

# **“Normal forms” and global phase portraits of quadratic and cubic integrable vector fields having two nonconcentric circles as invariant algebraic curves**

Jaume Llibre<sup>a</sup>, Marcelo Messias<sup>b\*</sup> and Alison C. Reinol<sup>b</sup>

<sup>a</sup>*Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain.*; <sup>b</sup>*Departamento de Matemática e Computação, Faculdade de Ciências e Tecnologia, Universidade Estadual Paulista, UNESP, P. Prudente, SP, Brasil.*

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**In this paper we give the normal form of all planar polynomial vector fields of degree  $d \leq 3$  having two nonconcentric circles  $C_1$  and  $C_2$  as invariant algebraic curves and the function  $H = C_1^\beta C_2^\alpha$ , with  $\alpha$  and  $\beta$  real values, as first integral. Moreover we classify all global phase portraits on the Poincaré disc of a subclass of these vector fields.**

**Keywords:** Quadratic and cubic vector fields; invariant algebraic curves; limit cycles; Poincaré compactification; global analysis; **normal forms.**

**MCS/CCS/AMS Classification/CR Category numbers:** 34C05, 34C07, 34C08, 34C20, 34C25, 37G10.

## **1. Introduction and statement of the main result**

In this work we deal with the following planar polynomial differential systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1)$$

where  $P$  and  $Q$  are polynomials in the real variables  $x$  and  $y$  with degree  $d = \max\{\deg(P), \deg(Q)\}$ , where  $\deg(P)$  and  $\deg(Q)$  are the degrees of  $P$  and  $Q$ , respectively. In system (1) the dot denotes derivative with respect to the time  $t$ . When  $d = 3$  we say that system (1) is *cubic* and when  $d = 2$  we say that it is *quadratic*. Here we consider  $d \leq 3$ . We can associate to system (1) the vector field  $X = (P, Q)$  defined on  $\mathbb{R}^2$ . Beyond its theoretical importance, systems like (1) appear frequently in many branches of applied mathematics, as for example in modeling physical and biological problems. An important question about system (1) is the determination and localization of its limit cycles, which is related to the well-known Hilbert 16<sup>th</sup> problem [4].

**As the whole class of systems (1) is very difficult to be studied, even in the quadratic case, one of the strategies to study them is to consider subclasses of these systems, satisfying some properties, as for instance the subclasses studied in [1, 5, 6] and references therein. In order to establish certain properties to system (1), we introduce some definitions of Darboux theory of**

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\*Corresponding author. Email: marcelo@fct.unesp.br

integrability. This kind of integrability provides a link between the integrability of vector fields and their invariant algebraic curves. More details about this theory can be found in Chapter 2 of [2] and in Chapter 8 of [3]. Denote by  $\mathbb{R}[x, y]$  the ring of polynomials in the real variables  $x$  and  $y$  and coefficients in  $\mathbb{R}$  and let  $f \in \mathbb{R}[x, y] \setminus \mathbb{R}$ , that is  $f$  is a nonconstant polynomial. The algebraic curve  $f(x, y) = 0$  is an *invariant algebraic curve* of system (1) if for some polynomial  $K \in \mathbb{R}[x, y]$  we have

$$X(f) = P \frac{\partial f}{\partial x} + Q \frac{\partial f}{\partial y} = Kf.$$

The polynomial  $K$  is called the *cofactor* of the invariant algebraic curve  $f = 0$ .

**In Theorem 2.1 of section 2 we shall prove that the most general normal form of planar polynomial vector fields having two invariant algebraic curves  $f_1 = 0$  and  $f_2 = 0$ , is**

$$X = \lambda_1 f_1 H_{f_2} + \lambda_2 f_2 H_{f_1} + f_1 f_2 \tilde{X}, \quad (2)$$

**where  $H_{f_i}$  denotes the Hamiltonian vector field  $(-\partial f_i / \partial y, \partial f_i / \partial x)$ ,  $\lambda_i$  are arbitrary rational functions, for  $i = 1, 2$ , and  $\tilde{X}$  is an arbitrary polynomial vector field. As an example, consider the polynomial vector field**

$$X_0 = (3y(3x^2 + 2xy + y^2 - 9x - 3y + 8), -6x^3 - 3x^2y + 3y^3 + 27x^2 + 9xy - 27x - 3y)$$

of degree 3 obtained from (2) taking

$$\begin{aligned} f_1(x, y) &= x^2 + y^2 - 1 = 0, \\ f_2(x, y) &= (x - 3)^2 + y^2 - 1 = 0, \\ \alpha &= -(x + y)/2, \\ \beta &= -(\alpha + 3/2), \\ \tilde{X} &= 0. \end{aligned}$$

