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PERIODIC ORBITS OF PERTURBED ELLIPTIC OSCILLATORS IN 6D VIA AVERAGING THEORY

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Abstract. We provide sufficient conditions on the energy levels to guarantee the existence of periodic orbits for the perturbed elliptic oscillators in 6D using the averaging theory. We give also an analytical estimation of the shape of these periodic orbits parameterized by the energy. The Hamiltonian system here studied comes either from the analyze of the galactic dynamics, or from the motion of the atomic particles in physics.

1. Introduction

The Hamiltonian studied in this paper consists of a three coupled harmonic oscillators known as perturbed elliptic oscillators.

$$H = \frac{1}{2}(x^2 + y^2 + z^2 + p_x^2 + p_y^2 + p_z^2) + \varepsilon(x^2y^2 + x^2z^2 + y^2z^2 - x^2y^2z^2). \tag{1}$$

Perturbed elliptic oscillators appears very often in several fields of nonlinear mechanics, as in galactic dynamics, and in atomic physics. During the last three decades, in galactic dynamics, in order to describe the local dynamics properties of galaxies we consider Hamiltonian systems. Many studies have been made, see mainly [2, 3, 4, 5, 6, 7, 9]. Despite of the simple form of the perturbation (generally, they are cubic and quartic polynomials) they lead to chaotic phenomena as was shown in the work of Henon and Heiles [8]. In this work we analyze the periodic orbits of the Hamiltonian system associated to the Hamiltonian (1) studied by Caranicolas and Zotos in [5]. We present conditions on the energy level to guarantee the existence of periodic orbits which, we hope, be useful information on the study of periodic motion not only in galactic dynamics but in the general field of nonlinear dynamics.

Our objective is to compute analytically the families of the periodic solutions of the Hamiltonian system defined by the Hamiltonian (1). The Hamiltonian system

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associated to (1) is

$$
\begin{aligned}
\dot{x} = & \quad \frac{\partial H}{\partial p_x} = p_x, \\
\dot{y} = & \quad \frac{\partial H}{\partial p_y} = p_y, \\
\dot{z} = & \quad \frac{\partial H}{\partial p_z} = p_z, \\
\dot{p}_x = & \quad -\frac{\partial H}{\partial x} = -x - \varepsilon(2xy^2 + 2xz^2 - 2xy^2z^2), \\
\dot{p}_y = & \quad -\frac{\partial H}{\partial y} = -y - \varepsilon(2x^2y + 2yz^2 - 2x^2yz^2), \\
\dot{p}_z = & \quad -\frac{\partial H}{\partial z} = -z - \varepsilon(2x^2z + 2y^2z - 2x^2y^2z),
\end{aligned} \tag{2}
$$

where p_x, p_y, p_z are the components of the momentum per unit mass, ε is a small positive real parameter, in fact ε is the perturbation strength. The dot denotes derivative with respect to the independent variable t, the time.

We study the existence of families of periodic orbits of the Hamiltonian system (2) and we compute them analytically. The tool used for studying these families of periodic solutions is the *averaging theory*, for more details about this technique see section 2.

As it is well known the study of the periodic orbits is very interesting because they form with the equilibrium points the most simple solutions of the system and their stability determines the kind of motion in their neighborhood. In this paper the periodic orbits studied are isolated in every energy level.

The averaging method transforms the problem of finding periodic solutions of a differential system in finding zeros of some convenient finite dimensional function. Once we have the zeros of that function we check the conditions under which the averaging theory guarantees the existence of periodic orbits, and finally we estimate the analytical shape of the periodic orbits in function of the energy. The main results and statements of this research are the following theorem and preposition.

Theorem 1. *At every fixed energy level $H = h$ with $h > 0$, the perturbed elliptic oscillator Hamiltonian system* (2) *has at least*

(a) *Ten periodic orbits if $h \in (0, 6/5)$;*

(b) *Twenty periodic orbits if $h \in \Big(6/5, 9/5\Big) \cup \Big(9/5, 27/10\Big) \cup \Big(27/10, 3]$;*

(c) *Twenty two periodic orbits if $h \in (3, 6)$;*

(d) *Thirty two periodic orbits if $h \in (6, 5 + 2\sqrt{6})$;*

(e) *Forty two periodic orbits if $h \in (5 + 2\sqrt{6}, +\infty)$.*

Theorem 1 is proved in section 3.

Proposition 2. *The family of periodic orbits of system* (2) *generated by the zeros* $(r^*, \alpha^*, R^*, \beta^*)$ *of the averaged function* $(f_{11}, f_{12}, f_{13}, f_{14})$ *given in* (13) *at every* *fixed energy level $H = h > 0$ are given by*

$$\begin{aligned}
x(t,\varepsilon) = & \quad r^*\cos t + O(\varepsilon),\\
y(t,\varepsilon) = & \quad \sqrt{2h - r^{*2} - R^{*2}}\cos(\alpha^* + t) + O(\varepsilon),\\
z(t,\varepsilon) = & \quad R^*\cos(\beta^* + t) + O(\varepsilon),\\
p_x(t,\varepsilon) = & \quad r^*\sin t + O(\varepsilon),\\
p_y(t,\varepsilon) = & \quad \sqrt{2h - r^{*2} - R^{*2}}\sin(\alpha^* + t) + O(\varepsilon),\\
p_z(t,\varepsilon) = & \quad R^*\sin(\beta^* + t) + O(\varepsilon).
\end{aligned}$$

Proposition 2 is proved at the end of section 3.

2. The averaging theory of First Order

In this section we remember the main results of averaging theory that we shall use for proving Theorem 1 .

We deal with the differential system

$$\dot{\mathbf{x}} = \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{3}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Additionally we assume that the functions $F_1(t,\mathbf{x})$ and $F_2(t,\mathbf{x},\varepsilon)$ are $T-$periodic in t. The averaged differential system in D is defined as follows

$$\dot{\mathbf{y}} = \varepsilon f_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{4}$$

where

$$f_1(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t,\mathbf{y})dt. \tag{5}$$

Later on we see under convenient hypotheses that the equilibria solutions of the averaged system will provide $T-$periodic solutions of system (3).

Theorem 3. *Consider the two initial value problems* (3) *and* (4)*. Suppose that*

(i) *the functions F_1, $\partial F_1/\partial x$, $\partial^2 F_1/\partial x^2$, F_2 and $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty)\times D$ and $\varepsilon \in (0,\varepsilon_0]$;*

(ii) *the functions F_1 and F_2 are $T-$periodic in t (T independent of ε).*

Then the following statements hold.

(a) *If p, an equilibrium point of the averaged system* (4)*, satisfies*

$$\det\left(\frac{\partial f_1}{\partial \mathbf{y}}\right)\Big|_{\mathbf{y}=p} \neq 0, \tag{6}$$

then there is a $T-$periodic solution $\varphi(t,\varepsilon)$ of system (3) *such that $\varphi(0,\varepsilon) \to p$ as $\varepsilon \to 0$.*

(b) *If p, an equilibrium point of the averaged system* (4)*, is hyperbolic then it has the stability behavior of the Poincaré map associated to the periodic solution $\varphi(t,\varepsilon)$.*

For a proof of Theorem 3, see Theorems 11.5 and 11.6 of Verhulst [10].

We proceed as follow, first we do some changes in the variables of the Hamiltonian differential system (2) in order to write it into the normal form of system (3) to apply the averaging theory. For doing that we use some generalized polar coordinates in

$\mathbb{R}^6$. After, we need the periodicity of the differential system so we take an angle coordinate as a new independent variable, instead of the time, t. Then, fixing the energy level and omitting a redundant variable in every energy level, we will obtain a differential system written in the normal form for applying the averaging theory (Theorem 3). Finally we apply the averaging theory to prove the existence of some isolated periodic solutions in every positive energy level.

3. Proof of Theorem 1

The periodicity of the independent variable of the differential system is needed to apply the averaging theory, so we change the Hamiltonian system (2) to a kind of generalized polar coordinates $(r, \theta, \rho, \alpha, R, \beta)$ in $\mathbb{R}^6$ defined by

$$\begin{array}{lll} x = r\cos\theta, & y = \rho\cos(\theta+\alpha), & z = R\cos(\theta+\beta), \\ p_x = r\sin\theta, & p_y = \rho\sin(\theta+\alpha), & p_z = R\sin(\theta+\beta), \end{array} \tag{7}$$

where $r \geq 0$, $\rho \geq 0$ and $R \geq 0$.

The first integral H in the new variables is

$$\begin{aligned} H = \;& \frac{1}{2}(\rho^2 + r^2 + R^2) + \varepsilon\Big[r^2\cos^2\theta\Big(\rho^2\cos^2(\theta+\alpha) \\ & \big(1 - R^2\cos^2(\theta+\beta)\big) + R^2\cos^2(\theta+\beta)\Big) \\ & + \rho^2R^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta)\Big], \end{aligned}$$

and the equations of motion (2) become

$$\begin{aligned} \dot r = \;& 2r\varepsilon\sin\theta\cos\theta\Big[\rho^2\cos^2(\theta+\alpha)\big(R^2\cos^2(\theta+\beta)-1\big) \\ & -R^2\cos^2(\theta+\beta)\Big], \\ \dot\theta = \;& -1 + 2\varepsilon\cos^2\theta\Big[\rho^2\cos^2(\theta+\alpha)\big(R^2\cos^2(\theta+\beta)-1\big) \\ & -R^2\cos^2(\theta+\beta)\Big], \\ \dot\rho = \;& 2\rho\varepsilon\sin(\theta+\alpha)\cos(\theta+\alpha)\Big[R^2\cos^2(\theta+\beta)\big(r^2\cos^2\theta-1\big) \\ & -r^2\cos^2\theta\Big], \\ \dot\alpha = \;& \varepsilon\Big[\cos^2\theta\Big((r^2-\rho^2)\cos^2(\theta+\alpha)\big(R^2\cos 2(\theta+\beta)+R^2-2\big) \\ & +2R^2\cos^2(\beta+\theta)\Big) - 2R^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta)\Big], \\ \dot R = \;& 2R\varepsilon\sin(\theta+\beta)\cos(\theta+\beta)\Big[\rho^2\cos^2(\theta+\alpha)\Big(r^2\cos^2(\theta)-1\Big) \\ & -r^2\cos^2(\theta)\Big], \\ \dot\beta = \;& \varepsilon\Big[-2\rho^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta) + 2\cos^2\theta \\ & \Big(\rho^2\cos^2(\theta+\alpha) + (r^2-R^2)\cos^2(\theta+\beta)\big(\rho^2\cos^2(\theta+\alpha)-1\big)\Big)\Big]. \end{aligned} \tag{8}$$

