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On primitive constant dimension codes and a geometrical sunflower bound

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Abstract

In this paper we study subspace codes with constant intersection dimension (SCIDs). We investigate the largest possible dimension spanned by such a code that can yield non-sunflower codes, and classify the examples attaining equality in that bound as one of two infinite families. We also construct a new infinite family of primitive SCIDs.

Keywords: subspace codes, constant intersection dimension codes, rank codes, finite geometry, Galois geometry

MSC 2010 codes: 51E20, 94B60

1 Introduction

A (k, m)-*SCID* is a set of k-dimensional subspaces in a vector space V, pairwise intersecting in an m-dimensional subspace (SCID: *Subspaces with Constant Intersection Dimension*). This term was coined in [2]. The $(k, 0)$-SCIDs in an ℓ-dimensional vector space are called *partial* k*-spreads* and are a classic object in finite geometry (so for V defined over a finite field), see e.g. [5, 6].

A (k, m)-SCID $\mathcal{S}$ in the vector space V is called *primitive* ([2]) if it satisfies the following properties:

1. $\langle \mathcal{S} \rangle = V$;
2. no nonzero vector is contained in all of the elements of $\mathcal{S}$ (or $\cap_{\pi \in \mathcal{S}} \pi = \{0\}$);
3. each element π of $\mathcal{S}$ is spanned by $\{\pi \cap \sigma | \sigma \in \mathcal{S} \setminus \{\pi\}\}$;
4. $\dim(V) \geq 2k$.

An easy way to construct a (k, m)-SCID in the vector space V is by fixing an m-space V' and by considering k-spaces through V' that have no point in common outside of V'. Such a (k, m)-SCID is called an m*-sunflower*. Section 2 is devoted to SCIDs that are sunflowers.

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It is easy to see that an $(n, n-1)$-SCID is an $(n-1)$-sunflower or a set of n-spaces in a fixed $(n+1)$-space. In [2], the $(n, n-2)$-SCIDs are studied, following the work of [1] wherein $(2,0)$-SCIDs were studied. Further constructions of SCIDs were presented in [3]. Section 3 contains new constructions of primitive SCIDs.

Recently, SCIDs gained attention as they are *equidistant constant dimension codes*. The influential paper [4] marked the beginning of the theory of *random network coding* which was developed for transmission of information in networks with a number of sources, inner nodes and sinks, with a varying topology, typically a network where users come and go. Unlike classical coding theory where vectors are sent as codewords, in random network coding vector subspaces are used as codewords. These codes are therefore called *subspace codes*. The distance $d(U, V)$ between two subspaces U and V is commonly defined as $d(U, V) = \dim(U) + \dim(V) - 2\dim(U \cap V)$. The subspace codes whose elements all have the same dimension are called *constant dimension codes* and are the most studied subspace codes as they are *q-analogues* of the classical codes. SCIDs correspond to equidistant constant dimension codes, constant dimension codes whose pairwise distances of codewords are all equal, the q-analogues of classical equidistant codes.

2 A bound for the sunflower property

In [3], the following theorem is proved (stated as a result on subspace codes).

Theorem 2.1 ([3, Theorem 1])**.** If a (k, t)-SCID in a vector space V over the field $\mathbb{F}_q$ has more than $\left(\frac{q^k-q^t}{q-1}\right)^2 + \frac{q^k-q^t}{q-1} + 1$ elements, then it is a sunflower.

So, the largest SCIDs are sunflowers. However, from a random network coding point of view the only interesting sunflowers are the 0-sunflowers. In other terms, Theorem 2.1 sets an upper bound on the size of the 'interesting' subspace codes. Note that a k-spread (a maximal partial spread) in the vector space $\mathbb{F}_q^n$, with $k \mid n$, contains $\frac{q^n-1}{q^k-1}$ elements.

In this section we look at SCIDs that span a large subspace. We will prove that again sunflowers are the 'largest' SCIDs. Of course, not all t-sunflowers with the same number of elements span a subspace of the same dimension. A *sunflower of maximal dimension* is a t-sunflower such that any element π meets the subspace generated by all the other elements in precisely the common t-space. It is easy to see that this name is well-chosen.

