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About non equivalent completely regular codes with identical intersection array

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Abstract

We obtain several classes of completely regular codes with different parameters, but identical intersection array. Given a prime power q and any two natural numbers a, b, we construct completely transitive codes over different fields with covering radius $\rho = \min\{a, b\}$ and identical intersection array, specifically, we construct one code over $\mathbb{F}_{q^r}$ for each divisor r of a or b. As a corollary, for any prime power q, we show that distance regular bilinear forms graphs can be obtained as coset graphs from several completely regular codes with different parameters.

Keywords: Completely regular codes, coset graphs, distance regular graphs

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1 Introduction

Let $\mathbb{F}_q$ be a finite field of the order q and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. A q-ary linear code C of length n is a k-dimensional subspace of $\mathbb{F}_q^n$. Given any vector $\mathbf{v} \in \mathbb{F}_q^n$, its *distance to the code* C is $d(\mathbf{v}, C) = \min_{\mathbf{x}\in C}\{d(\mathbf{v}, \mathbf{x})\}$, the *minimum distance* of the code is $d = \min_{\mathbf{v}\in C}\{d(\mathbf{v}, C\setminus\{\mathbf{v}\})\}$ and the *covering radius* of the code C is $\rho = \max_{\mathbf{v}\in\mathbb{F}_q^n}\{d(\mathbf{v}, C)\}$. Two vectors $\mathbf{x}$ and $\mathbf{y}$ are *neighbors* if $d(\mathbf{x}, \mathbf{y}) = 1$. We say that C is a $[n, k, d; \rho]_q$-code. Let $D = C + \mathbf{x}$ be a *coset* of C, where $+$ means the component-wise addition in $\mathbb{F}_q$. For a given q-ary code C we define $C(i) = \{\mathbf{x} \in \mathbb{F}_q^n : d(\mathbf{x}, C) = i\}, \quad i = 1, 2, ..., \rho$.

Definition 1.1 [6] A q-ary code C is completely regular, if for all $l \geq 0$ every vector $x \in C(l)$ has the same number c_l of neighbors in $C(l-1)$ and the same number b_l of neighbors in $C(l+1)$. Define $a_l = (q-1)n - b_l - c_l$ and set $c_0 = b_\rho = 0$. Denote by $(b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho)$ the intersection array of C.

Let M be a monomial matrix, i.e. a matrix with exactly one nonzero entry in each row and column. If q is a power of a prime number, then $\mathrm{Aut}(C)$ consist of all monomial $(n \times n)$-matrices M over $\mathbb{F}_q$ such that $\mathbf{c}M \in C$ for all $\mathbf{c} \in C$ and also contains any field automorphism of $\mathbb{F}_q$ which preserves C. The group $\mathrm{Aut}(C)$ acts on the set of cosets of C in the following way: for all $\sigma \in \mathrm{Aut}(C)$ and for every vector $\mathbf{v} \in \mathbb{F}_q^n$ we have $(\mathbf{v} + C)^\sigma = \mathbf{v}^\sigma + C$.

Definition 1.2 [4,10] A linear code C over $\mathbb{F}_q$ with covering radius ρ is completely transitive if $\mathrm{Aut}(C)$ has $\rho + 1$ orbits when acts on the cosets of C.

Since two cosets in the same orbit should have the same weight distribution, it is clear, that any completely transitive code is completely regular.

Completely regular and completely transitive codes are classical subjects in algebraic coding theory, which are closely connected with graph theory, combinatorial designs and algebraic combinatorics. Existence, construction and enumeration of all such codes are open hard problems (see [1,3,5,6] and references there).

In a recent paper [8] we described an explicit construction, based on the Kronecker product of parity check matrices, which provides, for any natural number ρ and for any prime power q, an infinite family of q-ary linear completely regular codes with covering radius ρ. In [9] we presented another class of q-ary linear completely regular codes with the same property, based on lifting of perfect codes. Here, we extend the Kronecker product construction to the case when component codes have different alphabets and connect the resulting completely regular codes with the codes obtained by lifting q-ary perfect codes. This gives several different infinite classes of completely regular codes with different parameters and with identical intersection arrays.

Definition 1.3 For two matrices $A = [a_{r,s}]$ and $B = [b_{i,j}]$ over $\mathbb{F}_q$ define a new matrix H which is the Kronecker product $H = A \otimes B$, where H is obtained by changing any element $a_{r,s}$ in A by the matrix $a_{r,s}B$.

Definition 1.4 Let C be the $[n, k, d]_q$ code with parity check matrix H where $1 \leq k \leq n-1$ and $d \geq 3$. Denote by C_r the $[n, k, d]_{q^r}$ code over $\mathbb{F}_{q^r}$ with the same parity check matrix H. Say that code C_r is obtained by lifting C to $\mathbb{F}_{q^r}$.

