i+1}}(i^*).$$

Then,

$$\left|\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))\right| \leq 2^{-|k|}.$$

Proof. The lemma holds trivially when $|k+1| = |i+1|$. Thus, we may assume that $|k+1| \neq |i+1|$. Then by Lemma 5.4, $|k| < |i|$, $|k+1| < |i+1|$ and

$$\gamma_{|k+1|}(R_\omega(\theta)) = \gamma_{|k+1|+1}(R_\omega(\theta)) = \cdots = \gamma_{|i+1|-1}(R_\omega(\theta)).$$

By assumption we have

$$\theta \in \begin{cases} B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*) & \text{when } i \geq 0, \text{ and} \\ B_{\widetilde{i}}[i^*] \setminus \{i^*\} = B_{\alpha_{|i+1|}}[i^*] \setminus \{i^*\} & \text{when } i < 0, \end{cases}$$

and, hence,

$$R_\omega(\theta) \in \begin{cases} B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*) & \text{when } i \geq 0, \text{ and} \\ B_{\alpha_{|i+1|}}[(i+1)^*] \setminus \{(i+1)^*\} & \text{when } i < 0. \end{cases}$$

Thus, in the case $i \geq 0$ we have

$$R_\omega(\theta) \notin B_{\alpha_{i+1}}((i+1)^*) \cup B_{\widetilde{-(i+1)}}[(-(i+1))^*] \cup Z^*_{i+1}$$

by Definition 3.4(R.2) and Remark 3.5(R.2). So, by Definition 3.4,

$$\gamma_{i+1}(R_\omega(\theta)) = \gamma_i(R_\omega(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)).$$

This ends the proof of the lemma in this case.

Assume now that $i < 0$. By Lemma 3.6(c,d,f) and Definition 3.4(R.2) and Remark 3.5(R.2),

$$\begin{aligned} \left|\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))\right| &= \left|\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|i+1|-1}(R_\omega(\theta))\right| \\ &\leq \mathsf{diam}(\mathcal{R}((i+1)^*)) \leq 2^{-|i+1|} \leq 2^{-|k|} \end{aligned}$$

(observe that $|i+1| > |k+1| \geq |k| - 1$). □

Lemma 10.2. *Let $s, t \in \mathbb{Z}$, $s \neq t$ be such that $\theta \in B_{\widetilde{s}}(s^*) \setminus B_{\alpha_{|s|}}(s^*)$, and either $t < 0$ and $\theta \in B_{\alpha_{|t|}}(t^*)$ or $t \geq 0$ and $\theta \in B_{\alpha_{t+1}}(t^*)$. Then, the following statements hold:*

(a) $R_\omega(\theta) \in B_{\alpha_{|s+1|}}((s+1)^*) \cap B_{\alpha_{|t+1|}}((t+1)^*)$.

(b) *Let $u, v \in \mathbb{Z}$ be such that $\{u, v\} = \{s, t\}$ and $|u+1| \leq |v+1|$. Then, $\mathbb{I}_{v+1, R_\omega(\theta)} \subset \mathbb{I}_{u+1, R_\omega(\theta)}$.*

(c)

$$|x - y| \leq 2 \cdot 2^{-|u|}$$

for every $x \in \mathbb{I}_{t+1, R_\omega(\theta)}$ and $y \in \mathbb{I}_{s+1, R_\omega(\theta)}$.

Proof. By assumption we have

$$\theta \in \begin{cases} B_{\alpha_{t+1}}(t^*) & \text{when } t \geq 0, \text{ and} \\ B_{\alpha_{|t|}}(t^*) \subset B_{\widetilde{t}}(t^*) = B_{\alpha_{|t+1|}}(t^*) & \text{when } t < 0. \end{cases}$$

Hence, $R_\omega(\theta) \in B_{\alpha_{|t+1|}}((t+1)^*)$. Moreover, as in the proof of Lemma 10.1, $s < 0$ and $R_\omega(\theta) \in B_{\alpha_{|s+1|}}((s+1)^*)$. This proves (a).

Now we prove (b). From (a) we have

$$\begin{aligned} R_\omega(\theta) &\in B_{\alpha_{|u+1|}}((u+1)^*) \cap B_{\alpha_{|v+1|}}((v+1)^*) \\ &\subset B_{\alpha_{|u+1|}}((u+1)^*) \cap B_{\widetilde{v+1}}[(v+1)^*]. \end{aligned}$$

Moreover, $s \neq t$ implies $u+1 \neq v+1$ and we have $|u+1| \leq |v+1|$ by assumption. Consequently, by Lemma 3.6(g,d) and Definition 3.4(R.2) and Remark 3.5(R.2), $|u+1| < |v+1|$ and

$$\mathcal{R}((v+1)^*) \subset \mathsf{Int}\left(\mathcal{R}((u+1)^*) \setminus \upuparrows\{(u+1)^*\}\right)$$

which implies (b).

Thus, $x, y \in \mathbb{I}_{u+1, R_\omega(\theta)}$ and, by Lemma 3.6(f),

$$|x - y| \leq \mathsf{diam}(\mathcal{R}((u+1)^*)) \leq 2^{-|u+1|} \leq 2^{-(|u|-1)} = 2 \cdot 2^{-|u|}.$$

□

Now we are ready to start the proof of Proposition 7.4.