Theorem 2.2. Let $\mathcal{S}$ be a $(k, k-t)$-SCID in a vector space V, with $|\mathcal{S}| \geq 3$ and $3 \leq t \leq k-1$. If $\dim\langle\mathcal{S}\rangle \geq k + (t-1)(n-1) + 2$, then $\mathcal{S}$ is a $(k-t)$-sunflower.

Proof. We assume that $\mathcal{S}$ is not a sunflower. We denote $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ and we consider the differences $\delta_i = \dim\langle\pi_1, \ldots, \pi_i\rangle - \dim\langle\pi_1, \ldots, \pi_{i-1}\rangle$ for $i = 1, \ldots, n$; so, $\delta_1 = k$, $\delta_2 = t$, ... (we considered the span of the empty set as the empty subspace). We can sort the spaces in $\mathcal{S}$ in such a way that the sequence $(\delta_1, \ldots, \delta_n)$ is nonincreasing; without loss of generality we can say that $\pi_1, \ldots, \pi_n$ are sorted such that this property is met. Note that such an ordering is not necessarily unique.

Let m be the largest index for which $\delta_m = t$. Then, the spaces $\pi_1, \ldots, \pi_m$ form a $(k-t)$-sunflower of maximal dimension. Obviously, $m \geq 2$. Denote the common $(k-t)$-space of $\pi_1, \ldots, \pi_m$ by V'. By the assumption $\mathcal{S}$ is not a sunflower, so we can find a subspace $\pi_r \in \mathcal{S}$ not containing V'. We denote $k - t - \dim(\pi_r \cap V')$ by ε; it is immediate that $\varepsilon \geq 1$. In the quotient vector space $\Pi = \langle \mathcal{S} \rangle / V'$, we then see that $\dim_\Pi \pi_r = t+\varepsilon$ and that $\dim_\Pi(\pi_r \cap \pi_i) = \varepsilon$ for $1 \le i \le m$. Moreover, the subspaces $(\pi_r \cap \pi_i)/V'$ in Π are linearly independent. Hence,

$$\delta_r \le t + \varepsilon - m \cdot \varepsilon \le t - m + 1 \,. \tag{1}$$

Since $(\delta_1, \ldots, \delta_n)$ is nonincreasing, we find that

$$\begin{aligned}
\dim\langle \mathcal{S} \rangle &= \sum_{i=1}^{n} \delta_i \\
&\le k + (m-1)t + (r-m-1)(t-1) + (n-r+1)(t-m+1) && (2)\\
&= k + (n-1)(t-1) - (n-r)(m-2) + 1 \\
&\le k + (n-1)(t-1) + 1 \,, && (3)
\end{aligned}$$

which proves the theorem. $\square$

To show that the bound in the previous theorem is sharp, we will now present two families of SCIDs that are not sunflowers, but where equality in the bound $\dim\langle \mathcal{S} \rangle \le k+(n-1)(t-1)+1$ is attained.

