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Weakly Motzkin predecomposable sets

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Dedicated to Michel Théra on the occasion of his 70th birthday

Abstract

We introduce and study the class of weakly Motzkin predecomposable sets, which are those sets in $\mathbb{R}^n$ that can be expressed as the Minkowski sum of a bounded convex set and a convex cone, none of them being necessarily closed. This class contains that of Motzkin predecomposable sets, for which the bounded components are compact, which in turn contains the class of Motzkin decomposable sets, for which the bounded components are compact and the conic components are closed.

Keywords: Motzkin decomposable sets · Convex sets · Convex cones
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1 Introduction

The remote origin of the research topic of this paper dates back to 1936, when in his PhD thesis [7] Motzkin studied the structure of the solution set of an arbitrary linear inequality system; his main result can be stated by saying that convex polyhedra in $\mathbb{R}^n$ are characterized as Minkowski sums of polytopes and polyhedral convex cones. Since polytopes are compact convex sets and polyhedral cones are convex, the natural question arises of characterizing the wider class of closed convex sets that can be expressed as sums of compact convex sets and closed convex cones. These sets, which are called Motzkin decomposable (M-decomposable, in short) were introduced and studied in [1] and further investigated in [2], [3] and [4]. The class of M-decomposable sets enjoys many nice properties; nevertheless some other desirable properties fail to hold. For instance, there are easy examples showing that the sum of two M-decomposable sets need not be M-decomposable. One easily realizes that the reason for this lies in the fact that the sum of two closed convex cones is not necessarily closed; thus, the sum of two M-decomposable sets can be expressed as the sum of a compact convex set (because the class of compact convex sets is closed under Minkowski addition) and a convex cone which need not be closed. In the same way, the linear image of an M-decomposable set is not necessarily M-decomposable, though it admits a decomposition of the type just mentioned for sums of M-decomposable sets. This suggests to consider the wider class of convex sets admitting such a representation. This was done in [5], where such sets were called Motzkin predecomposable (M-predecomposable, in short). As shown in [5], this class shares many nice properties with that of M-decomposable sets, but there are important differences too. A nice feature of the new class is that, unlike that of M-decomposable sets, it is closed under Minkowski addition and linear images.

Since removing the closedness assumption on the convex cones entering in Motzkin decompositions gave rise to the interesting class of M-predecomposable sets, it seems worth to investigate the widest possible class to be considered in the spirit of Motzkin decomposability, namely, that of convex sets obtained by removing not only the closedness assumption on the convex cones but also that on the compact convex components, that is, by replacing the compactness assumption with just boundedness. This is precisely what we do in this paper: We introduce the class of convex sets in $\mathbb{R}^n$ that can be expressed as the sum of a bounded convex set and a convex cone, none of them being necessarily closed. We call such sets weakly Motzkin predecomposable (wM-predecomposable, in short). We study their fundamental properties and provide two characterizations, one of them in terms of recession cones and exposed faces, and the other one in terms of truncations, that is, intersections with closed halfspaces. As shown in [3], truncation is an important operation on M-decomposable sets. In connection with this operation, it is worth mentioning that we give an example exhibiting an M-predecomposable set which has no unbounded M-predecomposable truncation other than itself but has infinitely many unbounded wM-predecomposable truncations. Thus, wM-predecomposable sets appear also in a natural way when dealing with truncations. Let us mention that an interesting class of wM-predecomposable sets, namely, the one consisting of sums of open bounded convex sets and closed convex cones, has been recently introduced and studied in [6].

We will adopt the terminology and notation of the classical Rockafellar's monograph Convex Analysis [8], with only one exception: By a cone we mean a non-empty set closed under non-negative scalar multiplication. Thus, unlike in [8], all our cones contain the origin.

2 Weakly Motzkin predecomposable sets

The notion of weakly Motzkin predecomposable set, which is introduced next, generalizes that of Motzkin predecomposable set [5], which in turn generalizes that of Motzkin decomposable set [1, 2, 3].

