This is a preprint of: "Centers for the Kukles homogeneous systems with even degree", Jaume Giné, Jaume Llibre, Clàudia Valls, *J. Appl. Anal. Comp.*, vol. 7(4), 1534–1548, 2017.
DOI: [10.11948/2017093]

Journal of Applied Analysis and Computation
Volume *, Number *, * *, 1–15

Website:http://jaac-online.com/
DOI:10.11948/****

CENTERS FOR THE KUKLES HOMOGENEOUS SYSTEMS WITH EVEN DEGREE

Jaume Giné[1,†], Jaume Llibre[2] and Claudia Valls[3]

Abstract For the polynomial differential system $\dot{x} = -y$, $\dot{y} = x + Q_n(x, y)$, where $Q_n(x, y)$ is a homogeneous polynomial of degree n there are the following two conjectures done in 1999. (1) Is it true that the previous system for $n \geq 2$ has a center at the origin if and only if its vector field is symmetric about one of the coordinate axes? (2) Is it true that the origin is an isochronous center of the previous system with the exception of the linear center only if the system has even degree? We give a step forward in the direction of proving both conjectures for all n even. More precisely, we prove both conjectures in the case $n = 4$ and for $n \geq 6$ even under the assumption that if the system has a center or an isochronous center at the origin, then it is symmetric with respect to one of the coordinate axes, or it has a local analytic first integral which is continuous in the parameters of the system in a neighborhood of zero in the parameters space. The case of n odd was studied in [8].

Keywords Center-focus problem, isochronous center, Poincaré-Liapunov constants, Gröbner basis of polynomial systems.

MSC(2010) 34C05, 37C10.

1. Introduction and statement of the main results

The conditions under which the origin for the differential system of the form

$$\dot{x} = -y, \quad \dot{y} = x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3, \qquad (1.1)$$

is a center were found in [11]. During may years it had been thought that these conditions were necessary and sufficient conditions, but some new centers have been found later on, see [2,10]. In [4] the center problem for the class of system (1.1) with $a_7 = 0$ was solved, and it was proved that at most five limit cycles bifurcate from the origin. In [12] it was solved the center problem for system (1.1) when $a_2 = 0$ and it was proved that at most six limit cycles bifurcate from the origin. The first complete solution of the center-focus problem of system (1.1) was obtained in [13]. Using the Cherkas' method of passing to a Liénard equation, in [16] it was also given the complete solution of the center-focus problem of system (1.1), see also the works [14,18]. The study of this family is the first step to the full classification of the cubic polynomial differential systems with a center.

[†]the corresponding author. Email address: gine@matematica.udl.cat (J. Giné)
[1]Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69; 25001 Lleida, Catalonia, Spain
[2]Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
[3]Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001, Lisboa, Portugal

In this paper we continue the classification of the centers for a linear center plus a homogeneous polynomial, more precisely for systems of the form

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x, y), \tag{1.2}$$

where $Q_n(x, y)$ is a homogeneous polynomial of degree n, i.e.

$$Q_n(x, y) = \sum_{j=0}^{n} c_j x^j y^{n-j}, \quad c_j \in \mathbb{R}. \tag{1.3}$$

In [7] these systems were called *Kukles homogeneous systems*, also called *lopsided systems* in others works, see [15, 17]. Another main objective is to characterize the isochronous centers of systems of the form (1.2). We recall that a center is *isochronous* if all the periodic solutions have the same constant period in a neighborhood of the origin. See [19] and references therein for a survey about the isochronicity. In the paper [19] the following two conjectures were stated.

Conjecture 1. *Is it true that a system* (1.2) *with nonlinearities of degree higher than two has a center at the origin if and only if its vector field is symmetric about one of the coordinate axes?*

Conjecture 2. *Is it true that the origin is an isochronous center of system* (1.2) *with the exception of the linear center only if the system has even degree and is reduced to system*

$$\dot{x} = -y, \quad \dot{y} = x + x^{2m-1} y(x^2 + y^2), \tag{1.4}$$

after a change of variables an a possible scaling of the time?

From [19] it is known that system (1.2) when $n = 2$ and $n = 3$ has no isochronous centers at the origin. Also in [19] it is shown that there is exactly one isochronous system for $n = 4$ which is system (1.4) with $m = 1$, and that if $n = 5$ or $n = 7$ then the origin is never an isochronous center. In [7] it is given a positive answer to the Conjecture 1 for $n = 4$ and $n = 5$, see also [15,17]. In [8] the authors give a positive answer to both Conjectures 1 and 2 for $n \geq 5$ odd.

We note that there are very few results classifying centers for polynomial systems of arbitrary degree. A family where its centers are classified is the Liénard polynomial differential systems of arbitrary degree, see [3]. In the next theorem we prove both conjectures in the case $n = 4$ and for $n \geq 6$ even under the assumption that if the system has a center or an isochronous center at the origin, then it is symmetric with respect to one of the coordinate axes, or it has a local analytic first integral which is continuous in the parameters of the system in a neighborhood of zero in the parameters space. We recall that a monodromic system such as (1.2) has a center at the origin if and only if there exists an analytic first integral in the variables (x, y) defined in a neighborhood of the origin.

We define the *Property A*: If system (1.2) has a center at the origin, then it is symmetric with respect to one of the coordinate axes, or it has a local analytic first integral in the variables x and y and continuous in the parameters of the system in a neighborhood of zero in the parameters space formed by the free parameters once we have a fixed center.

Theorem 1.1. *System* (1.2) *with* $n \geq 6$ *even satisfies property A if and only if one of the following two conditions hold: either* $c_{2j} = 0$ *for* $j = 0, 1, \ldots, n/2$*, or* $c_{2j+1} = 0$ *for* $j = 0, 1, \ldots, (n-2)/2$.

Theorem 1.1 is equivalent to say that system (1.2) satisfies property A if and only if it is invariant under the symmetry $(x, y, t) \to (-x, y, -t)$, or under the symmetry $(x, y, t) \to (x, -y, -t)$. Thus the phase portrait of the system is either symmetric with respect to the x–axis, or with respect to the y–axis. So Theorem 1.1 goes in direction of proving Conjecture 1 for n even.