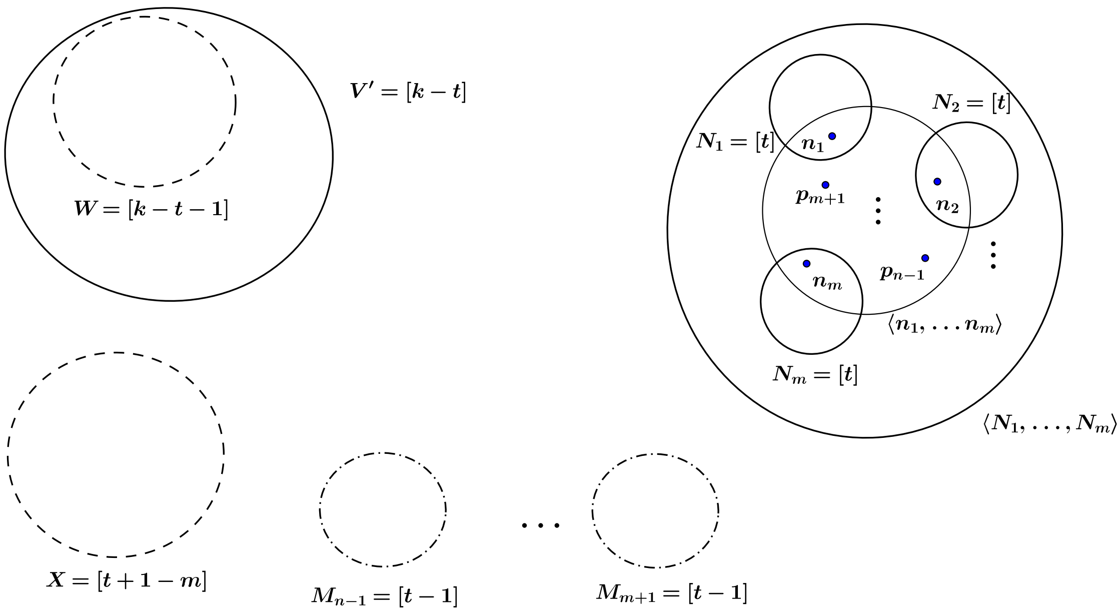

Figure 1: The $(k, k-t)$-SCID described in Example 2.3

Example 2.3. Choose integers $n \ge 3$ and k, t such that $3 \le t \le k-1$ and let m be an integer with $2 \le m \le \min\{t+1, n-1\}$. Let V be a vector space over a field $\mathbb{F}$ which is either infinite or else a finite field $\mathbb{F}_q$ with q such that $\frac{q^m-1}{q-1} + 1 \ge n$. Let V', X, $N_1, \ldots, N_m$ and $M_{m+1}, \ldots, M_{n-1}$ be linearly independent subspaces of V such that $\dim V' = k-t$, $\dim X = t+1-m$, $\dim N_i = t$ and $\dim M_i = t-1$.

Let $n_1, \ldots, n_m$ be 1-spaces in $N_1, \ldots, N_m$ respectively. Let $p_{m+1}, \ldots, p_{n-1}$ be distinct 1-spaces in $\langle n_1, \ldots, n_m \rangle \setminus \{n_1, \ldots, n_m\}$; here we need the bound on q in case $\mathbb{F}$ is a finite field. Let W be a $(k-t-1)$-space in V' (see Figure 1). Then we define the sets $\pi_1, \ldots, \pi_n$ as follows.

- $\pi_1 = \langle V', N_1 \rangle$, $\pi_2 = \langle V', N_2 \rangle$, ..., $\pi_m = \langle V', N_m \rangle$,
- $\pi_{m+1} = \langle V', M_{m+1}, p_{m+1} \rangle$, ..., $\pi_{n-1} = \langle V', M_{n-1}, p_{n-1} \rangle$,
- $\pi_n = \langle W, X, n_1, \ldots, n_m \rangle$.

The pairwise intersection of the subspaces π_i and π_j, $i, j = 1, \ldots, n-1$, with $i \neq j$, equals V' because p_j is not contained in N_i. Since each of the spaces $\pi_1, \ldots, \pi_{n-1}$ contains a unique 1-space from the set $\{n_1, \ldots, n_m, p_{m+1}, \ldots, p_{n-1}\}$ (and these 1-spaces are pairwise different), also $\dim(\pi_i \cap \pi_n) = k - t$ for all $i = 1, \ldots, n-1$. Hence, the set $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ is a set of n distinct k-spaces pairwise meeting in a $(k-t)$-space. As not all pairwise intersections equal the same $(k-t)$-space, $\mathcal{S}$ is not a sunflower.

The 1-spaces $n_1, \ldots, n_m, p_{m+1}, \ldots, p_{n-1}$ are contained in $\langle N_1, \ldots, N_m \rangle$ and also $W \subset V'$. Hence,

$$\langle \mathcal{S} \rangle = \langle \pi_1, \ldots, \pi_n \rangle = \langle V', N_1, \ldots, N_m, M_{m+1}, \ldots, M_{n-1}, X \rangle \ .$$

Since V', X, $N_1, \ldots, N_m$ and $M_{m+1}, \ldots, M_{n-1}$ are linearly independent subspaces of V, we find that

$$\begin{aligned} \dim\langle \mathcal{S} \rangle &= (k-t) + m \cdot t + (n-1-m) \cdot (t-1) + (t+1-m) \\ &= k + (n-1)(t-1) + 1 \ . \end{aligned}$$

Considering the integers δ_i as introduced in the proof of Theorem 2.2, and using the ordering $\pi_1, \ldots, \pi_n$, we find that

$$(\delta_2, \ldots, \delta_n) = (\underbrace{t, \ldots, t}_{m-1 \text{ times}}, \underbrace{t-1, \ldots, t-1}_{n-m-1 \text{ times}}, t+1-m) \ .$$

Note that in this example it is allowed to choose $m = n-1$ in which case there are no k-spaces of the second kind.