Proof of Proposition 7.4. We start by showing that $\{T_m\}_{k=0}^{\infty}$ is a Cauchy sequence, assuming that the bound (16) holds for every $m \geq 2$ and $\theta \in \mathbb{S}^1$.

We start by estimating $\mathsf{d}_{\infty}(T_m, T_{m+1})$ for every $m \in \mathbb{N}$. From (16) and the definition of μ_m

$$\mathsf{d}_{\infty}(T_m, T_{m+1}) = \sup_{\theta\in\mathbb{S}^1} \|f_{m,\theta} - f_{m+1,\theta}\| \leq 2 \cdot \sup_{\theta\in\mathbb{S}^1} 2^{-\left|\mathsf{b}^{\frown}(\theta,m)\right|} \leq 2 \cdot 2^{-\mu_m}.$$

By Lemma 5.3(a) $\{\mu_m\}_{m=0}^{\infty}$ is strictly increasing (and $\lim_{m\to\infty} \mu_m = \infty$). Therefore, for every $\varepsilon > 0$, there exists $N \geq 2$, such that $4 \cdot 2^{-\mu_m} < \varepsilon$ for every $m \geq N$. Hence,

$$\begin{aligned}\mathsf{d}_{\infty}(T_m, T_{m+i}) &\leq \sum_{\ell=m}^{m+i-1} \mathsf{d}_{\infty}(T_\ell, T_{\ell+1}) \leq 2 \cdot \sum_{\ell=m}^{m+i-1} 2^{-\mu_\ell} \\ &\leq 2 \cdot 2^{-\mu_m} \sum_{\ell=0}^{\infty} 2^{-\ell} = 4 \cdot 2^{-\mu_m} \leq 4 \cdot 2^{-\mu_N} < \varepsilon\end{aligned}$$

for every $m \geq N$ and $i \in \mathbb{N}$. So, $\{T_m\}_{k=0}^{\infty}$ is a Cauchy sequence.

Now we prove (16). That is,

$$\|f_{m,\theta} - f_{m-1,\theta}\| \leq 2 \cdot 2^{-\left|\mathsf{b}^{\frown}(\theta,m-1)\right|}$$

for every $m \geq 2$ and $\theta \in \mathbb{S}^1$.

From Definition 7.2 and Lemma 9.4 we know that $f_{m,\theta} = f_{m-1,\theta}$ for every $\theta \in \left(\mathbb{S}^1 \setminus \mathbb{B}^{\frown}_m\right) \cup \mathbb{EB}^{\frown}_m$. Then, (16) holds in this case.

In the rest of the involved proof we assume that $\theta \in \mathbb{B}^{\frown}_m \setminus \mathbb{EB}^{\frown}_m$. Thus, by Lemmas 5.2(a,b), 3.6(g) and 5.4,

$\theta \in B^{\frown}_i(i^*) \subset B^{\frown}_k(k^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup \{k^*\}\right)$ *where*
$i = \mathsf{b}^{\frown}(\theta, m) \in \mathfrak{D}_m$, $k = \mathsf{b}^{\frown}(\theta, m-1) \in \mathfrak{D}_{m-1}$,
$|k| < |i|$, *and* $|k+1| \leq |i+1|$.

Moreover, $\mathsf{V}^{\frown}_{i^*} \subset \mathsf{V}^{\frown}_{k^*} \subset \mathbb{V}^{\frown}_{m-1}$. Consequently, by Lemma 9.2(a,b), the maps $f_{m,\theta}$ and $f_{m-1,\theta}$ are well defined, continuous, piecewise affine and non-increasing, and $f_{m,\theta}(2) = f_{m-1,\theta}(2) = -2$ and $f_{m,\theta}(-2) = f_{m-1,\theta}(-2) = 2$ (see Figures 6, 7 and 8 for some examples in generic cases).

We split the proof into three cases according to whether θ belongs to

$$B^{\frown}_i(i^*) \setminus B_{\alpha_{|i|}}(i^*), \; B_{\alpha_{|i|}}(i^*) \subset B^{\frown}_k(k^*) \setminus B_{\alpha_{|k|}}[k^*] \text{ or } B_{\alpha_{|i|}}(i^*) \subset B_{\alpha_{|k|}}(k^*).$$

Case 1. $\theta \in B^{\frown}_i(i^*) \setminus B_{\alpha_{|i|}}(i^*)$.

We have $i < 0$ because $B^{\frown}_i(i^*) = B_{\alpha_i}(i^*)$ for $i \geq 0$. Moreover, by Definition 6.2, $\theta \in \mathbb{WB}^{\frown}_m$.

To deal with this case we consider three subcases.

Subcase 1.1. $\theta \in \left(B^{\frown}_i(i^*) \setminus B_{\alpha_{|i|}}(i^*)\right) \setminus \mathbb{WIB}_m$.