We will prove a more general result that will lead to the proof of Theorem 1.1.

Consider the following system

$$\dot{x} = -y, \quad \dot{y} = x + (x^2 + y^2)^{(d-n)/2} Q_n(x, y), \tag{1.5}$$

where $Q_n(x, y)$ is the homogeneous polynomial of degree n, given in (1.3) and $d \geq n$ is even.

Theorem 1.2. *System (1.5) with $n \geq 4$ even and $d \geq n$ even satisfies property A if and only if either $c_{2j} = 0$ for $j = 0, 1, \ldots, n/2$, or $c_{2j+1} = 0$ for $j = 0, 1, \ldots, (n-2)/2$.*

Of course Theorem 1.2 coincides with Theorem 1.1 when $d = n$. The proof of Theorem 1.2 will be done by induction over n and is given in section 2.

The second main result in this paper is the following, which goes in the direction of proving Conjecture 2 for n even.

We define the *Property B*: If system (1.2) has an isochronous center at the origin, then it is symmetric about one of the coordinate axes, or it has a local analytic first integral in the variables x and y and continuous in the parameters of the system in a neighborhood of zero in the parameter space formed by the free parameters once we have a fixed isochronous center.

Theorem 1.3. *System (1.2) with $n \geq 6$ even satisfies property B if and only if $Q_n(x, y) = (x^2 + y^2)x^{n-3}y$.*

Again we will prove a more general result that will lead to the proof of Theorem 1.3.

We consider once again system (1.5). In view of Theorem 1.2 in order that system (1.5) satisfies property A, we must have that either $c_{2j} = 0$ for $j = 0, 1, \ldots, n/2$, or $c_{2j+1} = 0$ for $j = 0, 1, \ldots, (n-2)/2$. Hence, in the first case system (1.5) can be written as

$$\dot{x} = -y, \quad \dot{y} = x + (x^2 + y^2)^{(d-n)/2} x \sum_{j=0}^{(n-2)/2} c_{2j+1} x^{2j} y^{n-2j-1}. \tag{1.6}$$

and in the second case system (1.5) can be written as

$$\dot{x} = -y, \quad \dot{y} = x + (x^2 + y^2)^{(d-n)/2} \sum_{j=0}^{n/2} c_{2j} x^{2j} y^{n-2j}. \tag{1.7}$$

We will prove the following two theorems.

Theorem 1.4. *System (1.7) with $n \geq 4$ even and $d \geq n$ even does not satisfy property B.*

Theorem 1.5. *System* (1.6) *with* $n \geq 4$ *even and* $d \geq n$ *even satisfies property B if and only if it has the form*

$$\dot{x} = -y, \quad \dot{y} = x + (x^2 + y^2)^{(d-n+2)/2} x^{n-3} y.$$

Note that from Theorems 1.4 and 1.5 it follows Theorem 1.3 when $d = n$. The proofs of Theorems 1.4 and 1.5 will be done by induction over n and are given, respectively, in sections 3 and 4.

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2 by induction over n. We start with the case $n = 4$. To prove Theorem 1.2 when $n = 4$, we have to prove the following result.

Proposition 2.1. *System* (1.5) *with* $n = 4$ *and* $d \geq 4$ *even has a center at the origin if and only if one of the following two conditions hold: either* $c_{2j} = 0$ *for* $j = 0, 1, 2$, *or* $c_{2j+1} = 0$ *for* $j = 0, 1$.

Note that when $c_{2j} = 0$ for $j = 0, 1, 2$ the system is invariant under the symmetry $(x, y, t) \to (-x, y, -t)$, and when $c_{2j+1} = 0$ for $j = 0, 1$ the system is invariant under the symmetry $(x, y, t) \to (x, -y, -t)$. Thus when $n = 4$, among the centers which are symmetric about one of the coordinate axes it has no other center whose local analytic first integral is also continuous in the parameters of the system in a neighborhood of zero in the parameter space.

Proof. [Proof of Proposition 2.1] We first prove the sufficiency. If $c_{2j} = 0$ for $j = 0, 1, 2$ then system (1.5) becomes

$$\dot{x} = -y, \quad \dot{y} = x + (x^2 + y^2)^{(d-4)/2} (c_1 x y^3 + c_3 x^3 y). \tag{2.1}$$

System (2.1) is invariant under the symmetry $(x, y, t) \mapsto (-x, y, -t)$ and it is clear that in this situation system (2.1) has a center at the origin.

If $c_{2j+1} = 0$ for $j = 0, 1$, then system (1.5) becomes

$$\dot{x} = -y, \quad \dot{y} = x + (x^2 + y^2)^{(d-4)/2} (c_0 y^4 + c_2 x^2 y^2 + c_4 x^4). \tag{2.2}$$

System (2.2) is invariant under the symmetry $(x, y, t) \mapsto (x, -y, -t)$ and it is clear that in this situation system (2.2) has a center at the origin.

Now we shall prove the necessity. To do that we first write the system in complex variation as

$$\dot{z} = iz + (z\bar{z})^{\frac{d-4}{2}} (A_1 z^4 + A_2 z^3 \bar{z} + A_3 z^2 \bar{z}^2 + A_4 z \bar{z}^3 + A_5 \bar{z}^4), \tag{2.3}$$

where $z = x + iy$, $d \geq 4$ is an arbitrary even integer, and

$$\begin{aligned} A_1 =& a_1 + i a_2, \;\; A_2 = a_3 + i a_4, \;\; A_3 = a_5 + i a_6, \\ A_4 =& a_7 + i a_8, \;\; A_5 = a_9 + i a_{10}, \end{aligned}$$

where $a_i \in \mathbb{R}$. Now we write system (2.3) in the real variables (x, y) and impose that it has the form (1.5). This implies that

$$a_5 = 0, \quad a_7 = -a_3, \quad a_8 = a_4, \quad a_9 = -a_1, \quad a_{10} = a_2. \tag{2.4}$$