Definition 1 *A non-empty set $F \subset \mathbb{R}^n$ is weakly Motzkin predecomposable (wM-predecomposable in short) if there exists a bounded convex set C and a convex cone D such that $F = C + D$. The pair (C, D) will be said to be a weak Motzkin decomposition of F, and the sets C and D will be said to be a bounded component and a conic component, respectively, of F.*

The following proposition provides an equivalent definition of wM-predecomposable set. In contrast with the original definition, in this equivalent definition F is assumed a priori to be convex, but the convexity assumption on the bounded and conic components is removed.

Proposition 2 *A non-empty set $F \subset \mathbb{R}^n$ is wM-predecomposable if and only if it is convex and there exist a bounded set C and a cone D such that $F = C + D$. In this case, $(convC, convD)$ is a weak Motzkin decomposition of F.*

Proof. The proof is virtually identical to that of [5, Proposition 3]. We only need to prove the "if" statement. It follows from the equalities $F = convF = conv(C + D) = convC + convD$ and the fact that the convex hull of a bounded set (a cone) is bounded (a cone, respectively). ■

The following proposition shows that there is no loss of generality to assume that the conic component of a wM-decomposable set is its recession cone.

Proposition 3 *If (C, D) is a weak Motzkin decomposition of a non-empty set $F \subset \mathbb{R}^n$, then $D \subset 0^+F$ and $(C, 0^+F)$ is another weak Motzkin decomposition of F.*

Proof. From the equalities $F + D = C + D + D = C + D = F$ the inclusion $D \subset 0^+F$ follows. Hence $F = C + D \subset C + 0^+F \subset F + 0^+F = F$, whereby $F = C + 0^+F$, which shows that $(C, 0^+F)$ is a weak Motzkin decomposition of F. ■

We will use the following elementary lemma to show both that the conic component of a wM-decomposable set need not be unique and that an M-predecomposable set with a compact component having a non-empty interior admits weak Motzkin decompositions with non closed bounded components.

Lemma 4 *If U and V are two subsets of $\mathbb{R}^n$ and U is open, then $U + V = U + clV$.*

Proof. Let $u \in U$ and $v \in clV$, and take $\epsilon > 0$ such that $B(u, \epsilon) \subset U$ and $v' \in V \cap B(v, \epsilon)$. Since $u + v - v' \in B(u, \epsilon) \subset U$, we have $u + v = u + v - v' + v' \in U + V$. ■

Remark 5 *In the case of Motzkin predecomposable sets, the conic component* D *in a Motzkin decomposition* (C, D) *is uniquely determined by* $D = 0^+ (C + D)$ *[5, Proposition 6], but this is not the case for decompositions of wM-predecomposable sets. Indeed, if* C *is a bounded open convex set and* D *is an arbitrary convex cone then, according to Lemma 4, both* (C, D) *and* (C, clD) *are weak Motzkin decompositions of* $C + D$*, and they are different if* D *is not closed.*

Remark 6 *A weak Motzkin decomposition of a Motzkin predecomposable set is not necessarily a Motzkin decomposition. Indeed, if* C *is a compact convex set and* $D \neq \{0\}$ *is a convex cone then, for* $d \in D \setminus \{0\}$*, the pair* $(C + [0, 1)\, d, D)$ *is a weak Motzkin decomposition of the Motzkin predecomposable set* $C + D$*, but it is not a Motzkin decomposition because* $C + [0, 1)\, d$ *is not closed.*

Proposition 7 *If* $F \subset \mathbb{R}^n$ *is wM-predecomposable, then* clF *is M-decomposable and* $0^+ clF = cl0^+ F$*.*

Proof. The proof follows the same pattern as that of [5, Proposition 8]. It is a consequence of the equalities

$$clF = cl\left(C + 0^+ F\right) = clC + cl0^+ F \tag{1}$$

together with [5, Proposition 6]. ■

Corollary 8 *A closed set* $F \subset \mathbb{R}^n$ *is wM-predecomposable if and only if it is M-decomposable.*

Following an interesting suggestion made by a referee, we introduce the following notion.