By Lemmas 5.4, 9.3, 9.5 and 10.1,

$$\begin{aligned}\mathbb{I}_{i,\theta} &= \{m_i(\theta)\} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}, \\ f_{m,\theta}(m_i(\theta)) &= \gamma_{|i+1|}(R_\omega(\theta)), \\ f_{m-1,\theta}(m_i(\theta)) &= \gamma_{|k+1|}(R_\omega(\theta)), \text{ and} \\ \|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}} - f_{m-1,\theta}\big|_{\mathbb{I}_{i,\theta}} \right\| = \left| f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta)) \right| \\ &= \left| \gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta)) \right| \leq 2^{-\left|\mathsf{b}^{\frown}(\theta,m-1)\right|}.\end{aligned}$$

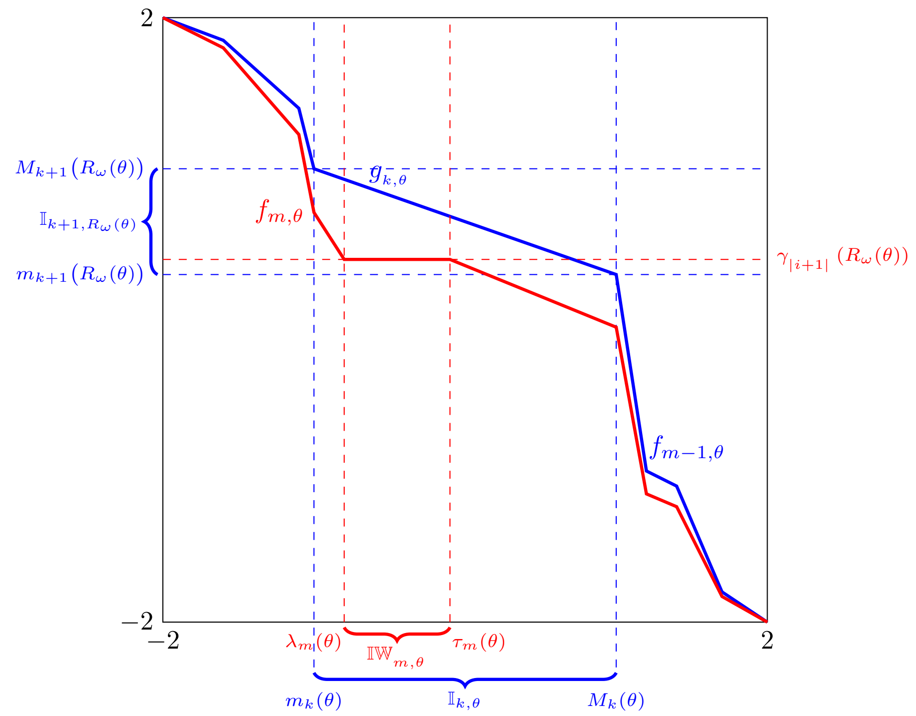


FIGURE 6. A symbolic representation of the maps $f_{m,\theta}$ and $f_{m-1,\theta}$ in Subcase 1.3 of Proposition 7.4 ($\theta \in \left(B_{\overset{\frown}{i}}(i^*) \setminus B_{\alpha_{|i|}}(i^*)\right) \cap \mathbb{WIB}_m$ and $B_{\overset{\frown}{i}}(i^*) \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$). The map $f_{m-1,\theta}$ and the corresponding intervals $\mathbb{I}_{k,\theta}$ and $\mathbb{I}_{k+1,R_\omega(\theta)}$ are drawn in blue. The map $f_{m,\theta}$, the interval $\mathbb{IW}_{m,\theta}$ and the point $\gamma_{|i+1|}(R_\omega(\theta))$ are drawn in red.

Subcase 1.2. $\theta \in \left(B_{\overset{\frown}{i}}(i^*) \setminus B_{\alpha_{|i|}}(i^*)\right) \cap \mathbb{WIB}_m$ *and* $B_{\overset{\frown}{i}}(i^*) \subset B_{\overset{\frown}{k}}(k^*) \setminus B_{\alpha_{|k|}}[k^*]$.

In this subcase, by Definition 6.2 we have

$$\theta \in B_{\overset{\frown}{k}}(k^*) \setminus B_{\alpha_{|k|}}[k^*] \subset \mathbb{WB}^{\frown}_{m-1}$$

(recall that $i < 0$). Then, by Lemmas 5.4 and 6.3(b,c), Definition 7.2 and Lemmas 9.5 and 10.1,

$$\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{IW}_{m,\theta} = \mathbb{IW}_{m-1,\theta},$$
$$f_{m,\theta}(x) = \gamma_{|i+1|}(R_\omega(\theta)) \text{ for every } x \in \mathbb{IW}_{m,\theta},$$
$$f_{m-1,\theta}(x) = \gamma_{|k+1|}(R_\omega(\theta)) \text{ for every } x \in \mathbb{IW}_{m-1,\theta}, \text{ and}$$
$$\begin{aligned}\|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}\big|_{\mathbb{IW}_{m,\theta}} - f_{m-1,\theta}\big|_{\mathbb{IW}_{m,\theta}} \right\| \\ &= \left|\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))\right| \le 2^{-\left|\overset{\frown}{\mathsf{b}}(\theta,m-1)\right|}.\end{aligned}$$

Observe that since $B_{\overset{\frown}{i}}(i^*)$ is connected and

$$B_{\overset{\frown}{i}}(i^*) \subset B_{\overset{\frown}{k}}(k^*) \setminus \left(\mathsf{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup \{k^*\}\right),$$

$B_{\overset{\frown}{i}}(i^*) \not\subset B_{\overset{\frown}{k}}(k^*) \setminus B_{\alpha_{|k|}}[k^*]$ implies $B_{\overset{\frown}{i}}(i^*) \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$.