Now we write system (1.5) with $n = 4$ and satisfying conditions (2.4) in polar coordinates, i.e., doing the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, and we obtain

$$\dot{r} = F(\theta)\, r^d, \qquad \dot{\theta} = 1 + G(\theta)\, r^{d-1}, \tag{2.5}$$

where $F(\theta)$ and $G(\theta)$ are the following homogeneous trigonometric polynomials

$$\begin{aligned} F(\theta) &= \sin\theta\,(a_6 + 2a_4\cos(2\theta) + 2a_2\cos(4\theta) + 2a_3\sin(2\theta) + 2a_1\sin(4\theta)),\\ G(\theta) &= (a_4 + a_6)\cos\theta + (a_2 + a_4)\cos(3\theta) + a_2\cos(5\theta)\\ &\quad + a_3\sin\theta + (a_1 + a_3)\sin(3\theta) + a_1\sin(5\theta). \end{aligned}$$

System (2.5) is equivalent to

$$\frac{dr}{d\theta} = \frac{F(\theta)\, r^d}{1 + G(\theta)\, r^{d-1}}. \tag{2.6}$$

It is clear that the differential equation (2.6) is well defined in a sufficient small neighborhood of the origin. The transformation $(r, \theta) \to (\rho, \theta)$ introduced by Cherkas [1] and defined by

$$\rho = \frac{r^{d-1}}{1 + G(\theta) r^{d-1}}, \quad \text{whose inverse is} \quad r = \frac{\rho^{1/(d-1)}}{(1 - \rho G(\theta))^{1/(d-1)}}, \tag{2.7}$$

is a diffeomorphism from the region $\dot{\theta} > 0$ into its image. If we transform equation (2.6) using the transformation (2.7), we obtain the Abel equation

$$\frac{d\rho}{d\theta} = -(d-1)G(\theta)F(\theta)\rho^3 + [(d-1)(F(\theta) - G'(\theta)]\rho^2. \tag{2.8}$$

The solution $\rho(\theta, \rho_0)$ of (2.8) satisfying that $\rho(0, \rho_0) = \rho_0$ can be expanded in a convergent series of $\rho_0 \geq 0$ sufficiently small of the form

$$\rho(\theta, \rho_0) = \rho_1(\theta)\rho_0 + \rho_2(\theta)\rho_0^2 + \rho_3(\theta)\rho_0^3 + \cdots \tag{2.9}$$

with $\rho_1(\theta) = 1$ and $\rho_k(0) = 0$ for $k \geq 2$. Let $P : [0, \tilde{\rho}_0] \to \mathbb{R}$ be the Poincaré return map defined by $P(\tilde{\rho}_0) = \rho(2\pi, \tilde{\rho}_0)$ for a convenient $\tilde{\rho}_0$. System (2.3) has a center at the origin if and only if $\rho_k(2\pi) = 0$ for every $k \geq 2$. If we assume that $\rho_2(2\pi) = \cdots = \rho_{m-1}(2\pi) = 0$ we say that $v_m = \rho_m(2\pi) \neq 0$ is the *m-th Poincaré-Liapunov-Abel* constant of system (2.3). If we compute these constants when system (2.5) has a center then all the v_m are zero.

Due to the Hilbert Basis theorem, the ideal $J =< v_1, v_2, ... >$ generated by the Poincaré-Liapunov-Abel constants is finitely generated, i.e. there exist $w_1, w_2, ..., w_k$ in J such that $J =< w_1, w_2, ..., w_k >$. This set of generators is called a *basis* of J and the conditions $w_j = 0$ for $j = 1, \ldots, k$ provide a finite set of necessary conditions to have a center. The set of coefficients for which all the Poincaré-Liapunov-Abel constants v_k vanish is called the *center variety* of the family and it is an algebraic set. First we determine a number of Poincaré-Liapunov-Abel constants assuming that inside this number there is the set of generators. The next step is to decompose this algebraic set into its irreducible components. We must use a computer algebra system. The computational tool which we use is the routine `minAssGTZ` [5] of the computer algebra system SINGULAR [9] which is based on the Gianni-Trager-Zacharias algorithm [6]. The computations in this case can be completed in the field of rational numbers. Hence all the points of the center variety have been found. That is, we know that all the encountered points belong to the decomposition of the center variety and we know that the given decomposition is complete.

We have computed v_k for $k = 1, \ldots, 9$. The decomposition of the ideal $J_9 =< v_1, v_2, ..., v_9 >$ gives two unique cases which are $a_1 = a_3 = 0$ and $a_2 = a_4 = a_6 = 0$. In the first case we obtain system (2.1) and in the second case we obtain system (2.2). This completes the proof of the proposition. □

It follows from Proposition 2.1 that Theorem 1.2 holds for $n = 4$ and $d \geq 4$ even. Now we assume that it holds for $n = 4, 6, \ldots, \ell$ with $\ell \geq 4$ even and we shall prove it for $n = \ell + 2$. Hence we have that system

$$\dot{x} = -y, \quad \dot{y} = x + (x^2 + y^2)^{(d-\ell)/2} \sum_{j=0}^{\ell} c_j x^j y^{\ell-j}, \quad c_j \in \mathbb{R}, \tag{2.10}$$

with $d \geq \ell$ even satisfies property A if and only if either $c_{2j} = 0$ for $j = 0, 1, \ldots, \ell/2$, or $c_{2j+1} = 0$ for $j = 0, 1, \ldots, (\ell - 2)/2$. Now we consider the system

$$\dot{x} = -y, \quad \dot{y} = x + (x^2 + y^2)^{(d-\ell-2)/2} \sum_{j=0}^{\ell+2} c_j x^j y^{\ell+2-j}, \quad c_j \in \mathbb{R}. \tag{2.11}$$