If we take the variable θ as the new independent variable instead of t in the system (8), we obtain the necessary periodicity to have the system of equations of motion in the normal form of the averaging theory. From now on the independent variable will be θ. Then the new differential system will have only five equations. Expanding system (8) in Taylor series in ε we have

$$
(9)\quad
\begin{aligned}
r' = &\ -2r\varepsilon\sin\theta\cos\theta\Big[\rho^2\cos^2(\theta+\alpha)\Big(R^2\cos^2(\theta+\beta)-1\Big)\\
&-R^2\cos^2(\theta+\beta)\Big]+O(\varepsilon^2),\\
\rho' = &\ -2\rho\varepsilon\sin(\theta+\alpha)\cos(\theta+\alpha)\Big[R^2\cos^2(\theta+\beta)(r^2\cos^2\theta-1)\\
&-r^2\cos^2\theta\Big]+O(\varepsilon^2),\\
\alpha' = &\ \varepsilon\Big[2R^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta)-\cos^2\theta\Big((r^2-\rho^2)\cos^2(\theta+\alpha)\\
&\Big(R^2\cos2(\theta+\beta)+R^2-2\Big)+2R^2\cos^2(\theta+\beta)\Big)\Big]+O(\varepsilon^2),\\
R' = &\ -2R\varepsilon\sin(\theta+\beta)\cos(\theta+\beta)\Big[\rho^2(r^2\cos^2\theta-1)\\
&\cos^2(\theta+\alpha)-r^2\cos^2\theta\Big]+O(\varepsilon^2),\\
\beta' = &\ 2\varepsilon\Big[\rho^2\cos^2(\theta+\beta)\cos^2(\theta+\alpha)-\cos^2\theta\Big(\rho^2\cos^2(\theta+\alpha)\\
&+(r^2-R^2)(\rho^2\cos^2(\theta+\alpha)-1)\cos^2(\theta+\beta)\Big)\Big]+O(\varepsilon^2).
\end{aligned}
$$

The prime denotes derivative with respect to the angle variable θ. System (9) is 2π-periodic respect to the variable θ, i.e. it is written in the normal form (3) of the averaging theory but we should fix the value of the first integral $H = h$ with $h \in \mathbb{R}^+$ to make it ready for applying the averaging theory. Otherwise the Jacobian (6) will be zero because the periodic orbits are non-isolated leaving on cylinders parameterized by the energy, see for more details [1].

We fix the energy level and we solve $H = h$ with respect to ρ, we get two solutions, but we take only the one with physical meaning and we expanded it in Taylor series in ε

$$
(10)\quad \rho = \sqrt{2h - r^2 - R^2} + O(\varepsilon).
$$

Since $\rho \geq 0$ we need that $2h - r^2 - R^2 \geq 0$.

Substituting ρ in system (9) we get the differential system written in the normal form of the averaging theory

$$
(11)\quad
\begin{aligned}
r' = &\ -2r\varepsilon\sin\theta\cos\theta\Big[\cos^2(\theta+\alpha)(2h-r^2-R^2)(R^2\cos^2(\theta+\beta)-1)\\
&-R^2\cos^2(\theta+\beta)\Big]+O(\varepsilon^2),\\
\alpha' = &\ \varepsilon\Big[\cos^2\theta\Big(\cos^2(\theta+\alpha)(2h-2r^2-R^2)(R^2\cos2(\theta+\beta)+R^2-2)\\
&-2R^2\cos^2(\theta+\beta)\Big)+2R^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta)\Big]+O(\varepsilon^2),\\
R' = &\ 2R\varepsilon\sin(\theta+\beta)\cos(\theta+\beta)\Big[r^2\cos^2\theta-\cos^2(\theta+\alpha)(2h-r^2-R^2)\\
&(r^2\cos\theta^2-1)\Big]+O(\varepsilon^2),\\
\beta' = &\ 2\varepsilon\Big[(R^2+r^2-2h)\cos^2\theta\cos^2(\theta+\alpha)-\cos^2(\theta+\beta)\\
&\Big((R^2-r^2)\cos^2\theta-(r^2+R^2-2h)((r^2-R^2)\cos^2\theta-1)\\
&\cos^2(\theta+\alpha)\Big)\Big]+O(\varepsilon^2).
\end{aligned}
$$

Following the notation of the averaging theory given in section 2, the function $F_1 = (F_{11}, F_{12}, F_{13}, F_{14})$ of (3) is

$$
\begin{aligned}
F_{11} = \;& 2r\sin\theta\cos\theta\Big[(2h - r^2 - R^2)\big(1 - R^2\cos^2(\theta+\beta)\big)\\
& \cos^2(\theta+\alpha) + R^2\cos^2(\theta+\beta)\Big],\\
F_{12} = \;& 2R^2\cos^2(\theta+\alpha)\cos^2(\theta+\beta) - \cos^2\theta\Big[\cos^2(\theta+\alpha)\\
& (-2h + 2r^2 + R^2)\big(R^2\cos 2(\theta+\beta) + R^2 - 2\big)\\
& +2R^2\cos^2(\theta+\beta)\Big]\\
F_{13} = \;& 2R\sin(\theta+\beta)\cos(\theta+\beta)\Big[\cos^2(\theta+\alpha)(2h - r^2 - R^2)\\
& (1 - r^2\cos^2\theta) + r^2\cos^2\theta\Big],\\
F_{14} = \;& 2\cos^2(\theta+\beta)\Big[(2h - r^2 - R^2) - 2\cos^2(\theta+\alpha)\\
& (1 - \cos^2\theta) - (r^2 - R^2)\cos^2\theta\Big((2h - r^2 - R^2) - 1\Big)\\
& \cos^2(\theta+\alpha)\Big],
\end{aligned}
\tag{12}
$$

where $F_{1j} = F_{1j}(\theta, r, \alpha, R, \beta)$ for $j = 1, 2, 3, 4$.

From (5) and (12) we calculate the averaged function $f_1 = (f_{11}, f_{12}, f_{13}, f_{14})$ and we obtain

$$
\begin{aligned}
f_{11} = \;& \frac{1}{8}r\Big[(2 - R^2)(R^2 + r^2 - 2h)\sin 2\alpha - R^2(R^2 + r^2 - 2h + 2)\\
& \sin 2\beta\Big],\\
f_{12} = \;& \frac{1}{8}\Big[(2h - 2r^2 - R^2)\Big(2R^2\cos\alpha\cos(\alpha - 2\beta) + (R^2 - 2)\cos 2\alpha\Big)\\
& +4h(R^2 - 2) - 2\Big(2r^2(R^2 - 2) + R^4\Big) + 2R^2(\cos 2(\alpha - \beta) + 2)\\
& -2R^2\cos 2\beta\Big],\\
f_{13} = \;& \frac{1}{8}R\Big[(2 - r^2)(-2h + r^2 + R^2)\sin 2(\alpha - \beta)\\
& +r^2(-2h + r^2 + R^2 + 2)\sin 2\beta\Big],\\
f_{14} = \;& \frac{1}{8}\Big[(r^2 - R^2 - 2)\cos 2(\alpha - \beta)(-2h + r^2 + R^2)+\\
& \cos 2\alpha(r^2 - R^2 + 2)(-2h + r^2 + R^2)+\\
& (r^2 - R^2)(-2h + r^2 + R^2 + 2)(\cos 2\beta + 2)\Big],
\end{aligned}
$$

where $f_{1j} = f_{1j}(r, \alpha, R, \beta)$ for $j = 1, 2, 3, 4$.

According to Theorem 3 our objective is to find the zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ of

$$
f_{1i}(r, \alpha, R, \beta) = 0 \quad \text{for } i = 1, 2, 3, 4,
\tag{13}
$$

and after we must check that the Jacobian determinant (6) evaluated at these zeros are nonzero. Note that in all numerical calculations the numerical value h, the energy, is treated as a parameter.

Solving $f_{11}(r, \alpha, R, \beta) = 0$ we obtain three cases:

(a) $r = 0$;

(b) $r = \sqrt{\dfrac{(2h-R^2)(2-R^2)\sin 2\alpha + R^2(2-2h+R^2)\sin 2\beta}{2\sin 2\alpha - R^2(\sin 2\beta + \sin 2\alpha)}}$

if $\quad 2\sin 2\alpha - R^2(\sin 2\beta + \sin 2\alpha) \neq 0$;

(c) $2\sin 2\alpha - R^2(\sin 2\beta + \sin 2\alpha) = 0$.

Case 1: $r = 0$. Substituting r in f_{12}, f_{13} and f_{14} we have

$$\begin{aligned} f_{12}(0,\alpha,R,\beta) = {} & \frac{1}{8}\Big[(2h-R^2)\Big(2R^2\cos\alpha\cos(\alpha-2\beta) + (R^2-2)(2+\cos 2\alpha)\Big) \\ & -4R^2\sin\alpha\sin(\alpha-2\beta)\Big], \\ f_{13}(0,\alpha,R,\beta) = {} & \frac{1}{4}R(R^2-2h)\sin 2(\alpha-\beta), \\ f_{14}(0,\alpha,R,\beta) = {} & \frac{1}{8}\Big[(2h-R^2)(R^2-2)\cos 2\alpha + (2h-R^2)(2+R^2)\cos 2(\alpha-\beta) \\ & -R^2(2-2h+R^2)(2+\cos 2\beta)\Big]. \end{aligned}$$

Solving $f_{13}(0,\alpha,R,\beta) = 0$ we obtain three subcases: $R = 0$, $R = \sqrt{2h}$, $\alpha = \beta + \dfrac{k\pi}{2}$ with $k \in \mathbb{Z}$.

Subcase 1.1: $R = 0$, f_{12} and f_{14} become

$$\begin{aligned} f_{12}(0,\alpha,0,\beta) = {} & -\frac{h}{2}(2+\cos 2\alpha), \\ f_{14}(0,\alpha,0,\beta) = {} & h\sin(2\alpha-\beta)\sin\beta. \end{aligned}$$

The system $f_{12} = f_{14} = 0$ has no solution in this subcase. The averaging theory does not give results.

Subcase 1.2: $R = \sqrt{2h}$, Then f_{12} and f_{14} become

$$\begin{aligned} f_{12}(0,\alpha,\sqrt{2h},\beta) = {} & -h\sin(\alpha-2\beta)\sin\alpha, \\ f_{14}(0,\alpha,\sqrt{2h},\beta) = {} & -\frac{h}{2}(2+\cos 2\beta). \end{aligned}$$

As in the previous case, the averaging theory does not give information.

Subcase 1.3: $\alpha = \beta + \dfrac{k\pi}{2}$ with k $\in \mathbb{Z}$. Due to the periodicity of the cosinus and the sinus we study two subcases, either ($k = 0$ and $k = 2$) or ($k = 1$ and $k = 3$).