Subcase 1.3. $\theta \in \left(B_{\overset{\frown}{i}}(i^*) \setminus B_{\alpha_{|i|}}(i^*)\right) \cap \mathbb{WIB}_m$ *and* $B_{\overset{\frown}{i}}(i^*) \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ (see Figure 6 for a symbolic representation of this case).

By Lemmas 5.4 and 6.3(b) and Definition 7.2,

$$\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{IW}_{m,\theta}, \text{ and}$$
$$f_{m,\theta}(x) = \gamma_{|i+1|}(R_\omega(\theta)) \text{ for every } x \in \mathbb{IW}_{m,\theta}.$$

On the other hand, by Definition 6.2 and Lemma 6.1(a,b), $\theta \in \mathbb{WIB}_m \subset \mathbb{WDB}_m$, and

$$\theta \in B_{\alpha_{|\ell|}}[\ell^*] \subset B_i^{\frown}(i^*) \setminus B_{\alpha_{|i|}}[i^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$$

with $\ell = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m)) \in \mathfrak{WFD}_m$ and $|\ell| > |i| > |k|$. Then, by Lemma 3.6(g) and Definition 6.2, $\mathcal{R}(\ell^*) \subset \mathsf{Int}\left(\mathcal{R}(k^*) \setminus \upuparrows k^*\right)$ and

$$\mathbb{IW}_{m,\theta} = \mathbb{I}_{\ell,\theta} \subset \mathbb{I}_{k,\theta}.$$

Moreover, since $\theta \in B_{\alpha_{|k|}}(k^*) \subset \mathbb{B}_m$, Definition 7.2, Lemmas 4.3(b) and 4.5(b), and the definition of the maps $g_{i,\theta}$ for $i \geq 0$ (Definition 4.2) give

$$f_{m-1,\theta}\left(\mathbb{IW}_{m,\theta}\right) \subset f_{m-1,\theta}\left(\mathbb{I}_{k,\theta}\right)$$
$$\subset \begin{cases} \mathbb{I}_{k+1,R_\omega(\theta)} & \text{if } k < 0 \text{ or } k \geq 0 \text{ and } \theta \in B_{\alpha_{k+1}}(k^*), \\ \{\gamma_{k+1}(R_\omega(\theta))\} & \text{if } k \geq 0 \text{ and } \theta \in B_{\alpha_k}[k^*] \setminus B_{\alpha_{k+1}}(k^*). \end{cases}$$

Now, as before, we will use Lemma 9.5 to bound $\|f_{m,\theta} - f_{m-1,\theta}\|$. We start with the simplest case: $k \geq 0$ and $\theta \in B_{\alpha_k}[k^*] \setminus B_{\alpha_{k+1}}(k^*)$. By Lemma 10.1,

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \left\| f_{m,\theta}\big|_{\mathbb{IW}_{m,\theta}} - f_{m-1,\theta}\big|_{\mathbb{IW}_{m,\theta}} \right\|$$
$$= \left|\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))\right| \leq 2^{-\left|\mathsf{b}^{\frown}(\theta, m-1)\right|}.$$

Now we assume that $k < 0$ or $k \geq 0$ and $\theta \in B_{\alpha_{k+1}}(k^*)$. In this case Lemma 10.2 applies. By Lemmas 10.2, 3.6(d) and Definition 3.4(R.2) and Remark 3.5(R.2), and Lemma 9.5 we have

$$\gamma_{|i+1|}(R_\omega(\theta)) \in \mathbb{I}_{i+1,R_\omega(\theta)} \subset \mathbb{I}_{k+1,R_\omega(\theta)},$$
$$f_{m-1,\theta}(x) \in \mathbb{I}_{k+1,R_\omega(\theta)} \quad \text{for every } x \in \mathbb{IW}_{m,\theta}.$$

and

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \sup_{x \in \mathbb{IW}_{m,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)|$$
$$= \sup_{x \in \mathbb{IW}_{m,\theta}} \left|\gamma_{|i+1|}(R_\omega(\theta)) - f_{m-1,\theta}(x)\right|$$
$$\leq 2 \cdot 2^{-|k|} = 2 \cdot 2^{-\left|\mathsf{b}^{\frown}(\theta, m-1)\right|}.$$

This ends the proof of the proposition in this case.

Case 2. $\theta \in B_{\alpha_{|i|}}(i^*) \subset B_k^{\frown}(k^*) \setminus B_{\alpha_{|k|}}[k^*]$ (see Figure 7 for a symbolic representation of this case).

In this case we will use Lemma 9.5 with $\mathbb{I}_{i,\theta}$. Thus, we need to compare the maps $f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}}$ and $f_{m-1,\theta}\big|_{\mathbb{I}_{i,\theta}}$.