Definition 9 *A non-empty set* $F \subset \mathbb{R}^n$ *is weakly Motzkin decomposable if there exists a bounded convex set* C *and a closed convex cone* D *such that* $F = C + D$*.*

The next pproposition characterizes weakly Motzkin decomposable sets as those wM-predecomposable sets that have a closed recession cone.

Proposition 10 *A non-empty set* $F \subset \mathbb{R}^n$ *is weakly Motzkin decomposable if and only if it is wM-predecomposable and* $0^+ F$ *is closed. In this case, the conic component of* F *is uniquely determined, namely it is* $0^+ F$*.*

Proof. The "if" statement is an immediate consequence of Proposition 3. Conversely, if F is weakly Motzkin decomposable then it is clearly wM-predecomposable. Let (C, D) be a weak Motzkin decomposition of F. Taking closures in the equality $F = C + D$ and using that C is bounded, we get $clF = clC + clD = clC + D$. By Proposition 7, the set clF is M-decomposable; hence, by Propositions 3 and 7 and [1, Proposition 13.(vi)], we have $cl0^+ F = 0^+ clF = D \subset 0^+ F$, which proves that $0^+ F$ is closed and $D = 0^+ F$. ■

The preceding proposition shows that weakly Motzkin decomposable sets still satisfy a property of Motzkin decomposable sets, namely the uniqueness of their conic components, which does not hold for general wM-predecomposable sets (see Remark 5).

Weakly Motzkin predecomposable sets arise sometimes when considering truncations of Motzkin predecomposable sets. The following example exhibits an M-predecomposable set which has no unbounded M-predecomposable truncation other than itself but has infinitely many unbounded weakly Motzkin decomposable truncations. It is also worth noticing that in this example the M-predecomposable set and its proper truncations have different recession cones.

Example 11 *Let* $F := int\mathbb{R}_+^2 \cup \left\{0^2\right\}$*. Since* F *is a convex cone, it is wM-predecomposable. Clearly, every compact intersection of* F *with a hyperplane reduces to* $\left\{0^2\right\}$*, and hence* F *has no unbounded M-predecomposable truncation other than itself. For* $a, b > 0$*, let us consider the hyperplane*

$$H =: \left\{(x, y) \in \mathbb{R}^2 : ax + by = 1\right\}$$

and its associated halfspace

$$H^+ =: \left\{(x, y) \in \mathbb{R}^2 : ax + by \geq 1\right\}.$$

Clearly, $F \cap H$ *is bounded and one has* $F \cap H^+ = F \cap H + F = F \cap H + \mathbb{R}_+^2$*, which shows that* $F \cap H^+$ *is weakly Motzkin decomposable. On the other hand,* $0^+ \left(F \cap H^+\right) = \mathbb{R}_+^2 \neq F = 0^+ F$.

Remark 12 *A truncation of an M-decomposable set* F *need not be wM-predecomposable, even if* 0^+F *is pointed. As observed in [3, p. 37], if* F *is the "ice-cream cone" with vertex at the origin and axis* $(0, 0, 1)$*, its intersection with one of the closed halfspaces determined by a vertical hyperplane not containing the origin is not M-decomposable. Since this intersection is closed, by Corollary 8 it is not wM-predecomposable either.*

The proof of the following proposition is an easy exercise (see [5, Proposition 13] for an analogous result on M-predecomposability).

Proposition 13 *(i) If* F_1 *and* F_2 *are wM-predecomposable then* $F_1 + F_2$ *is wM-predecomposable.*

(ii) If F *is wM-predecomposable and* $E : \mathbb{R}^n \to \mathbb{R}^n$ *is affine, then* $E(F)$ *is wM-predecomposable.*

The preceding proposition shows that weak Motzkin predecomposability is preserved by sums and linear images. These properties are not satisfied by weakly Motzkin decomposable sets, as one can easily realize by taking into account that closed convex cones are particular instances of weakly Motzkin decomposable sets.