Example 2.4. Choose integers $n \geq 3$ and k, t such that $3 \leq t \leq k-1$. Let V be a vector space over a field $\mathbb{F}$ which is either infinite or else a finite field $\mathbb{F}_q$ with q such that $\frac{q^{k-t+2}-1}{q-1} \geq n$, and let $X_1, \ldots, X_n$ and V' be linearly independent subspaces of V such that $\dim V' = k-t+2$ and $\dim X_i = t-1$ (see Figure 2).

Let $W_1, \ldots, W_n$ be n distinct $(k-t+1)$-spaces in V', not all through a common $(k-t)$-space. We define $\pi_1 = \langle X_1, W_1 \rangle$, $\pi_2 = \langle X_2, W_2 \rangle$, ..., $\pi_n = \langle X_n, W_n \rangle$. It is clear that $\pi_1, \ldots, \pi_n$ all are k-spaces. For all $i, j = 1, \ldots, n$, with $i \neq j$, we know that $\pi_i \cap \pi_j = W_i \cap W_j$ and any two distinct $(k-t+1)$-spaces in V' meet in a $(k-t)$-space. Hence, the set $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ is a set of n distinct k-spaces pairwise meeting in a $(k-t)$-space. As there is no $(k-t)$-space common to $W_1, \ldots, W_n$, the set $\mathcal{S}$ is not a sunflower.

It is easy to see that $\langle \mathcal{S} \rangle = \langle V', X_1, \ldots, X_n \rangle$, which by the linear independence of the spaces $X_1, \ldots, X_n$ and V' yields

$$\dim\langle \mathcal{S} \rangle = (k-t+2) + n \cdot (t-1) = k + (n-1)(t-1) + 1 \ .$$

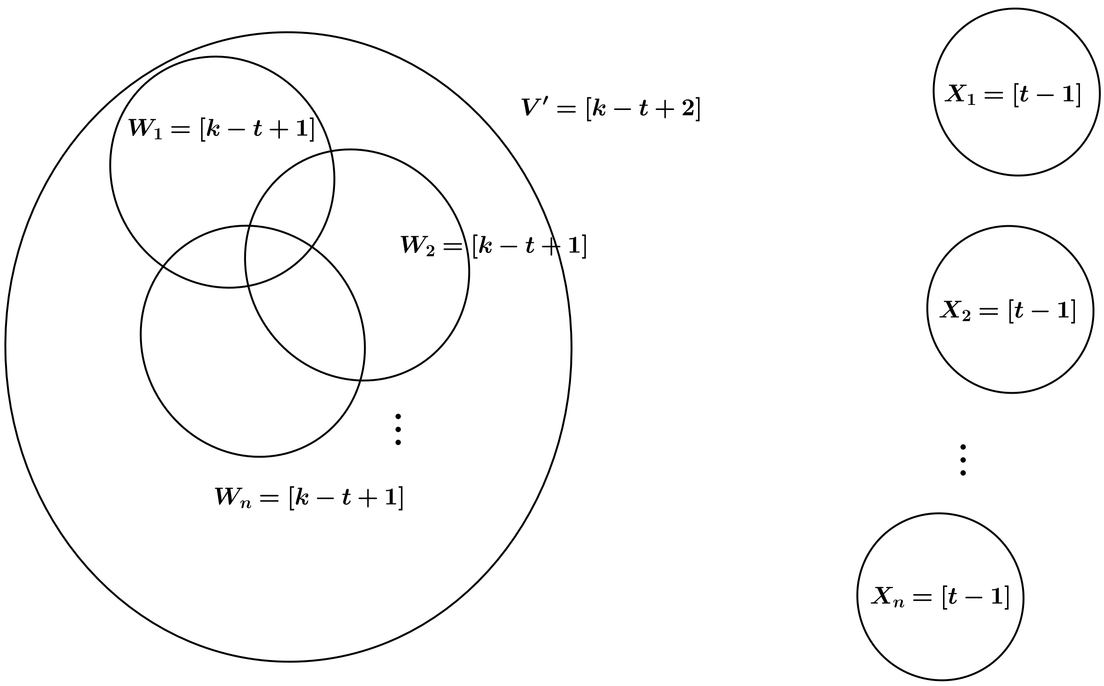

Figure 2: The $(k, k-t)$-SCID described in Example 2.4

Again, considering the integers δ_i as introduced in the proof of Theorem 2.2, we find that

$$(\delta_2, \dots, \delta_n) = (t, \underbrace{t-1, \dots, t-1}_{n-2 \text{ times}})$$

for every possible ordering of $\pi_1, \dots, \pi_n$.