Directly from the definitions we get $k < 0$, $B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}_m$ and $B_{\alpha_{|k|}}[k^*] \subset \mathbb{B}_{m-1}$. Consequently, by Lemma 5.2(b) and Definition 6.2,

$$\theta \in \mathbb{B}_m \quad \text{and} \quad \theta \in \mathbb{B}^{\frown}_{m-1} \setminus \mathbb{B}_{m-1} \subset \mathbb{WDB}_{m-1} \subset \mathbb{WB}^{\frown}_{m-1}.$$

Moreover, $\mathsf{led}(\theta, m-1) = m$, $i = \mathsf{b}^{\frown}(\theta, m) = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m-1)) \in \mathfrak{WFD}_{m-1}$ and, by Definition 6.2, $\theta \in \mathbb{WIB}_{m-1}$, and

$$\mathbb{IW}_{m-1,\theta} = \mathbb{I}_{i,\theta}.$$

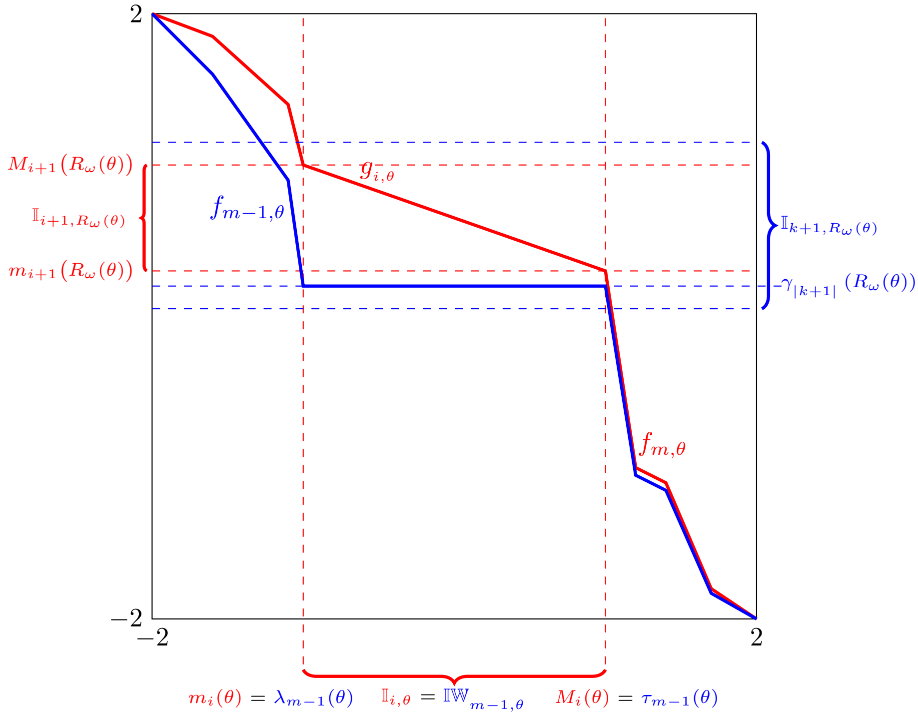


FIGURE 7. A symbolic representation of the maps $f_{m,\theta}$ and $f_{m-1,\theta}$ in Case 2 ($\theta \in B_{\alpha_{|i|}}(i^*) \subset B_{\widetilde{k}}(k^*) \setminus B_{\alpha_{|k|}}[k^*]$) of Proposition 7.4. The map $f_{m-1,\theta}$ and the corresponding intervals $\mathbb{IW}_{m-1,\theta}$ and $\mathbb{I}_{k+1,R_\omega(\theta)}$ are drawn in blue. The map $f_{m,\theta}$ and the corresponding intervals $\mathbb{I}_{i,\theta} = \mathbb{IW}_{m-1,\theta}$ and $\mathbb{I}_{i+1,R_\omega(\theta)}$ are drawn in red.

Furthermore, since $k < 0$, as in the proof of Lemma 10.1, $R_\omega(\theta) \in B_{\alpha_{|k+1|}}((k+1)^*)$. Thus, Definition 7.2, Lemma 3.6(d) and Definition 3.4(R.2) and Remark 3.5(R.2), give

$$f_{m-1,\theta}(x) = \gamma_{|k+1|}(R_\omega(\theta)) \in \mathbb{I}_{k+1,R_\omega(\theta)}$$

for every $x \in \mathbb{I}_{i,\theta} = \mathbb{IW}_{m-1,\theta}$.

Now we will use Lemma 9.5 to bound the norm $\|f_{m,\theta} - f_{m-1,\theta}\|$. By Definition 6.2 and Lemma 9.5, $\theta \in \mathbb{B}_m \subset \mathbb{B}_{\widetilde{m}} \setminus \mathbb{WIB}_m$, and

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| = \sup_{x\in\mathbb{I}_{i,\theta}} \left|f_{m,\theta}(x) - \gamma_{|k+1|}(R_\omega(\theta))\right|.$$

Next we will compute $f_{m,\theta}(\mathbb{I}_{i,\theta})$. We start with the simplest case: $i \geq 0$ and $\theta \in B_{\alpha_i}(i^*) \setminus B_{\alpha_{i+1}}(i^*)$. By Definition 7.2, the definition of the maps $g_{i,\theta}$ for $i \geq 0$ (Definition 4.2) and Lemma 10.1,

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \sup_{x\in\mathbb{I}_{i,\theta}} \left|f_{m,\theta}(x) - \gamma_{|k+1|}(R_\omega(\theta))\right| = \left|\gamma_{i+1}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))\right| \leq 2^{-\left|\widetilde{\mathsf{b}}(\theta,m-1)\right|}.$$