The following example shows that the convex hull of a union of wM-predecomposable sets need not be wM-predecomposable.

Example 14 *Let*

$$F_1 := \left\{(x,y,0) \in \mathbb{R}^3 : x > 0,\ y \geq 0\right\} \cup \left\{0^3\right\}$$

and

$$F_2 =: \left\{(x,y,1) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0\right\}.$$

Since F_1 is a convex cone, it is M-predecomposable. Since F_2 is the translate of a closed convex cone, it is M-decomposable. However the set

$$\begin{aligned} F :=\ & conv\,(F_1 \cup F_2) \\ =\ & \left\{(x,y,z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ 0 \leq z \leq 1\right\} \setminus \left\{(0,y,0) \in \mathbb{R}^3 : y > 0\right\} \end{aligned} \quad (2)$$

is not wM-predecomposable. Indeed, if we had $F = C + D$, with C convex and bounded and D being a cone, then, by Proposition 3, we would have $D \subset cl0^+F = \left\{(x,y,0) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0\right\}$, which, together with the inclusion $F \subset \mathbb{R}^3_+$ and the fact that $0^3 \in F$, implies that $0^3 \in C$. Let

$$\overline{y} := \sup\left\{y \in \mathbb{R} : (x,y,z) \in C\right\},$$

and take $y > \overline{y}$. Since $(0,y,1) \in F \setminus C$, there exist $c \in C$ and $d \in D \setminus \{0\}$ such that $(0,y,1) = c + d$. The third component of d is 0, and since both c and d have non-negative components, the first component of d must be 0 too. Consequently, $(0,1,0) \in D \subset F$, which contradicts (2). This proves that F is not wM-predecomposable.

The following proposition is analogous to [5, Proposition 14]; the proofs of both results are almost identical.

Proposition 15 *Every non-empty face of a wM-predecomposable set is wM-predecomposable, too. More specifically, if C is a bounded component of a wM-decomposable set F and G is a non-empty face of F, then $C \cap G = C \cap affG$ is a bounded component of G.*

Proof. Let F, C and G be as in the statement. We will prove that $G = C \cap G + 0^+G$. Since $C \cap G \subset C \cap affG \subset F \cap affG = G$, the inclusion $\supset$ is obvious; moreover, by taking the intersection with C it follows that $C \cap G = C \cap affG$. For proving the inclusion $G \subset C \cap G + 0^+G$, let $x \in G$. Since $G \subset F = C + 0^+F$ (Proposition 3), there exist $c \in C$ and $d \in 0^+F$ such that $x = c + d$. For every $\lambda > 1$ we have $x = \left(1 - \frac{1}{\lambda}\right)c + \frac{1}{\lambda}(c + \lambda d)$; hence, given that $c, c + \lambda d \in F$, we conclude that $c \in G$ and $c + \lambda d \in G$, from which we deduce that $x + (\lambda - 1)\,d \in G$. This shows that $d \in 0^+G$. Since $c \in C \cap G$, it follows that $x \in C \cap G + 0^+G$. We have thus proved that $C \cap G$ is a bounded component of G. ■

However, the analogy between wM-predecomposable sets and M-predecomposable sets regarding faces does not go much beyond the preceding proposition. In the case of an M-predecomposable set F, the recession cone of a face is contained in that of F [5, Theorem 15]; however this is not the case of wM-predecomposable sets, as the following example shows.

