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Very general monomial valuations of $\mathbb{P}^2$ and a Nagata type conjecture

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Abstract

It is well known that multi-point Seshadri constants for a small number t of points in the projective plane are submaximal. It is predicted by the Nagata conjecture that their values are maximal for $t \geq 9$ points. Tackling the problem in the language of valuations one can make sense of t points for any real $t \geq 1$. We show somewhat surprisingly that a Nagata-type conjecture should be valid for $t \geq 8 + 1/36$ points and we compute explicitly all Seshadri constants (expressed here as the asymptotic maximal vanishing element) for $t \leq 7+1/9$. In the range $7+1/9 \leq t \leq 8+1/36$ we are able to compute some sporadic values.

Keywords Nagata Conjecture, SHGH Conjecture, Seshadri constants, monomial valuations, anticanonical divisor

Mathematics Subject Classification (2000) MSC 14C20, 13A18

1 Introduction

The main purpose of this work is to formulate an analogue of Nagata's conjecture which makes sense for real values $t \geq 1$ of the number of points blown up instead of integral ones. Using quasi-monomial valuations of the plane we construct a $\hat{\mu} : [1, \infty) \to \mathbb{R}$, such that if $\hat{\mu}(t) = \sqrt{t}$ for all integers $t \geq 9$ then Nagata's conjecture is true. Moreover we show that $\hat{\mu}$ is a continuous function and we propose a conjecture that asserts the equality $\hat{\mu}(t) = \sqrt{t}$ for $t \geq 8+1/36$ (Conjecture 2.4Def.2.4). This fits well with the expected behavior of linear systems on blow-ups of $\mathbb{P}^2$, as it would follow from a stronger open conjecture by G. M. Greuel, C. Lossen and E. Shustin. On the other hand, the behavior of the function $\hat{\mu}$ in the range $7 + 1/9 \leq t \leq 8 + 1/36$ is somewhat mysterious.

By continuity, it suffices to verify the conjecture at rational square values of t, which boils down to verifying nefness of appropriate divisor classes with selfintersection zero. Thus Nagata's conjecture is reduced to proving a statement of a kind which has shown to be tractable, see [9], [3]. These selfintersection zero classes live on blow-up configurations that have not been known earlier to shed light on the original conjecture for ten or more points.

Going further down this road we discuss the value of $\hat{\mu}$ at many non-integral cases and compute it in a wide range including all $t \leq 7 + 1/9$. As a tool and also as a result of independent interest, we describe the Mori cone of the related blown up surfaces whenever they are anticanonical.

In our approach valuations are considered as a generalization of points, a natural step taken in many situations ever since Zariski's pioneering work. In the context of linear systems defined by multiple base points on projective varieties, positivity, and Seshadri constants, it is a point of view which seems to have been explored explicitly only recently. In [7] and [6], S. Boucksom, M. Dumnicki, A. Küronya, C. Maclean, and T. Szemberg introduced the constant a_{max} of a valuation (here denoted $\hat\mu$), analogous to the s-invariant introduced by L. Ein, S. D. Cutkosky and R. Lazarsfeld in [10] for ideals (see also [24, 5.4]). For a valuation v centered at the origin of $A^2 = \operatorname{Spec} C[x, y]$, one has by definition

$$\hat\mu(v) = \lim_{d\to\infty} \frac{\max\{v(f) \mid f \in C[x,y], \deg f \leq d\}}{d} .$$

All such invariants encode essentially the same information as the Seshadri constant does in the case of points and, as is the case for Seshadri constants, they turn out to be extremely hard to compute.

The last decade has also seen the blossoming of a geometric study of spaces of real valuations (see C. Favre–M. Jonsson [14]) or spaces of seminorms, usually called Berkovich spaces [2], which essentially coincide in dimension two (see M. Jonsson [21, section 6] for a description in the plane case). Being compact and arcwise connected, the topology of such spaces has very interesting and useful properties. The work of S. Boucksom, C. Favre and M. Jonsson [4], [5] implicitly reveals connections between such valuation spaces, positivity, and birational geometry.

In this paper the invariant $\hat\mu$ is studied as a function on the space V of plane valuations of real rank 1. This invariant turns out to be lower semicontinuous and continuous along arcs in V (Theorem 2.21Def.2.21). There is no difficulty in extending the definition of $\hat\mu$ to other varieties; one obtains a function-invariant for line bundles whose geometric significance would deserve further study. Motivated by what is known in the case of points and by the conjectures of Nagata and Segre–Harbourne–Gimigliano–Hirschowitz, our focus will be on valuations along a very general half-line in V.

$\mu(t) = \hat\mu(v_t)$ with $t \in [1, \infty)$, where v_t is a very general quasimonomial valuation with characteristic exponent t (see Section 2Preliminariessection.2 for precise definitions).

Our main results, Theorems 3.4Def.3.4 and 5.10Def.5.10, are the first steps toward the computation of $\hat\mu$. Divisorial valuations are dense in each arc of the valuation space; our tools provide a good grip on such valuations, and we work on the minimal proper birational model X_t where the center of v_t is a divisor. When X_t supports an effective anticanonical divisor, extensive knowledge of its geometry is available, see [18], [19]. In section 3Anticanonical surfacessection.3 we determine the range of t for which X_t is anticanonical, and study the Mori cone of X_t in that range. The following theorem sums up the main results of section 3Anticanonical surfacessection.3.

Theorem A. Let v_t be a very general quasimonomial valuation on P^2 with characteristic exponent $t \in Q$, and let X_t be the minimal model where v_t has divisorial center. X_t supports an effective anticanonical divisor if and only if $1 \leq t \leq 7, t = 7 + 1/n$ for some natural number n, or $t = 9$.

If $1 \leq t \leq 7$, then the Mori cone $\overline{NE}(X_t)$ is a polyhedral cone, spanned by the classes of the exceptional components of $X_t \to P^2$, the class of a particular nodal cubic, and finitely many (-1)-curves (whose number is explicitly bounded, see 3.4Def.3.4).

If $t = 7 + 1/n$ for natural n, then the only prime divisors C in X_t with $C^2 \leq -2$ are exceptional components, and $\overline{NE}(X_t)$ is a polyhedral cone if and only if $n \leq 8$.

If $1 \leq t \leq 3$, $t = 3+1/n$ for natural n, or $t = 5$, then the monoid of effective classes can be generated by the classes of the components of the exceptional divisor, a particular conic, and the (-1)-curves.

It is not hard to see that $\hat\mu(t) \geq \sqrt{t}$, and one should expect the equality to hold unless there is a good geometric reason, in the form of a (-1)-curve C^t on X_t with value higher than $\deg C_s \cdot \sqrt{t}$.

Conjecture B. For every $t \geq 8 + 1/36$, $\hat\mu(t) = \sqrt{t}$.

In section 4A variation on Nagata's conjecturesection.4 we explore the relations of conjecture BTI.2 and existing conjectures, showing in particular that Nagata's conjecture is just a special case of conjecture BTI.2. If t is an integer, then it is the number of points that have been blown up to construct X_t, and we look at $\hat\mu(t)$ as a continuous function that interpolates between the inverses of Seshadri constants at t very general points, whose values at non-integer t also have geometric meaning. In addition, it is not hard to show (Proposition 2.22Def.2.22) that for integer values of t that are squares, $\hat\mu(t) = \sqrt{t}$ holds. A further, stronger conjecture, motivated by our main results, is proposed at the end of section 5Supraminimal curvessection.5.

Knowing the cone of curves allows to compute $\hat\mu$, which for small t is done in section 5Supraminimal curvessection.5. Denote $F_{-1} = 1$, $F_0 = 0$ and $F_{i+1} = F_i + F_{i-1}$ the Fibonacci numbers, and $\varphi = (1+\sqrt{5})/2 = \lim F_{i+1}/F_i$ the "golden ratio".

Theorem C. The value of $\hat\mu(t)$ for $t \in [1, \varphi^4]$ is given by

$$\hat\mu(t) = \begin{cases} \frac{F_{i-2}}{F_i}\, t & \text{if } t \in \left[\frac{F_i^2}{F_{i-2}^2}, \frac{F_{i+2}}{F_{i-2}}\right], \\ \frac{F_{i+2}}{F_i} & \text{if } t \in \left[\frac{F_{i+2}}{F_{i-2}}, \frac{F_{i+2}^2}{F_i^2}\right], \end{cases}$$

where $i \geq 1$ takes all odd values. For $t \in [\varphi^4, 7 + 1/9]$,

$$\hat\mu(t) = \begin{cases} \frac{1+t}{3} & \text{if } t \in \left[\varphi^4, 7\right], \\ \frac{8}{3} & \text{if } t \in [7, 7 + 1/9]. \end{cases}$$

In particular there is a sequence of rational squares $t < 8$ with $\hat\mu(t) = \sqrt{t}$, with an accumulation point at φ^4; we suspect that at least some rational squares $t > 9$ can be dealt with by existing techniques, which by continuity of $\hat\mu$ would allow to compute $\hat\mu(t)$ for nonsquare t.

For anticanonical X_t there exists a (-1)-curve computing $\hat\mu(t)$. This implies that $\hat\mu$ is piecewise linear near t. We describe a (countably infinite) family of (-1)-curves from which Theorem CTI.3 follows, and also determine $\hat\mu(t)$ for other small values of t (see Figure 1In red, the known behavior of $\hat\mu(t)$ for $t \leq 9$; in yellow, the lower bound $\sqrt{t}$figure.1). We conjecture that this list is complete. If that is indeed so, then in particular $\hat\mu(t) = \sqrt{t}$ for $t \geq 8 + 1/36$. Except for

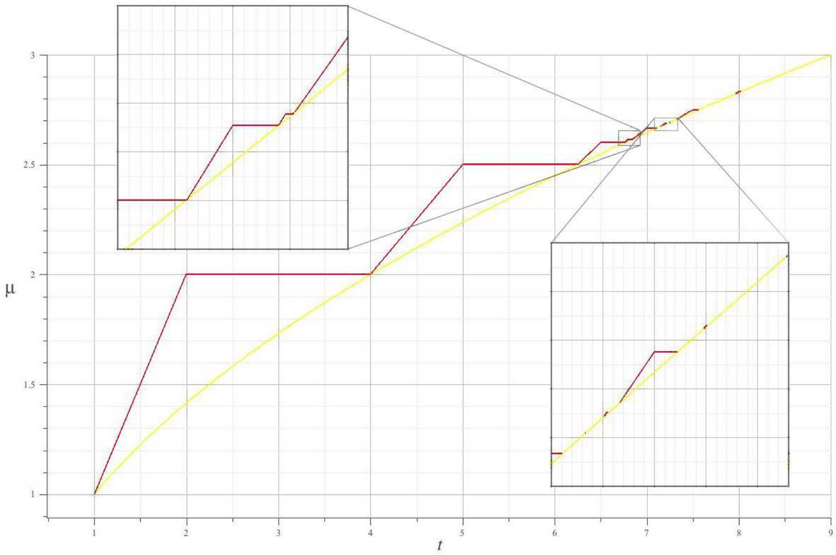

Figure 1: In red, the known behavior of $\hat{\mu}(t)$ for $t \leq 9$; in yellow, the lower bound $\sqrt{t}$.

9 cases, the (-1)-curves of Section 5Supraminimal curvessection.5 are the same unicuspidal curves which are known to give the asymptotically extremal ratio between degree and multiplicity, as explained in Y. Orevkov's work [26] (see also the overview [15]).

In what follows we work over the field of complex numbers.

2 Preliminaries

We refer to the references O. Zariski–P. Samuel [27, Chapter VI. and Appendix 5.] and E. Casas–Alvero [8, Chapter 8] for the general theory of valuations and complete ideals on surfaces. Let us now briefly recall the definitions and facts needed for the definition of $\hat{\mu}$ and the statement of the conjecture.

Let v be a rank 1 valuation (meaning that the value group is an ordered subgroup of $\mathbb{R}$) on the field of functions F of a projective algebraic surface S. For every effective divisor $D \subset S$, denote $v(D)$ the value of any equation of $D \cap U$, where U is an affine chart intersecting D.