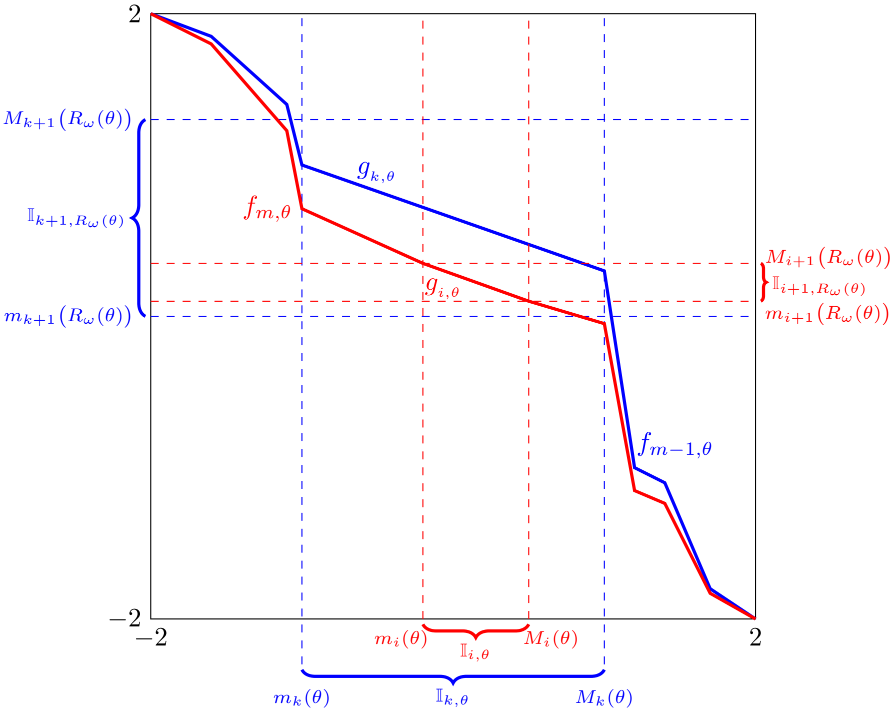


FIGURE 8. A symbolic representation of the maps $f_{m,\theta}$ and $f_{m-1,\theta}$ in Subcase 3.1 from the proof of Proposition 7.4 ($\theta \in B_{\alpha_{|i|}}(i^*)$ and $\mathbb{I}_{i,\theta} \subset \mathbb{I}_{k,\theta}$ and either $k < 0$ or $k \geq 0$ and $i^* \in B_{\alpha_{k+1}}[k^*]$). The map $f_{m-1,\theta}$ and the corresponding intervals $\mathbb{I}_{k,\theta}$ and $\mathbb{I}_{k+1,R_\omega(\theta)}$ are drawn in blue. The map $f_{m,\theta}$ and the corresponding intervals $\mathbb{I}_{i,\theta}$ and $\mathbb{I}_{i+1,R_\omega(\theta)}$ are drawn in red.

Assume that $i < 0$ or $i \geq 0$ and $\theta \in B_{\alpha_{i+1}}(i^*)$. Then, again by Definition 7.2 and Lemmas 4.3(b), 4.5(b) and 10.2,

$$f_{m,\theta}(x) \in \mathbb{I}_{i+1,R_\omega(\theta)} \subset \mathbb{I}_{k+1,R_\omega(\theta)} \quad \text{for every} \quad x \in \mathbb{I}_{i,\theta},$$

and

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x \in \mathbb{I}_{i,\theta}} \left| f_{m,\theta}(x) - \gamma_{|k+1|}\left(R_\omega(\theta)\right) \right| \\ &\leq 2 \cdot 2^{-|k|} = 2 \cdot 2^{-\left|\breve{\mathsf{b}}(\theta, m-1)\right|}. \end{aligned}$$

This ends the proof of the proposition in Case 2.

Case 3. $\theta \in B_{\alpha_{|i|}}(i^*) \subset B_{\alpha_{|k|}}(k^*)$.

In this case we have $B_{\alpha_{|i|}}(i^*) \subset \mathbb{B}_m$ and $B_{\alpha_{|k|}}(k^*) \subset \mathbb{B}_{m-1}$ so that, $\theta \in \mathbb{B}_m \cap \mathbb{B}_{m-1}$. Moreover, by Lemma 3.6(g), $\mathcal{R}(i^*) \subset \operatorname{Int}\left(\mathcal{R}(k^*) \setminus \Uparrow k^*\right)$ and, hence,

$$\mathbb{I}_{i,\theta} \subset \mathbb{I}_{k,\theta}.$$

Since $\theta \in \mathbb{B}_m$, by Definition 6.2 and Lemma 9.5, $\theta \in \breve{\mathbb{B}}_m \setminus \mathbb{WIB}_m$, and

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \left\| f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}} - f_{m-1,\theta}\big|_{\mathbb{I}_{i,\theta}} \right\| = \sup_{x \in \mathbb{I}_{i,\theta}} \left| f_{m,\theta}(x) - f_{m-1,\theta}(x) \right|.$$

Thus, we need to compare the maps $f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}}$ and $f_{m-1,\theta}\big|_{\mathbb{I}_{i,\theta}}$. To do this we consider two subcases.

Subcase 3.1. *Either* $k < 0$ *or* $k \geq 0$ *and* $\theta \in B_{\alpha_{k+1}}(k^*)$
(see Figure 8 for a symbolic representation of this case).