2 Extending the Kronecker product construction

Recall that by $C(H)$ we denote the code defined by the parity check matrix H, by H^q_m we denote the parity check matrix of the q-ary Hamming $[n, n-m, 3]_q$ code $C = C(H^q_m)$ of length $n = (q^m - 1)/(q-1)$, and by $C_r(H^q_m)$ we denote the code (with the same length n) obtained by lifting $C(H^q_m)$ to the field $\mathbb{F}_{q^r}$.

Considering the Kronecker construction obtained in [8] we could see that the alphabets of both matrices $A = [a_{i,j}]$ and B should be compatible to each other in the sense that the multiplication $a_{i,j}B$ can be carried out. To have this compatibility it is enough that, say, the matrix A is over $\mathbb{F}_{q^u}$ and B is over $\mathbb{F}_q$. First, we consider the covering radius of the resulting codes.

Lemma 2.1 *Let $C(H^{q^u}_{m_a})$ and $C(H^q_{m_b})$ be two Hamming codes with parameters $[n_a, n_a - m_a, 3]_{q^u}$ and $[n_b, n_b - m_b, 3]_q$, respectively, where $n_a = (q^{u\,m_a} - 1)/(q^u - 1)$, $n_b = (q^{m_b} - 1)/(q-1)$, q is a prime power, $m_a, m_b \geq 2$, and $u \geq 1$. Then the code C with parity check matrix $H = H^{q^u}_{m_a} \otimes H^q_{m_b}$, the Kronecker product of $H^{q^u}_{m_a}$ and $H^q_{m_b}$, has covering radius $\rho = \min\{u\,m_a,\ m_b\}$.*

The following statement generalizes the results of [8,9].

Theorem 2.2 *Let $C(H^{q^u}_{m_a})$ and $C(H^q_{m_b})$ be two Hamming codes with parameters $[n_a, n_a - m_a, 3]_{q^u}$ and $[n_b, n_b - m_b, 3]_q$, respectively, where $n_a = (q^{u\,m_a} - 1)/(q^u - 1)$, $n_b = (q^{m_b} - 1)/(q-1)$, q is a prime power, $m_a, m_b \geq 2$, and $u \geq 1$. Let C be the code with parity check matrix $H = H^{q^u}_{m_a} \otimes H^q_{m_b}$. Then*

(i) The code C is a completely transitive, and completely regular, $[n, k, d; \rho]_{q^u}$ code with parameters:

$$n = n_a\, n_b, \quad k = n - m_a\, m_b, \quad d = 3, \quad \rho = \min\{u\, m_a,\ m_b\}.$$

(ii) The intersection numbers of the code C are:

$$b_\ell = \frac{(q^{u\,m_a} - q^\ell)(q^{m_b} - q^\ell)}{(q-1)},\ (0 \leq \ell \leq \rho - 1);\ \text{and } c_\ell = q^{\ell-1}\frac{q^\ell - 1}{q-1},\ (1 \leq \ell \leq \rho).$$

(iii) The lifted code $C_{m_b}(H^q_{um_a})$ is a completely regular code with the same intersection array as C.

Remark 2.3 We have to remark here that in the statement (iii) we can not choose the code $C_{m_b}(H^{q^u}_{m_a})$ (instead of $C_{m_b}(H^q_{um_a})$), which seems to be natural. We emphasize that the codes $C_{m_b}(H^q_{um_a})$ and $C_{m_b}(H^{q^u}_{m_a})$ are not only different completely regular codes, but they induce different distance-regular graphs with different intersection arrays. So, the code $C_{m_b}(H^q_{um_a})$ suits to the codes from (i) in the sense that it has the same intersection array. For example, the code $C_2(H^{2^2}_3)$ induces a distance-regular graph with intersection array $(315, 240; 1, 20)$ and the code $C_2(H^2_6)$ gives a distance-regular graph with intersection array $(189, 124; 1, 6)$.

Remark 2.4 Theorem 2.2 above can not be extended to the more general case when the alphabets $\mathbb{F}_{q^a}$ and $\mathbb{F}_{q^b}$ of component codes C_A and C_B, respectively, neither $\mathbb{F}_{q^a}$ is a subfield of $\mathbb{F}_{q^b}$ or vice versa $\mathbb{F}_{q^b}$ is a subfield of $\mathbb{F}_{q^a}$. We illustrate it by considering the smallest nontrivial example. Take two Hamming codes, the $[5, 3, 3]$ code C_A over $\mathbb{F}_{2^2}$ with parity check matrix $H^{2^2}_2$, and the $[9, 7, 3]$ code C_B over $\mathbb{F}_{2^3}$ with parity check matrix $H^{2^3}_2$. Then the resulting $[45, 41, 3]$ code $C = C(H^{2^2}_2 \otimes H^{2^3}_2)$ over $\mathbb{F}_{2^6}$ is not even uniformly packed in the wide sense, since it has the covering radius $\rho = 3$ and the outer distance $s = 7$, which can be checked by considering the parity check matrix of C.