Following [6], we denote

$$\mu_D(v) = \max\{v(D^0) \mid D^0 \in |D|\}, \quad \text{and} \quad \hat{\mu}_D(v) = \lim_{k\to\infty} \frac{\mu_{kD}(v)}{k}.$$

For every non-negative $m \in \mathbb{R}$, the ideal sheaves

$$\mathbf{I}_m = \{f \in \mathcal{O}_S \mid v(f) \geq m\}, \quad \text{and} \quad \mathbf{I}_m^+ = \{f \in \mathcal{O}_S \mid v(f) > m\}$$

are called valuation ideals. The closed subscheme defined by $\mathbf{I}_0^+$ is an irreducible subvariety, called center of the valuation, center(v). If R_v denotes the valuation ring of v, the generic point of the center is the image of the closed point under the unique map $\operatorname{Spec} R_v \to S$ that exists by the valuative criterion of properness. Of course, all this continues to apply if we substitute S by another projective model S' (i.e., a smooth projective surface with a fixed isomorphism $K(S') \cong F$).

If the center(v) is a curve C, then v is (up to a constant $c \in \mathbb{R}$) the order of vanishing along C; thus, $v(D) = c \cdot \operatorname{ord}_C D = c \cdot \max\{k \mid D - kC \geq 0\}$.

We are mostly interested in valuations of $S = \mathbb{P}^2$ such that the center(v) is a closed point. In this case the volume of v, as defined in [12], is

$$\operatorname{vol}(v) := \lim_{m\to\infty} \frac{\dim_{\mathbb{C}} (\mathcal{O}_S/\mathbf{I}_m)}{m^2/2}$$

(note that $\mathcal{O}_S/\mathbf{I}_m$ is an artinian $\mathbb{C}$-algebra supported at the center of the valuation) and the volume of a divisor class D on a surface S of dimension d is defined as

$$\operatorname{vol}(D) := \limsup_{k\to\infty} \frac{h^0(S, kD)}{k^2/2}.$$

Boucksom-Küronya-MacLean-Szemberg show that the invariant $\hat\mu$ can be bounded in terms of values in arbitrary dimension; let us recall their result in the case of surfaces:

Proposition 2.1 ([6, Proposition 2.9]). *Let D be a big divisor and v a real valuation centered at a point $p \in S$. Then*

$$\mu_D(v) \geq \sqrt{\operatorname{vol}(D)/\operatorname{vol}(v)}\ .$$

When D is ample this is equivalent to the bound $\mu_D(v) \geq \sqrt{D^2/\operatorname{vol}(v)}$. Valuations which satisfy the equality in Proposition 2.1[6, Proposition 2.9]Def.2.1, with $D \subset S = \mathbb{P}^2$ a line, will be called minimal.

For the sake of simplicity we recall the notion of quasimonomial valuations specializing to the case when $S = \mathbb{P}^2$ and the center of v is the origin $(0,0) \in \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x,y] \subset \mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[X,Y,Z]$, with $x = X/Z, y = Y/Z$. In this situation we write

$$\mu_d(v) = \max\{v(f) | f \in \mathbb{C}[x,y], \deg f \leq d\}\ , \quad \text{and} \quad \hat\mu(v) = \lim_{d\to\infty} \frac{\mu_d(v)}{d}\ .$$

Definition 2.2. Given a series $\xi(x) \in \mathbb{C}[[x]]$ with $\xi(0) = 0$ and a real number $t \geq 1$, let

$$v(\xi, t; f) := \operatorname{ord}_x(f(x, \xi(x) + \theta x^t))\ ,$$

where the symbol θ is transcendental over $\mathbb{C}$.

Equivalently, expand f as a Laurent series

$$f(x,y) = \sum a_{ij} x^i (y - \xi(x))^j\ ,$$

and put

$$v(\xi, t; f) := \min\{i + tj | a_{ij=0}\}\ . \qquad (*)$$

Then $f \mapsto v(\xi, t; f)$ is a valuation which we denote $v(\xi, t)$. Such valuations are called monomial if $\xi = 0$, and quasimonomial in general. Slightly abusing language, t will be called the characteristic exponent of $v(\xi, t)$ (even if it is an integer). For simplicity we also write

$$\mu(\xi, t) = \hat\mu(v(\xi, t))\ .$$

Remark 2.3. The valuation $v(\xi, t)$ depends only on the $\lfloor t \rfloor$-th jet of ξ, so for fixed t this series can be safely assumed to be a polynomial; however, later on we'll let t vary for a fixed ξ.

It is not difficult to see directly using (*equation.2.1), and will be proved using geometric considerations in the next subsection that $\mathrm{vol}(v(\xi, t)) = t^{-1}$. The precise statement of Conjecture BTI.2 is now:

Conjecture 2.4. For a sufficiently general choice of ξ, and every $t \geq 8 + 1/36$, the valuation $v(\xi, t)$ is minimal.

Cluster of centers of a valuation

Next we introduce the geometric structures attached to valuations $v(\xi, t)$ which allow us to study $\hat{v}(\xi, t)$ and justify the conjecture.

Each valuation with 0-dimensional center naturally determines a cluster of centers, as follows. To begin with, let $p_1 = \mathrm{center}(v)$ in the projective surface S. Consider the blowup $\pi_1 : S_1 \to S$ centered at p_1 and let E_1 be the corresponding exceptional divisor. The center of v on S_1 may be E_1 or a point $p_2 \in E_1$.

Iteratively blowing up the centers $p_1, p_2, \ldots$ of v either ends with a model where the center of v is an exceptional divisor E_n, in which case

$$v(f) = c \cdot \mathrm{ord}_{E_n} f$$

for some constant c, and v is called a divisorial valuation, or this process goes on indefinitely. For each center p_i of v, general curves through p_i and smooth at p_i have the same value $v_i = v(E_i)$.

Following [8, Chapter 4], we call the sequence $K = (p_1, p_2, \ldots)$, with weights $v_i = v(E_i)$, a weighted cluster of points, which completely determines v. Indeed, for every effective divisor $D \subset S$,

$$v(D) = \sum_i v_i \cdot \mathrm{mult}_{p_i} \tilde{D}_i, \qquad (\dagger)$$

where $\tilde{D}_i$ denotes proper transform at S_i. The sum may be infinite, but for valuations with real rank 1, which are the ones we consider here, $\tilde{D}$ can have positive multiplicity at only a finite number of centers [8, 8.2].

Sometimes we shall say that a divisor goes through an infinitely near point to mean that its proper transform on the appropriate surface goes through it.

Definition 2.5. With notation as above, given indices $j < i$, the center p_i is called proximate to p_j (p_i p_j) if p_i belongs to the proper transform $\tilde{E}_j$ of the exceptional divisor of p_j. Each p_i with $i > 0$ is proximate to p_{i-1} and to at most one other center p_j, $j < i - 1$; in this case $p_i = \tilde{E}_j \cap E_{i-1}$ and p_i is called a satellite point. A point which is not a satellite point is called free.

Remark 2.6. The irreducible components of exceptional divisors can be computed as proper transforms if the proximity relations are known: $\tilde{E}_j = E_j - \sum_{p_i \ p_j} E_i$

Remark 2.7. For every valuation v, and every center p_i such that v is not the divisorial valuation associated to p_i, equation (†Cluster of centers of a valuationequation.2.2) applied to $D = E_j$ gives rise to the so-called proximity equality

$$v_j = \sum_{p_i \to p_j} v_i \ .$$

For effective divisors D on S, the intersection number $\tilde{D} \cdot \tilde{E}_j \geq 0$ together with remark 2.6Def.2.6 yield the proximity inequality

$$\mathrm{mult}_{p_j}(\tilde{D}_j) \geq \sum_{p_i \to p_j} \mathrm{mult}_{p_i}(\tilde{D}_i) \ .$$

Assume now that $v = \mathrm{ord}_{E_s}$ is the divisorial valuation with cluster of centers $K = (p_1, \ldots, p_s)$, while $\pi_K : S_K \to S$ denotes the composition of the blowups of all points of K. Then, for every $m > 0$, the valuation ideal sheaf $\mathbf{I}_m$ can be described as

$$\mathbf{I}_m = (\pi_K)_*(\mathcal{O}_{S_K}(-mE_s)) \ .$$

Remark 2.8. As soon as $s > 1$, the negative intersection number $-mE_s \cdot \tilde{E}_{s-1} = -m$ implies that all global sections of $\mathcal{O}_{S_K}(-mE_s)$ vanish along $\tilde{E}_{s-1}$, and therefore

$$\mathbf{I}_m = (\pi_K)_*(\mathcal{O}_{S_K}(-mE_s - \tilde{E}_{s-1})) = (\pi_K)_*(\mathcal{O}_{S_K}(-E_{s-1} - (m-1)E_s)) \ .$$

This unloads a unit of multiplicity from p_s to p_{s-1}. The finite process of subtracting all exceptional components that are met negatively, (i.e., starting from a divisor $D_0 = -m_1E_1 - \cdots - m_sE_s$ and successively replacing D_i by $D_i - \tilde{E}_j$, starting with $i = 0$, whenever $D_i \cdot \tilde{E}_j < 0$ for some j, until one obtains a D_i such that $D_i \cdot \tilde{E}_j \geq 0$ for all j) is classically called unloading the weights of the cluster. The final uniquely determined system of weights $\bar{m}_i$ satisfies

$$D_m = -\sum \bar{m}_i E_i \quad \text{is nef relative to } \pi_K$$

(recall that a divisor is nef relative to a morphism f when it intersects nonnegatively every curve mapping to a point [24, 1.7.11]) and

$$\mathbf{I}_m = (\pi_K)_*(\mathcal{O}_{S_K}(D_m)) \ .$$

In this case, general sections of $\mathbf{I}_m$ have multiplicity exactly $\bar{m}_i$ at p_i, and no other singularity. More precisely, for any ample divisor class A on S, the complete system $|kA + D_m|$ for $k \gg 0$ is base-point-free (were we denote $A = (\pi_K)^*(A)$) and its general elements are smooth and meet each E_j transversely at $\bar{m}_j - \sum_{p_i \to p_j} \bar{m}_i$ distinct points. Note that relative nefness of D_m is equivalent to the proximity inequality $\bar{m}_j \geq \sum_{p_i \to p_j} \bar{m}_i$.

It follows using (†Cluster of centers of a valuationequation.2.2) that the valuation of an effective divisor D on S can be computed as a local intersection multiplicity

$$v(D) = I_{p_1}(D, C)$$

where C is the image in S of a general element of $|kA + D_m|$.

The unloading procedure just described also yields the following.

Lemma 2.9. Let $v = \mathrm{ord}_{E_s}$ be the divisorial valuation whose cluster of centers is $K = (p_1, \dots, p_s)$ with weights v_i, and for every $m > 0$ denote $D_m = -\sum \bar{m}_i E_i$ the unique nef divisor relative to π_K with $\mathbf{I}_m = (\pi_K)_*(\mathcal{O}_{S_K}(D_m))$. If $m = k \sum v_i^2$ for some integer k, then $\bar{m}_i = kv_i$ for all i.

Proof. It is clear that $-\sum \bar{m}_i E_i$ is nef relative to π_K because of the proximity equalities from remark 2.7Def.2.7. Moreover, because every effective divisor D satisfies the proximity inequalities, if $\mathrm{mult}_{p_1} D < \bar{m}_1$ then $\mathrm{mult}_{p_i} D < \bar{m}_i$ for all i, and by equation (†Cluster of centers of a valuationequation.2.2), $v(D) < m$. Arguing by induction on s, one sees that $\mathbf{I}_m = (\pi_K)_*(-\sum \bar{m}_i E_i)$. □

Remark 2.10. Write m_0 for $\sum v_i^2$. Then, in the context of Zariski's theory of factorizations of complete ideals, lemma 2.9Def.2.9 translates into

$$\mathbf{I}_{km_0} = \mathbf{I}_{m_0}^k ,$$

and to the fact that $\mathbf{I}_{m_0}$ is a simple complete ideal. For other values of m one has instead

$$\mathbf{I}_{km_0+\delta} = \mathbf{I}_{m_0}^k \mathbf{I}_\delta .$$

Non-divisorial valuations can be considered to be limits of divisorial valuations and their valuation ideals turn out to be complete as well, determined by finitely many centers. The ideal $\mathbf{I}_{km}$ is then never a power of $\mathbf{I}_m$, rather there exists $\delta > 0$ such that

$$\mathbf{I}_m^k \subset \mathbf{I}_{km} \subset \mathbf{I}_{m-\delta}^k$$

for all m and k. Such bounds actually hold in greater generality, namely for Abhyankar valuations in arbitrary dimension; see [12] by L. Ein, R. Lazarsfeld and K. Smith.