First we prove sufficiency for system (2.11). If $c_{2j} = 0$ for $j = 0, \ldots (\ell + 2)/2$, then system (2.11) becomes

$$\dot{x} = -y, \quad \dot{y} = x + (x^2 + y^2)^{(d-\ell-2)/2} \sum_{j=0}^{\ell/2} c_{2j+1} x^{2j+1} y^{\ell+1-2j}. \tag{2.12}$$

If $c_{2j+1} = 0$ for $j = 0, \ldots \ell/2$, then system (2.11) becomes

$$\dot{x} = -y, \quad \dot{y} = x + (x^2 + y^2)^{(d-\ell-2)/2} \sum_{j=0}^{(\ell+2)/2} c_{2j} x^{2j} y^{\ell+2-2j}. \tag{2.13}$$

Note that system (2.12) is invariant under the symmetry $(x, y, t) \mapsto (-x, y, -t)$, and system (2.13) is invariant under the symmetry $(x, y, t) \mapsto (x, -y, -t)$. It is clear that both systems (2.12) and (2.13) have a center at the origin and that system (2.11) among the centers which are invariant about one of the coordinate axes it has no other center having a local analytic first integral which is also continuous in a neighborhood of zero in the parameter space.

Now we shall prove necessity for system (2.11). Note that we can write

$$\sum_{j=0}^{\ell+2} c_j x^j y^{\ell+2-j} = \sum_{j=0}^{(\ell+2)/2} c_{2j} x^{2j} y^{\ell+2-2j} + \sum_{j=0}^{\ell/2} c_{2j+1} x^{2j+1} y^{\ell+1-2j}. \tag{2.14}$$

Dividing the first summand on the right-hand side of (2.14) by $x^2 + y^2$ we get

$$\sum_{j=0}^{(\ell+2)/2} c_{2j} x^{2j} y^{\ell+2-2j} = (x^2 + y^2) \sum_{j=0}^{\ell/2} d_{2j} x^{2j} y^{\ell-2j} + d_{\ell+1} y^{\ell+2},$$

and dividing the second summand on the right-hand side of (2.14) by x^2+y^2 we get

$$\sum_{j=0}^{\ell/2} c_{2j+1}x^{2j+1}y^{\ell+1-2j} = (x^2+y^2)\sum_{j=0}^{(\ell-2)/2} d_{2j+1}x^{2j+1}y^{\ell-1-2j} + d_{\ell+2}xy^{\ell+1}.$$

Hence it follows from (2.14) that

$$\begin{aligned}\sum_{j=0}^{\ell+2} c_jx^jy^{\ell+2-j} &= (x^2+y^2)\sum_{j=0}^{\ell/2} d_{2j}x^{2j}y^{\ell-2j} + d_{\ell+1}y^{\ell+2}\\ &\quad + (x^2+y^2)\sum_{j=0}^{(\ell-2)/2} d_{2j+1}x^{2j+1}y^{\ell-1-2j} + d_{\ell+2}xy^{\ell+1}\\ &= (x^2+y^2)\sum_{j=0}^{\ell} d_jx^jy^{\ell-j} + d_{\ell+1}y^{\ell+2} + d_{\ell+2}xy^{\ell+1}.\end{aligned}$$

Thus we write (2.11) as

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + (x^2+y^2)^{(d-\ell)/2}\sum_{j=0}^{\ell} d_jx^jy^{\ell-j} + d_{\ell+1}(x^2+y^2)^{(d-\ell-2)/2}y^{\ell+2}\\ &\quad + d_{\ell+2}(x^2+y^2)^{(d-\ell-2)/2}xy^{\ell+1}\end{aligned} \tag{2.15}$$

with $d_j \in \mathbb{R}$ for $j = 0,\dots,\ell+2$.

Now assume that system (2.15) has a center at the origin whose local analytic first integral is continuous in a neighborhood of zero in the parameter space and either some $d_{2j} \neq 0$ for $j = 0,\dots,\ell/2$, or $d_{2j+1} \neq 0$ for $j = 0,\dots,(\ell-2)/2$. By assumptions there exists a local analytic first integral in a neighborhood of the origin $H = H_{d_{\ell+1},d_{\ell+2}}(x,y)$ which is also continuous in the parameters $d_{\ell+1}$ and $d_{\ell+2}$ around $d_{\ell+1} = 0$ and $d_{\ell+2} = 0$. Hence, setting $d_{\ell+1} \to 0$ and $d_{\ell+2} \to 0$ we conclude that $H_{0,0}(x,y)$ is a local analytic first integral of system (2.15) restricted to $d_{\ell+1} = d_{\ell+2} = 0$, i.e., of system

$$\dot{x} = -y, \quad \dot{y} = x + (x^2+y^2)^{(d-\ell)/2}\sum_{j=0}^{\ell} d_jx^jy^{\ell-j}. \tag{2.16}$$

Moreover $H_{0,0}(x,y)$ is continuous in the parameters d_j for $j = 0,\dots,\ell$. This would imply that system (2.16) satisfies property A, but this is not possible because by induction hypotheses system (2.16) satisfies property A if and if and only if either $d_{2j} = 0$ for $j = 0,1,\dots,\ell/2$, or $d_{2j+1} = 0$ for $j = 0,1,\dots,(\ell-2)/2$.

In short, if system (2.15) satisfies property A then either $d_{2j} = 0$ for $j = 0,\dots,\ell/2$, or $d_{2j+1} = 0$ for $j = 0,\dots,(\ell-2)/2$.