Subcase 1.3.1: Assume that either $k = 0$ or $k = 2$, i.e. either $\alpha = \beta$ or $\alpha = \beta + \pi$. Then

$$\begin{aligned} f_{12}(0,\alpha,R,\beta) = {} & \frac{1}{8}\big[6(h+1)R^2 - 2\cos 2\beta\big(R^4 - 2h(R^2-1)\big) - 8h - 3R^4\big], \\ f_{14}(0,\alpha,R,\beta) = {} & \frac{1}{8}\big[6(h-1)R^2 - 2\cos 2\beta\big(R^4 - 2h(R^2-1)\big) + 4h - 3R^4\big]. \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ we obtain $R = \sqrt{h}$ and $\beta = \pm\dfrac{1}{2}\arccos\dfrac{2-3h}{2(h-2)}$. Substituting R in formula (10): $\rho = \sqrt{2h - r^2 - R^2}$, we get $\rho = \sqrt{h}$.

Supposing that

$$h > 0 \quad \text{and} \quad \left|\frac{2-3h}{2(h-2)}\right| < 1, \tag{14}$$

which is equivalent to $0 < h < \dfrac{6}{5}$.

System (13) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{h}$ given by

$$\begin{aligned} S^*_{1,2} = & \left(0, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}, \sqrt{h}, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}\right), \\ S^*_{3,4} = & \left(0, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}, \sqrt{h}, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)} + \pi\right), \end{aligned} \tag{15}$$

which reduce to two solutions if $h > 0$ and $\left|\dfrac{2-3h}{2(h-2)}\right| = 1$.

Now we check if the Jacobian of f_1 evaluated at these solutions is different from zero. By definition the Jacobian is

$$J_{f_1} = \quad |D_{r\alpha R\beta} f_1(S^*)| = \left|\begin{matrix} \dfrac{\partial f_{11}}{\partial r} & \dfrac{\partial f_{11}}{\partial \alpha} & \dfrac{\partial f_{11}}{\partial R} & \dfrac{\partial f_{11}}{\partial \beta} \\ \dfrac{\partial f_{12}}{\partial r} & \dfrac{\partial f_{12}}{\partial \alpha} & \dfrac{\partial f_{12}}{\partial R} & \dfrac{\partial f_{12}}{\partial \beta} \\ \dfrac{\partial f_{13}}{\partial r} & \dfrac{\partial f_{13}}{\partial \alpha} & \dfrac{\partial f_{13}}{\partial R} & \dfrac{\partial f_{13}}{\partial \beta} \\ \dfrac{\partial f_{14}}{\partial r} & \dfrac{\partial f_{14}}{\partial \alpha} & \dfrac{\partial f_{14}}{\partial R} & \dfrac{\partial f_{14}}{\partial \beta} \end{matrix}\right|_{(r,\alpha,R,\beta)=S^*}.$$

So $J_{f_1(S^*)} = -\dfrac{3}{64}h^4(h+2)(5h-6)$.

If (14) hold, then $J_{f_1(S^*)} \neq 0$ and the four solutions (15) of system (13) can provide four periodic solutions of differential system (11). However, going back from the differential system (11) to the differential system (2) and using the Proposition 2 we obtain only two different periodic orbits S_1^* and S_3^*, because S_1^* and S_2^* are in the same family of periodic orbits as well as S_3^* and S_4^*. Note that these two solutions exist for $h \in \left(0, \dfrac{6}{5}\right)$ and they constituted two of the six periodic orbits mentioned in (a) of Theorem 1.

Subcase 1.3.2: Assume that either $k = 1$ or $k = 3$, i.e. either $\alpha = \beta + \dfrac{\pi}{2}$ or $\alpha = \beta + \dfrac{3\pi}{2}$. Then

$$\begin{aligned} f_{12}(0,\alpha,R,\beta) = & \frac{1}{8}\big[-8h + 2(1+h)R^2 - R^4 + 4(h-R^2)\cos 2\beta\big], \\ f_{14}(0,\alpha,R,\beta) = & \frac{1}{8}\big[-4h + 2(-1+h)R^2 - R^4 + 4(h-R^2)\cos 2\beta\big]. \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ we do not have solutions. Therefore the averaging theory does not give information.

Case 2: $r = \sqrt{\dfrac{(2h-R^2)(2-R^2)\sin 2\alpha + R^2(2-2h+R^2)\sin 2\beta}{(2-R^2)\sin 2\alpha - R^2\sin 2\beta}}$ under the condition $D = 2\sin 2\alpha - R^2(\sin 2\alpha + \sin 2\beta) \neq 0$.

We cannot study this general case due to the difficulty of the calculations and to the huge expressions obtained when replacing r in f_{12}, f_{13}, f_{14}, but from the expression of D we can study the following two particular subcases, either $(2-R^2)\sin 2\alpha = 0$ and $R^2\sin 2\beta \neq 0$ or $(2-R^2)\sin 2\alpha \neq 0$ and $R^2 \sin 2\beta = 0$.

Subcase 2.1: $(2-R^2)\sin 2\alpha = 0$ and $R^2\sin 2\beta \neq 0$.

Subcase 2.1.1: $R = \sqrt{2}$ and $R^2 \sin 2\alpha \sin 2\beta \neq 0$. Replacing R in r we get $r = \sqrt{2(h-2)}$ then substituting R and r in formula (10): $\rho = \sqrt{2h - R^2 - r^2}$ we get $\rho = \sqrt{2}$. Substituting r in f_{13} we obtain $f_{13} = \dfrac{\sqrt{2}}{2}(h-3)\sin 2(\alpha-\beta)$. When $f_{13} = 0$, we have either $\alpha = \beta + \dfrac{m\pi}{2}$ with $m \in \mathbb{Z}$ or $h = 3$. Due to the periodicity of the cosinus we study the subcases $m = 0$ and $m = 2$, and the subcases $m = 1$ and $m = 3$ together.

Subcase 2.1.1.1: either $m = 0$ or $m = 2$ i.e. either $\alpha = \beta$ or $\alpha = \beta + \pi$. Replacing α in f_{12} and f_{14} we get

$$\begin{aligned} f_{12} = & -\frac{1}{2}\big[(h-4) + (h-2)\cos 2\beta\big], \\ f_{14} = & -\frac{1}{2}\big[(h-4) + (h-2)\cos 2\beta\big]. \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ we get $\beta = \pm \arccos \dfrac{4-h}{h-2}$.

With the condition

$$h > 0, \quad h-2 > 0 \quad \text{and} \quad \left|\frac{4-h}{h-2}\right| < 1, \tag{16}$$

which is equivalent to $h > 3$, system (13) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{2}$ given by

$$\begin{aligned} S^*_{1,2} = & \left(\sqrt{2(h-2)}, \pm\frac{1}{2}\arccos\frac{4-h}{h-2}, \sqrt{2}, \pm\frac{1}{2}\arccos\frac{4-h}{h-2}\right), \\ S^*_{3,4} = & \left(\sqrt{2(h-2)}, \pm\frac{1}{2}\arccos\frac{4-h}{h-2}, \sqrt{2}, \pm\frac{1}{2}\arccos\frac{4-h}{h-2} + \pi\right), \end{aligned} \tag{17}$$

If $h > 0$, $h-2 > 0$ and $\left|\dfrac{4-h}{h-2}\right| = 1$ i.e. $h = 3$ we have only two zeros.
The Jacobian evaluated on these solutions is $J_{f_1(S^*)} = 8(h-3)^3$.

We conclude that if (16) hold we get $J_{f_1(S^*)} \neq 0$ and the four solutions (17) of system (13) can provide four periodic solutions of differential system (11). Nevertheless, going back to the differential system (2) and using the Proposition 2 we obtain only two different periodic orbits S_1^* and S_3^* since S_1^* and S_2^* are in the same family of periodic orbits as well as S_3^* and S_4^*. These two periodic orbits exits for $h > 3$ and they constituted two of the ten periodic orbits cited in (c) of Theorem 1.

Subcase 2.1.1.2: Assume that either $m = 1$ and $m = 3$, i.e. either $\alpha = \beta + \dfrac{\pi}{2}$ or $\alpha = \beta + \dfrac{3\pi}{2}$. Then

$$\begin{aligned} f_{12} = &\ \frac{1}{2}\big[(h-4) - (h-2)\cos 2\beta\big], \\ f_{14} = &\ \frac{1}{2}\big[(h-4) + (h-2)\cos 2\beta\big]. \end{aligned}$$

The averaging theory does not give information because solving $f_{12} = f_{14} = 0$, we do not obtain any solutions.

Subcase 2.1.1.3: $h = 3$, $\sin 2(\alpha - \beta) \neq 0$ and $R^2 \sin 2\beta \neq 0$. We obtain $r = \sqrt{2}$ and $R = \sqrt{2}$ and f_{12} and f_{14} become

$$\begin{aligned} f_{12} = &\ -\sin\alpha \sin(\alpha - 2\beta), \\ f_{14} = &\ \sin\beta \sin(2\alpha - \beta). \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ with respect to (α, β) we obtain eight solutions. System (13) has eight zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{2}$ given by

$$\begin{aligned} S_1^* = &\ \left(\sqrt{2}, -\frac{2\pi}{3}, \sqrt{2}, -\frac{\pi}{3}\right), \\ S_2^* = &\ \left(\sqrt{2}, -\frac{2\pi}{3}, \sqrt{2}, \frac{2\pi}{3}\right), \\ S_3^* = &\ \left(\sqrt{2}, -\frac{\pi}{3}, \sqrt{2}, -\frac{2\pi}{3}\right), \\ S_4^* = &\ \left(\sqrt{2}, -\frac{\pi}{3}, \sqrt{2}, \frac{\pi}{3}\right), \\ S_5^* = &\ \left(\sqrt{2}, \frac{\pi}{3}, \sqrt{2}, -\frac{\pi}{3}\right), \\ S_6^* = &\ \left(\sqrt{2}, \frac{\pi}{3}, \sqrt{2}, \frac{2\pi}{3}\right), \\ S_7^* = &\ \left(\sqrt{2}, \frac{2\pi}{3}, \sqrt{2}, -\frac{2\pi}{3}\right), \\ S_8^* = &\ \left(\sqrt{2}, \frac{2\pi}{3}, \sqrt{2}, \frac{\pi}{3}\right), \end{aligned} \tag{18}$$

The Jacobian evaluated at these eight solutions $J_{f_1(S^*)} = -\dfrac{81}{16} \neq 0$.

Despite the eight solutions (18) of system (13) which can provide eight periodic solutions of differential system (11), we have only four different periodic orbits S_1^*, S_2^*, S_3^* and S_4^*. Because going back to the differential system (2) and using the Proposition 2 we get that S_1^* and S_8^* are in the same family of periodic orbits as well as S_2^* and S_7^*, S_3^* and S_6^*, S_4^* and S_5^*.