It is not difficult to show that the cubic polynomial vector field  $X_0$  has the circles  $f_1 = 0$  and  $f_2 = 0$  as limit cycles, i.e. these two circles are periodic orbits isolated in the set of all periodic orbits of **vector field** (2). The phase portrait of  $X_0$  in the Poincaré disc is given in Figure 1. Note that  $X_0$  is an example of a cubic non-integrable vector field having two nonconcentric circles as invariant algebraic curves. This kind of vector fields may have a complicated behavior.

**Since this is a first work on quadratic and cubic vector fields having two nonconcentric invariant circles  $C_1$  and  $C_2$ , then we will restrict our goal to study the integrable ones, whose first integral is constructed uniquely using the nonconcentric invariant circles, that is the first integral is the function  $H = C_1^\beta C_2^\alpha$ , with  $\alpha$  and  $\beta$  real values.** After an appropriate change of coordinates we can suppose without loss of generality that the invariant circles  $C_1$  and  $C_2$  are given by

$$C_1(x, y) = x^2 + y^2 - 1 = 0 \quad \text{and} \quad C_2(x, y) = (x - a)^2 + y^2 - r^2 = 0, \quad (3)$$

with  $a > 0$  and  $r > 0$ .

**In Proposition 2.3 of section 2 we shall prove that the normal form of all planar polynomial vector fields of degree  $d \leq 3$  having two nonconcentric invariant circles  $C_1$  and  $C_2$ , given by (3), and a first integral given by  $H = C_1^\beta C_2^\alpha$ ,**

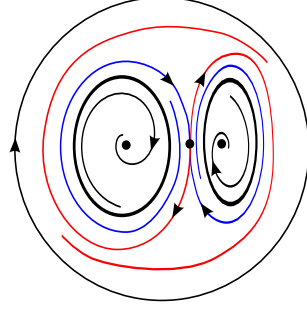


Figure 1. Phase portrait having two non-nested invariant circles which are limit cycles.

with  $\alpha$  and  $\beta$  real values, is

$$X = \lambda_1 C_1 H_{C_2} + \lambda_2 C_2 H_{C_1}, \quad (4)$$

where  $H_{C_i}$  denotes the Hamiltonian vector field  $(-\partial C_i/\partial y, \partial C_i/\partial x)$ , for  $i = 1, 2$ , and  $\lambda_1$  and  $\lambda_2$  are arbitrary rational functions such that  $\beta\lambda_1 = \alpha\lambda_2$ . Finally we do a study of the global phase portraits of the vector fields

$$X = \alpha C_1 H_{C_2} + \beta C_2 H_{C_1}, \quad (5)$$

obtained as a subclass of vector fields (4) taking  $\lambda_1 = \alpha$  and  $\lambda_2 = \beta$ . As we shall see in Proposition 2.4 these vector fields have no limit cycles.

Here all the phase portraits of vector field (5) presented in this note are drawn on the Poincaré disc. **More precisely**, the Poincaré compactification extends a polynomial vector field  $X = (P, Q)$  to a vector field  $\pi(X)$  defined on the compact manifold  $S^2$  (the unit sphere of  $\mathbb{R}^3$  centered at the origin), giving two copies of the vector field  $X$ , one in the open northern hemisphere of  $S^2$  and the other on the open southern hemisphere, in such a way that the flow on the equator corresponds to the flow at infinity of the vector field  $X$ . Using this technique, the global phase portrait of  $X$  become completely determined whenever we consider only the phase portraits of  $\pi(X)$  restricted to the northern closed hemisphere and we project it on the closed unit disc on  $\mathbb{R}^2$ , called the Poincaré disc. **For more details and definition on the Poincaré disc see Chapter 5 of [3].**

Aiming to determine the global phase portraits of vector field (5) it is important to consider the possible relative positions between the circles  $C_1$  and  $C_2$  given in (3) (see Figure 2), since the dynamical behavior of the solutions are directly related to their relative position.