Given the bound in Theorem 2.2, a natural next step is to classify all examples where the bound is sharp. We prove that the only two families of examples meeting the bound are the ones presented in Examples 2.3 and 2.4.

Theorem 2.5. Let $\mathcal{S}$ be a $(k, k-t)$-SCID in a vector space V, with $|\mathcal{S}| \geq 3$ and $3 \leq t \leq k-1$. If $\dim\langle \mathcal{S} \rangle = k + (t-1)(n-1) + 1$, then $\mathcal{S}$ is a sunflower or $\mathcal{S}$ is described by Example 2.3 or Example 2.4.

Proof. We denote the elements of $\mathcal{S}$ by $\pi_1, \dots, \pi_n$ and we assume that $\mathcal{S}$ is not a sunflower. Given this ordering, we define the differences $\delta_i = \dim\langle \pi_1, \dots, \pi_i \rangle - \dim\langle \pi_1, \dots, \pi_{i-1} \rangle$ for $i = 1, \dots, n$, as in Theorem 2.2. We will consider all possible orderings of the elements of $\mathcal{S}$ such that $(\delta_2, \dots, \delta_n)$ is nonincreasing. Since $\dim\langle \mathcal{S} \rangle = k + (t-1)(n-1) + 1$, we have equality in (2) and (3). Hence,

$$(\delta_2, \dots, \delta_n) = (\underbrace{t, \dots, t}_{m-1 \text{ times}}, \underbrace{t-1, \dots, t-1}_{n-m-1 \text{ times}}, t+1-m) \tag{4}$$

for some $m \geq 3$ or

$$(\delta_2, \dots, \delta_n) = (t, \underbrace{t-1, \dots, t-1}_{n-2 \text{ times}}) \,. \tag{5}$$

We distinguish between two cases.

- First we assume that we can find a permutation of $\mathcal{S}$ such that $(\delta_2, \dots, \delta_n)$ is as in (4) for a value $m \geq 3$. Then for the set $\mathcal{S}' = \{\pi_1, \dots, \pi_{n-1}\}$, we find that $\dim\langle \mathcal{S}' \rangle = k + (n-2)(t-1) + m - 1 \geq k + (n-2)(t-1) + 2$. So, by Theorem 2.2, $\mathcal{S}'$ is a sunflower. Let V' be the common $(k-t)$-dimensional intersection space of $\pi_1, \dots, \pi_{n-1}$.

  Denote $k - t - \dim(\pi_n \cap V')$ by ε as in the proof of Theorem 2.2. Since $\mathcal{S}$ is not a sunflower, $\varepsilon \geq 1$. It now follows from the equality in (1) that $\varepsilon = 1$. We denote the $(k-t-1)$-space $V' \cap \pi_n$ by W. Since $\dim(\pi_i \cap \pi_n) = k - t$ for $i \leq n-1$, π_n must contain 1-spaces $n_1 \in \pi_1, \dots, n_m \in \pi_m$, meeting V' trivially and linearly independent. Since $\delta_n = t + 1 - m$, it follows that $\pi_n = \langle W, n_1, \dots, n_m, X \rangle$ for some $(t+1-m)$-dimensional space X linearly independent of $\langle \pi_1, \dots, \pi_{n-1} \rangle$. Now we can choose t-spaces $N_1, \dots, N_m$ such that $\pi_1 = \langle V', N_1 \rangle$, ..., $\pi_m = \langle V', N_m \rangle$ and such that $n_i \in N_i$, $i = 1, \dots, m$. Note that $N_1, \dots, N_m, V'$ are necessarily linearly independent subspaces.