Subcase 2.1.2: $\alpha = \dfrac{l\pi}{2}$ with $l \in \mathbb{Z}$ and $(2 - R^2)R^2 \sin 2\beta \neq 0$. Due to the periodicity of the cosinus and sinus we study the subcases $l = 0$ and $l = 2$, and the subcases $l = 1$ and $l = 3$ together.

Subcase 2.1.2.1: either $l=0$ or $l=2$. i.e. either $\alpha=0$ or $\alpha=\pi$. Substituting α in r we have $r=\sqrt{2h-2-R^2}$. Replacing r and α in f_{13} we get $f_{13}=\frac{1}{4}R(4-2h+R^2)\sin 2\beta$. Solving $f_{13}=0$ we get $R=\sqrt{2h-4}$ because by hypothesis $R^2\sin 2\beta\neq 0$. Then $\rho=\sqrt{2}$. So replacing r and R in f_{12}, f_{14} we get

$$\begin{aligned} f_{12} = & \quad 0, \\ f_{14} = & \quad -\frac{1}{2}\big[h-4+(h-2)\cos 2\beta\big]. \end{aligned}$$

Solving $f_{14}=0$ we have $\beta=\pm\frac{1}{2}\arccos\frac{4-h}{h-2}$.

Assuming that

$$h>0, \quad h-2>0 \quad \text{and} \quad \left|\frac{4-h}{h-2}\right|<1, \tag{19}$$

which is equivalent to $h>3$.

Then system (13) has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho=\sqrt{2}$ given by

$$\begin{aligned} S^*_{1,2} = & \left(\sqrt{2}, 0, \sqrt{2h-4}, \pm\frac{1}{2}\arccos\frac{4-h}{h-2}\right), \\ S^*_{3,4} = & \left(\sqrt{2}, \pi, \sqrt{2h-4}, \pm\frac{1}{2}\arccos\frac{4-h}{h-2}\right), \end{aligned} \tag{20}$$

The Jacobian evaluated on these solutions $J_{f_1(S^*)}=8(h-3)^2$.

When (19) hold we have $J_{f_1(S^*)}\neq 0$ and the four solutions (20) of system (13) can provide four periodic solutions of differential system (11). But going back to the differential system (2) and by the Proposition 2 we get only two different periodic orbits S_1^* and S_3^* since S_1^* and S_2^* are in the same family of periodic orbits likewise S_3^* and S_4^*. So for $h>3$ adding these two new periodic orbits to the two ones found in the subcase 2.1.1.1, we obtain four periodic orbits of the ten periodic orbits cited in (c) of Theorem 1.

Subcase 2.1.2.2: $l=0$ and $l=3$. i.e. $\alpha=\frac{\pi}{2}$ and $\alpha=\frac{3\pi}{2}$. Replacing α in r we get $r=\sqrt{2h-2-R^2}$. Substituting r and α in f_{13} we obtain $f_{13}=-\frac{1}{4}R(4-2h+R^2)\sin 2\beta$. Solving $f_{13}=0$ we have $R=\sqrt{2h-4}$ because by hypothesis $R\sin 2\beta\neq 0$. Then we get $r=\sqrt{2}$ and $\rho=\sqrt{2}$. f_{12} and f_{14} become

$$\begin{aligned} f_{12} = & \quad (2-h)\cos 2\beta, \\ f_{14} = & \quad -\frac{1}{2}\big[h-4+(h-2)\cos 2\beta\big]. \end{aligned}$$

Solving $f_{12}=f_{14}=0$ we do not have solutions for β. The averaging theory does not give information.

Subcase 2.1.3: $\sin 2\alpha=0$, $(2-R^2)=0$ and $R^2\sin 2\beta\neq 0$.

Subcase 2.1.3.1: Either $\alpha=0$ or $\alpha=\pi$ and $R=\sqrt{2}$. Replacing α and $R=\sqrt{2}$ in r we get $r=\sqrt{2h-4}$. Substituting r in f_{13} we obtain $f_{13}=-\frac{\sqrt{2}}{2}(h-3)\sin 2\beta$.

Solving $f_{13}=0$ we have $h=3$ because $\sin 2\beta \neq 0$. f_{12}, f_{14} become

$$\begin{aligned} f_{12} &= (3-h)\cos 2\beta, \\ f_{14} &= -\frac{1}{2}\big[h-2+(h-4)\cos 2\beta\big]. \end{aligned}$$

System $f_{12}=f_{14}=0$ does not give solutions for β.

Subcase 2.1.3.2: Either $\alpha = \dfrac{\pi}{2}$ or $\alpha = \dfrac{3\pi}{2}$ and $R=\sqrt{2}$. Substituting α and $R=\sqrt{2}$ in r we obtain $r=\sqrt{2h-4}$. Replacing r in f_{13} we have $f_{13} = -\dfrac{\sqrt{2}}{2}(h-3)\sin 2\beta$. Solving $f_{13}=0$ we have $h=3$ because $\sin 2\beta \neq 0$.

$$\begin{aligned} f_{12} &= -\cos 2\beta, \\ f_{14} &= \frac{1}{2}\big[h-2+(h-4)\cos 2\beta\big]. \end{aligned}$$

System $f_{12}=f_{14}=0$ does not provide solutions for β.

Subcase 2.2: $(2-R^2)\sin 2\alpha \neq 0$ and $R^2 \sin 2\beta = 0$.

Subcase 2.2.1: $R=0$, $\sin 2\beta \neq 0$ and $(R^2-2)\sin 2\alpha \neq 0$. Then $r=\sqrt{2h}$ and

$$\begin{aligned} f_{12} &= \frac{h}{2}(2+\cos 2\alpha), \\ f_{14} &= \frac{h}{2}(2+\cos 2\beta). \end{aligned}$$

The system $f_{12}=f_{14}=0$ does not have solutions. The averaging theory does give information.

Subcase 2.2.2: $R\neq 0$, $\sin 2\beta = 0$ and $(R^2-2)\sin 2\alpha \neq 0$. Because of the periodicity of the sinus and the cosinus we study $\beta = \dfrac{k\pi}{2}$ for $k=0$ and $k=2$ together and for $k=1$ and $k=3$ together.

Subcase 2.2.2.1: either $k=0$ or $k=2$, i.e. either $\beta=0$ or $\beta=\pi$. So we have $r=\sqrt{2h-R^2}$ and

$$\begin{aligned} f_{12} &= \frac{1}{8}\Big[8h-6(1+h)R^2+3R^4+2\big(R^4-2h(R^2-1)\big)\cos 2\alpha\Big], \\ f_{13} &= 0, \\ f_{14} &= \frac{3}{2}(h-R^2). \end{aligned}$$

Solving $f_{14}=0$ we obtain $R=\sqrt{h}$ then $\rho=0$. Replacing R in f_{12} and solving $f_{12}=0$ we have $\alpha = \pm \arccos \dfrac{2-3h}{2(h-2)}$.

If

$$h>0 \quad \text{and} \quad \left|\frac{2-3h}{2(h-2)}\right| < 1, \tag{21}$$

which is equivalent to $0<h<\dfrac{6}{5}$.

System (13) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = 0$ given by

$$
\begin{aligned}
S^*_{1,2} &= \left(\sqrt{h}, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}, \sqrt{h}, 0\right),\\
S^*_{3,4} &= \left(\sqrt{h}, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}, \sqrt{h}, \pi\right),
\end{aligned} \tag{22}
$$

which reduce to two solutions if $h > 0$ and $\left|\dfrac{2-3h}{2(h-2)}\right| = 1$.

The Jacobian read $J_{f_1(S^*)} = -\dfrac{3}{64}h^4(h+2)(5h-6)$.

If (21) hold, then $J_{f_1(S^*)} \neq 0$ and the four solutions (22) of system (13) can provide four periodic solutions of differential system (11). However going back to the differential system (2) and by the Proposition 2 we obtain only two different periodic orbits S^*_1 and S^*_3 which they formed with the two ones find in the subcase 1.3.1, four periodic orbits of the ten mentioned in (a) of Theorem 1.

Subcase 2.2.2.2: either $k = 1$ or $k = 3$. i.e. either $\beta = \dfrac{\pi}{2}$ or $\beta = \dfrac{3\pi}{2}$. Then $r = \sqrt{2h - R^2}$ and

$$
\begin{aligned}
f_{12} &= \frac{1}{8}\Big[8h - 2(1+h)R^2 + R^4 + 4(h - R^2)\cos 2\alpha\Big],\\
f_{13} &= 0,\\
f_{14} &= \frac{1}{2}(h - R^2).
\end{aligned}
$$

Solving $f_{14} = 0$ we get $R = \sqrt{h}$. Replacing R in f_{12} we have $f_{12} = \dfrac{h}{8}(6-h) =$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 2.2.2.3.1: $R = 0$, $\beta = 0$ and $(R^2 - 2)\sin 2\alpha \neq 0$. We have $r = \sqrt{2h}$ and

$$
\begin{aligned}
f_{12} &= \frac{h}{2}(2 + \cos 2\alpha),\\
f_{14} &= \frac{3h}{2}.
\end{aligned}
$$

We have $f_{14} =$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 2.2.2.3.2: $R = 0$, $\beta = \dfrac{\pi}{2}$ and $(R^2 - 2)\sin 2\alpha \neq 0$. So $r = \sqrt{2h}$ and

$$
\begin{aligned}
f_{12} &= \frac{h}{2}(2 + \cos 2\alpha),\\
f_{14} &= \frac{h}{2}.
\end{aligned}
$$

As in the anterior subcase $f_{14} =$ constant. The Jacobian will be zero and the averaging theory does not give information.

Case 3: $D = 2\sin 2\alpha - R^2(\sin 2\alpha + \sin 2\beta) = 0$.