In this situation we aim at proving that

$$f_{m-1,\theta}\left(\mathbb{I}_{i,\theta}\right), f_{m,\theta}\left(\mathbb{I}_{i,\theta}\right) \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

We start with $f_{m-1,\theta}\left(\mathbb{I}_{i,\theta}\right)$. By Definition 7.2 and Lemmas 4.3(b) and 4.5(b) we obtain

$$f_{m-1,\theta}\left(\mathbb{I}_{i,\theta}\right) \subset f_{m-1,\theta}\left(\mathbb{I}_{k,\theta}\right) = g_{k,\theta}\left(\mathbb{I}_{k,\theta}\right) \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

Next we show that $f_{m,\theta}\left(\mathbb{I}_{i,\theta}\right) \subset \mathbb{I}_{k+1,R_\omega(\theta)}$.

Since $k < 0$ or $k \geq 0$ and $\theta \in B_{\alpha_{k+1}}(k^*)$, by Definition 3.4(R.1) we obtain

$$R_\omega(\theta) \in \begin{cases} R_\omega\left(B_{\alpha_{|k|}}(k^*)\right) = B_{\alpha_{|k|}}((k+1)^*) \subset B_{\alpha_{|k+1|}}((k+1)^*) & \text{if } k < 0, \\ R_\omega\left(B_{\alpha_{k+1}}(k^*)\right) = B_{\alpha_{k+1}}((k+1)^*) & \text{if } k \geq 0 \text{ and } \theta \in B_{\alpha_{k+1}}(k^*). \end{cases} \tag{26}$$

Assume that $i < 0$ or $i \geq 0$ and $\theta \in B_{\alpha_{i+1}}(i^*)$. By (26) with k replaced by i,

$$R_\omega(\theta) \in B_{\alpha_{|i+1|}}\left((i+1)^*\right) \cap B_{\alpha_{k+1}}\left((k+1)^*\right) \subset B^{\frown}_{i+1}\left[(i+1)^*\right] \cap B^{\frown}_{k+1}\left[(k+1)^*\right].$$

Therefore, since $|k+1| \leq |i+1|$ and $k+1 \neq i+1$, from Lemma 3.6(g) we obtain $|k+1| < |i+1|$,

$$B_{\alpha_{|i+1|}}\left[(i+1)^*\right] \subset B_{\alpha_{|k+1|}}\left((k+1)^*\right) \setminus \{(k+1)^*\}, \text{ and}$$
$$\mathcal{R}((i+1)^*) \subset \operatorname{Int}\left(\mathcal{R}((k+1)^*) \setminus \upuparrows (k+1)^*\right).$$

Thus, by Definition 7.2 and Lemmas 4.3(b) and 4.5(b),

$$f_{m,\theta}\left(\mathbb{I}_{i,\theta}\right) = g_{i,\theta}\left(\mathbb{I}_{i,\theta}\right) \subset \mathbb{I}_{i+1,R_\omega(\theta)} \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

Now we will consider the case $i \geq 0$ and $\theta \in B_{\alpha_i}(i^*) \setminus B_{\alpha_{i+1}}(i^*)$. The fact that $|k| < |i| = i$ implies $|k+1| \leq |k| + 1 \leq i$. We claim that

$$B_{\alpha_i}\left((i+1)^*\right) \subset B_{\alpha_{|k+1|}}\left((k+1)^*\right) \setminus \{(k+1)^*\}.$$

To prove the claim note that, by (26),

$$R_\omega(\theta) \in R_\omega\left(B_{\alpha_i}\left(i^*\right)\right) \cap B_{\alpha_{|k+1|}}\left((k+1)^*\right) \subset B_{\alpha_i}\left((i+1)^*\right) \cap B^{\frown}_{k+1}\left[(k+1)^*\right].$$

Moreover, the interval $B_{\alpha_i}((i+1)^*)$ is disjoint from $B^{\frown}_{i}[i^*]$ and $B^{\frown}_{-i}[(-i)^*]$ by Definition 3.4(R.2). Thus, $i \neq k+1, -(k+1)$ and, hence, $|k+1| < i$ (that is, $k+1 \in Z_{i-1}$). So, there exists $q \in Z_{i-1}$ such that $B_{\alpha_i}[(i+1)^*] \cap B^{\frown}_{q}[q^*] \neq \emptyset$ and $|q| \geq |k+1|$ is maximal verifying these conditions. By Definition 3.4(R.4),

$$B_{\alpha_i}\left((i+1)^*\right) \subset B^{\frown}_{q}\left(q^*\right) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|q|}}\left[q^*\right]\right) \cup \{q^*\}\right).$$

So, the claim holds when $q = k+1$. Assume that $q \neq k+1$. Then,

$$R_\omega(\theta) \in B_{\alpha_i}\left((i+1)^*\right) \cap B_{\alpha_{|k+1|}}\left((k+1)^*\right) \subset B^{\frown}_{q}\left(q^*\right) \cap B_{\alpha_{|k+1|}}\left((k+1)^*\right).$$

Hence, by Lemma 3.6(g), $|q| > |k+1|$ and

$$B_{\alpha_i}\left((i+1)^*\right) \subset B^{\frown}_{q}\left[q^*\right] \subset B_{\alpha_{|k+1|}}\left((k+1)^*\right) \setminus \{(k+1)^*\}.$$

This ends the proof of the claim.