Let  $X$  and  $Y$  be two polynomial vector fields of  $\mathbb{R}^2$ . We say that these vector fields are *topologically equivalent* if there exists a homeomorphism  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $h$  carries orbits under the flow of  $X$  onto orbits under the flow of  $Y$ , preserving or reversing the sense of all orbits.

We state below one of the main results of this paper, which give all the possible phase portraits on the Poincaré disc of vector fields (5), under topological equivalence.

**Theorem 1.1:** *The phase portrait on the Poincaré disc of vector fields (5) is topologically equivalent to one of the 22 phase portraits described in Figure 3.*

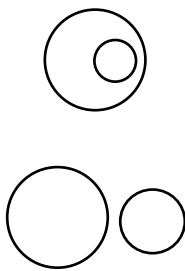
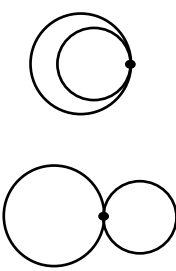
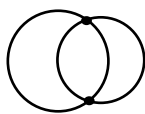
$0 < r <  l-a $ or $r > l+a$	$r =  l-a $ or $r = l+a$	$ l-a  < r < l+a$
		

Figure 2. Possible relative positions between the circles  $C_1$  and  $C_2$  given in (3).

This paper is devoted to prove Theorem 1.1. The phase portraits 1, 2 and 3 of Figure 3 correspond to quadratic systems and all the others correspond to cubic systems.

In the rest of this paper we present a global study of vector field (5) as follows. In section 2 we derive the normal form (5), **as a subclass of vector fields** (4), and, as a consequence, we prove that vector field (5) has no limit cycles. In section 3 we consider **quadratic vector fields** (5), obtained by taking  $\alpha = -\beta$ , and we describe all the possible global phase portraits for this subclass. In section 4 we do a global analysis of vector field (5), with  $\alpha \neq -\beta$ , considering three cases:  $\alpha = 0$  and  $\beta \neq 0$ ;  $\beta = 0$  and  $\alpha \neq 0$ ; and  $\alpha\beta \neq 0$ .

## 2. Normal forms

Let  $h_1$  and  $h_2$  be functions defined in an open subset  $U$  of  $\mathbb{R}^2$ . We define the *Jacobian matrix* of  $h_1$  and  $h_2$  as

$$J = \begin{pmatrix} \partial h_1 / \partial x & \partial h_1 / \partial y \\ \partial h_2 / \partial x & \partial h_2 / \partial y \end{pmatrix}.$$

The *Jacobian of  $J$* , i.e. the determinant of  $J$ , is denoted here by

$$|J| := \{h_1, h_2\}.$$

The next result is inspired in Theorem 1 of [7], which characterizes all vector fields having  $f_i = 0$ , for  $i = 1, \dots, n$ , as invariant algebraic curves such that  $\{f_1, \dots, f_n\} \neq 0$ . Here we present a shorter and a direct proof for the particular case of planar polynomial vector fields having two invariant algebraic curves.

**Theorem 2.1:** *Let  $f_1$  and  $f_2$  be irreducible polynomials on  $\mathbb{R}[x, y]$  such that the Jacobian  $\{f_1, f_2\} \neq 0$ . Then any planar polynomial vector field which admits  $f_1 = 0$  and  $f_2 = 0$  as invariant algebraic curves can be written in the form (2).*

*Proof.* Suppose that  $X = (P, Q)$  is a polynomial vector field which admits  $f_1 = 0$