Subcase 3.1: $\sin 2\alpha + \sin 2\beta \neq 0$. Then solving $D = 0$ we have

$R = \sqrt{\dfrac{2\sin 2\alpha}{\sin 2\alpha+\sin 2\beta}}$. The new averaged function $f_1 = (f_{11}, f_{12},\ f_{13}, f_{14})$ becomes

$$
\begin{aligned}
f_{11} = & -\frac{r\sin 2\alpha \sin 2\beta}{4(\cos\alpha\sin\alpha+\cos\beta\sin\beta)},\\
f_{12} = & \frac{1}{4(\sin 2\alpha+\sin 2\beta)^2}\Bigg[\sin 2\beta\Big((h-r^2+1)\sin(4\alpha-2\beta)+2(h-r^2)\\
& \sin 2(\alpha-\beta)+\sin 2\alpha\big(-2\cos 2\beta-3h+3r^2+3\big)\Big)+\sin 2\alpha\Big(-2\sin\alpha\\
& \cos(\alpha+2\beta)+(h-r^2)\sin(4\alpha-2\beta)+(2h-2r^2-3)\sin 2(\alpha-\beta)\Big)\\
& +\big(-3h+3r^2+1\big)\sin^2 2\beta\Bigg],\\
f_{13} = & \sqrt{\frac{\sin\alpha\cos\alpha}{16(\sin 2\alpha+\sin 2\beta)}}\Bigg[(2-r^2)\sin 2(\alpha-\beta)\Big(r^2-2h+\frac{2\sin 2\alpha}{\sin 2\alpha+\sin 2\beta}\Big)\\
& +r^2\sin 2\beta\Big(2-2h+r^2+\frac{2\sin 2\alpha}{\sin 2\alpha+\sin 2\beta}\Big)\Bigg],\\
f_{14} = & \frac{1}{8}\Bigg[\cos 2(\alpha-\beta)\big(r^2-2-\frac{2\sin 2\alpha}{\sin 2\alpha+\sin 2\beta}\big)\big(r^2-2h+\frac{2\sin 2\alpha}{\sin 2\alpha+\sin 2\beta}\big)\\
& +\cos 2\alpha\big(2+r^2-\frac{2\sin 2\alpha}{\sin 2\alpha+\sin 2\beta}\big)\big(r^2-2h+\frac{2\sin 2\alpha}{\sin 2\alpha+\sin 2\beta}\big)\\
& +(2+\cos 2\beta)\big(r^2-\frac{2\sin 2\alpha}{\sin 2\alpha+\sin 2\beta}\big)\big(2-2h+r^2+\frac{2\sin 2\alpha}{\sin 2\alpha+\sin 2\beta}\big)\Bigg].
\end{aligned}
$$

Solving $f_{11} = 0$ we obtain three main subcases: $r = 0$ (studied in case 1), $\alpha = \dfrac{k\pi}{2}$ with $k \in \mathbb{Z}$ and $\beta = \dfrac{m\pi}{2}$ with $m \in \mathbb{Z}$.

Subcase 3.1.1: $\alpha = \dfrac{k\pi}{2}$ with $k \in \mathbb{Z}$. Due to the periodicity of the cosinus and sinus we study the subcases $k = 0$ and $k = 2$, and the subcases $k = 1$ and $k = 3$ together.

Subcase 3.1.1.1: either $k = 0$ or $k = 2$. i.e. either $\alpha = 0$ or $\alpha = \pi$. Substituting α in f_{12} we have $f_{12} = -\dfrac{3}{2}(h - r^2)$. Solving $f_{12} = 0$ we obtain $r = \sqrt{h}$. Replacing r in f_{14} and solving $f_{14} = 0$ we get $\beta = \pm\dfrac{1}{2}\arccos\dfrac{2-3h}{2(h-2)}$. Then $R = 0$ and $\rho = \sqrt{h}$.

Supposing

$$h > 0 \quad \text{and} \quad \Big|\frac{2-3h}{2(h-2)}\Big| < 1. \tag{23}$$

which is equivalent to $0 < h < \dfrac{6}{5}$.

System (13) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{h}$ given by

$$
\begin{aligned}
S^*_{1,2} &= \left(\sqrt{h}, 0, 0, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}\right), \\
S^*_{3,4} &= \left(\sqrt{h}, \pi, 0, \pm\frac{1}{2}\arccos\frac{2-3h}{2(h-2)}\right),
\end{aligned} \tag{24}
$$

which reduce to two solutions if $h > 0$ and $\left|\frac{2-3h}{2(h-2)}\right| = 1$.

The Jacobian evaluated on these solutions $J_{f_1(S^*)} = -\frac{3}{64}h^4(h+2)(5h-6)$.

When (23) hold, $J_{f_1(S^*)} \neq 0$ and the four solutions (24) of system (13) provide only two periodic solutions of differential system (11) because when $R = 0$ the two solutions of β provide the same initial conditions in (7). Then when $0 < h < \frac{6}{5}$ we obtained two new periodic orbits which they formed with the four ones find in the subcases 1.3.1 and 2.2.2.1 six of the ten periodic orbits mentioned in (a) of Theorem (1).

Subcase 3.1.1.2: either $k = 1$ or $k = 3$. i.e. either $\alpha = \frac{\pi}{2}$ or $\alpha = \frac{3\pi}{2}$. Substituting α in f_{12} we obtain $f_{12} = \frac{1}{2}(r^2 - h)$. Solving $f_{12} = 0$ we get $r = \sqrt{h}$. Replacing r in f_{14} we have $f_{14} = -\frac{h}{8}(h-6) =$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 3.1.2: $\beta = \frac{m\pi}{2}$ with $m \in \mathbb{Z}$. Due to the periodicity of the cosinus and sinus we study the subcases $m = 0$ and $m = 2$, and the subcases $m = 1$ and $m = 3$ together.

Subcase 3.1.2.1: either $m = 0$ or $m = 2$. i.e. either $\beta = 0$ or $\beta = \pi$. Substituting β in f_{13} we get $f_{13} = -\frac{\sqrt{2}}{8}(r^2 - 2)(r^2 + 2 - 2h)\sin 2\alpha$. Solving $f_{13} = 0$ under the hypothesis $\sin 2\alpha + \sin 2\beta \neq 0$ we get two subcases $r = \sqrt{2}$ and $r = \sqrt{2h-2}$.

Subcase 3.1.2.1.1: $r = \sqrt{2}$. Replacing r in f_{12} and solving $f_{12} = 0$ we obtain $\alpha = \pm\arccos\frac{4-h}{h-2}$. Then we get $R = \sqrt{2}$ and $\rho = \sqrt{2h-4}$.

With the condition

$$
h > 0, \quad h - 2 > 0 \quad \text{and} \quad \left|\frac{4-h}{h-2}\right| < 1. \tag{25}
$$

which is equivalent to $h > 3$, system (13) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho = \sqrt{2h-4}$ given by

$$
\begin{aligned}
S^*_{1,2} &= \left(\sqrt{2}, \pm\arccos\frac{4-h}{h-2}, \sqrt{2}, 0\right), \\
S^*_{3,4} &= \left(\sqrt{2}, \pm\arccos\frac{4-h}{h-2}, \sqrt{2}, \pi\right),
\end{aligned} \tag{26}
$$

which reduce to two solutions if $h > 0$, $h - 2 > 0$ and $\left|\frac{4-h}{h-2}\right| = 1$.

The Jacobian evaluated on these solutions is $J_{f_1(S^*)} = 8(h-3)^3$.

If (25) hold, then $J_{f_1(S^*)} \neq 0$ and the four solutions (26) of system (13) can provide four periodic solutions of differential system (11). But going back to the differential system (2) and using the Proposition 2 we have only two different periodic orbits S_1^* and S_3^*.

Subcase 3.1.2.1.2: $r = \sqrt{2h-2}$. Substituting r in f_{14} we have $f_{14} = \dfrac{3}{2}(h-2)=$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 3.1.2.2: either $m = 1$ or $m = 3$, i.e. $\beta = \dfrac{\pi}{2}$ and $\beta = \dfrac{3\pi}{2}$. Replacing β in f_{13} we have $f_{13} = \dfrac{\sqrt{2}}{8}(r^2-2)(r^2-2h+2)\sin 2\alpha$. Solving $f_{13} = 0$ we get either $r = \sqrt{2}$ or $r = \sqrt{2h-2}$.

Subcase 3.1.2.2.1: $r = \sqrt{2}$. Then

$$
\begin{aligned}
f_{12} = &\ \frac{1}{2}\Big(4 - h - (h-2)\cos 2\alpha\Big),\\
f_{14} = &\ -(h-2)\cos 2\alpha.
\end{aligned}
$$

$f_{12} = f_{14} = 0$ does not have solution. So the averaging theory is not available.

Subcase 3.1.2.2.2: $r = \sqrt{2h-2}$. We have $f_{14} = \dfrac{1}{2}(h-2)=$ constant. The averaging theory does not give results.

Subcase 3.2: $\sin 2\alpha + \sin 2\beta = 0$. Solving $D = 2\sin 2\alpha - R^2(\sin 2\alpha + \sin 2\beta) = 0$ we obtain $\alpha = \dfrac{m\pi}{2}$ with $m \in \mathbb{Z}$ and $\beta = \dfrac{n\pi}{2}$ with $n \in \mathbb{Z}$.

Subcase 3.2.1: For the values of $(\alpha, \beta) = (0,0), (\pi,\pi), (0,\pi), (\pi,0)$ we have

$$
\begin{aligned}
f_{12} = &\ \frac{1}{8}(2h - 2r^2 - R^2)(5R^2 - 6),\\
f_{14} = &\ \frac{1}{8}(r^2 - R^2)(5R^2 + 5r^2 - 10h + 6).
\end{aligned}
$$

Solving $f_{12} = f_{14} = 0$ we obtain four pairs of (r, R) :,

When $r_1 = \sqrt{\dfrac{6}{5}}$, $R_1 = \sqrt{\dfrac{2(5h-6)}{5}}$ then $\rho_1 = \sqrt{\dfrac{6}{5}}$.

Assuming that

$$
h > 0 \quad \text{and} \quad 5h - 6 > 0, \tag{27}
$$

which is equivalent to $h > \frac{6}{5}$, system (13) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_1 = \sqrt{\frac{6}{5}}$ given by

$$
\begin{aligned}
S_1^* &= \left(\sqrt{\frac{6}{5}}, 0, \sqrt{\frac{2(5h-6)}{5}}, 0\right),\\
S_2^* &= \left(\sqrt{\frac{6}{5}}, 0, \sqrt{\frac{2(5h-6)}{5}}, \pi\right),\\
S_3^* &= \left(\sqrt{\frac{6}{5}}, \pi, \sqrt{\frac{2(5h-6)}{5}}, 0\right),\\
S_4^* &= \left(\sqrt{\frac{6}{5}}, \pi, \sqrt{\frac{2(5h-6)}{5}}, \pi\right),
\end{aligned} \tag{28}
$$

The Jacobian evaluated at these solutions is

$$J_{f_1(S^*)} = -\frac{36}{15625}(5h-9)^2(5h-6)(10h-27).$$

Under the assumption (27), we get $J_{f_1(S^*)} \neq 0$. Therefore the four zeros (28) of system (13) provide four periodic solutions of differential system (11).

If $r_2 = \sqrt{\frac{2h}{3}}$, $R_2 = \sqrt{\frac{2h}{3}}$ we have $\rho_2 = \sqrt{\frac{2h}{3}}$.