In the first case we have that system (2.15) becomes

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + (x^2+y^2)^{(d-\ell)/2} \sum_{j=0}^{(\ell-2)/2} d_{2j+1} x^{2j+1} y^{\ell-1-2j} \\ &\quad + d_{\ell+1}(x^2+y^2)^{(d-\ell-2)/2} y^{\ell+2} + d_{\ell+2}(x^2+y^2)^{(d-\ell-2)/2} x y^{\ell+1},\end{aligned} \tag{2.17}$$

Note that we are assuming that there exists at least one $d_{2j+1} \neq 0$ for $j = 0, \ldots, (\ell-2)/2$. Without loss of generality we can assume $d_1 \neq 0$ because the argument does not depend on the particular coefficients which are different from zero. Now assume that system (2.17) has a center at the origin whose local analytic first integral is also continuous in the parameters of the system in a neighborhood of zero in the parameter space and that $d_{\ell+1} \neq 0$. Hence, there exists a first integral $H = H_{d_1,\ldots,d_{\ell-1},d_{\ell+1},d_{\ell+2}}$ defined in a neighborhood of the origin which is continuous in $d_1, \ldots, d_{\ell-1}, d_{\ell+1}, d_{\ell+2}$. Setting $d_3, d_5, \ldots, d_{\ell-1}, d_{\ell+2} \to 0$, the first integral $G_{d_1,d_{\ell+1}}(x,y) = H_{d_1,0,\ldots,0,d_{\ell+1},0}(x,y)$ is a first integral of system (2.17) restricted to $d_2 = \cdots = d_{\ell-1} = 0$ and $d_{\ell+2} = 0$, that is, of system

$$\dot{x} = -y, \quad \dot{y} = x + (x^2+y^2)^{(d-\ell)/2} d_1 x y^{\ell-1} + d_{\ell+1}(x^2+y^2)^{(d-\ell-2)/2} y^{\ell+2}. \tag{2.18}$$

with $d_1 \neq 0$. So, in particular system (2.18) has a center at the origin. Computing the first nonzero Poincaré-Liapunov constant for this system (see section 2 for an explanation on how to do the computations) and the decomposition of the ideal generated by these constants we obtain that either $d_1 = 0$ or $d_{\ell+1} = 0$, getting a contradiction in both cases. Hence a necessary condition for system (2.11) to have a center at the origin whose local analytic first integral is continuous in a neighborhood of the origin in the parameter space is that $c_{2j} = 0$ for $j = 0, \ldots, (\ell+2)/2$. This concludes the proof of the induction process and completes the proof of Theorem 2 in this case.

In the second case we have that system (2.15) becomes

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + (x^2+y^2)^{(d-\ell)/2} \sum_{j=0}^{\ell/2} d_{2j} x^{2j} y^{\ell-2j} \\ &\quad + d_{\ell+1}(x^2+y^2)^{(d-\ell-2)/2} y^{\ell+2} + d_{\ell+2}(x^2+y^2)^{(d-\ell-2)/2} x y^{\ell+1}.\end{aligned} \tag{2.19}$$

Note that we are assuming that there exists at least $d_{2j} \neq 0$ for $j = 0, \ldots, \ell/2$ and we can assume without loss of generality that $d_0 \neq 0$ because the arguments does not depend on the particular coefficient that is different from zero. Now assume that system (2.19) has a center at the origin which either is symmetric about one of the coordinate axes or has a local analytic first integral which is also continuous in the parameters of the system in a neighborhood of zero in the parameter space and that $d_{\ell+2} \neq 0$. Hence, there exists a first integral $H = H_{d_0,\ldots,d_\ell,d_{\ell+1},d_{\ell+2}}$ defined in a neighborhood of the origin which is continuous in $d_0, \ldots, d_\ell, d_{\ell+1}, d_{\ell+2}$. Setting $d_2, d_4, \ldots, d_\ell, d_{\ell+1} \to 0$ we obtain that $G_{d_0,d_{\ell+2}}(x,y) = H_{d_0,0,\ldots,0,0,d_{\ell+2}}(x,y)$ is a first integral of system (2.19) restricted to $d_2 = \cdots = d_\ell = d_{\ell+1} = 0$, that is, of system

$$\dot{x} = -y, \quad \dot{y} = x + d_0 y^\ell + d_{\ell+2}(x^2+y^2)^{(d-\ell-2)/2} x y^{\ell+1},$$

with $d_0 \neq 0$. Computing the first nonzero Poincaré-Liapunov constants and the decomposition of the ideal generated by these constants we obtain that either $d_0 = 0$ or $d_{\ell+2} = 0$, getting a contradiction in both cases. Hence a necessary condition for system (2.11) to have a center at the origin which is either symmetric about one of the coordinate axes or has a local analytic first integral that is also continuous in the parameters of the system in a neighborhood of zero in the parameter space is that $c_{2j+1} = 0$ for $j = 0, \ldots, \ell/2$.

In particular, this implies that among the centers that are symmetric about one of the coordinate axes there is no other center whose local analytic first integral is also continuous in the parameters in a neighborhood of the origin of the parameter space. This concludes the proof of the induction process and completes the proof of Theorem 2.

3. Proof of Theorem 1.4

In this section we prove Theorem 1.4 by induction over n. We start with the case $n = 4$.

Proposition 3.1. *System* (1.7) *with* $n = 4$ *and* $d \geq 4$ *even has no isochronous centers with the exception of the linear center.*

Proof. From the second equation of (2.5) we have

$$T = \int_0^{2\pi} \frac{d\theta}{\dot\theta} = \int_0^{2\pi} \frac{1}{1 + G(\theta) r(\theta)^{d-1}} d\theta.$$

Using the change (2.7) the previous integral becomes

$$T = \int_0^{2\pi} (1 - G(\theta)\rho(\theta)) d\theta = 2\pi - \int_0^{2\pi} G(\theta)\rho(\theta) d\theta,$$

where $\rho(\theta) = \sum_{j\geq 1} \rho_j(\theta)\rho_0^j$ is the solution given in (2.9). System (2.3) has an isochronous center at the origin if the origin is a center and satisfies

$$\int_0^{2\pi} G(\theta)\rho(\theta) d\theta = \sum_{j\geq 1} \left(\int_0^{2\pi} G(\theta)\rho_j(\theta) d\theta \right) \rho_0^j = 0.$$