Lemma 2.11. Let $v = \mathrm{ord}_{E_n}$ be the divisorial valuation with cluster of centers $K = (p_1, \dots, p_n)$ and weights v_i. Then

$$\mathrm{vol}(v) = \left(\sum v_i^2 \right)^{-1} .$$

Proof. For $m = k \sum v_i^2$, $\dim_{\mathbb{C}}(\mathcal{O}_X/\mathbf{I}_m) = \sum kv_i(kv_i + 1)/2$ by [8, 4.7]. □

Remark 2.12. It is proven by Cutkosky and Srinivas in [11, Corollary 1] that divisorial valuations on surfaces have rational volume under mild conditions. On the other hand [22, Theorem 1.1] shows that this is not the case in higher dimensions.

Consider the group of numerical equivalence classes of $\mathbb{R}$-divisors $N_1(S_K)$, where S_K is the blowup at the cluster of centers of v. One calls a rational ray in $N_1(S_K)$ effective, if it is generated by an effective class. The Mori cone $\mathrm{NE}(S_K)$ is the closure in $N_1(S_K)$ of the set $\mathrm{NE}(S_K)$ of all effective rays, and it is the dual of the nef cone $\mathrm{Nef}(S_K)$ which is the closed cone described by all nef rays.

A (-1)-ray in $N_1(S_K)$ is a ray generated by a (-1)-curve, i.e., a smooth, irreducible, rational curve C with $C^2 = -1$ (hence $C \cdot \kappa = -1$, where κ denotes the canonical class). Mori's Cone Theorem says that

$$\overline{\mathrm{NE}}(S_K) = \overline{\mathrm{NE}}(S_K)^{<} + \mathbf{R}_n ,$$

where $\overline{NE}(S_K)^{<}$ denotes the subset of $\overline{NE}(S_K)$ described by rays generated by nonzero classes η such that $\eta \cdot \kappa \geq 0$ with κ being the canonical class, and

$$R_n = \sum_{\rho \text{ a } (-1)-\text{ray}} \rho \subseteq \overline{NE}(S_K)^{4} .$$

Remark 2.13. In cases when $\overline{NE}(S_K)$ is a polyhedral cone, Proposition 2.1[6, Proposition 2.9]Def.2.1 yields that $\hat{\mu}_D(v)$ is a rational number, and therefore v can be minimal only if $\sqrt{D^2/\mathrm{vol}(v)}$ is rational. In fact, all examples of divisorial minimal valuations included here correspond to rational values of $\sqrt{D^2/\mathrm{vol}(v)}$, even for nonpolyhedral $NE(S_K)$. For some examples of non-divisorial minimal valuations, see Remark 5.8Def.5.8; for these, vol(v) defines a quadratic extension of Q in which it is a square (i.e., $\sqrt{\mathrm{vol}(v)} \in \mathbb{Q}(\mathrm{vol}(v))$).

Centers of a quasimonomial valuation

Quasimonomial valuations are exactly the valuations whose cluster of centers consists of a few free points followed by satellites, which may be finite or infinite in number, but not infinitely many proximate to the same center. We will work with very general quasimonomial valuations on P^2. The genericity condition refers to the position of the free centers; it will be made precise below, after describing the continuity and semicontinuity properties of $\hat{\mu}$ on the space of quasimonomial valuations.

Remark 2.14. [8] The cluster K of centers of $v(\xi, t)$ can be easily described from the continued fraction expansion

$$t = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots}}} .$$

K consists of $s = \sum n_i$ centers; if $t = n_1$ then they all lie on the proper transform of the germ

$$\Gamma : y = \xi(x) ,$$

otherwise the first $n_1 + 1$ lie on Γ and the rest are satellites: starting from p_{n_1+1} there are $n_2 + 1$ points proximate to p_{n_1}, the last of which starts a sequence of $n_3 + 1$ points proximate to $p_{n_1+n_2}$ and so on. If the continued fraction is finite, with r terms, then the last n_r points (not $n_r + 1$) are proximate to $p_{n_1+\cdots+n_{r-1}}$. The weights are $v_i = 1$ for $i = 1, \ldots, n_1$, then $v_i = t - n_1$ for $i = n_1 + 1, \ldots, n_1 + n_2$, and $v_i = v_{n_1+\cdots+n_{j-1}} - n_j v_{n_1+\cdots+n_j}$ for $i = n_1 + \cdots + n_j + 1, \ldots, n_1 + \cdots + n_{j+1}$.

If t is rational, there are only finitely many coefficients $n_1, \ldots, n_r$, so $K = (p_1, p_2, \ldots, p_s)$ is finite and the valuation is divisorial. More precisely,

$$v(\xi, t; f) = v_s \cdot \mathrm{ord}_{E_s}(f) .$$

The prime divisor components $\tilde{E}_i$ of E_1 on S_K can then be described as follows (where $\tilde{E}_i$, as in Remark 2.6Def.2.6, is the proper transform in S_K of the blowup of the point p_i). Note that $s = n_1 + \ldots + n_r$; let s_i be the sum $n_1 + \cdots + n_i$, so $s = s_r$. The only i with $(\tilde{E}_i)^2 = -1$ is $i = s$, and in this case $\tilde{E}_s = E_s$. For each $1 \leq i < r - 1$, we have: $\tilde{E}_{s_i} = E_{s_i} - E_{s_i+1} - \cdots - E_{s_{i+1}+1}$, so $(\tilde{E}_{s_i})^2 = -2 - n_{i+1}$;

for $i = r-1$ we have $\hat{E}_{s_{r-1}} = E_{s_{r-1}} - E_{s_{r-1}+1} - \cdots - E_s$, so $(\hat{E}_{s_{r-1}})^2 = -1 - n_r$; and for every $1 \le j \le s$ not in the set $\{s_1, \ldots, s_r\}$ we have $\hat{E}_j = E_j - E_{j+1}$, so $(\hat{E}_j)^2 = -2$.

If t is irrational, then the sequence of centers is infinite and the group of values has rational rank 2. There is no surface S_K, but denoting S_j the blowup of the first j points of K, the above description of the divisors $\hat{E}_i$ holds whenever it makes sense; for instance, $\hat{E}_{s_i} = E_{s_i} - E_{s_i+1} - \cdots - E_{s_{i+1}+1}$ in every S_j with $j \ge s_{i+1} + 1$.

Corollary 2.15. Let $v(\xi, t)$ be a quasimonomial valuation as above. Then

$$\mathrm{vol}(v(\xi, t)) = t^{-1}, \quad \mu_d(\xi, t) \ge d\sqrt{t},$$

and

$$\text{þ}(\xi, t) \le \sqrt{t},$$

so $v(\xi, t)$ is minimal whenever $\text{þ}(\xi, t) = \sqrt{t}$.

Proof. The only point that needs proving is the value of $\mathrm{vol}(v(\xi, t))$, which follows from Lemma 2.11Def.2.11, taking into account the values v_i computed above and using induction on the number of terms in the continued fraction of t. □

The space of valuations

Remark 2.16. In definition 2.2Def.2.2 one may allow formal series $\xi(x) = \sum_{j\ge1} a_j x^{\beta_j}$ whose exponents β_j form an arbitrary increasing sequence of rational numbers, and one still obtains valuations $v(\xi, t)$ (no longer quasimonomial). It is even possible to allow $t = \infty$, except when ξ is the (convergent) Puiseux series of a branch of curve going through the center p_1. In this way, all real valuations with center at p_1 are obtained (up to a normalizing constant factor, see [8, 8.2] or [14, Chapter 4]).

The most natural topology in the set $\mathcal{T}$ of all real valuations with center at p_1 is the coarsest such that for all $f \in F$, $v \mapsto v(f)$ is a continuous map $\mathcal{T} \to \mathbb{R}$. It is called the weak topology. For a fixed ξ, the map $t \mapsto v(\xi, t)$ is then continuous. There is in $\mathcal{T}$ a finer topology of interest: namely, the finest topology such that $t \mapsto v(\xi, t)$ is continuous for all ξ. It is called the strong topology. With the strong topology, $\mathcal{T}$ is a profinite $\mathbb{R}$-tree, rooted at the p_1-adic valuation (see [14] for precise definitions and proofs). To avoid confusion with branches of curve, we call arcs the branches in $\mathcal{T}$. Maximal arcs are homeomorphic to the interval $[1, \infty]$ (respectively $[1, \infty)$) and parameterized by $t \mapsto v(\xi, t)$ where ξ is not (respectively, is) the Puiseux series of a branch of curve at p_1.

The arcs of $\mathcal{T}$ share the obvious segments given by coincident jets, and separate at rational values of t; these correspond to divisorial valuations (also in this general case).

Proposition 2.17. Fix a real number $t > 1$ and a natural number d. Set $k = \lceil t \rceil$ and denote by $\mathcal{J}_k \subset \mathbb{C}[[x]]$ the space of $(k-1)$-jets of power series with $\xi(0) = 0$, endowed with the Zariski topology coming from the coefficients map $\mathcal{J}_k \cong \mathbb{A}^{k-1}$, $\sum a_i x^i \mapsto (a_1, \ldots, a_{k-1})$.

Then the function $\xi \mapsto \mu_d(\xi, t)$ descends to an upper semicontinuous function

$$\mathbf{J}_k \to \langle 1, t\rangle_{\mathbb{Q}} \subset \mathbb{R}$$

which takes on only finitely many values.

It follows that for fixed t, $\mu(\xi, t)$ takes its smallest value for ξ with very general jet ξ_{n-1} (i.e., in a countable intersection of Zariski-open subsets of $\mathbf{J}_k = \mathbb{A}^{k-1}$).

Proof. Because only the k free centers of $v(\xi, t)$ depend on ξ ($k = n_1$ in the continued fraction expansion if t is an integer and $k = n_1 + 1$ otherwise), it is clear that the valuation only depends on the $(k-1)$-th jet of ξ, and the existence of the function

$$\mathbf{J}_k \to \langle 1, t\rangle_{\mathbb{Q}} \subset \mathbb{R}$$

is clear. We will prove that it only takes on a finite number of values and that for fixed m, the preimage of $[m, \infty)$ is Zariski-closed.

Given fixed t and d, there exists $m_{t,d} \in \langle 1, t\rangle_{\mathbb{Q}}$ such that $f \in \mathbb{C}[x, y]$, $v(\xi, t; f) \geq m_{t,d}$ implies $f \in (x, y)^{d+1}$ independently on ξ (by unloading, or using the definition (*equation.2.1)). Thus

$$\mu_d(\xi, t) < m_{t,d}$$

for all ξ.

Similarly, there exists $i_{t,d}$ such that no $f \in \mathbb{C}[x, y]_d$ has a proper transform going through any center p_i of $v(\xi, t)$ with $i > i_{t,d}$. Therefore for every $f \in \mathbb{C}[x, y]_d$, the value $v(\xi, t; f)$ belongs to the finite set

$$\left(\bigoplus_{i=1} \mathbb{N} v_i\right) \cap [1, m_{t,d}) ,$$

and the $\mu_d(\xi, t)$ belong to this set.

Now let V be the $\mathbb{C}$-subspace of $\mathbb{C}[\theta, x, x^t]$ consisting of polynomials P with $\deg_\theta(P) \leq d$ and $\deg_x(P) < m_{t,d}$. The space V is obviously finite-dimensional, $V \cong \mathbb{C}^N$ after taking the basis given by monomials.

Consider the composition of the substitution map

$$\mathbf{J}_k \times \mathbb{C}[x, y]_d \to \mathbb{C}[\theta][[x, x^t]] ,$$

given by $(\xi, f) \mapsto f(x, \xi(x) + \theta x^t)$, with truncation $\mathbb{C}[\theta][[x, x^t]] \to V$, seen as an algebraic morphism of $\mathbb{C}$-schemes.