  The k-spaces $\pi_{m+1}, \dots, \pi_{n-1}$ contain the $(k-t)$-space V', meet each other in a $(k-t)$-space and meet all of the spaces $\pi_1, \dots, \pi_m, \pi_n$ in a $(k-t)$-space. The space π_i, $m+1 \leq i \leq n-1$, meets $\langle n_1, \dots, n_m \rangle$ in a 1-space p_i since it meets π_n in a $(k-t)$-space that contains W. This 1-space p_i is distinct from each 1-space n_j, $j = 1, \dots, m$, as π_i meets $\pi_1, \dots, \pi_m$ in the $(k-t)$-space V'. Since $\delta_i = t - 1$, it is immediate that $\pi_i = \langle V', p_i, M_i \rangle$, $i = m+1, \dots, n-1$, with M_i a $(t-1)$-space and such that V', $N_1, \dots, N_m, M_{m+1}, \dots, M_{n-1}$ and X are linearly independent. Finally, the requirement that $\dim(\pi_i \cap \pi_j) = k - t$, $m+1 \leq i, j \leq n-1$, with $i \neq j$, yields that all 1-spaces p_i are different. Hence, $\mathcal{S}$ is isomorphic to Example 2.3.

- If there is a permutation of $\mathcal{S}$ such that $\delta_n \leq t-2$, then we can find a permutation of $\mathcal{S}$ such that $(\delta_2, \dots, \delta_n)$ is nonincreasing and such that $\delta_n \leq t-2$; this case has been covered in the preceding bullet point. So we may assume that for any permutation of $\mathcal{S}$ the tuple $(\delta_2, \dots, \delta_n)$ is as in (5). Hence, as every space in $\mathcal{S}$ could be the final k-space in the ordering, each space in $\mathcal{S}$ contains a $(t-1)$-space linearly independent of the span of all the other spaces in $\mathcal{S}$. So, $\pi_i = \langle V_i, M_i \rangle$, for $i = 1, \dots, n$, where $M_1, \dots, M_n$ and $\langle V_1, \dots, V_n \rangle$ are linearly independent subspaces. From

$$\begin{aligned} k + (n-1)(t-1) + 1 = \dim\langle S \rangle &= \dim\langle V_1, \dots, V_n, M_1, \dots, M_n \rangle \\ &= \dim\langle V_1, \dots, V_n \rangle + n(t-1), \end{aligned}$$

  we find that $\dim\langle V_1, \dots, V_n \rangle = (k + (n-1)(t-1) + 1) - (n(t-1)) = k - t + 2$. Hence, $\mathcal{S}$ is isomorphic to Example 2.4. □

3 A new family of primitive SCIDs

In this section we first present a new construction for primitive SCIDs, and afterwards we apply it to describe some new examples. At the end we describe some other new examples. There is always a particular attention for vector spaces over finite fields, given the interpretation of SCIDs as subspace codes in random network coding.

Theorem 3.1. Let $\mathcal{L}$ be a set of k-subsets of a finite set Ω, with $|\mathcal{L}| = t \geq n - k + 1$ and $|\Omega| = m \geq n + k + 1$ for an integer $n \geq k + 1$, such that every element of Ω is contained in at least two elements of $\mathcal{L}$, such that any two different elements in $\mathcal{L}$ have exactly one element of Ω in common, and such that no element of Ω is contained in all the elements of $\mathcal{L}$. If V is an $(m+n-k+1)$-dimensional vector space over an infinite field or a finite field $\mathbb{F}_q$ with $\frac{q^{n-k+1}-1}{q-1} \geq t$, then a primitive $(n, n-k)$-SCID of size t in V exists.

Proof. We denote the set $\mathcal{L}$ of subsets by $\{L_1, L_2, \ldots, L_t\}$ and the element set Ω by $\bigcup_{L \in \mathcal{L}} L = \{p_1, p_2, \ldots, p_m\}$. Let $e_1, e_2, \ldots, e_{m+n-k+1}$ be a basis of V. Define $W = \langle e_{m+1}, \ldots, e_{m+n-k+1} \rangle$ and $W' = \langle e_1, \ldots, e_m \rangle$. Let $W_1, \ldots, W_t$ be different hyperplanes in W such that $\bigcap_{i=1}^{t} W_i$ is the zero vector; note that it is possible to choose such a set of hyperplanes since $t \geq n-k+1$ and since the underlying field of V is either an infinite field or else a finite field $\mathbb{F}_q$ with $\frac{q^{n-k+1}-1}{q-1} \geq t$.