If

$$h > 0, \tag{29}$$

system (13) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_2 = \sqrt{\frac{2h}{3}}$ given by

$$
\begin{aligned}
S{*}_1 &= \left(\sqrt{\frac{2h}{3}}, 0, \sqrt{\frac{2h}{3}}, 0\right),\\
S_2^* &= \left(\sqrt{\frac{2h}{3}}, 0, \sqrt{\frac{2h}{3}}, \pi\right),\\
S_3^* &= \left(\sqrt{\frac{2h}{3}}, \pi, \sqrt{\frac{2h}{3}}, 0\right),\\
S_4^* &= \left(\sqrt{\frac{2h}{3}}, \pi, \sqrt{\frac{2h}{3}}, \pi\right),
\end{aligned} \tag{30}
$$

The Jacobian evaluated at these solutions is

$$J_{f_1(S^*)} = -\frac{h^4}{729}(h-3)^2(5h-9)^2.$$

If $(5h-9)(h-3) \neq 0$ and (29) hold. The set of conditions on the energy level h is not empty because for $h = 1$ it satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (30) of system (13) provide four periodic solutions of differential system (11). Then for $h > 0$, $h \neq 3$, and $h \neq 9/5$ we obtained four periodic orbits which they formed with the six ones obtained from the subcases 1.3.1, 2.2.2.1, 3.1.1.1 the ten periodic orbits mentioned in (a) of Theorem 1 .

When $r_3 = \sqrt{\frac{6}{5}}$, $R_3 = \sqrt{\frac{6}{5}}$ we obtain $\rho_3 = \sqrt{\frac{2(5h-6)}{5}}$.

Assuming that

$$h > 0 \quad \text{and} \quad 5h - 6 > 0, \tag{31}$$

which is equivalent to $h > \dfrac{6}{5}$, then system (13) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_3 = \sqrt{\dfrac{2(5h-6)}{5}}$ given by

$$\begin{aligned} S_1^* &= \left(\sqrt{\frac{6}{5}}, 0, \sqrt{\frac{6}{5}}, 0\right), \\ S_2^* &= \left(\sqrt{\frac{6}{5}}, 0, \sqrt{\frac{6}{5}}, \pi\right), \\ S_3^* &= \left(\sqrt{\frac{6}{5}}, \pi, \sqrt{\frac{6}{5}}, 0\right), \\ S_4^* &= \left(\sqrt{\frac{6}{5}}, \pi, \sqrt{\frac{6}{5}}, \pi\right), \end{aligned} \tag{32}$$

The Jacobian evaluated on these solutions

$$J_{f_1(S^*)} = -\frac{36}{15625}(5h-9)^2(5h-6)(10h-27).$$

If $(5h-9)(10h-27) \neq 0$ and (31) hold. The set of conditions on the energy level h is not empty because for $h = 3/2$ it satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (32) of system (13) provide four periodic solutions of differential system (11).

Finally if $r_4 = \sqrt{\dfrac{2(5h-6)}{5}}$, $R_4 = \sqrt{\dfrac{6}{5}}$ we get $\rho_4 = \sqrt{\dfrac{6}{5}}$.

Assuming that

$$h > 0 \quad \text{and} \quad 5h - 6 > 0, \tag{33}$$

which is equivalent to $h > \dfrac{6}{5}$, then system (13) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_4 = \sqrt{\dfrac{6}{5}}$ given by

$$\begin{aligned} S_1^* &= \left(\sqrt{\frac{2(5h-6)}{5}}, 0, \sqrt{\frac{6}{5}}, 0\right), \\ S_2^* &= \left(\sqrt{\frac{2(5h-6)}{5}}, 0, \sqrt{\frac{6}{5}}, \pi\right), \\ S_3^* &= \left(\sqrt{\frac{2(5h-6)}{5}}, \pi, \sqrt{\frac{6}{5}}, 0\right), \\ S_4^* &= \left(\sqrt{\frac{2(5h-6)}{5}}, \pi, \sqrt{\frac{6}{5}}, \pi\right), \end{aligned} \tag{34}$$

The Jacobian evaluated at these solutions is

$$J_{f_1(S^*)} = -\frac{36}{15625}(5h-9)^2(5h-6)(10h-27).$$

If $(5h-9)(10h-27)\neq 0$ and (33) hold. The set of conditions on the energy level h is not empty because for $h=3/2$ it satisfy it. Then $J_{f_1(S^*)}\neq 0$ and the four solutions (34) of system (13) provide four periodic solutions of differential system (11).

Subcase 3.2.2: If $(\alpha,\beta)=(0,\frac{\pi}{2}),(0,\frac{3\pi}{2}),(\pi,\frac{\pi}{2}),(\pi,\frac{3\pi}{2})$ we get

$$\begin{aligned} f_{12} = &\ \frac{1}{8}(2h-2r^2-R^2)(R^2-6),\\ f_{14} = &\ \frac{1}{8}\Big(r^4+6r^2-2hr^2+2R^2(1+h)-R^4-8h\Big). \end{aligned}$$

Solving $f_{12}=f_{14}=0$ we obtain four pairs of (r,R) :,

When $r_1=\sqrt{\dfrac{1+h-\sqrt{h^2-10h+1}}{3}}$, $R_1=\sqrt{\dfrac{2(2h-1+\sqrt{h^2-10h+1})}{3}}$ we get $\rho_1=r_1$.

If

$$\begin{aligned} &h>0,\quad 1+h-\sqrt{h^2-10h+1}>0,\quad h^2-10h+1>0,\\ &\text{and}\quad 2h-1+\sqrt{h^2-10h+1}>0, \end{aligned} \tag{35}$$

which is equivalent to $h>5+2\sqrt{6}$, then system (13) has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho_1=\sqrt{\dfrac{1+h-\sqrt{h^2-10h+1}}{3}}$ given by

$$\begin{aligned} S_1^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},0,\sqrt{\frac{2(2h-1+\sqrt{h^2-10h+1})}{3}},\frac{\pi}{2}\right),\\ S_2^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},0,\sqrt{\frac{2(2h-1+\sqrt{h^2-10h+1})}{3}},\frac{3\pi}{2}\right),\\ S_3^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},\pi,\sqrt{\frac{2(2h-1+\sqrt{h^2-10h+1})}{3}},\frac{\pi}{2}\right),\\ S_4^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},\pi,\sqrt{\frac{2(2h-1+\sqrt{h^2-10h+1})}{3}},\frac{3\pi}{2}\right), \end{aligned} \tag{36}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = &\ \frac{\sqrt{h^2-10h+1}}{34992}(1+h-\sqrt{h^2-10h+1})^2(5-h+\sqrt{h^2-10h+1})\\ &(2h-10+\sqrt{h^2-10h+1})(2h-1+\sqrt{h^2-10h+1})\\ &(h^2+17h-14+(14-h)\sqrt{h^2-10h+1}). \end{aligned}$$

The $J_{f_1(S^*)}$ does not vanish for $h\neq 5\pm 2\sqrt{6}$. Therefore with the condition (35), the four solutions (36) of system (13) can provide four periodic solutions of differential system (11). Although, going back to the differential system (2) and using the Proposition 2 we obtain only two different periodic orbits S_1^* and S_3^*.

If $r_2=\sqrt{\dfrac{1+h+\sqrt{h^2-10h+1}}{3}}$, $R_2=\sqrt{\dfrac{2(2h-1-\sqrt{h^2-10h+1})}{3}}$ then we obtain $\rho_2=r_2$.

With the condition

$$\begin{aligned} &h>0,\quad 1+h+\sqrt{h^2-10h+1}>0,\quad h^2-10h+1>0,\\ &\text{and}\quad 2h-1-\sqrt{h^2-10h+1}>0, \end{aligned} \tag{37}$$

which is equivalent to $h > 5+2\sqrt{6}$, system (13) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_2 = \sqrt{\dfrac{1+h+\sqrt{h^2-10h+1}}{3}}$ given by

$$
\begin{aligned}
S_1^* &= \left(\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, 0, \sqrt{\frac{2(2h-1-\sqrt{h^2-10h+1})}{3}}, \frac{\pi}{2}\right),\\
S_2^* &= \left(\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, 0, \sqrt{\frac{2(2h-1-\sqrt{h^2-10h+1})}{3}}, \frac{3\pi}{2}\right),\\
S_3^* &= \left(\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, \pi, \sqrt{\frac{2(2h-1-\sqrt{h^2-10h+1})}{3}}, \frac{\pi}{2}\right),\\
S_4^* &= \left(\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, \pi, \sqrt{\frac{2(2h-1-\sqrt{h^2-10h+1})}{3}}, \frac{3\pi}{2}\right),
\end{aligned}
\tag{38}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(S^*)} = \ & -\frac{\sqrt{h^2-10h+1}}{34992}(1+h+\sqrt{h^2-10h+1})^2(h-5+\sqrt{h^2-10h+1})\\
& (10-2h+\sqrt{h^2-10h+1})(2h-1-\sqrt{h^2-10h+1})\\
& (h^2+17h-14+(h-14)\sqrt{h^2-10h+1}).
\end{aligned}
$$

For $h \neq 5 \pm 2\sqrt{6}$, $J_{f_1(S^*)}$ does not vanish. Then with the condition (37), the four solutions (38) of system (13) can provide four periodic solutions of differential system (11). Nevertheless, going back to the differential system (2) and using the Proposition 2 we obtain only two different periodic orbits S_1^* and S_3^*.

When $r_3 = \sqrt{h-3-\sqrt{h^2-10h+33}}$ and $R_2 = \sqrt{6}$, then we get $\rho_3 = \sqrt{h-3+\sqrt{h^2-10h+33}}$.

Assuming

$$
\begin{aligned}
& h>0, \quad h-3-\sqrt{h^2-10h+33}>0, \quad h^2-10h+33>0,\\
& \text{and} \quad h-3-\sqrt{h^2-10h+33}>0,
\end{aligned}
\tag{39}
$$

which is equivalent to $h > 6$. Then system (13) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_3 = \sqrt{h-3+\sqrt{h^2-10h+33}}$ given by

$$
\begin{aligned}
S_1^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, 0, \sqrt{6}, \frac{\pi}{2}\right),\\
S_2^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, 0, \sqrt{6}, \frac{3\pi}{2}\right),\\
S_3^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \pi, \sqrt{6}, \frac{\pi}{2}\right),\\
S_4^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \pi, \sqrt{6}, \frac{3\pi}{2}\right),
\end{aligned}
\tag{40}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(S^*)} = \ & \frac{3}{2}(h-6)(h^2-10h+33)(3-h+\sqrt{h^2-10h+33})\\
& (h-3+\sqrt{h^2-10h+33}).
\end{aligned}
$$

For $h \neq 6$ the Jacobian does not vanish. So with the condition (39) we get $J_{f_1(S^*)} \neq 0$, therefore the four solutions (40) of system (13) can provide four periodic solutions of differential system (11). But, going back to the differential system (2) and using the Proposition 2 we obtain only two different periodic orbits S_1^* and S_3^*.