For each value m, the 'incidence' subset

$$\{(\xi, f) \in \mathbf{J}_k \times \mathbb{C}[x, y]_d \mid v(\xi, t; f) \geq m\}$$

is by definition the preimage of the Zariski-closed set

$$\{\eta \in V \mid \mathrm{ord}_x(\eta(x)) \geq m\}$$

hence Zariski-closed. It is also closed under scalar multiplication on the second component, so it determines a closed subset $I_m \subset \mathbf{J}_k \times \mathbb{P}(\mathbb{C}[x, y]_d)$.

The locus in $\mathbf{J}_k$ where $\mu_d(\xi, t) \geq m$ is the projection of I_m to $\mathbf{J}_k$, therefore it is Zariski-closed. □

Proposition 2.18. For every $\xi(x)$, the function $t \mapsto \mu(\xi, t)$ (for $t \in [1, \infty)$) is
Lipschitz continuous with Lipschitz constant 1.

Proof. For every $f \in \mathbb{C}[x, y]$, the function $t \mapsto v(\xi, t; f)$ is a tropical polynomial function of degree at most $\deg(f)$. Therefore, the scaled function $\mu_f : t \mapsto v(\xi, t; f)/\deg(f)$ is continuous concave and piecewise affine linear with slopes in
$\{0, 1/\deg(f), 2/\deg(f), \dots, 1\}$ (compare with [5, Corollary C]). In particular, it is Lipschitz continuous with Lipschitz constant at most 1.

The function $t \mapsto \mu(\xi, t)$ in the claim is $\sup_{f \in \mathbb{C}[x,y]}\{\mu_f\}$; therefore it is also Lipschitz continuous with Lipschitz constant at most 1 (and it is not hard to see that it is actually equal to 1). □

Remark 2.19. We proved in Proposition 2.17Def.2.17 that for a fixed t, very $\mu(\xi, t)$ which we denote $\hat{\mu}(t)$.

By the countability of the rational number field, it follows that very general series $\xi(x)$ give the same (minimal) function $\hat{\mu}(\xi, t)$ of $t \in \mathbb{Q}$. Continuity of the functions $\hat{\mu}(\xi, t)$ then imply that very general series give the same function over all of $\mathbb{R}$, and also the following:

Corollary 2.20. The function $t \mapsto \hat{\mu}(t)$ is Lipschitz continuous with Lipschitz constant 1.

It is immediate to extend the definition of μ and $\hat{\mu}$ to the tree $\mathcal{T}$ of all valuations centered at p_1. The continuity properties of the resulting function $\mu : \mathcal{T} \to \mathbb{R}$ —which we shall not need— are summarized as follows:

Theorem 2.21. The function $\hat{\mu} : \mathcal{T} \to \mathbb{R}$ is lower semicontinuous for the weak topology and continuous for the strong topology.

Proof. As in the proof of propositon 2.18Def.2.18, for all $f \in \mathbb{C}[x, y]$, let $\mu_f(v) = v(f)/\deg(f)$. By definition of the weak topology, μ_f is continuous for all f. Then, $\mu(v) = \sup_{f \in \mathbb{C}[x,y]}\{\mu_f(v)\}$, as the supremum of a family of continuous functions, is lower semicontinuous.

In order to prove continuity for the strong topology, one needs to show continuity along all arcs in the profinite tree $\mathcal{T}$. It is not hard to see that (with minor changes) the proof of propositon 2.18Def.2.18 works for series ξ with rational exponents as in Remark 2.16Def.2.16, showing the desired continuity.

Alternatively, given a strong neighbourhood U of a given valuation v_0, there is a model of the plane in which every $v \in U$ is quasimonomial. Then Proposition 2.18Def.2.18 shows that $\hat{\mu}$ is continuous in U. □

The next claim will show the first analogy to Nagata's conjecture.

Proposition 2.22. If t is the square of an integer, then a very general quasi-monomial valuation $v(\xi, t)$ is minimal.

Proof. For integral values of t, the cluster of centers of $v(\xi, t)$ consists of the first t points infinitely near to the origin along the branch $y = \xi(x)$, and for each integer $m = qt + r$ (with $0 \le r < t$)

$$\mathbf{I}_m = (\pi_K)_*(\mathcal{O}_{S_K}(-q(E_1 + \cdots + E_t) - (E_1 + \cdots + E_r))) .$$

For $d > 0$ and very general ξ, we want to prove that $\mu_d(\xi, t) \le d\sqrt{t}$ or, in other words, that for every integer $m > d\sqrt{t}$, the valuation ideal $\mathbf{I}_m$ has no sections of degree d:

$$H^0(\mathcal{O}_{S_K}(dL - q(E_1 + \cdots + E_t) - (E_1 + \cdots + E_r))) = 0 ,$$

where L denotes the pullback of a line to S_K. By semicontinuity (Proposition 2.17Def.2.17) it will be enough to see this for a particular choice of ξ, e.g., an irreducible polynomial of degree $a = \sqrt{t}$. But the proper transform on S_K of the projectivized curve

$$D : YZ^{a-1} = Z^a \xi(X/Z)$$

defined by ξ is then an irreducible curve of self-intersection zero, therefore nef, and

$$D \cdot (dL - q(E_1 + \cdots + E_t) - (E_1 + \cdots + E_r))) = d\sqrt{t} - m < 0 . \qquad \square$$

3 Anticanonical surfaces

This section contains a complete description of the Mori cone of S_K for $v = v(\xi, t)$ with $t \le 7$ (see Theorem 3.4Def.3.4 and Proposition 3.6Def.3.6), and substantial information for $t = 7 + \frac{1}{n_2}$, $n_2 \in \mathbb{N}$ (see Proposition 3.8Def.3.8 and Corollary 3.11Def.3.11). In these cases the rational surface S_K obtained by blowing up the cluster of centers of a valuation v on the plane is anticanonical, meaning it has an effective anticanonical divisor. Under this hypothesis, adjunction becomes a very powerful tool to study the geometry of S_K.

We begin by justifying that S_K is anticanonical in these cases.

Proposition 3.1. *Let $v(\xi, t)$ be a divisorial quasimonomial valuation (so t is rational), and S_K the blowup of its cluster of centers. Let $A = [1, 7] \cup \{7 + \frac{1}{n}\}_{n\in\mathbb{N}} \cup \{9\} \subset \mathbb{R}$.*

1. *If $t \in A$, then S_K is anticanonical.*
2. *If S_K is anticanonical for very general ξ, then $t \in A$.*

Proof. The question is whether the anticanonical class $-\kappa = 3L - \sum E_i$ on S_K (where L denotes the pullback of a line) has nonzero global sections.

Suppose t is an integer. Then K consists of t free points; if $t \le 9$, there is a cubic going through them all, so $-\kappa$ is effective. On the other hand, for an integer $t > 9$, there is no such plane cubic for general K. Thus (1) and (2) hold when t is an integer.

Now suppose $t = n_1 + \frac{1}{n_2}$ is a nonintegral rational. Then $K = (p_1, \ldots, p_{n_1+n_2})$ has $n_1 + 1$ free centers and $n_2 - 1 > 0$ satellites, all of them proximate to p_{n_1}; so $\tilde{E}_{n_1} = E_{n_1} - E_{n_1+1} - \cdots - E_{n_1+n_2}$. A simple unloading computation (see Remark 2.8Def.2.8 and 2.14Def.2.14) then shows that

$$\begin{aligned} H^0(\mathcal{O}_{S_K}(-\kappa)) &= H^0(\mathcal{O}_{S_K}(3L - \sum E_i)) \\ &\cong H^0(\mathcal{O}_{S_K}(3L - 2E_1 - (E_2 + \cdots + E_{n_1})) \end{aligned}$$

(the divisors $\tilde{E}_{n_1}, \tilde{E}_{n_1-1}, \dots, \tilde{E}_1$ intersect negatively and have been subtracted). Consequently, S_K is anticanonical exactly when there exists a cubic singular at p_1 and going through the free points $p_2, \dots, p_{n_1}$. (If $n_1 > 1$ then both p_2 and p_3 are free, so if the cubic is irreducible its singularity is a node.) For $n_1 \le 7$, there is always such a cubic, so (1) holds, while for a general choice of the free points we must have $n_1 \le 7$, so (2) holds.

If the continued fraction for t has more than 2 coefficients $n_1, n_2, \dots, n_r$, the corresponding unloading computation consists in subtracting, for $i = r-1, r-2, \dots, 1$, the divisors $\tilde{E}_{n_1+\cdots+n_i}, \tilde{E}_{n_1+\cdots+n_i-1}, \dots \tilde{E}_{n_1+\cdots+n_{i-1}+1}$, and leads to $H^0(\mathcal{O}_{S_K}(3L - 2E_1 - (E_2 + \cdots + E_{n_1+1}))$, so S_K is anticanonical exactly when there exists a cubic singular at p_1 and going through the free points $p_2, \dots, p_{n_1+1}$. Such a cubic always exists if $n_1 \le 6$, so (1) holds, and for a general choice of the free points, we must have $n_1 + 1 \le 7$, so (2) holds. □

The next lemma is needed for the proof of Proposition 3.6Def.3.6.

Lemma 3.2. Let $v(\xi, t)$ be a divisorial quasimonomial valuation (so t is rational), and S_K the blowup of its cluster of centers. Let $B = [1,3] \cup \{3 + \frac{1}{n}\}_{n\in\mathbb{N}} \cup \{4,5\} \subset \mathbb{R}$.

1. If $t \in B$, then $-\kappa - L$ is effective on S_K.

2. If $-\kappa - L$ is effective on S_K for very general ξ, then $t \in B$.

Proof. The proof is similar to the one for Proposition 3.1Def.3.1. The integer cases are well known and easy to see.

Next say $t = n_1 + \frac{1}{n_2}$ is a nonintegral rational. Then $(-\kappa - L) \cdot \tilde{E}_{n_1} < 0$, so unloading (as in the proof of Proposition 3.1Def.3.1) gives $H^0(\mathcal{O}_{S_K}(-\kappa - L)) \cong H^0(\mathcal{O}_{S_K}(-\kappa - L - \tilde{E}_{n_1}) \cong H^0(\mathcal{O}_{S_K}(2L - 2E_1 - (E_2 + \cdots + E_{n_1}))$. The class of the proper transform of the line through p_1 in the direction of p_2 is $\tilde{L} = L - E_1 - \cdots - E_i$ for some $2 \le i \le n_1 + 1$. Thus $(2L - 2E_1 - (E_2 + \cdots + E_{n_1})) \cdot \tilde{L} < 0$. Therefore, if $n_1 \le 3$, subtracting $\tilde{L}$ and unloading we have $H^0(\mathcal{O}_{S_K}(2L - 2E_1 - (E_2 + \cdots + E_{n_1}))) \cong H^0(\mathcal{O}_{S_K}(L - E_1 - E_2)) = 0$. Thus (1) holds for such t, while for a general choice of the free points we must have $n_1 \le 3$, so (2) holds for such t.

Finally, if the continued fraction for t has more than 2 coefficients n_i, the corresponding unloading computation leads to $H^0(\mathcal{O}_{S_K}(-\kappa - L)) \cong H^0(\mathcal{O}_{S_K}(2L - 2E_1 - (E_2 + \cdots + E_{n_1+1}))$. Again $\tilde{L} = L - E_1 - \cdots - E_i$ for some $2 \le i \le n_1 + 1$, so subtracting $\tilde{L}$ and unloading gives $H^0(\mathcal{O}_{S_K}(2L - 2E_1 - (E_2 + \cdots + E_{n_1+1})))$ $H^0(\mathcal{O}_{S_K}(L - E_1 - \cdots - E_{n_1+2-i}))$. The latter is clearly nonzero if $n_1 \le 3$, so (1) holds, and for a general choice of the free points, we must have $n_1 \le 2$, so (2) holds. □

Remark 3.3. Note that if $t \le 7$, then K has at most 7 free centers, so there is always a divisor $\tilde{\Gamma}$ in $|3L - 2E_1 - \sum_{i>1, p_i \text{ free}} E_i|$. For general ξ, p_1, p_2, p_3 are not aligned and $p_1, \dots, p_6$ do not belong to a conic, so $\tilde{\Gamma}$ can be assumed to be the proper transform of an irreducible nodal cubic Γ, and $\Gamma_K = \tilde{\Gamma} + \sum \tilde{E}_i$ on S_K is a particular anticanonical divisor which contains all exceptional components (independently of t). For nongeneral ξ, $\tilde{\Gamma}$ may be reducible, but $\Gamma_K = \tilde{\Gamma} + \sum \tilde{E}_i$ still determines an effective anticanonical divisor which contains all exceptional components.