Example 16 *Let*

$$C := \left\{ (x,y,0) \in \mathbb{R}^3 : 0 \le x \le 1,\ 0 \le y \le 1 \right\} \setminus \{(1,0,0), (1,1,0)\},$$

$D := \left\{ (0,y,z) \in \mathbb{R}^3 : z > |y| \right\} \cup \left\{ 0^3 \right\}$, $F := C + D$, *H be the hyperplane defined by the equation $x = 1$, and $G := F \cap H$. Clearly, F is wM-predecomposable and, since H is a supporting hyperplane of F, the set G is an exposed face of F. It is easy to see that* $G = \left\{ (1,y,z) : z \ge 0,\ z > \left| y - \frac{1}{2} \right| - \frac{1}{2} \right\}$. *One can also check that $(0,-1,1) \in 0^+G$; however $(0,-1,1) \notin 0^+F$, since $0^3 \in F$ but $(0,-1,1) \notin F$, as it is readily seen.*

Our next theorem gives a characterization of wM-predecomposable sets involving recession cones and exposed faces.

Theorem 17 *Let $F \subset \mathbb{R}^n$ be a non-empty convex set. Then F is wM-predecomposable if and only if the following conditions hold:*

(a) $cl0^+F = 0^+clF$.

(b) There exists a bounded convex set $C \subset F$ such that

(i) every linear function which is bounded from below on F attains its minimum over clF on clC.

(ii) for every supporting hyperplane H of F one has

$$F \cap H = C \cap H + 0^+F \cap (H - H). \tag{3}$$

In such a case, a bounded convex set $C \subset F$ is a bounded component of F if and only if it satisfies the properties stated in (i) and (ii).

Proof. Assume that F is wM-predecomposable. By Proposition 7, condition (a) holds. Take a bounded component C of F. By (1), the set clC is a compact component of the M-decomposable set clF; hence, by [1, Proposition 16], the set C satisfies (i). Let H be a supporting hyperplane of F. The inclusion $\supset$ clearly holds in (3). To prove the opposite one, let $x \in F \cap H$. Then $x = c + d$, where $c \in C$ and $d \in 0^+F$. Since the case $d = 0^n$ is trivial, assume that $d \neq 0^n$. Since $F \cap H$ is a face of F, from $x = \frac{1}{2}(c + (c + 2d))$ and $c, c + 2d \in F$ it follows that $c \in H$. Hence $d = x - c \in H - H$, which shows that $x \in C \cap H + 0^+F \cap (H - H)$. This proves the inclusion $\subset$ in (3) and hence the equality.

Conversely, let us assume that (a) and (b) hold. We shall prove that $F = C + 0^+F$. By [1, Proposition 16], the set clF is M-decomposable and $clF = clC + 0^+clF$. Using (a), we obtain $C + 0^+F \subset F \subset clF = clC + 0^+clF = clC + cl0^+F = cl(C + 0^+F)$. From these inclusions we easily get the equality $ri(C + 0^+F) = riF$. Let $x \in F \setminus riF$, and take a supporting hyperplane H of F at x. By (3), we have $x \in C \cap H + 0^+F \cap (H - H) \subset C + 0^+F$. We thus conclude that $F = C + 0^+F$.

In view of the above proof of the "only if" statement, every bounded component C of F satisfies properties (i) and (ii). Conversely, according to the proof of the "if" statement, if (a) holds and a bounded convex set $C \subset F$ satisfies properties (i) and (ii), then C is a bounded component of F. ■

Remark 18 *The above proof of the "only if" implication shows that (3) actually holds for every affine manifold H, not necessarily a hyperplane, such that $F \cap H$ is a face of F. Hence, since every non-empty face G of F coincides with the intersection of its affine hull $aff(G)$ with F, by applying (3) with H replaced by $aff(G)$, one obtains*

$$G = C \cap aff(G) + 0^+F \cap (aff(G) - aff(G)). \tag{4}$$

From this equality, the inclusion $0^+G \subset 0^+F$ immediately follows. Furthermore, if F is weakly Motzkin decomposable, using the uniqueness of its conic component (Proposition 10) one obtains the equality $0^+G = 0^+F \cap (aff(G) - aff(G))$. Another immediate consequence of (4) is that every non-empty face of a weakly Motzkin decomposable set is weakly Motzkin decomposable, too.