That is $T = \displaystyle\int_0^{2\pi} d\theta/\dot\theta = 2\pi - \sum_{j\geq 1} T_j \rho_0^j = 2\pi$, where

$$T_j = \int_0^{2\pi} G(\theta)\rho_j(\theta) d\theta, \tag{3.1}$$

are called the *period Abel constants.* System (1.7) with $n = 4$ and $d \geq 4$ even is, in fact, system (2.2) that using the parametrization of system (2.3) takes the form

$$\begin{aligned} \dot x &= -y, \\ \dot y &= x + (x^2 + y^2)^{(d-4)/2}((2a_2 + 2a_4 + a_6)x^4 \\ &\quad + (2a_6 - 12a_2)x^2y^2 + (2a_2 - 2a_4 + a_6)y^4). \end{aligned} \tag{3.2}$$

For system (3.2) the first two non-zero period Abel constants are

$$T_2 = -32a_2^2 - 30a_2a_4 - 40a_4^2 - 30a_4a_6 + 2a_2^2d + 10a_4^2d - 15a_6^2d,$$
$$T_3 = a_4\,(-32a_2^2 - 30a_2a_4 - 40a_4^2 - 30a_4a_6 + 2a_2^2d + 10a_4^2d - 15a_6^2d).$$

The decomposition of the ideal generated by the period Abel constants gives a unique case which is $a_2 = a_4 = a_6 = 0$, and we obtain that the unique isochronous system is the linear one. This completes the proof of the proposition. □

It follows from Proposition 3.1 that Theorem 1.4 holds for $n = 4$ and $d \geq 4$ even. Now assume that it holds for $n = 4, 6 \ldots, \ell$ with $\ell \geq 4$ even and we shall prove it for $n = \ell + 2$. We thus have that system

$$\dot{x} = -y, \quad \dot{y} = x + (x^2+y^2)^{(d-\ell)/2} \sum_{j=0}^{\ell/2} c_{2j} x^{2j} y^{\ell-2j}, \tag{3.3}$$

with $d \geq \ell$ even does not satisfy property B. Now we consider the system

$$\dot{x} = -y, \quad \dot{y} = x + (x^2+y^2)^{(d-\ell-2)/2} \sum_{j=0}^{(\ell+2)/2} c_{2j} x^{2j} y^{\ell+2-2j}. \tag{3.4}$$

We will prove that system (3.4) does not satisfy property B.

As in the proof of Theorem 1.2 we write

$$\sum_{j=0}^{(\ell+2)/2} c_{2j} x^{2j} y^{\ell+2-2j} = (x^2+y^2) \sum_{j=0}^{\ell/2} d_{2j} x^{2j} y^{\ell-2j} + d_{\ell+1} y^{\ell+2},$$

and system (3.4) becomes

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + (x^2+y^2)^{(d-\ell)/2} \sum_{j=0}^{\ell/2} d_{2j} x^{2j} y^{\ell-2j} + d_{\ell+1}(x^2+y^2)^{(d-\ell-2)/2} y^{\ell+2}, \end{aligned} \tag{3.5}$$

with $d_j \in \mathbb{R}$ for $j = 0, \ldots, \ell + 1$.

Now assume that system (3.5) satisfies property B and some $d_{2j} \neq 0$ for $j = 0, \ldots, \ell/2$. Note that the period of the system is constant and the solution $r(\theta)$ depends continuously on the parameter $d_{\ell+1}$, we get that $T = T(d_{\ell+1})$ is constant and only depends (continuously) on $d_{\ell+1}$. Setting $d_{\ell+1} \to 0$ we get that $T(0)$ is constant and it is indeed the period of the system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + (x^2+y^2)^{(d-\ell)/2} \sum_{j=0}^{\ell/2} d_{2j} x^{2j} y^{\ell-2j}. \end{aligned} \tag{3.6}$$

This implies that system (3.6) satisfies property B, but this is not possible because by hypothesis (see (3.3)) system (3.6) satisfies property B if $d_{2j} = 0$ for $j = 0, \ldots, \ell/2$.

In short, if system (3.5) satisfies property B then $d_{2j} = 0$ for $j = 0, \ldots, \ell/2$ and so system (3.5) becomes

$$\dot{x} = -y, \quad \dot{y} = x + d_{\ell+1}(x^2 + y^2)^{(d-\ell-2)/2} y^{\ell+2}. \tag{3.7}$$

For such system we have the following proposition.

Proposition 3.2. *If system* (3.7) *satisfies property B, then* $d_{\ell+1} = 0$.

Proof. We take polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ and system (3.7) takes the form

$$\dot{r} = d_{\ell+1}\, r^d \sin^{3+\ell}\theta, \quad \dot{\theta} = 1 + d_{\ell+1}\, r^{d-1}\cos\theta\sin^{2+\ell}\theta.$$

Next we apply the Cherkas' transformation to arrive to the associated Abel equation (2.8). Then we compute the solution $\rho(\theta, \rho_0)$ of (2.8) satisfying that $\rho(0, \rho_0) = \rho_0$ up to certain order. Using these expansion and equation (3.1) we find that the first period Abel constant are $T_1 = 0$ and

$$T_2 = -\frac{\big(1 + (-1)^\ell\big)\, d_{\ell+1}\, \pi^{1/2}\Gamma(\ell + \frac{5}{2})}{(\ell+3)^2\Gamma(\ell+2)} - \frac{e^{i\pi\ell} d_{\ell+1}\, \pi^{3/2}(2 + d + \ell)}{(\ell+3)\Gamma(-\frac{3}{2} - \ell)\Gamma(4+\ell)}.$$

Taking into account that ℓ is even we obtain that $d_{\ell+1}$ must be zero. □

It follows from Proposition 3.2 that a necessary condition in order that system (1.7) satisfies property B is that $c_{2j} = 0$ for $j = 0, \ldots, \ell/2$ but then system (1.7) becomes

$$\dot{x} = -y, \quad \dot{y} = x,$$

which is not possible. This concludes the proof of Theorem 4.

4. Proof of Theorem 1.5

In this section we prove Theorem 1.5 by induction over n. We start with the case $n = 4$ and we prove the following result.