On the other hand, by Definition 3.4(R.2) and Remark 3.5(R.2),

$$\left(B_{\alpha_i}\left[(i+1)^*\right] \setminus B_{\alpha_{i+1}}\left((i+1)^*\right)\right) \cap Z_{i+1} = \emptyset.$$

Thus, by the claim,

$$\begin{aligned}R_\omega(\theta) \in R_\omega\left(B_{\alpha_i}(i^*)\setminus B_{\alpha_{i+1}}(i^*)\right) &= B_{\alpha_i}((i+1)^*)\setminus B_{\alpha_{i+1}}((i+1)^*)\\ &\subset B_{\alpha_{|k+1|}}((k+1)^*)\setminus Z_{i+1}.\end{aligned}$$

By Definition 7.2, the definition of the maps $g_{i,\theta}$ for $i \geq 0$ (Definition 4.2) and Lemma 3.6(d) (with $\ell = k+1$ and $n = i+1$),

$$f_{m,\theta}(\mathbb{I}_{i,\theta}) = g_{i,\theta}(\mathbb{I}_{i,\theta}) = \left\{\gamma_{i+1}(R_\omega(\theta))\right\} \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

Summarizing, we have proved that

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}), f_{m,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

So, by Lemma 3.6(f) (and the fact that $|k+1| \geq |k| - 1$),

$$\begin{aligned}\|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq \operatorname{diam}\left(\mathbb{I}_{k+1,R_\omega(\theta)}\right)\\ &\leq \operatorname{diam}(\mathcal{R}((k+1)^*)) \leq 2^{-|k+1|} \leq 2\cdot 2^{-|k|} = 2\cdot 2^{-\left|\widetilde{\mathfrak{b}}(\theta,m-1)\right|}.\end{aligned}$$

This ends the proof of the proposition in this subcase.

Subcase 3.2. *$k \geq 0$ and $\theta \in B_{\alpha_k}(k^*)\setminus B_{\alpha_{k+1}}(k^*)$.*

We start by computing $f_{m-1,\theta}(\mathbb{I}_{i,\theta})$. By Definition 7.2 and the definition of the maps $g_{k,\theta}$ for $k \geq 0$ (Definition 4.2),

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}) \subset f_{m-1,\theta}(\mathbb{I}_{k,\theta}) = g_{k,\theta}(\mathbb{I}_{k,\theta}) = \{\gamma_{k+1}(R_\omega(\theta))\}.$$

Analogously, if $i \geq 0$ and $\theta \in B_{\alpha_i}(i^*)\setminus B_{\alpha_{i+1}}(i^*)$,

$$f_{m,\theta}(\mathbb{I}_{i,\theta}) = g_{i,\theta}(\mathbb{I}_{i,\theta}) = \{\gamma_{i+1}(R_\omega(\theta))\}.$$

Then, by Lemma 10.1,

$$\begin{aligned}\|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}} - f_{m-1,\theta}\big|_{\mathbb{I}_{i,\theta}} \right\|\\ &= \left|\gamma_{i+1}(R_\omega(\theta)) - \gamma_{k+1}(R_\omega(\theta))\right| \leq 2^{-\left|\widetilde{\mathfrak{b}}(\theta,m-1)\right|}.\end{aligned}$$

Assume now that $i < 0$ or $i \geq 0$ and $\theta \in B_{\alpha_{i+1}}(i^*)$. By (26), Definition 7.2 and Lemmas 4.3(b) and 4.5(b)

$$\begin{gathered}R_\omega(\theta) \in B_{\alpha_{|i+1|}}((i+1)^*), \text{ and}\\ f_{m,\theta}(\mathbb{I}_{i,\theta}) = g_{i,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{i+1,R_\omega(\theta)}.\end{gathered}$$

Moreover, if $k+1 < |i+1|$, by Lemmas 5.4(a) and 3.6(c), we have

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}) = \left\{\gamma_{k+1}(R_\omega(\theta))\right\} = \left\{\gamma_{|i+1|-1}(R_\omega(\theta))\right\} \subset \mathbb{I}_{i+1,R_\omega(\theta)}.$$

Therefore, by Lemma 3.6(f),

$$\begin{aligned}\|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)|\\ &= \sup_{x\in\mathbb{I}_{i,\theta}} \left|f_{m,\theta}(x) - \gamma_{|i+1|-1}(R_\omega(\theta))\right|\\ &\leq \operatorname{diam}\left(\mathbb{I}_{i+1,R_\omega(\theta)}\right) \leq \operatorname{diam}(\mathcal{R}((i+1)^*)) \leq 2^{-|i+1|}\\ &< 2^{-(k+1)} < 2^{-\left|\widetilde{\mathfrak{b}}(\theta,m-1)\right|}.\end{aligned}$$

So, to end the proof of the proposition we have to show that, in this subcase, $k+1 < |i+1|$. To prove this, notice that when $i \geq 0$, $k+1 = |k|+1 < |i|+1 = |i+1|$.

So, assume by way of contradiction that $i < 0$ and $k+1 = |i+1|$ (recall that $k+1 \le |i+1|$). Then, $k+1 = -(i+1)$ and, hence,

$$R_\omega(\theta) \in R_\omega\left(B_{\alpha_k}\left(k^*\right)\right) = B_{\alpha_k}\left((k+1)^*\right), \text{ and}$$

$$R_\omega(\theta) \in B_{\alpha_{|i+1|}}\left((i+1)^*\right) = B_{\alpha_{k+1}}\left((-(k+1))^*\right) \subset B_{\widetilde{-(k+1)}}\left((-(k+1))^*\right),$$

which is a contradiction by Definition 3.4(R.2). □

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