Remark 19 *In view of Proposition 10, Theorem 17 still holds true if one replaces "wM-predecomposable" and "cl0^+F" with "weakly Motzkin decomposable" and "0^+F," respectively, in its statement.*

The following examples show that none of conditions (a), (b)(i) and (b)(ii) is superfluous in the statement of Theorem 17.

Example 20 *The convex set*

$$F := \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\ y > 0\right\} \cup \left\{(x,0) \in \mathbb{R}^2 : 0 \leq x \leq 1\right\}$$

satisfies condition (b) of Theorem 17, with $C := \left\{(x,0) \in \mathbb{R}^2 : 0 \leq x \leq 1\right\}$, but not condition (a), because $0^+clF = \left\{(0,y) \in \mathbb{R}^2 : y \geq 0\right\}$ and $0^+F = \{0^2\}$. Thus F is not wM-predecomposable. This shows that condition (a) is not superfluous in the statement of Theorem 17.

Example 21 *(see [5, Example 18]) The convex set*

$$F := \left\{(x,y) \in \mathbb{R}^2 : x > 0,\ y > 0,\ xy > 1\right\} \cup \{(1,1)\}$$

satisfies conditions (a) and (b)(ii) of Theorem 17 with $C := \{(1,1)\}$, since the only supporting hyperplane H of F is, $H := \left\{(x,y) \in \mathbb{R}^2 : x + y = 1\right\}$, but there is no bounded convex set C satisfying condition (b)(i), because, for instance, the linear function $(x,y) \mapsto x$ is bounded from below on F but attains no minimum on cl$F$. Thus F is not wM-predecomposable. This shows that condition (b)(i) is not superfluous in the statement of Theorem 17.

Example 22 *(see [5, Example 19]) The convex set*

$$F := \left\{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ y \geq 0,\ z \geq 0\right\} \backslash \left\{(x,y,0) \in \mathbb{R}^3 : x \in \{0,1\},\ y > 0\right\}$$

satisfies conditions (a) and (b)(i) of Theorem 17, with

$$C := \left\{(x,0,0) \in \mathbb{R}^3 : 0 \leq x \leq 1\right\},$$

but since the exposed face

$$\left\{(x,y,0)\in\mathbb{R}^3 : 0<x<1,\ y\geq 0\right\}\cup\left\{(0,0,0),(1,0,0)\right\}$$

is not wM-predecomposable (see Example 20), by Proposition 15 the set F is not wM-predecomposable either. This shows that condition (b)(ii) is not superfluous in the statement of Theorem 17.

As a consequence of Theorem 17, we get the following characterization of M-predecomposable sets.

Corollary 23 *Let $F\subset\mathbb{R}^n$ be a non-empty convex set. Then F is M-predecomposable if and only if the following conditions hold:*

(a) $cl0^+F=0^+clF$.

(b) There exists a compact convex set $C\subset F$ such that

(i) every linear function which is bounded from below on F attains its minimum over F on C.

(ii) for every supporting hyperplane H of F one has

$$F\cap H=C\cap H+0^+F\cap(H-H).$$

In such a case, a compact convex set $C\subset F$ is a compact component of F if and only if it satisfies the properties stated in (i) and (ii).

Proof. Assume that F is M-predecomposable and let C be a compact component. Then F is wM-predecomposable, and C is a bounded component; hence, by Theorem 17, (a) holds and C satisfies (i) and (ii). Conversely, assume that conditions (a) and (b) hold. Then, by Theorem 17, the set F is wM-predecomposable and has C as a bounded component, which implies that F is actually M-predecomposable and has C as a compact component. ■

The main difference between the preceding characterization of M-predecomposable sets and the one given given in [5, Theorem 15] is that the latter involves two properties of faces: one involving recession cones of general faces and the other one involving compact components of exposed faces. In Corollary 23(b)(ii) only exposed faces are considered, and the stated condition actually has exactly the same meaning as the corresponding one in [5, Theorem 15] regarding compact components but is more explicit regarding recession cones, as it means that the one of a exposed face of an M-predecomposable set coincides with the intersection of that of the set with the one of the supporting hyperplane that determines the given exposed face.