Proposition 4.1. *System* (1.6) *with* $n = 4$ *and* $d \geq 4$ *even has an isochronous center if and only if it has the form*

$$\dot{x} = -y, \quad \dot{y} = x + a(x^2 + y^2)^{(d-2)/2} xy,$$

for $a \in \mathbb{R}$.

Proof. Using the arguments of Proposition 3.1 we first write system (1.6) with $n = 4$ and $d \geq 4$ using the parametrization of system (2.3). In this case it takes the form

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + (x^2 + y^2)^{(d-4)/2}((8a_1 + 4a_3)x^3 y - (8a_1 - 4a_3)xy^3). \end{aligned} \tag{4.1}$$

For system (4.1) the first two non-zero period Abel constants (using (3.1)) are

$$\begin{aligned} T_2 &= -16a_1^2 - 15a_1a_3 - 20a_3^2 + a_1^2 d + 5a_3^2 d, \\ T_3 &= a_3(2880a_1^3 - 5040a_1^2a_3 - 4474a_1a_3^2 - 9976a_3^3 - 495a_1a_3^2 d \\ &\quad + 1974a_3^3 d - 323a_1a_3^2d^2 - 14a_3^3d^2 + 12a_1a_3^2d^3 + 36a_3^3d^3). \end{aligned}$$

The decomposition of the ideal generated by the period Abel constants gives a unique case which is $a_1 = 0$, and we obtain that the unique isochronous system is given by

$$\begin{aligned}\dot{x} &= -y\\ \dot{y} &= x + (x^2+y^2)^{(d-4)/2}(4a_3x^3y + 4a_3xy^3)\\ &= x + 4a_3(x^2+y^2)^{(d-2)/2}xy.\end{aligned}$$

This completes the proof of the proposition. □

It follows from Proposition 4.1 that Theorem 1.5 holds for $n = 4$ and $d \geq 4$ even. Now assume it holds for $n = 4, 6 \ldots, \ell$ with $\ell \geq 4$ even and we shall prove it for $n = \ell + 2$. We thus have that system

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + (x^2+y^2)^{(d-\ell)/2}\sum_{j=0}^{(\ell-2)/2} c_{2j+1}x^{2j+1}y^{\ell-1-2j} = x + R_0(x,y),\end{aligned} \tag{4.2}$$

with $d \geq \ell$ even satisfies property B if and only if

$$R_0(x,y) = a(x^2+y^2)^{(d-\ell+2)/2}x^{\ell-3}y, \tag{4.3}$$

and system (4.2) becomes

$$\dot{x} = -y, \quad \dot{y} = x + a(x^2+y^2)^{(d-\ell+2)/2}x^{\ell-3}y. \tag{4.4}$$

Now we consider the system

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + (x^2+y^2)^{(d-\ell-2)/2}\sum_{j=0}^{\ell/2} c_{2j+1}x^{2j+1}y^{\ell+1-2j} = x + R_1(x,y).\end{aligned} \tag{4.5}$$

We will prove that system (4.5) satisfies property B if and only if

$$R_1(x,y) = a(x^2+y^2)^{(d-\ell)/2}x^{\ell-1}y.$$

We write

$$\begin{aligned}&\sum_{j=0}^{\ell/2} c_{2j+1}x^{2j+1}y^{\ell+1-2j} = (x^2+y^2)\sum_{j=0}^{(\ell-2)/2} d_{2j+1}x^{2j+1}y^{\ell-1-2j} + d_{\ell+2}xy^{\ell+1}\\ &= \sum_{j=0}^{(\ell-4)/2} d_{2j+1}x^{2j+1}y^{\ell-1-2j} + d_{\ell-1}(x^2+y^2)x^{\ell-1}y + d_{\ell+2}xy^{\ell+1},\end{aligned}$$

and thus system (4.5) becomes

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + (x^2+y^2)^{(d-\ell)/2}\sum_{j=0}^{(\ell-4)/2} d_{2j+1}x^{2j+1}y^{\ell-1-2j}\\ &\quad + d_{\ell-1}(x^2+y^2)^{(d-\ell)/2}x^{\ell-1}y + d_{\ell+2}(x^2+y^2)^{(d-\ell-2)/2}xy^{\ell+1},\end{aligned} \tag{4.6}$$

with $d_j \in \mathbb{R}$ for $j = 0, \dots, \ell$ and $j = \ell + 2$.

Now assume that system (4.6) satisfies property B and denote

$$R_2 := (x^2 + y^2)^{(d-\ell)/2} \sum_{j=0}^{(\ell-2)/2} d_{2j+1} x^{2j+1} y^{\ell-1-2j}$$

We observe since R_2 does not contain the term $d_{\ell-1}(x^2+y^2)^{(d-\ell)/2} x^{\ell-1} y$, if $R_2 \neq 0$, it cannot be of the form given in (4.3). By assumptions, the period of the center is constant and since the solution $r(\theta)$ depends continuously on the parameters $d_{\ell-1}$ and $d_{\ell+2}$, we get that $T = T(d_{\ell-1}, d_{\ell+2})$ is constant and only depends (continuous) on $d_{\ell-1}$ and $d_{\ell+2}$. Setting $d_{\ell-1}, d_{\ell+2} \to 0$ we get that $T(0, 0)$ is constant, and it is the period of the system

$$\dot{x} = -y, \quad \dot{y} = x + R_2(x, y). \tag{4.7}$$

This implies that system (4.7) satisfies property B, but this is not possible because by hypothesis (see (4.2)), system (4.7) satisfies property B if and only if $R_2 = 0$ and then system (4.6) becomes

$$\begin{aligned} &\dot{x} = -y, \\ &\dot{y} = x + d_{\ell-1}(x^2 + y^2)^{(d-\ell)/2} x^{\ell-1} y + d_{\ell+2}(x^2 + y^2)^{(d-\ell-2)/2} x y^{\ell+1}. \end{aligned} \tag{4.8}$$

For such system we have the following proposition.