Theorem 3.4. Let $v(\xi, t)$ be a divisorial quasimonomial valuation with $t \leq 7$, and S_K the blowup of its cluster of centers. Let s be the number of centers. Then the number of (-1)-curves other than $E_s = \tilde{E}_s$ is at most s, and $NE(S_K)$ is a polyhedral cone, spanned by the classes of the $\tilde{E}_i$, $\tilde{\Gamma}$ and the (-1)-curves, where Γ is a nodal cubic as above.

Proof. Let Γ_K be an effective anticanonical divisor containing all exceptional components $\tilde{E}_i$; for general ξ we can write $\Gamma_K = \tilde{\Gamma} + \sum \tilde{E}_i$, where Γ is a nodal cubic. Particular cases in which the cubic is reducible are treated similarly and we leave the details to the reader. We claim that every irreducible curve $C \subset S_K$ which is not a component of Γ_K lies in $\overline{NE}(S_K)$. Indeed, C is the proper transform of a curve $\pi_K(C) \subset P^2$; if $\pi_K(C)$ does not go through the origin p_1 of K, then C intersects $\tilde{\Gamma}$ and so

$$C \cdot \kappa = -(C \cdot (\Gamma_K)) = -(C \cdot \tilde{\Gamma}) < 0 .$$

Otherwise, C intersects some $\tilde{E}_i$ and so

$$C \cdot \kappa = -(C \cdot (\Gamma_K)) \leq -(C \cdot \tilde{E}_i) < 0 .$$

Thus by Mori's cone theorem, $\overline{NE}(S_K)$ is generated by the rays spanned by the components of Γ_K and the (-1)-curves, so it only remains to bound the number of (-1)-curves.

But a (-1)-curve C satisfies $C \cdot \kappa = -1$, so if it is not a component of Γ_K, it must intersect it in exactly one component. Write $C = dL - \sum m_i E_i$. If C meets only $\tilde{E}_k$, it must satisfy $m_j = \sum_{p_i \succ p_j} m_i$ (i.e., $C \cdot \tilde{E}_j = 0$) for all $j \neq k$, $m_k = \sum_{p_i \succ p_k} m_i + 1$ (i.e., $C \cdot \tilde{E}_k = 1$) and $3d - \sum m_i = 1$ (i.e., $C \cdot \Gamma_K = 1$). These are $s+1$ linearly independent conditions which uniquely determine the class of C; so there is at most one (-1)-curve meeting $\tilde{E}_k$. On the other hand, C cannot meet only $\tilde{\Gamma}$, because then $C \cdot \tilde{E}_j = 0$ for all j, which implies $m_j = C \cdot E_j = 0$ for all j, and hence $1 = C \cdot \Gamma_K = 3d - \sum m_i = 3d$. Thus the number of (-1)-curves not components of Γ_K is at most s. □

Remark 3.5. Along the way we proved that there are finitely many curves with negative selfintersection when $t \leq 7$. Indeed, if C is such a curve, and it is not a component of $\Gamma_K = \tilde{\Gamma} + \sum \tilde{E}_i$ then $C \cdot \kappa < 0$, which implies $0 > C^2 + C \cdot \kappa = 2g - 2$, so C is a rational curve and in fact a (-1)-curve, of which there are at most s.

For $t \in B$, one can be a bit more precise: not only do the negative curves generate the Mori cone over R, they generate the monoid of effective classes (over N).

Proposition 3.6. Let $v(\xi, t)$ be a divisorial quasimonomial valuation with $t \in B$, $t > 1$, and S_K the blowup of its cluster of centers. Let s be the number of centers. Then the monoid in $\mathrm{Pic}\, S_K \cong Z^{s+1}$ of the effective classes has a minimal (finite) set of generators consisting of the classes of the $\tilde{E}_i$, the (-1)-curves, and the components of $-\kappa - L$ meeting $-\kappa - L$ negatively.

Proof. Thanks to Lemma 3.2Def.3.2, we can apply [17, Proposition III.ii.1]. □

Remark 3.7. A similar result holds for any divisorial quasimonomial valuation $v(\xi, t)$ when $t = 4 + \frac{1}{n_2}$, namely the effective monoid in Pic S_K is generated by $\tilde{E}_i$, $i = 1, \dots, s = n_1 + n_2$, and the proper transform $\tilde{E}_0$ of $L - E_1 - E_2$ (and $Q = 2L - E_1 - \cdots - E_5$ if $\tilde{E}_0 = L - E_1 - E_2$, i.e., if p_3 does not belong to the line through p_1 in the direction of p_2). We sketch the argument in case $\tilde{E}_0 = L - E_1 - E_2$. Take the basis $D_0, \dots, D_s$ for the divisor class group of S_K, satisfying $D_i \cdot \tilde{E}_j = \delta_{ij}$, where δ_{ij} is Kronecker's delta, hence $\delta_{ij} = 0$ for $i = j$ and 1 if $i = j$. (Thus D_i is just the basis dual to $\tilde{E}_j$, specifically: $D_0 = L$, $D_1 = L - E_1$, $D_2 = 2L - E_1 - E_2$, $D_3 = 2L - E_1 - E_2 - E_3$, $D_4 = 2L - E_1 - E_2 - E_3 - E_4$, $D_5 = Q = 2L - E_1 - E_2 - E_3 - E_4 - E_5$, $D_6 = 4L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - E_5 - E_6$, $D_7 = 6L - 3E_1 - 3E_2 - 3E_3 - 3E_4 - E_5 - E_6 - E_7$, $D_8 = 8L - 4E_1 - 4E_2 - 4E_3 - 4E_4 - E_5 - E_6 - E_7 - E_8$, and so on, so for $4 < i \le s - 4$ we have $D_{i+4} = 2iL - iE_1 - iE_2 - iE_3 - iE_4 - E_5 - \dots - E_{i+4}$.) Every prime divisor D not among the $\tilde{E}_j$ is $\sum_i (D \cdot \tilde{E}_i)D_i$, hence it suffices to check the divisors D_i. It is easy to write down the classes D_i explicitly and then to check that each D_i is a nonnegative integral sum of classes $\tilde{E}_j$, $j \ge 0$, when $i < 5$, and a nonnegative integral sum of the classes Q and $\tilde{E}_j$, $j \ge 1$, when $i \ge 5$. (Essentially the same argument works when $\tilde{E}_0 = L - E_1 - \cdots - E_l$ for $l > 2$, except the result is that D_i is a nonnegative integral sum of the classes $\tilde{E}_j$, $j \ge 0$, for all i. In this case we note that $Q \cdot \tilde{E}_0 < 0$ so Q is no longer prime, and $\tilde{E}_0 \cdot \tilde{E}_4$ having to be nonnegative forces $l \le 5$.)

In fact, we can show that a similar result holds for $t = n_1 + \frac{1}{n_2}$ also for $n_1 = 5$ and 6, namely that there are only finitely many prime divisors of negative self-intersection on S_K, and they generate the effective monoid. The proof is more involved, however, since, for $n_1 = 5$, $D_6 = 2L - E_1 - \cdots - E_6$ need not be effective but it could be and if it is, it may but might not be a prime divisor. Likewise, for $n_1 = 6$, additional cases arise: $3L - 2E_1 - E_2 - \cdots - E_7$ and $5L - 2E_1 - \cdots - 2E_6 - E_7 - E_8$ are effective but may or might not be prime, and $2L - E_1 - \cdots - E_6$ and $2L - E_1 - \cdots - E_7$ may or might not be effective. Nonetheless, the proof follows similar lines (in each of the several cases, find an explicit finite set of generators for the effective monoid, and then show each generator is a sum of negative curves). Because checking the various cases is somewhat lengthy, we do not include the proof here.

For $7 < t < 8$, it is not clear which values of t give polyhedral Mori cones, but C. Galindo and F. Monserrat [16] give some positive results in this context. In particular, their Corollary 5, (1) shows that for $t = 7+1/n_2$ with $n_2 = 1, 2, \dots, 8$, $NE(S_K)$ is polyhedral. We show this result is sharp, in the sense that $NE(S_K)$ is not polyhedral for $n_2 > 8$, provided that ξ is very general (see Corollary 3.11Def.3.11). On the other hand, parts (2) and (3) of [16, Corollary 5] are sharpened by Theorem 3.4Def.3.4 above.

In preparation for proving Corollary 3.11Def.3.11, we first prove a result concerning prime divisors C with $C^2 < -1$.

Proposition 3.8. Let $v(\xi, t)$ be a very general divisorial quasimonomial valuation with $t = 7 + 1/n_2$ for $n_2 \ge 1$, and let S_K be the blowup of its cluster of centers. The only prime divisors C in S_K with $C^2 \le -2$ are components of the exceptional divisors E_i.

Proof. As before, let Γ be a nodal cubic curve which has its node at the origin and goes through six additional free centers, $p_2, \dots, p_7 \in K$. Then $\Gamma_K = \tilde{\Gamma} + \sum_{i=1}^{7} \tilde{E}_i$ on S_K is the unique effective anticanonical divisor.

By adjunction we have $C^2 + \kappa_{S_K} \cdot C = 2g - 2$, so $C^2 < -2$ implies $\Gamma_K \cdot C < 0$, hence C is a component of Γ_K. Computing the self-intersection of each of them shows that the only possibility is $C = \tilde{E}_7 = E_7 - E_8 - \cdots - E_s$ where again s is the total number of blowups and $C^2 = -1 - n_2$.

By adjunction again, if $C^2 = -2$, then C is rational and $\kappa_{S_K} \cdot C = 0$, i.e., it 2)-curve. Thus the question is what (-2)-curves can occur on S_K. The exceptional components $\tilde{E}_i$ for $i = 7, s$ are (-2)-curves. Now assume that C is not one of them. Then $\Gamma_K \cdot C = 0$ implies $C \cdot \tilde{E}_i = 0$ for $i = 0, \dots, 7$, and $C \cdot \tilde{E}_i \geq 0$ for $i > 7$.

Write $C = dL - m_1 E_1 - \cdots - m_s E_s$. The constraint $C \cdot \tilde{E}_7 = 0$ gives $m_7 = m_8 + \cdots + m_s$. The constraints $C \cdot \tilde{E}_i = 0$ for $i = 1, \dots, 6$ give $m_1 = \cdots = m_7$. Taking $m = m_1$, $C \cdot \tilde{\Gamma} = 0$ gives $3d = 7m + m_8 + \cdots + m_s = 8m$, so $d = 8m/3$. Note that d is an integer.

Consider the case that $n_2 = 1$. Then $-2 = C^2 = (8m/3)^2 - 8m^2 = -8m^2/9$. This has no integer solutions, so no C exists.

Next consider the case that $n_2 = 2$, so $s = 9$. The possible solutions C to $C^2 = -2$, $C \cdot \kappa_{S_K} = 0$ with $C \cdot L \geq 0$ are known (see the second half of the proof $E_j) -$ $i, j \leq 9$, $i = j$, 9, i, j, k distinct, $r \geq 0$; $(2L - E_{i_1} - \cdots - E_{i_6}) - r\kappa_{S_K}$ with $1 \leq i_j \leq 9$, i_j distinct for $1 \leq j \leq 6$, $r \geq 0$; and $(3L - 2E_{i_1} - E_{i_2} - \cdots - E_{i_8}) - r\kappa_{S_K}$, $1 \leq i_j \leq 9$, i_j distinct for $1 \leq j \leq 8$, $r \geq 0$. An exhaustive check shows that each of these divisors intersects some exceptional component or Γ negatively, and thus is either itself a component of an exceptional curve, or is not reduced or irreducible.