Our next characterization of wM-predecomposable sets is in term of truncations.

Theorem 24 *Let $F\subset\mathbb{R}^n$ be a non-empty convex set such that $cl0^+F$ is pointed. Then the following statements are equivalent:*

(a) F is wM-predecomposable.

(b) There exists a hyperplane H with associated halfspaces H^- and H^+ such that $F \cap H^-$ is non-empty and bounded and $F \cap H^+ = F \cap H + 0^+F$.

(c) There exists a hyperplane H with associated halfspaces H^- and H^+ such that $F \cap H^-$ is bounded and $F \cap H^+$ is wM-predecomposable with a conic component contained in 0^+F.

Proof. (a) $\Longrightarrow$ (b) Let C be a bounded component. Since $cl0^+F$ is pointed, there exists a hyperplane H with associated halfspaces H^- and H^+ such that $F \cap H^-$ is bounded and contains C, which implies that $0^+F \cap (H^- - H) = \{0^n\}$. Let $x \in F \cap H^+$; then $x = c + d$, where $c \in C$ and $d \in 0^+F$. Since $c \in F \cap H^-$ and $c + d \in F \cap H^+$, there exist $\overline{c} \in F \cap H$ and $\lambda \in [0, 1]$ such that $x = \overline{c} + \lambda d$; therefore $x \in F \cap H + 0^+F$. This proves the inclusion $F \cap H^+ \subset F \cap H + 0^+F$. To prove the opposite inclusion, let $x \in F \cap H + 0^+F$; then $x = c + d$, where $c \in F \cap H$ and $d \in 0^+F$. If $d = 0^n$, then $x \in F \cap H \subset F \cap H^+$. If $d \neq 0^n$, then, by $0^+F \cap (H^- - H) = \{0^n\}$, we have $d \notin H^- - H$; since $d = x - c$ and $c \in H$, it follows that $x \notin H^-$, and hence $x \in H^+$. This proves the inclusion $F \cap H^+ \supset F \cap H + 0^+F$ and hence the equality.

Implication (b) $\Longrightarrow$ (c) is obvious.

(c) $\Longrightarrow$ (a) Let (C, D) be a weak Motzkin decomposition of $F \cap H^+$ with $D \subset 0^+F$. We shall prove that the equality $F = conv\left((F \cap H^-) \cup C\right) + D$ holds. The inclusion $\supset$ is obvious, since $conv\left((F \cap H^-) \cup C\right) \subset F$. To prove $\subset$, let $x \in F$. If $x \in H^-$, we are done. If $x \in H^+$, then, by the assumption $F \cap H^+ = C + D$, we have $x \in conv\left((F \cap H^-) \cup C\right) + D$, which proves the required inclusion. Therefore, $F = conv\left((F \cap H^-) \cup C\right) + D$, which, as $F \cap H^-$ and C are bounded, proves that F is wM-predecomposable. ∎

Concerning minimal decompositions, the situation with weakly Motzkin decomposable sets is quite different from the case of M-predecomposable sets. Indeed, if an M-predecomposable set F has a recession cone with pointed closure, a smallest compact component exists and is the closed convex hull of the set of extreme points of F [5, Theorem 29]. On the contrary, every open bounded convex set provides an example of a weakly Motzkin decomposable set with no extreme point but with a smallest bounded component, which is obviously itself. Furthermore, a minimal bounded component does not necessarily exist even if the recession cone has a pointed closure, as the weakly Motzkin decomposable set $\left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\ y > 0\right\}$ shows.

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