Proposition 4.2. *If system* (4.8) *satisfies property B then* $d_{\ell+2} = 0$.

Proof. We take polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ and system (4.8) takes the form

$$\begin{aligned} \dot{r} &= r^d (d_{\ell-1} \cos^{\ell-1}\theta \sin^2\theta + d_{\ell+2} \cos\theta \sin^{\ell+2}\theta), \\ \dot{\theta} &= 1 + r^{d-1}(d_{\ell-1} \cos^{\ell}\theta \sin\theta + d_{\ell+2} \cos^2\theta \sin^{\ell+1}\theta). \end{aligned}$$

Now as in Proposition 3.2 we apply the Cherkas' transformation to arrive to the associated Abel equation (2.8). Then we compute the solution $\rho(\theta, \rho_0)$ of (2.8) satisfying that $\rho(0, \rho_0) = \rho_0$ up to certain order. Using these expansion and equation (3.1) we find that the first period Abel constants is $T_1 = 0$ and the second is

$$T_2 = -\frac{d_{\ell+2}}{2(\ell+3)} \mathcal{F}(\ell, d_{\ell-1}, d_{\ell+2}),$$

where $\mathcal{F}(\ell, d_{\ell-1}, d_{\ell+2})$ is a not null huge function of ℓ, $d_{\ell-1}$ and $d_{\ell+2}$ that we do not write here. Hence we obtain that $d_{\ell+2}$ must be zero. □

It follows from Proposition 4.2 that a necessary condition in order that system (1.6) satisfies property B is that it is of the form

$$\dot{x} = -y, \quad \dot{y} = x + d_{\ell-1}(x^2 + y^2)^{(d-\ell)/2} x^{\ell-1} y.$$

This completes the induction process and concludes the proof of Theorem 1.5.

We note that taking $d = \ell + 2 = n$ we conclude that system (1.6) satisfies property B if and only if it is of the form

$$\dot{x} = -y, \quad \dot{y} = x + b(x^2 + y^2) x^{n-3} y,$$

for $b \in \mathbb{R}$.

Acknowledgements

The first author is partially supported by a MINECO grant number MTM2014-53703-P and an AGAUR (Generalitat de Catalunya) grant number 2014SGR 1204. The second author is partially supported by a MINECO grant number MTM2013-40998-P, an AGAUR grant number 2014SGR 568, and two FP7-PEOPLE-2012-IRSES numbers 316338 and 318999. The third author is partially supported by FCT/Portugal through the project UID/MAT/04459/2013.

References

[1] L.A. Cherkas, *The number of limit cycles of a certain second order autonumous system* (Russian), Differencial'nye Uravnenija, 1976, 12, 944–946.

[2] L.A. Cherkas, *Conditions for the equation* $yy' = \sum_{i=0}^{3} p_i(x)y^i$ *to have a centre*, Differential Equations, 1978, 14, 1133–1138.

[3] C.J. Christopher, *An algebraic approach to the classification of centers in polynomial Liénard systems*, J. Math. Anal. Appl., 1999, 229, 319–329.

[4] C.J. Christopher, N.G. Lloyd, *On the paper of Jin and Wang concerning the conditions for a centre in certain cubic systems*, Bull. London Math. Soc., 1990, 22, 5–12.

[5] W. Decker, S.Laplagne, G. Pfister, H.A. Schonemann, *SINGULAR*, 3-1 library for computing the prime decomposition and radical of ideals, primdec.lib, 2010.

[6] P. Gianni, B. Trager, G. Zacharias, *Gröbner bases and primary decompositions of polynomials*, J. Symbolic Comput., 1988, 6, 146–167.

[7] J. Giné, *Conditions for the existence of a center for the Kukles homogenenous systems*, Comput. Math. Appl., 2002, 43, 1261–1269.

[8] J. Giné, J. LLibre, C. Valls, *Centers for the Kukles homogeneous systems with odd degree*, Bull. London Math. Soc., 2015, 47, 315–324.

[9] G.M. Greuel, G. Pfister, H.A. Schönemann, `SINGULAR 3.0` *A Computer Algebra System for Polynomial Computations*, Centre for Computer Algebra, University of Kaiserlautern, 2005. `http://www.singular.uni-kl.de.`

[10] X. Jin, D. Wang, *On the conditions of Kukles for the existence of a centre*, Bull. London Math. Soc., 1990, 22, 1–4.

[11] I.S. Kukles, *Sur quelques cas de distinction entre un foyer et un centre*, Dokl. Akad. Nauk. SSSR, 1944, 42, 208–211.

[12] N.G. Lloyd, J.M. Pearson, *Conditions for a centre and the bifurcation of limit cycles*, in Bifurcations of Planar Vector Fields, J.P. Françoise, R. Roussarie (Eds.), Lecture Notes Math., vol. 1455, Springer, 1990, pp. 230–242.

[13] N.G. Lloyd, J.M. Pearson, *Computing centre conditions for certain cubic systems*, J. Comp. Appl. Math., 1992, 40, 323–336.

[14] J.M. Pearson, N.G. Lloyd, *Kukles revisited: advances in computing techniques*, Comput. Math. Appl., 2010, 60, 2797–2805.

[15] R. Pons, N. Salih, *Center conditions for a lopsided quintic polynomial vector field*, Qual. Theory Dyn. Syst., 2002, 3, 331–343.

[16] A.P. Sadovskii, *Solution of the center and focus problem for a cubic system of nonlinear oscillations*, Differential Equations, 1997, 33, 236–244.

[17] N. Salih, R. Pons, *Center conditions for a lopsided quartic polynomial vector field*, Bull. Sci. Math., 2002, 42 (5), 369–378.

[18] A.S. Shubè, *On the Kukles and Cherkas center conditions for a cubic system*, Differential Equations 1993, 29, 625–627.

[19] E. P. Volokitin, V. V. Ivanov, *Isochronicity and Commutation of polynomial vector fields*, Siberian Mathematical Journal, 1999, 40, 22–37.