Now consider the case that $n_2 \geq 3$, so $s \geq 10$, and we can write $C = dL - m(E_1 + \cdots + E_7) - m_8 E_8 - \cdots - m_s E_s = (8m/3)L - m(E_1 + \cdots + E_7) - m_8 E_8 - \cdots - m_s E_s$. Let $m = 3b$, so $C = 8bL - 3b(E_1 + \cdots + E_7) - m_8 E_8 - \cdots - m_s E_s$. Then $\Gamma_K \cdot C = 0$ gives $3b - m_8 - \cdots - m_s = 0$ and $C^2 = -2$ gives $b^2 - m_8^2 - \cdots - m_s^2 = -2$. Numerical considerations no longer suffice; there are many solutions to $3b - m_8 - \cdots - m_s = 0$ and $b^2 - m_8^2 - \cdots - m_s^2 = -2$. For example, we have $C = 8L - 3(E_1 + \cdots + E_7) - E_8 - E_9 - E_{10}$ (i.e., $s = 10$, $n_2 = 3$, $b = 1$, and $m_8 = m_9 = m_{10} = 1$). The following lemma however shows that such C can not be the class of a prime divisor, and finishes the proof. □

Lemma 3.9. *Let S_K be as in Proposition 3.8Def.3.8. Then there is no prime divisor C on S_K with $C \cdot \kappa_{S_K} = 0$ other than $\tilde{E}_i$ for $i = 7$.*

Proof. By the end of the proof of Proposition 3.8Def.3.8, if such a C exists it must be $C = dL - m(E_1 + \cdots + E_7) - m_8 E_8 - \cdots - m_s E_s$, where $d = 8m/3$, $m = 3b = m_8 + \cdots + m_s$ for some b, and $m_8 \geq \cdots \geq m_s > 0$. The divisor class $8L - 3(E_1 + \cdots + E_7)$ is effective and base point free, and has irreducible global sections; in fact it is the class of a homaloidal net, see Proposition 5.4Def.5.4 below. In particular it is nef. Pick an irreducible $B \in |8L - 3(E_1 + \cdots + E_7)|$. Since $B \cdot \tilde{\Gamma} = B \cdot \tilde{E}_i = 0$ for $i < 7$, we see $B|_{\Gamma_K}$ is a divisor which vanishes on each component $\tilde{E}_i$, $i < 7$ of Γ_K, and consists of a divisor B^0 of degree 3 on the interior of component $\tilde{E}_7$. Since $E_i|_{\Gamma_K} = E_s|_{\Gamma_K}$ for $i \geq 8$ and E_i is disjoint from $\tilde{E}_j$ for $i \geq 8$ and $j < 7$, we see $(-m_8 E_8 - \cdots - m_s E_s)|_{\Gamma_K}$ is a divisor which is trivial on each component of Γ_K except $\tilde{E}_7$, and on $\tilde{E}_7$ it gives the divisor $(m_8+\cdots+m_s)p_8 = mp_8 = 3bp_8$. Thus $\mathcal{O}_{\Gamma_K}(C)$ is the same as $\mathcal{O}_{\Gamma_K}(bB^0 - 3bp_8)$.

Consider the restriction exact sequence

$$0 \to \mathcal{O}_{S_K}(C - \Gamma_K) \to \mathcal{O}_{S_K}(C) \to \mathcal{O}_{\Gamma_K}(C) \to 0.$$

Then, since C is by assumption a prime divisor, we have $h^0(S_K, \mathcal{O}_{S_K}(C-\Gamma_K)) < h^0(\mathcal{O}_{S_K}(C))$, which by taking cohomology of the short exact sequence implies $h^0(\mathcal{O}_{\Gamma_K}(bB^0 - 3bp_8)) > 0$. But $\deg(bB^0 - 3bp_8) = 0$ so $h^0(\mathcal{O}_{\Gamma_K}(bB^0 - 3bp_8)) > 0$ implies $bB^0 - 3bp_8 \sim 0$ (where $\sim$ denotes linear equivalence). Since the class B^0 is fixed of positive degree but p_8 is very general, this would imply that $3b(p-q)$ for every pair of interior points $p, q \in \tilde{E}_7$, contradicting the fact that the identity component of $\mathrm{Pic}(\Gamma_K)$ is isomorphic to the multiplicative group $\mathbb{C}$ of the ground field (and so not every element is a torsion element). Thus there is no such prime divisor C. □

Remark 3.10. When $8 \le s \le 15$, it is enough for p_8 to be a general, rather than very general, point of $\tilde{E}_7$ in order to conclude that S_K has no (-2)-curves other than those arising as components of the exceptional loci of the points blown up. To see this, consider a prime divisor $C \subset S_K$ such that $K_{S_K}\cdot C = 0$ and $C\cdot L > 0$. Write $C \sim dL - m_1E_1 - \cdots - m_sE_s$. Then, as above, $C = dL - m(E_1 + \cdots + E_7) - m_8E_8 - \cdots - m_sE_s = b(8L - 3(E_1+\cdots+E_7)) - m_8E_8 - \cdots - m_sE_s$ and $m = m_8 + \cdots + m_s$, so

$$-2 = C^2 = b^2 - m_8^2 - \cdots - m_s^2 \le b^2 - \frac{m^2}{(s-7)^2}(s-7) = b^2\frac{s-16}{s-7},$$

hence for $8 \le s \le 15$ we have

$$d^2 = 8b^2 \le 8\frac{2s-14}{16-s}.$$

Thus for $8 \le s \le 15$ we have $d^2 \le 128$, so $d \le 11$.

I.e., for $8 \le s \le 15$ we see that d is bounded (i.e., $C\cdot L \le 11$) and hence that there are only finitely many possible (-2)-classes C. Since it is only for these classes that we must avoid $C|_{-K_X} = 0$ in order for C not to be effective, it is enough for p_8 to be general, in order to know that every (-2)-class is a component of the exceptional locus of a blow up.

Corollary 3.11. Let $v(\xi, t)$ be a very general divisorial quasimonomial valuation with $t = 7 + 1/n_2$ for $n_2 \ge 1$, and let S_K be the blowup of its cluster of centers. Then $\overline{NE}(S_K)$ is a cone with at most countably many extremal rays, spanned by the classes of the $\tilde{E}_i$, $\tilde{\Gamma}$ and the (-1)-curves, where Γ is a nodal cubic as above. Moreover, when $n_2 > 8$, there are infinitely many (-1)-curves.

Proof. Because of Mori's theorem, and because every divisor C in $\overline{NE}(S_K)^{<}$ either is a component of $\Gamma_K = \tilde{\Gamma} + \sum_{i=1}^{7} \tilde{E}_i$ or satisfies $C\cdot\Gamma_K = 0$, it is enough by Proposition 3.8Def.3.8 to show that the only prime divisors with $C\cdot\Gamma_K = 0$ are the (-2)-curves of the form $\tilde{E}_i$. But this follows from Lemma 3.9Def.3.9.

There will indeed be infinitely many extremal rays when $n_2 \ge 9$, because in this situation there are infinitely many (-1)-curves C. Briefly, we reduce to the case that S_K is the blow up of a cluster of 9 infinitely near points coming from blowing up 9 times at a very general point of a nodal cubic. In this situation, the only restrictions for a divisor C with $C^2 = C \cdot \kappa = -1$ to be a (-1)-curve follow from the proximity inequalities, which impose restrictions only to the monotonicity of the multiplicities of C at the centers of the blowups.

In more detail, apply the degree 8 Cremona map Φ_8 given by $|8L - 3(E_1 + \cdots + E_7)|$ (see Proposition 5.4Def.5.4), which maps S_K to P^2, mapping E_7 to a nodal cubic Γ^0 and representing S_K as a blowup of P^2 of two clusters of points. One is a cluster of 7 points $p^0_1, \dots, p^0_7$ on Γ^0 infinitely near the node, and the other is a cluster of n_2 points $p^0_8, \dots, p^0_{7+n_2}$ on Γ^0 infinitely near p^0_8, which is a very general point of Γ^0. If $n_2 \geq 9$, the blowup of $p^0_8, \dots, p^0_{16}$ gives a surface S with infinitely many (-1)-curves. Blowing up the remaining points p^0_i does not affect this, since none of the remaining points p^0_i can be on any of the (-1)-curves on S. (This is because the generality of $v(\xi, t)$ causes every (-1)-curve on S except E^0_{16} to meet the proper transform $\tilde{\Gamma}^0$ at points not infinitely near to either p^0_1 and p^0_8. For the fact that S has infinitely many (-1)-curves, using the notation of Remark 3.12Def.3.12, note that there are infinitely many classes $C = dL^0 - m_8E^0_8 - \cdots - m_{16}E^0_{16}$ with $C^2 = C\kappa_S = -1$ such that $m_8 \geq \cdots \geq m_{16}$, where κ_S is the canonical class of S. In fact it is not hard to see that all C with $C^2 = C\kappa_S = -1$ are precisely the classes $C = E^0_{16} + N + \frac{N^2}{2}\kappa_S$ where N is an arbitrary class satisfying $N \cdot \kappa_S = 0$ and $N \cdot E^0_{16} = 0$, hence N is any integer linear combination of $L^0 - E^0_8 - E^0_9 - E^0_{10}, \tilde{E}^0_8 = E^0_8 - E^0_9, \dots, \tilde{E}^0_{14} = E^0_{14} - E^0_{15}$. Clearly there are not only infinitely many such C but also infinitely many also satisfying $m_8 \geq \cdots \geq m_{16}$. Any divisor D on S with $D \cdot \kappa_S = 0$ is by linear algebra an integer linear combination of $L^0 - E^0_8 - E^0_9 - E^0_{10}, \tilde{E}^0_8, \dots \tilde{E}^0_{15}$. If D is in addition a prime divisor but not one of the $\tilde{E}^0_i$ nor $-\kappa_S$, then D is in the kernel of the functorial homomorphism $\pi : \mathrm{Pic}(S) \to \mathrm{Pic}(\tilde{\Gamma}^0)$, but the expression of D as a linear combination of $L^0 - E^0_8 - E^0_9 - E^0_{10}, \tilde{E}^0_8, \dots, \tilde{E}^0_{15}$ must involve $L^0 - E^0_8 - E^0_9 - E^0_{10}$, which implies that the image of $L^0 - E^0_8 - E^0_9 - E^0_{10}$ under π has finite order, contradicting the cluster $p^0_8, \dots, p^0_{7+n_2}$ being very general. Thus the only prime divisors satisfying $D \cdot \kappa_S = 0$ are $\tilde{E}^0_8, \dots \tilde{E}^0_{15}$ and $-\kappa_S$. It now follows by [23, Proposition 3.3] that every class $C = dL^0 - m_8 E^0_8 - \cdots - m_{16} E^0_{16}$ with $C^2 = C\kappa_S = -1$ such that $m_8 \geq \cdots \geq m_{16}$ is the class of a (-1)-curve.) $\square$

Remark 3.12. Here we explain the action of Φ_8, used in the proof of Corollary 3.11Def.3.11, in terms of the components of E_1. With $s = 7+n_2$, the components are $\tilde{E}_1 = E_1 - E_2$, $\tilde{E}_2 = E_2 - E_3$, $\tilde{E}_3 = E_3 - E_4$, $\tilde{E}_4 = E_4 - E_5$, $\tilde{E}_5 = E_5 - E_6$, $\tilde{E}_6 = E_6 - E_7$, $\tilde{E}_7 = E_7 - E_8 - \cdots - E_s$, $\tilde{E}_8 = E_8 - E_9, \dots, \tilde{E}_{s-1} = E_{s-1} - E_s$ and $\tilde{E}_s = E_s$. Applying Φ_8 is equivalent to blowing down the (-1)-curve $3L - 2E_1 - E_2 - \cdots - E_7$, followed by $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$, $\tilde{E}_4$, $\tilde{E}_5$, and $\tilde{E}_6$. Under this blow down, $\tilde{E}_7$ maps to a nodal cubic C^0 whose node is the image of the contracted curves, while $E_s, \tilde{E}_{s-1}, \cdots, \tilde{E}_8$ contract to a smooth point on this cubic. Reversing this blow down gives a blow up of P^2 at two clusters of points, the first $p^0_1, \dots, p^0_7$, and the second $p^0_8, \dots, p^0_s$ where $p^0_1, p^0_8 \in P^2$ and all of the points are free but lie on the proper transform of C^0. In terms of the exceptional divisors E^0_i of the centers p^0_i we have $\tilde{E}^0_1 = E^0_1 - E^0_2 = E_6 - E_7$, $\tilde{E}^0_2 = E^0_2 - E^0_3 = E_5 - E_6$, $\tilde{E}^0_3 = E^0_3 - E^0_4 = E_4 - E_5$, $\tilde{E}^0_4 = E^0_4 - E^0_5 = E_3 - E_4$, $\tilde{E}^0_5 = E^0_5 - E^0_6 = E_2 - E_3$, $\tilde{E}^0_6 = E^0_6 - E^0_7 = E_1 - E_2$, $E^0_7 = \tilde{E}^0_7 = 3L - 2E_1 - E_2 - \cdots - E_7$, and also $\tilde{E}^0_8 = E^0_8 - E^0_9 = E_8 - E_9, \dots, \tilde{E}^0_{s-1} = E^0_{s-1} - E^0_s = E_{s-1} - E_s$ and $E^0_s = E_s$.