Now we define $V_i = \langle e_j | p_j \in L_i \rangle$ and $\pi_i = \langle V_i, W_i \rangle$, for all $i = 1, \ldots, t$. It is immediate that $\dim \pi_i = k + (n-k) = n$, for all $i = 1, \ldots, t$. We also know that $\dim(\pi_i \cap \pi_j) = 1 + (n-k-1) = n-k$ for all $i, j \in \{1, \ldots, t\}$, with $i \neq j$, since the hyperplanes W_i and W_j of W meet in an $(n-k-1)$-space. Hence, $\mathcal{S} = \{\pi_1, \ldots, \pi_t\}$ is an $(n, n-k)$-SCID.

We check that $\mathcal{S}$ is primitive. Firstly, it is obvious that $\langle \mathcal{S} \rangle = V$. Secondly, we show that the intersection of all π_i is trivial. Each vector in $V = W \oplus W'$ can uniquely be written as the sum of a vector in W and a vector in W'. Assume that $v + v' \in \bigcap_{i=1}^{t} \pi_i$, with $v \in W$ and $v' \in W'$. Since $v + v' \in \pi_i$ for all $i \in \{1, \ldots, t\}$, we know that $v \in W_i$ for all $i \in \{1, \ldots, t\}$ and that $v' \in V_i$ for all $i \in \{1, \ldots, t\}$. From the former observation it follows that $v \in \bigcap_{i=1}^{t} W_i = \{0\}$, hence $v = 0$. From the latter observation it follows that $v' \in \bigcap_{i=1}^{t} V_i \subset W'$; as there is no element of Ω contained in all L_i, all coefficients of v' with respect to the basis $e_1, \ldots, e_m$ of W' must be 0, hence $v' = 0$. We find that $v + v' = 0$, so the intersection of all π_i is trivial.

Thirdly, we will show that each $\pi \in \mathcal{S}$ is generated by its pairwise intersections with the other spaces, i.e. $\pi = \langle \pi \cap \sigma | \sigma \in \mathcal{S} \setminus \{\pi\} \rangle$. Note that the inclusion $\pi \supseteq \langle \pi \cap \sigma | \sigma \in \mathcal{S} \setminus \{\pi\} \rangle$ is trivial. Fix an arbitrary π_i with $i \in \{1, \ldots, t\}$. Recall that no element of Ω is contained in exactly one of the elements of $\mathcal{L}$, hence each element of L_i is contained in the intersection of L_i and another $L \in \mathcal{L}$. Consequently, for each j such that $p_j \in L_i$, the vector e_j is contained in the intersection of V_i and a subspace V_j with $j \neq i$. So, on the one hand,

$$V_i = \langle V_i \cap V_j | j = 1, \ldots, i-1, i+1, \ldots, t \rangle \subseteq \langle \pi_i \cap \pi_j | j = 1, \ldots, i-1, i+1, \ldots, t \rangle \ .$$

On the other hand, two hyperplanes in W meet each other in an $(n-k-1)$-space. Since only the zero vector is contained in all the subspaces $W_1, \ldots, W_t$, not all intersections in the set $\{W_i \cap W_j | j = 1, \ldots, i-1, i+1, \ldots, t\}$ can be equal. Hence,

$$W_i = \langle W_i \cap W_j | j = 1, \ldots, i-1, i+1, \ldots, t \rangle \subseteq \langle \pi_i \cap \pi_j | j = 1, \ldots, i-1, i+1, \ldots, t \rangle \ .$$

Since $\pi_i = \langle V_i, W_i \rangle$, we find that $\pi_i = \langle \pi_i \cap \pi_j | j = 1, \ldots, i-1, i+1, \ldots, t \rangle$.

Finally, $\dim V = m + n - k + 1 \geq 2n + 2$ by the assumption on m. This concludes the proof that $\mathcal{S}$ is primitive. $\square$

We present some applications of this main theorem. We start with an easy example.

Corollary 3.2. If $n \geq 2$, there is a primitive $(n, 1)$-SCID in the $(n^2 - 2n + 3)$-dimensional vector space over the field $\mathbb{F}$, with $\mathbb{F}$ an infinite field or a finite field $\mathbb{F}_q$, with $q \geq 2n - 3$.