3 CR-codes with the same intersection array

In [9, Theo. 2.11] it is proved that lifting a q-ary Hamming code $C(H^q_m)$ to $\mathbb{F}_{q^s}$ we obtain a completely regular code $C_s(H^q_m)$ which is not necessarily isomorphic to the code $C_m(H^q_s)$. However, both codes $C_s(H^q_m)$ and $C_m(H^q_s)$ have the same intersection array. As we saw above, the code obtained by the Kronecker product construction, or our extension for the case when the component codes have different alphabets, can have the same intersection array. The next statement is one of the main results of our paper.

Theorem 3.1 *Let q be any prime power and let a, b, u be any natural numbers. Then:*

(i) There exist the following completely regular $[n, k, d; \rho]$-codes with different parameters, where $d = 3$ and $\rho = \min\{ua, b\}$:

(a) *$C_{ua}(H^q_b)$ over $\mathbb{F}_{q^{ua}}$ with $n = \frac{q^b-1}{q-1}$, $k = n - b$;*

(b) *$C_b(H^q_{ua})$ over $\mathbb{F}_{q^b}$ with $n = \frac{q^{ua}-1}{q-1}$, $k = n - ua$;*

(c) *$C(H^q_b \otimes H^q_{ua})$ over $\mathbb{F}_q$ with $n = \frac{q^{ua}-1}{q-1} \times \frac{q^b-1}{q-1}$, $k = n - bua$;*

(d) $C(H_b^q \otimes H_u^{q^a})$ *over* $\mathbb{F}_{q^a}$ *with* $n = \frac{q^b-1}{q-1} \times \frac{q^{ua}-1}{q^a-1}$, $k = n - bu$;

(e) $C(H_b^q \otimes H_a^{q^u})$ *over* $\mathbb{F}_{q^u}$ *with* $n = \frac{q^b-1}{q-1} \times \frac{q^{ua}-1}{q^u-1}$, $k = n - ba$;

(ii) All above codes have the same intersection numbers
$$b_\ell = \frac{(q^b-q^\ell)(q^{ua}-q^\ell)}{(q-1)}, (0 \leq \ell \leq \rho-1); \text{ and } c_\ell = q^{\ell-1}\frac{q^\ell-1}{q-1}, (1 \leq \ell \leq \rho).$$

(iii) All above codes from Kronecker constructions are completely transitive.

It is easy to see that the number of different completely transitive (and, therefore, completely regular) codes with different parameters and the same intersection array is growing with q.

4 Coset distance-regular graphs

Let C be a linear completely regular code with covering radius ρ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho)$. Let $\{B\}$ be the set of cosets of C. Define the graph Γ_C, which is called the *coset graph of* C, taking all different cosets $B = C + \mathbf{x}$ as vertices, with two vertices $\gamma = \gamma(B)$ and $\gamma' = \gamma(B')$ adjacent if and only if the cosets B and B' contain neighbor vectors.

Lemma 4.1 [1, 7] *Let C be a linear completely regular code with covering radius ρ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho)$ and let Γ_C be the coset graph of C. Then Γ_C is distance-regular of diameter $D = \rho$ with the same intersection array. If C is completely transitive, then Γ_C is distance-transitive.*

From all different completely transitive codes described in Theorem 3.1, we obtain distance-transitive graphs with classical parameters [1]. These graphs have q^{uab} vertices, diameter $D = \min\{ua, b\}$, and intersection array given by (ii) in Theorem 3.1.

Notice that bilinear forms graphs [1, Sec. 9.5] have the same parameters and are distance-transitive too. These graphs are uniquely defined by their parameters [1, Sec. 9.5]. Therefore, all graphs coming from the completely regular and completely transitive codes described in Theorem 3.1 are bilinear forms graphs. We did not find in the literature (in particular in [2], where the association schemes, formed by bilinear forms, have been introduced) the description of these graphs, as many different coset graphs of completely regular codes. It is also known that these graphs are not antipodal and do not have antipodal covers [1, Sec. 9.5].

Theorem 4.2 *Let C_1, C_2, $\ldots$, C_k be a family of linear completely transitive codes constructed by Theorem 2.2 and let $\Gamma_{C_1}, \Gamma_{C_2}, \ldots, \Gamma_{C_k}$ be their corresponding coset graphs. Then:*

(i) *Any graph* Γ_{C_i} *is a distance-transitive graph, induced by bilinear forms.*

(ii) *If any two codes* C_i *and* C_j *have the same intersection array, then the graphs* Γ_{C_i} *and* Γ_{C_j} *are isomorphic.*

(iii) *If the graph* Γ_{C_i} *has* q^m *vertices, where* m *is not a prime, then it can be presented as a coset graph by several different ways, depending on the number of factors of* m*.*

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