If $r_4 = \sqrt{h-3+\sqrt{h^2-10h+33}}$ and $R_4 = \sqrt{6}$, we have $\rho_4 = \sqrt{h-3-\sqrt{h^2-10h+33}}$.

Supposing

$$\begin{aligned} &h>0, \quad h-3-\sqrt{h^2-10h+33}>0, \quad h^2-10h+33>0, \\ &\text{and} \quad h-3+\sqrt{h^2-10h+33}>0, \end{aligned} \tag{41}$$

which is equivalent to $h > 6$. Then system (13) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_4 = \sqrt{h-3-\sqrt{h^2-10h+33}}$ given by

$$\begin{aligned} S_1^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, 0, \sqrt{6}, \frac{\pi}{2}\right), \\ S_2^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, 0, \sqrt{6}, \frac{3\pi}{2}\right), \\ S_3^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \pi, \sqrt{6}, \frac{\pi}{2}\right), \\ S_4^* &= \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \pi, \sqrt{6}, \frac{3\pi}{2}\right), \end{aligned} \tag{42}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = \; & \frac{3}{2}(h-6)(h^2-10h+33)(3-h+\sqrt{h^2-10h+33}) \\ & (h-3+\sqrt{h^2-10h+33}). \end{aligned}$$

If $h \neq 6$ we have $J_{f_1(S^*)} \neq 0$. The condition (41) guarantees the fact that $J_{f_1(S^*)} \neq 0$. Moreover the four solutions (42) of system (13) can provide four periodic solutions of differential system (11). However, going back to the differential system (2) and using the Proposition 2 we obtain only two different periodic orbits S_1^* and S_3^*.

Subcase 3.2.3: If $(\alpha, \beta) = (\frac{\pi}{2}, 0), (\frac{3\pi}{2}, 0), (\frac{\pi}{2}, \pi), (\frac{3\pi}{2}, \pi)$ then f_{12}, f_{14} become

$$\begin{aligned} f_{12} &= -\frac{1}{8}\Big(R^4 - 2R^2(h-1-r^2) - 4r^2 + 4h\Big), \\ f_{14} &= \frac{1}{8}(r^2-R^2)(R^2+r^2-2h+6). \end{aligned}$$

Solving $f_{12} = f_{14} = 0$ we obtain four pairs of solutions (r, R),

$r_1 = R_1 = \sqrt{\dfrac{1+h-\sqrt{h^2-10h+1}}{3}}$ and $\rho_1 = \sqrt{\dfrac{2(2h-1+\sqrt{h^2-10h+1})}{3}}$.

Whenever

$$\begin{aligned} &h>0, \quad 1+h-\sqrt{h^2-10h+1}>0, \quad h^2-10h+1>0 \\ &\text{and} \quad 2h-1+\sqrt{h^2-10h+1}>0, \end{aligned} \tag{43}$$

which is equivalent to $h>5+2\sqrt{6}$, then system (13) has four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho_1=\sqrt{\dfrac{2(2h-1+\sqrt{h^2-10h+1})}{3}}$ given by

$$
\begin{aligned}
S_1^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},\frac{\pi}{2},\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},0\right),\\
S_2^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},\frac{3\pi}{2},\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},0\right),\\
S_3^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},\frac{\pi}{2},\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},\pi\right),\\
S_4^* &= \left(\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},\frac{3\pi}{2},\sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}},\pi\right),
\end{aligned}
\tag{44}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(S^*)} = & -\frac{(1+h-\sqrt{h^2-10h+1})^2}{17496}(2h-10+\sqrt{h^2-10h+1})(56-596h\\
&+413h^2-7h^3+7h^4-h^5-56\sqrt{h^2-10h+1}+316h\sqrt{h^2-10h+1}\\
&+9h^2\sqrt{h^2-10h+1}-2h^3\sqrt{h^2-10h+1}+h^4\sqrt{h^2-10h+1}).
\end{aligned}
$$

The $J_{f_1(S^*)}$ does not vanish for $h\neq 5\pm 2\sqrt{6}$. Therefore with the condition (43), the four zeros (44) of system (13) can provide four periodic solutions of differential system (11). But, going back to the differential system (2) and using the Proposition 2 we obtain only two different periodic orbits S_1^* and S_3^*.

$r_2=R_2=\sqrt{\dfrac{1+h+\sqrt{h^2-10h+1}}{3}}$ then $\rho_2=\sqrt{\dfrac{2(2h-1-\sqrt{h^2-10h+1})}{3}}$.

With the condition

$$
\begin{aligned}
&h>0,\quad 1+h+\sqrt{h^2-10h+1}>0,\quad h^2-10h+1>0\\
&\text{and}\quad 2h-1-\sqrt{h^2-10h+1}>0,
\end{aligned}
\tag{45}
$$

which is equivalent to $h>5+2\sqrt{6}$, system (13) provide four zeros $S^*=(r^*,\alpha^*,R^*,\beta^*)$ with $\rho_2=\sqrt{\dfrac{2(2h-1-\sqrt{h^2-10h+1})}{3}}$ given by

$$
\begin{aligned}
S_1^* &= \left(\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}},\frac{\pi}{2},\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}},0\right),\\
S_2^* &= \left(\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}},\frac{3\pi}{2},\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}},0\right),\\
S_3^* &= \left(\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}},\frac{\pi}{2},\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}},\pi\right),\\
S_4^* &= \left(\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}},\frac{3\pi}{2},\sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}},\pi\right),
\end{aligned}
\tag{46}
$$

The Jacobian evaluated at theses solutions is

$$
\begin{aligned}
J_{f_1(S^*)} = & -\frac{(1+h+\sqrt{h^2-10h+1})^2}{17496}(2h-10-\sqrt{h^2-10h+1})(56-596h\\
&+413h^2-7h^3+7h^4-h^5+56\sqrt{h^2-10h+1}-316h\sqrt{h^2-10h+1}\\
&-9h^2\sqrt{h^2-10h+1}+2h^3\sqrt{h^2-10h+1}-h^4\sqrt{h^2-10h+1}).
\end{aligned}
$$

For $h \neq 5 \pm 2\sqrt{6}$, $J_{f_1(S^*)}$ does not vanish. Then with the condition (45), the four solutions (46) of system (13) can provide four periodic solutions of differential system (11). Although, going back to the differential system (2) and using the Proposition 2 we obtain only two different periodic orbits S_1^* and S_3^*.

When $r_3 = \sqrt{h-3+\sqrt{h^2-10h+33}}$ and $R_3 = \sqrt{h-3-\sqrt{h^2-10h+33}}$ we have $\rho_3 = \sqrt{6}$.

Assuming

$$(47)\quad \begin{array}{l} h>0, \quad h-3+\sqrt{h^2-10h+33}>0, \quad h-3-\sqrt{h^2-10h+33}>0 \\ \text{and} \quad h^2-10h+33>0, \end{array}$$

which is equivalent to $h > 6$. Therefore system (13) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_3 = \sqrt{6}$ given by

$$(48)\quad \begin{array}{rl} S_1^* = & \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \dfrac{\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, 0\right), \\ S_2^* = & \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \dfrac{3\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, 0\right), \\ S_3^* = & \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \dfrac{\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, \pi\right), \\ S_4^* = & \left(\sqrt{h-3+\sqrt{h^2-10h+33}}, \dfrac{3\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, \pi\right), \end{array}$$

The Jacobian is

$$\begin{array}{rl} J_{f_1(S^*)} = & -\dfrac{3}{2}(h-6)(h^2-10h+33)(h-3-\sqrt{h^2-10h+33}) \\ & (h-3+\sqrt{h^2-10h+33}). \end{array}$$

For $h \neq 6$ we have $J_{f_1(S^*)} \neq 0$. With the condition (47), the four solutions (48) of system (13) can provide four periodic solutions of differential system (11). Nevertheless, going back to the differential system (2) and using the Proposition 2 we obtain only two different periodic orbits S_1^* and S_3^*.

If $r_4 = \sqrt{h-3-\sqrt{h^2-10h+33}}$ and $R_4 = \sqrt{h-3+\sqrt{h^2-10h+33}}$. We get $\rho_4 = \sqrt{6}$.

Assuming

$$(49)\quad \begin{array}{l} h>0, \quad h-3-\sqrt{h^2-10h+33}>0, \quad h-3+\sqrt{h^2-10h+33}>0 \\ \text{and} \quad h^2-10h+33>0, \end{array}$$

which is equivalent to $h > 6$ then system (13) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_4 = \sqrt{6}$ given by

$$
\begin{aligned}
S_1^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, 0\right),\\
S_2^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{3\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, 0\right),\\
S_3^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, \pi\right),\\
S_4^* &= \left(\sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{3\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, \pi\right),
\end{aligned} \tag{50}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(S^*)} = &\ -\frac{3}{2}(h-6)(h^2-10h+33)(3-h+\sqrt{h^2-10h+33})\\
&(h-3-\sqrt{h^2-10h+33}).
\end{aligned}
$$

If $h \neq 6$ we have $J_{f_1(S^*)} \neq 0$. Then with the condition (49), the four zeros (50) of system (13) can provide four periodic solutions of differential system (11). However, going back to the differential system (2) and using the Proposition 2 we obtain only two different periodic orbits S_1^* and S_3^*.

Subcase 3.2.4: If $(\alpha, \beta) = (\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{3\pi}{2}), (\frac{3\pi}{2}, \frac{\pi}{2}), (\frac{3\pi}{2}, \frac{3\pi}{2})$ then f_{12}, f_{14} become

$$
\begin{aligned}
f_{12} = &\ -\frac{1}{8}\Big(R^4 - 2R^2(h+3-r^2) - 4r^2 + 4h\Big),\\
f_{14} = &\ -\frac{1}{8}\Big(R^4 - 2R^2(h-3) - r^4 + 2r^2(1+h) - 8h\Big).
\end{aligned}
$$

Solving $f_{12} = f_{14} = 0$ we have four pairs of solutions (r, R),

If $r_1 = \sqrt{\frac{2}{3}(2h-1+\sqrt{h^2-10h+1})}$ and $R_1 = \sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}$, we get $\rho_1 = \sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}$.