We also have $L' = 8L - 3E_1 - \cdots - 3E_7$, and $\hat{E}_7 = E_7 - E_8 - \cdots - E_s = 3L' - 2E'_1 - E'_2 - \cdots - E'_s$.

4 A variation on Nagata's conjecture

In this section we elaborate on the close analogy with Nagata's conjecture.

Let K be a finite union of finite weighted clusters on $\mathbb{P}^2$, and assume that the proximity inequalities

$$m_p \geq \sum_{q \to p} m_q$$

are satisfied, with the sum taken over all points $q \in K$ proximate to p.

Then

$$\mathcal{H}_{K,m} = \pi_* \left(\mathcal{O}_{S_K} \left(- \sum_{p\in K} m_p E_p \right) \right)$$

is an ideal sheaf on $\mathbb{P}^2$ for which

$$h^0(\mathcal{H}_{K,m}(d)) = \frac{(d+1)(d+2)}{2} - \sum_{p\in K} \frac{m_p (m_p+1)}{2}$$

for $d \gg 0$, and its general member defines a degree d curve with multiplicity m_p at each $p \in K$.

It is expected that, if K is suitably general, then the dimension count is correct as soon as it gives a nonnegative value:

Conjecture 4.1 (Greuel-Lossen-Shustin, [13, Conjecture 6.3]). Let K be a finite union of weighted clusters on the plane, satisfying the proximity inequalities, and $\mathcal{H}_{K,m}$ the corresponding ideal sheaf. Assume that K is general among all clusters with the same proximities, and let d be an integer which is larger than the sum of the three biggest multiplicities of m. Then

$$h^0(\mathcal{H}_{K,m}(d)) = \max\left\{0, \frac{(d+1)(d+2)}{2} - \sum_{p\in K} \frac{m_p(m_p+1)}{2}\right\}.$$

Proposition 4.2. If the Greuel-Lossen-Shustin conjecture holds, then $\forall t \geq 9$ a very general quasimonomial valuation $v(\xi, t)$ is minimal.

Proof. By continuity of $\hat\mu(t)$, it is enough to consider rational $t > 9$. Let $K = (p_1, \ldots, p_s)$ be the sequence of centers, with weights $(v_1, \ldots, v_s)$. For each integer $k > 0$, set $m_k = \lceil kt/v_s \rceil$. We shall prove that there is a sequence of integers d_k with $m_k > d_k \sqrt{t}$ and $\lim_{k\to\infty} m_k/d_k = \sqrt{t}$ such that if ξ is very general, then the valuation ideal $\mathcal{I}_{m_k}$ has no sections of degree d. It will follow that $\mu(\xi, t) \leq \lim_{k\to\infty} m_k/d \leq \sqrt{t}$ and $v(\xi, t)$ is minimal.

By Lemma 2.9Def.2.9, the ideal $\mathcal{I}_{m_k} = (\pi_K)_*(\mathcal{O}_{S_K}(-\sum \bar{m}_i E_i))$ is simple and the three largest multiplicities are $\bar{m}_1 = \bar{m}_2 = \bar{m}_3 = k/v_s$. Hence $\bar{m}_1 + \bar{m}_2 + \bar{m}_3 = 3k/v_s < \sqrt{t}k/v_s$. Without loss of generality we may assume that k is large enough that there exist integers $d_k < m_k/\sqrt{t}$ which also satisfy $d_k > \bar{m}_1 + \bar{m}_2 + \bar{m}_3$. In this case the hypothesis in conjecture 4.1Greuel-Lossen-Shustin, [13, Conjecture 6.3]Def.4.1 is satisfied and $h^0(\mathcal{H}_{K,m}(d_k)) =$

$\max\{0,(d_k+1)(d_k+2)/2-\sum \bar m_i(\bar m_i+1)/2\}$. By way of contradiction, assume $\mathbf{I}_{m_k}$ has sections of degree d_k. Then $(d_k+1)(d_k+2)/2 \geq \sum \bar m_i(\bar m_i+1)/2$, which together with $d_k < m_k/\sqrt{t} = \sum \bar m_i^2$ implies $3d_k+2 > \sum \bar m_i \geq 10\bar m_1 > kt/v_s = m_k$, a contradiction. □

With this in mind, we propose the following:

Conjecture 4.3 (Nagata's Conjecture for quasimonomial valuations). For all $t \geq 9$, we have $\hat\mu(t) = \sqrt{t}$.

Proposition 4.4. Conjecture 4.3Nagata's Conjecture for quasimonomial valuationsDef.4.3 implies Nagata's conjecture.

Proof. Let $t > 9$ be a nonsquare integer. By a "collision de front" [20] and semicontinuity, Nagata's conjecture for t points would follow by showing that, for a very general $\xi(x) \in \mathbb{C}[[x]]$, and for every couple of integers d, m with $0 < d < m\sqrt{t}$, the ideal $(x^t, y-\xi(x))^m \cap \mathbb{C}[x,y]$ has no nonzero element in degree d. But this is an immediate consequence of $\hat\mu(t) = \sqrt{t}$. □

In view of the computations in next section, we expect that in fact the range of t for which $\hat\mu(t) = \sqrt{t}$ is larger, see Conjecture 5.11Def.5.11.

5 Supraminimal curves

If some valuation v is not minimal, this is due to the existence of a curve C (which may be taken irreducible and reduced) with larger valuation than what one would expect from the degree. These curves will be called supraminimal, and are the subject of this section. For simplicity, we fix $p_1 = (0,0) \in \mathbb{A}^2 \subset \mathbb{P}^2$ as before.

Lemma 5.1. If there is an irreducible polynomial $f \in \mathbb{C}[x,y]$ with

$$v(\xi,t;f) > \frac{1}{\sqrt{\operatorname{vol}(v(\xi,t))}}\deg(f)\ ,$$

then $v(\xi,t;f) = \mu(\xi,t)\deg(f)$.

Moreover, if $\hat\mu(\xi,t) > \frac{1}{\sqrt{\operatorname{vol}(v(\xi,t))}}$, then there is such an irreducible polynomial f.

In the case above we say that f computes $\hat\mu(\xi,t)$.

Proof. By continuity of $\hat\mu(\xi,t)$ as a function of t, it is enough to consider the case $t \in \mathbb{Q}$. Let $v = v(\xi,t)$.

Let f be as in the claim, and $d = \deg f$. It will be enough to prove that, for every polynomial g with degree e and $v(g) = w > \frac{e}{\sqrt{\operatorname{vol}(v)}}$, f divides g. Choose an integer k such that $kw \in \mathbb{N}$ is an integer multiple of t, and consider the ideal

$$I_{kw} = \{h \in \mathbb{C}[x,y] \mid v(h) \geq kw\}.$$

A general $h \in I_{kw}$ has kw/t Puiseux series roots, each of them of the form $\xi(x)+ax^t+\dots$; therefore the local intersection multiplicity of $h=0$ with $f=0$ is

$$I_0(h,f) \geq \frac{kw}{t}v(f) > \frac{kwd}{t\sqrt{\operatorname{vol}(v)}} = \frac{kwd}{\sqrt{t}}. \qquad (\ddagger)$$

Since obviously $g^k \in I$, the intersection multiplicity $I_0(g^k, f)$ is bounded below by (‡Supraminimal curvesequation.5.3), and therefore

$$I_0(g, f) > \frac{wd}{\sqrt{t}} = dw\sqrt{\mathrm{vol}(v)} > de,$$

so f is a component of g.

Now assume $\hat{\mu}(v) > \frac{1}{\sqrt{\mathrm{vol}(v)}}$. So there is a polynomial $g \in \mathbb{C}[x, y]$ of degree e with $v(g) > \frac{e}{\sqrt{\mathrm{vol}(v)}}$. Since $v(f_1 \cdot f_2) = v(f_1) + v(f_2)$, it follows that at least one irreducible component f of g, satisfies $v(f) > \frac{\deg f}{\sqrt{\mathrm{vol}(v)}}$. □

Proposition 5.2. Assume that $d \in \mathbb{N}$, $m_1/n_1, \dots, m_r/n_r \in \mathbb{Q}$, with $\gcd\{m_i, n_i\} = 1$ are such that, for a very general $\xi(x)$, there exists an irreducible $f \in \mathbb{C}[x, y]$ with $\deg(f) = d$ which decomposes in $\mathbb{C}[[x, y]]$ as a product of r irreducible series $f = f_1 \dots f_r$ with $\mathrm{ord}_x f_i(x, \xi(x)) = m_i$, $\mathrm{ord}_x f_i(x, y) = n_i$. Consider the tropical polynomial

$$\mu_f(t) = \sum_{i=1}^{r} \min(n_i t, m_i).$$

Then $\hat{\mu}(t) \geq \mu_f(t)/d$, with equality at all values of t such that $\mu_f(t) > d\sqrt{t}$.

Proof. It is immediate that $v(\xi, t; f) = \mu_f(t)$, so the inequality $\hat{\mu}(t) \geq \mu_f(t)/d$ is clear. Now assume that $\mu_f(t) > d\sqrt{t}$. This implies that $v(\xi, t)$ is not minimal, $\mu(v(\xi, t)) = \hat{\mu}(t)$. □

Example 5.3. The easiest examples of the situation described in Proposition 5.2Def.5.2 are given by (smooth) curves of degree 1 and 2.

Namely, for $d = 1$, $m_1/n_1 = 2$, it is trivial that for general $\xi(x)$, there exists a degree 1 polynomial f with $\mathrm{ord}_x f(x, \xi(x)) = 2$, $\mathrm{ord}\, f_i(x, y) = 1$; one simply has to take the equation of the tangent line to $y - \xi(x) = 0$, or $f = y - \xi_1(x)$ (where ξ_1 denotes the 1-jet).

In the same vein, for $d = 2$, $m_1/n_1 = 5$, it is easy to show that for general $\xi(x)$, there exists a degree 2 polynomial f with $\mathrm{ord}_x f(x, \xi(x)) = 5$, $\mathrm{ord}\, f_i(x, y) = 1$, which for general ξ is irreducible; one simply has to take the equation of the conic through the first five points infinitely near to $(0, 0)$ on the curve $y - \xi(x) = 0$ (more fancily, the curvilinear ideal $(y - \xi(x)) + (x, y)^5 \subset \mathbb{C}[x, y]$ has maximal Hilbert function and colength 5, and therefore a unique element in degree 2 up to a constant factor).

Proposition 5.2Def.5.2 then gives that

$$\hat{\mu}(t) = \begin{cases} t & \text{if } 1 \leq t \leq 2, \text{ computed by a line,} \\ 2 & \text{if } 2 \leq t \leq 4, \text{ computed by a line,} \\ t/2 & \text{if } 4 \leq t \leq 5, \text{ computed by a conic,} \\ 5/2 & \text{if } 5 \leq t \leq 25/4, \text{ computed by a conic.} \end{cases}$$

In order to construct the supraminimal curves in general position computing the function $\hat{\mu}$ for small values of t, we need certain Cremona maps, presumably well known, which have been used by Orevkov in [26] to show sharpness of his bound on the degree of cuspidal rational curves.

Proposition 5.4. Let $K = (p_1, \dots, p_7)$ be a general cluster with p_{i+1} infinitely near to p_i for $i = 1, \dots, 6$. There exists a degree 8 plane Cremona map Φ_8 whose cluster of fundamental points is K, with all points weighted with multiplicity 3, and satifying the following properties:

1. The characteristic matrix of Φ_8 is
$$\begin{pmatrix} 8 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -3 & -1 & -2 & -1 & -1 & -1 & -1 & -1 \\ -3 & -2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -2 & -1 & -1 & -1 \\ -3 & -1 & -1 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -2 & -1 \\ -3 & -1 & -1 & -1 & -1 & -2 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -1 & -2 \end{pmatrix}.$$

2. The inverse Cremona map is of the same type, i.e., it has the same characteristic matrix and its fundamental points are a sequence, each infinitely near to the preceding one.