Proof. We consider an $(n-1) \times (n-1)$-grid. The statement follows by applying Theorem 3.1 with Ω the set of $(n-1)^2$ points of the grid and $\mathcal{L}$ the set of $2(n-1)$ lines. $\square$

The *projective plane* $\mathrm{PG}(2,q)$ over the finite field $\mathbb{F}_q$ is the point-line geometry arising from the vector space $V(3,q)$ by considering the 1-spaces as *points* and the 2-spaces as *lines*, i.e. it is the geometry of the subspaces of the vector space $V(3,q)$. A *line set without tangent points* in $\mathrm{PG}(2,q)$ is a line set $\mathcal{S}$ in $\mathrm{PG}(2,q)$ such that any point of $\mathrm{PG}(2,q)$ is on zero or at least two lines of $\mathcal{S}$.

Corollary 3.3. Let $k-1$ be a prime power. Assume that $\mathrm{PG}(2,k-1)$ contains a line set $\mathcal{S}$ without tangent points, such that $|\mathcal{S}|=t$ and there are m points on the union of the lines in $\mathcal{S}$. If n is an integer such that $k+1\leq n\leq \min\{t+k-1,m-k-1\}$, then there is a primitive $(n,n-k)$-SCID in the $(m+n-k+1)$-dimensional vector space over the field $\mathbb{F}$, with $\mathbb{F}$ an infinite field or a finite field $\mathbb{F}_q$ with q such that $\frac{q^{n-k+1}-1}{q-1}\geq t$.

Proof. We take the points of $\mathrm{PG}(2,k-1)$ that are on the union of the lines in $\mathcal{S}$ as the set Ω. Now we consider the lines as sets of points, hence as subsets of Ω. The statement is a direct corollary of Theorem 3.1 by choosing $\mathcal{S}$ as the set $\mathcal{L}$. $\square$

We apply this corollary for two well-known line sets without tangent points. Many more applications are possible.

Corollary 3.4. Let $k-1$ be a prime power.

- For all integers n such that $k+1\leq n\leq \min\{t+k-1,m-k-1\}$, there is a primitive $(n,n-k)$-SCID in the $(k^2-2k+2+n)$-dimensional vector space over the infinite field $\mathbb{F}$.
- Let t be an integer such that $3k-3\leq t\leq k^2-k+1$. For all integers n with $\max\{k+1,\frac{\ln(t(q-1)+1)}{\ln(q)}+k-1\}\leq n\leq \min\{t+k-1,k^2-2k\}$, there is a primitive $(n,n-k)$-SCID in the $(k^2-2k+2+n)$-dimensional vector space over the finite field $\mathbb{F}_q$.

Proof. Let P_1, P_2 and P_3 be three noncollinear points in $\mathrm{PG}(2,k-1)$. Any line set that contains all $3k-3$ lines that pass through P_1, P_2 or P_3, is a set without tangent points. Now one can apply Corollary 3.3. $\square$

Corollary 3.5. Let $h\geq 1$ be an integer and denote $k=2^h+1$. For all integers n such that $k+1\leq n\leq \min\{2k,\frac{k^2-k}{2}-1\}$, there is a primitive $(n,n-k)$-SCID in the $(\frac{k^2-k}{2}+n+1)$-dimensional vector space over the field $\mathbb{F}$, with $\mathbb{F}$ an infinite field or a finite field $\mathbb{F}_q$ with q such that $\frac{q^{n-k+1}-1}{q-1}\geq k+1$.

Proof. We apply Corollary 3.3 for a dual hyperoval in $\mathrm{PG}(2,2^h)$, a set of 2^h+2 lines such that any point is contained on zero or two of them. $\square$

We conclude this article by presenting a primitive (5,2)-SCID that does not arise from Theorem 3.1.

Example 3.6. Consider the set $\{1,\dots,12\}$ and let $\mathcal{L}$ be the following set of subsets:

$$\begin{aligned}\mathcal{L}=\{&\{1,2,4,7,8\},\{1,2,5,9,10\},\{1,2,6,11,12\},\{1,3,4,9,12\},\\&\{1,3,5,7,11\},\{1,3,6,8,10\},\{2,3,4,5,6\}\}\ .\end{aligned}$$

Consider a 12-dimensional vector space V with basis $\{e_1,\dots,e_{12}\}$. Now, let $\mathcal{S}$ be the set

$$\mathcal{S}=\{\langle e_v,e_w,e_x,e_y,e_z\rangle\mid\{v,w,x,y,z\}\in\mathcal{L}\}\ .$$

It can straightforwardly be checked that $\mathcal{S}$ is a primitive (5,2)-SCID of V.

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