Whenever

$$
\begin{aligned}
&h > 0, \quad 2h-1+\sqrt{h^2-10h+1} > 0, \quad h^2-10h+1 > 0\\
&\text{and} \quad 1+h-\sqrt{h^2-10h+1} > 0,
\end{aligned} \tag{51}
$$

which is equivalent to $h > 5 + 2\sqrt{6}$, then system (13) has four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_1 = \sqrt{\dfrac{1+h-\sqrt{h^2-10h+1}}{3}}$ given by

(52)
$$\begin{aligned}
S_1^* &= \left(\sqrt{\frac{2}{3}(2h-1+\sqrt{h^2-10h+1})}, \frac{\pi}{2}, \sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, \frac{\pi}{2}\right),\\
S_2^* &= \left(\sqrt{\frac{2}{3}(2h-1+\sqrt{h^2-10h+1})}, \frac{\pi}{2}, \sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, \frac{3\pi}{2}\right),\\
S_3^* &= \left(\sqrt{\frac{2}{3}(2h-1+\sqrt{h^2-10h+1})}, \frac{3\pi}{2}, \sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, \frac{\pi}{2}\right),\\
S_4^* &= \left(\sqrt{\frac{2}{3}(2h-1+\sqrt{h^2-10h+1})}, \frac{3\pi}{2}, \sqrt{\frac{1+h-\sqrt{h^2-10h+1}}{3}}, \frac{3\pi}{2}\right),
\end{aligned}$$

The Jacobian evaluated on these zeros

$$\begin{aligned}
J_{f_1(S^*)} = \;& \frac{\sqrt{h^2-10h+1}}{34992}(1+h-\sqrt{h^2-10h+1})^2(5-h+\sqrt{h^2-10h+1})\\
& (2h-10+\sqrt{h^2-10h+1})(2h-1+\sqrt{h^2-10h+1})\\
& (h^2+17h-14+(14-h)\sqrt{h^2-10h+1}).
\end{aligned}$$

The $J_{f_1(S^*)}$ does not vanish for $h \neq 5 \pm 2\sqrt{6}$. Therefore with the condition (51), the four zeros (52) of system (13) can provide four periodic solutions of differential system (11). Although, going back to the differential system (2) and using the Proposition 2 we obtain only one periodic orbits S_1^*.

If $r_2 = \sqrt{\frac{2}{3}(2h-1-\sqrt{h^2-10h+1})}$ and $R_2 = \sqrt{\dfrac{1+h+\sqrt{h^2-10h+1}}{3}}$, we get $\rho_2 = \sqrt{\dfrac{1+h+\sqrt{h^2-10h+1}}{3}}$.

With the condition

$$\begin{aligned}
&h > 0, \quad 1+h+\sqrt{h^2-10h+1} > 0, \quad h^2-10h+1 > 0\\
&\text{and} \quad 2h-1-\sqrt{h^2-10h+1} > 0,
\end{aligned} \tag{53}$$

which is equivalent to $h > 5+2\sqrt{6}$, system (13) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_2 = \sqrt{\dfrac{1+h+\sqrt{h^2-10h+1}}{3}}$ given by

(54)
$$\begin{aligned}
S_1^* &= \left(\sqrt{\frac{2}{3}(2h-1-\sqrt{h^2-10h+1})}, \frac{\pi}{2}, \sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, \frac{\pi}{2}\right),\\
S_2^* &= \left(\sqrt{\frac{2}{3}(2h-1-\sqrt{h^2-10h+1})}, \frac{\pi}{2}, \sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, \frac{3\pi}{2}\right),\\
S_3^* &= \left(\sqrt{\frac{2}{3}(2h-1-\sqrt{h^2-10h+1})}, \frac{3\pi}{2}, \sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, \frac{\pi}{2}\right),\\
S_4^* &= \left(\sqrt{\frac{2}{3}(2h-1-\sqrt{h^2-10h+1})}, \frac{3\pi}{2}, \sqrt{\frac{1+h+\sqrt{h^2-10h+1}}{3}}, \frac{3\pi}{2}\right),
\end{aligned}$$

The Jacobian evaluated at theses solutions is

$$\begin{aligned} J_{f_1(S^*)} = & -\frac{\sqrt{h^2-10h+1}}{34992}(2h-1-\sqrt{h^2-10h+1})(h-5+\sqrt{h^2-10h+1}) \\ & (10-2h+\sqrt{h^2-10h+1})(1+h+\sqrt{h^2-10h+1})^2 \\ & (h^2+17h-14+(h-14)\sqrt{h^2-10h+1}). \end{aligned}$$

For $h \neq 5 \pm 2\sqrt{6}$, $J_{f_1(S^*)}$ does not vanish. Then with the condition (53), the four solutions (54) of system (13) can provide four periodic solutions of differential system (11). But, going back to the differential system (2) and using the Proposition 2 we obtain only one periodic orbits S_1^*.

When $r_3 = \sqrt{6}$ and $R_3 = \sqrt{h-3-\sqrt{h^2-10h+33}}$, we have $\rho_3 = \sqrt{h-3+\sqrt{h^2-10h+33}}$.

Assuming

$$\begin{aligned} & h>0, \quad h-3+\sqrt{h^2-10h+33}>0, \quad h-3-\sqrt{h^2-10h+33}>0 \\ & \text{and} \quad h^2-10h+33>0, \end{aligned} \tag{55}$$

which is equivalent to $h > 6$. Therefore system (13) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with $\rho_3 = \sqrt{h-3+\sqrt{h^2-10h+33}}$ given by

$$\begin{aligned} S_1^* = & \left(\sqrt{6}, \frac{\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{\pi}{2}\right), \\ S_2^* = & \left(\sqrt{6}, \frac{\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{3\pi}{2}\right), \\ S_3^* = & \left(\sqrt{6}, \frac{3\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{\pi}{2}\right), \\ S_4^* = & \left(\sqrt{6}, \frac{3\pi}{2}, \sqrt{h-3-\sqrt{h^2-10h+33}}, \frac{3\pi}{2}\right), \end{aligned} \tag{56}$$

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)} = & \frac{3}{2}\sqrt{h^2-10h+33}(h-6)(3-h+\sqrt{h^2-10h+33}) \\ & \left(h^2-10h+33+(h-3)\sqrt{h^2-10h+33}\right). \end{aligned}$$

For $h \neq 6$ we have $J_{f_1(S^*)} \neq 0$. Then with the condition (55), the four solutions (56) of system (13) can provide four periodic solutions of differential system (11). But, going back to the differential system (2) and using the Proposition 2 we obtain only one periodic orbits S_1^*.

If $r_4 = \sqrt{6}$ and $R_4 = \sqrt{h-3+\sqrt{h^2-10h+33}}$, we have $\rho_4 = \sqrt{h-3-\sqrt{h^2-10h+33}}$.

Assuming

$$\begin{aligned} & h>0, \quad h-3+\sqrt{h^2-10h+33}>0, \quad h-3-\sqrt{h^2-10h+33}>0 \\ & \text{and} \quad h^2-10h+33>0, \end{aligned} \tag{57}$$

which is equivalent to $h > 6$. Then system (13) provide four zeros $S^* = (r^*, \alpha^*, R^*, \beta^*)$ with

$\rho_4 = \sqrt{h-3-\sqrt{h^2-10h+33}}$ given by

$$
\begin{aligned}
S_1^* = & \left(\sqrt{6}, \frac{\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{\pi}{2}\right),\\
S_2^* = & \left(\sqrt{6}, \frac{\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{3\pi}{2}\right),\\
S_3^* = & \left(\sqrt{6}, \frac{3\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{\pi}{2}\right),\\
S_4^* = & \left(\sqrt{6}, \frac{3\pi}{2}, \sqrt{h-3+\sqrt{h^2-10h+33}}, \frac{3\pi}{2}\right),
\end{aligned} \tag{58}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(S^*)} = & -\frac{3}{2}\sqrt{h^2-10h+33}(h-6)(3-h-\sqrt{h^2-10h+33})\\
& \left(h^2-10h+33+(h-3)\sqrt{h^2-10h+33}\right).
\end{aligned}
$$

If $h \neq 6$ we have $J_{f_1(S^*)} \neq 0$. With the condition (57), the four zeros (58) of system (13) can provide four periodic solutions of differential system (11). Nevertheless, going back to the differential system (2) and using the Proposition 2 we obtain only one periodic orbits S_1^*.

Proof of Proposition 2. $(r^*, \alpha^*, R^*, \beta^*)$ is a periodic solution of (11) using the averaging theory means that

$$
\begin{aligned}
r(t,\varepsilon) = & \; r^* + O(\varepsilon),\\
\alpha(t,\varepsilon) = & \; \alpha^* + O(\varepsilon),\\
R(t,\varepsilon) = & \; R^* + O(\varepsilon),\\
\beta(0,\varepsilon) = & \; \beta^* + O(\varepsilon).
\end{aligned} \tag{59}
$$

Adding the expression of $\rho = \sqrt{2h - r^{*2} - R^{*2}}$ in system (59) we have

$$
\begin{aligned}
r(t,\varepsilon) = & \; r^* + O(\varepsilon),\\
\rho(t,\varepsilon) = & \; \sqrt{2h - r^{*2} - R^{*2}} + O(\varepsilon),\\
\alpha(t,\varepsilon) = & \; \alpha^* + O(\varepsilon),\\
R(t,\varepsilon) = & \; R^* + O(\varepsilon),\\
\beta(t,\varepsilon) = & \; \beta^* + O(\varepsilon).
\end{aligned} \tag{60}
$$

Instead of θ we reconsider the variable, t, the temps, then (60) becomes

$$
\begin{aligned}
r(t,\varepsilon) = & \; r^* + O(\varepsilon),\\
\theta(t,\varepsilon) = & \; t + O(\varepsilon),\\
\rho(t,\varepsilon) = & \; \sqrt{2h - r^{*2} - R^{*2}} + O(\varepsilon),\\
\alpha(t,\varepsilon) = & \; \alpha^* + O(\varepsilon),\\
R(t,\varepsilon) = & \; R^* + O(\varepsilon),\\
\beta(t,\varepsilon) = & \; \beta^* + O(\varepsilon).
\end{aligned} \tag{61}
$$

Finally, using the change of variables (7), the system (61) becomes

$$\begin{aligned}
x(t,\varepsilon) = &\quad r^*\cos t + O(\varepsilon),\\
y(t,\varepsilon) = &\quad \sqrt{2h - r^{*2} - R^{*2}}\cos(\alpha^* + t) + O(\varepsilon),\\
z(t,\varepsilon) = &\quad R^*\cos(\beta^* + t) + O(\varepsilon),\\
p_x(t,\varepsilon) = &\quad r^*\sin t + O(\varepsilon),\\
p_y(t,\varepsilon) = &\quad \sqrt{2h - r^{*2} - R^{*2}}\sin(\alpha^* + t) + O(\varepsilon),\\
p_z(t,\varepsilon) = &\quad R^*\sin(\beta^* + t) + O(\varepsilon).
\end{aligned}$$

□

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