3. The only curve contracted by Φ_8 is the nodal cubic which is singular at p_1 and goes through $(p_2, \dots, p_7)$. The only expansive fundamental point is p_7, whose relative principal curve is the nodal cubic going through the fundamental points of the inverse map, and singular at the first of them.

Recall that the characteristic matrix of a plane Cremona map is the matrix of base change in the Picard group of the blow up $\pi : S \to \mathbb{P}^2$ that resolves the map, from the natural base formed by the class of a line and the exceptional divisors, to the natural base in the image $\hat{\mathbb{P}}^2$, formed by the class of a line there (the homaloidal net in the original $\mathbb{P}^2$) and the divisors contracted by the map (which are the exceptional divisors of $\pi' : S \to \hat{\mathbb{P}}^2$), see [1]. We use it later on to compute images of curves under Φ_8.

Proof. This proof is taken from [26, p. 667]; the only modification lies in the remark that K can be taken general. Indeed, for K general, there exists a unique irreducible nodal cubic Γ with multiplicity 2 at p_1 and going through $p_2, \dots, p_7$. Φ_8 is then defined as follows: let $\pi_K : S_K \to \mathbb{P}^2$ be the blowup of all points on K. The (proper) exceptional divisors $\tilde{E}_1, \dots, \tilde{E}_6$ are (-2)-curves, E_7 is a (-1) curve. The proper transform $\tilde{\Gamma} \subset S_K$ is another (-1)-curve that meets the (proper) exceptional divisors $\tilde{E}_1$ and E_7. Blow down $\tilde{\Gamma}, \tilde{E}_1, \dots \tilde{E}_6$ to obtain another map $\pi'_K : S_K \to \hat{\mathbb{P}}^2$. Then take $\Phi_8 = \pi'_K \circ \pi_K^{-1}$. All the stated properties are easy to check. □

Denote $F_{-1} = 1$, $F_0 = 0$ and $F_{i+1} = F_i + F_{i-1}$ the Fibonacci numbers, and $\varphi = (1 + \sqrt{5})/2 = \lim F_{i+1}/F_i$ the "golden ratio".

Proposition 5.5. For each odd $i \geq 1$, there exist rational curves C_i of degree F_i with a single cuspidal singularity of characteristic exponent F_{i+2}/F_{i-2} whose six singular free points are in general position. These curves become (-1)-curves in their embedded resolution, and are supraminimal for t in the interval $\left[\frac{F_i^2}{F_{i-2}^2}, \frac{F_{i+2}^2}{F_i^2}\right]$.

Note that for $i = 1$ the line is actually not singular (the "characteristic exponent" is 2, an integer) but the statement in that case means that the line goes through the first two of six infinitely near points in general position, i.e., the exponent is interpreted as $m_i/n_i = 2$ in Proposition 5.2Def.5.2.

Proof. The existence of such curves, without the generality statement, is [26, Theorem C, (a) and (b)]. Since the construction goes by recursively applying the rational map Φ_8, and the free singular points of C_i are exactly the seven fundamental points of Φ_8, it follows from 5.4Def.5.4 that these can be chosen to be general. They are (-1)-curves after resolution because the starting point of the construction are the two lines tangent to the two branches of the nodal cubic Γ (which becomes an exceptional divisor after Φ_8) i.e., (-1)-curves (each is a line through a point and an infinitely near point).

Now, with notation as in Proposition 5.2Def.5.2,

$$\mu_f(t) = \begin{cases} \frac{F_{i-2}}{F_i} t & \text{if } t \le \frac{F_{i+2}}{F_{i-2}}, \\ \frac{F_{i+2}}{F_i} & \text{if } t \ge \frac{F_{i+2}}{F_{i-2}}. \end{cases}$$

supraminimality in the claimed interval follows. □

Corollary 5.6. *For every odd i,*

$$\hat\mu(t) = \begin{cases} \frac{F_{i-2}}{F_i} t & \text{if } t \in \left[\frac{F_i^2}{F_{i-2}^2}, \frac{F_{i+2}}{F_{i-2}}\right], \\ \frac{F_{i+2}}{F_i} & \text{if } t \in \left[\frac{F_{i+2}}{F_{i-2}}, \frac{F_{i+2}^2}{F_i^2}\right]. \end{cases}$$

Remark 5.7. We proved that supraminimal values of $\hat\mu$ are computed by a single irreducible curve. In contrast, we see that the minimal values at $t = F_{i+2}^2/F_i^2$ are computed both by C_i and C_{i+2}. In fact, the two divisors $F_{i+2}C_i$ and F_iC_{i+2} generate a pencil whose general members also compute $\hat\mu(t)$; they are again unicuspidal rational curves classified in [15, Theorem 1.1, (c)].

Remark 5.8. In addition to the preceding family of curves, nine additional (-1)-curves compute $\hat\mu(t)$ for some range of t (see Table 5.1Sporadic supraminimal curves. Here $(d; v_i)$ denotes the degree and multiplicities sequence, with $^{\times k}$ meaning k-tuple repetition, and m_i/n_i follows the notation of Proposition 5.2Def.5.2, with $^{\times 2}$ again meaning repetitiontable.5.1). The existence of these curves is proved as follows. D_1 and D_2 are well known. The rest are obtained by applying the Cremona map Φ_8 to already constructed curves (the names chosen indicate that curve X^* is built from curve X). Recall that, because the intervals where a degree d curve C computes $\hat\mu$ are those where $\mu_f(t) \ge d\sqrt{t}$ (Proposition μ) the endpoints correspond to values of t where $\hat\mu$ is minimal. Note that all such endpoints given in Table 5.1Sporadic supraminimal curves. Here $(d; v_i)$ denotes the degree and multiplicities sequence, with $^{\times k}$ meaning k-tuple repetition, and m_i/n_i follows the notation of Proposition 5.2Def.5.2, with $^{\times 2}$ again meaning repetitiontable.5.1 are squares in $\mathbb{Q}$ or in the quadratic field to which they belong. This is due to the fact that they satisfy $\mu_f(t) = d\sqrt{t}$, and $\mu_f(t)$ is a piecewise affine linear function of t with rational coefficients.

Name	$(d; v_i)$	m_i/n_i	t
D_1	$(3; 2, 1^{\times 6})$	1,7	$\left[\varphi^4, \left(\frac{8}{3}\right)^2\right]$
D_2^*	$(48; 18^{\times 7}, 3, 2^{\times 7})$	$7,\ \left(7+\frac{1}{8}\right)^{\times 2},\ 8$	$\left(\frac{24+\sqrt{457}}{17}\right)^2,\ \left(24-\sqrt{455}\right)^2$
C_1^{**}	$(64; 24^{\times 7}, 3^{\times 7}, 1^{\times 2})$	$7^{\times 2},\ 7+\frac{1}{7+1/2},\ 7+\frac{1}{7}$	$\left(\frac{32-\sqrt{177}}{7}\right)^2,\ \left(\frac{16+\sqrt{179}}{11}\right)^2$
D_1^*	$(24; 9^{\times 7}, 2, 1^{\times 6})$	$7,\ 7+\frac{1}{7},\ 8$	$\left(\frac{6+\sqrt{22}}{4}\right)^2,\ \left(12-\sqrt{87}\right)^2$
C_5^*	$(40; 15^{\times 7}, 2^{\times 6}, 1^{\times 2})$	$7^{\times 2},\ 7+\frac{1}{6+1/2}$	$\left(\frac{20+\sqrt{218}}{13}\right)^2,\ \left(\frac{107}{40}\right)^2$
C_3^*	$(16; 6^{\times 7}, 1^{\times 5})$	$7,\ 7+\frac{1}{5}$	$\left(\frac{8+\sqrt{29}}{5}\right)^2,\ \left(\frac{43}{16}\right)^2$
D_3	$(35; 13^{\times 7}, 4, 3^{\times 3})$	$7+\frac{1}{4},\ 8$	$\left(\frac{35}{13}\right)^2,\ \left(\frac{35-\sqrt{877}}{2}\right)^2$
C_1^*	$(8; 3^{\times 7}, 1^{\times 2})$	$7,\ 7+\frac{1}{2}$	$\left(\frac{4+\sqrt{2}}{2}\right)^2,\ \left(\frac{22}{8}\right)^2$
D_2	$(6; 3, 2^{\times 7})$	1, 8	$\left(\frac{3+\sqrt{7}}{2}\right)^2,\ \left(\frac{17}{6}\right)^2$

Table 5.1: Sporadic supraminimal curves. Here $(d; v_i)$ denotes the degree and multiplicities sequence, with $^{\times k}$ meaning k-tuple repetition, and m_i/n_i follows the notation of Proposition 5.2Def.5.2, with $^{\times 2}$ again meaning repetition.

Example 5.9. As an example, let us show the existence of D_1^*. Let $K = (p_1, \dots, p_8)$ be a general cluster with each point infinitely near to the preceding one; we want to show that there is an irreducible curve of degree 24 with three branches, two smooth, one of which goes through $(p_1, \dots, p_7)$ and the other through all of K, and one singular, with characteristic exponent 50/7. Because K is general, there exist a cubic curve D_1 with multiplicities $[2, 1^6, 0]$ on K and another cubic Γ through K that has a node at some other point q_1. Choose one of the branches of Γ and let $q_2, \dots, q_7$ be the points infinitely near to q_1 on that branch. Apply the Cremona map Φ_8 based on $(q_1, \dots, q_7)$: then $D_1^* = \Phi_8(D_1)$.

All these computations together show that indeed, (-1)-curves compute $\hat\mu$ in the anticanonical range:

Theorem 5.10. For $t \in A$, $\hat\mu(t)$ is computed by (-1)-curves; more precisely, the (infinitely many) curves C_i and 7 of the curves in table 5.1Sporadic supraminimal curves. Here $(d; v_i)$ denotes the degree and multiplicities sequence, with $^{\times k}$ meaning k-tuple repetition, and m_i/n_i follows the notation of Proposition 5.2Def.5.2, with $^{\times 2}$ again meaning repetitiontable.5.1.

Figure 1In red, the known behavior of $\hat\mu(t)$ for $t \geq 9$; in yellow, the lower bound $\sqrt{t}$figure.1 shows $\hat\mu(t)$ in the ranges where it is known, together with the lower bound $\sqrt{t}$.

1 5 μ in ranges of t which do not intersect

The two curves C^{**} and C^{*} compute b
the anticanonical locus A. We expect that there are no more curves with such behavior, and so propose the following strengthening of conjecture 4.3Nagata's Conjecture for quasimonomial valuationsDef.4.3:

Conjecture 5.11. Let $t \in \mathbb{R}$ be such that $\mu(t) > \sqrt{t}$. Then $\mu_C(t) > \sqrt{t}$ for a curve C which is either on the list of table b5.1Sporadic supraminimal curves. Here $(d; v_i)$ denotes the degree and multiplicities sequence, with $^{\times k}$ meaning k-tuple repetition, and m_i/n_i follows the notation of Proposition 5.2Def.5.2, with $^{\times 2}$ again meaning repetitiontable.5.1 or one of the C_i. Equivalently, if $t > 7 + 1/9$ is not contained in any one of the intervals of table 5.1Sporadic supraminimal curves. Here $(d; v_i)$ denotes the degree and multiplicities sequence, with $^{\times k}$ meaning k-tuple repetition, and m_i/n_i follows the notation of Proposition 5.2Def.5.2, with $^{\times 2}$ again meaning repetitiontable.5.1, then a very general valuation $v(\xi, t)$ is minimal.

Obviously, Conjecture 5.11Def.5.11 implies Conjecture 2.4Def.2.4.

Remark 5.12. For $t > (17/6)^2$, it is possible to show (using Cremona maps) that no (-1)-curve is ever supraminimal. Thus conjecture 5.11Def.5.11 splits naturally into two conjectures: first, that all supraminimal curves are (-1)-curves, and second, that the only supraminimal (-1)-curves in the interval $[7, 8]$ are the ones above. Our evidence for the latter statement is experimental, obtained by a computer search.

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