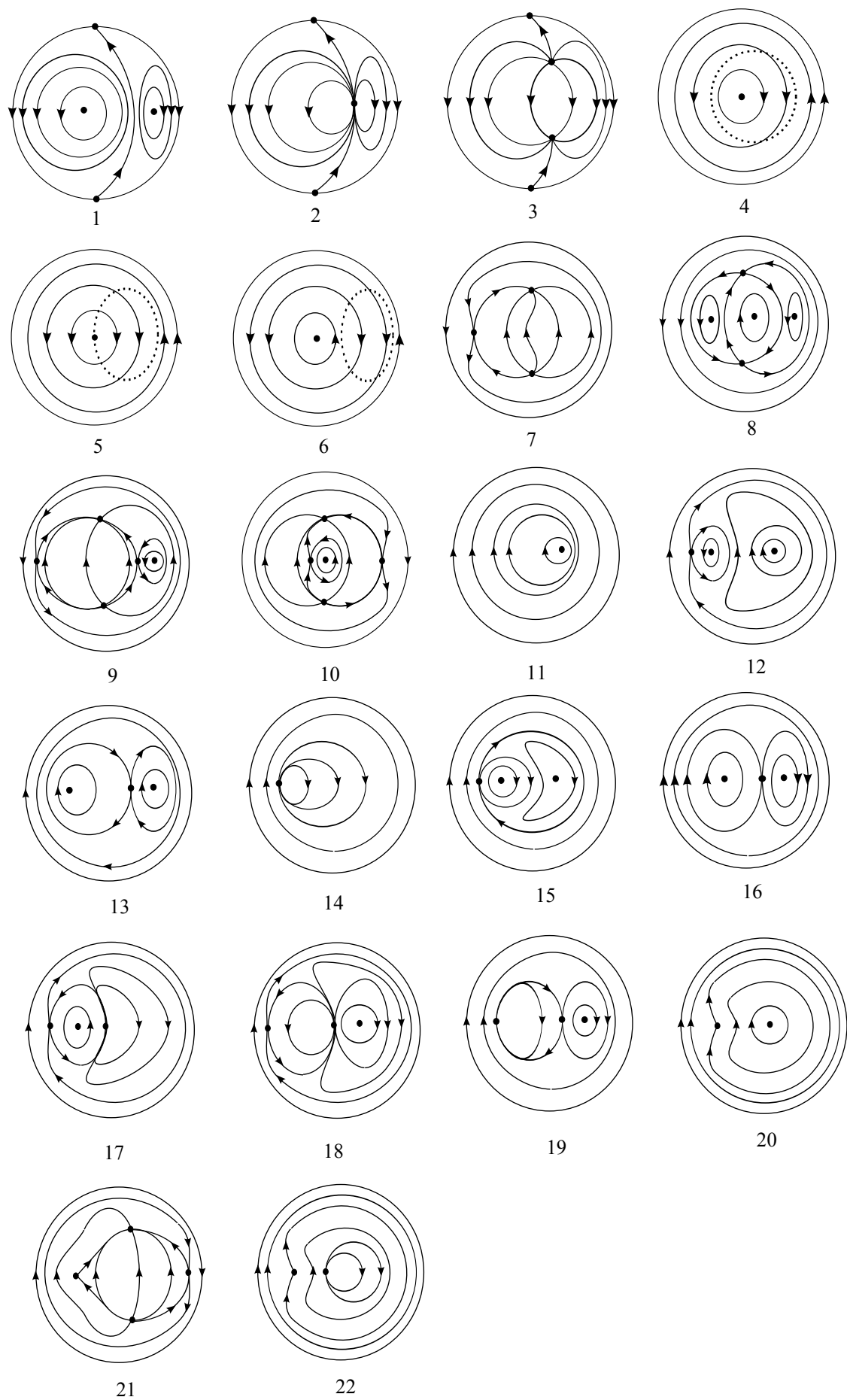


Figure 3. Possible phase portraits of vector field (5) on the Poincaré disc.

and  $f_2 = 0$  as invariant algebraic curves. Then by definition

$$X(f_1) = P \frac{\partial f_1}{\partial x} + Q \frac{\partial f_1}{\partial y} = K_1 f_1 \quad (6)$$

$$X(f_2) = P \frac{\partial f_2}{\partial x} + Q \frac{\partial f_2}{\partial y} = K_2 f_2, \quad (7)$$

where  $K_1$  and  $K_2$  are the cofactors of  $f_1 = 0$  and  $f_2 = 0$ , respectively. Hence multiplying equation (7) by  $\partial f_1 / \partial y$  and substituting  $Q \partial f_1 / \partial y$  from (6) into (7) we obtain

$$P = \frac{-K_1}{\{f_1, f_2\}} f_1 \left( -\frac{\partial f_2}{\partial y} \right) + \frac{K_2}{\{f_1, f_2\}} f_2 \left( -\frac{\partial f_1}{\partial y} \right).$$

Now multiplying equation (7) by  $\partial f_1 / \partial x$  and substituting  $P \partial f_1 / \partial x$  from (6) into (7) we obtain

$$Q = \frac{-K_1}{\{f_1, f_2\}} f_1 \frac{\partial f_2}{\partial x} + \frac{K_2}{\{f_1, f_2\}} f_2 \frac{\partial f_1}{\partial x}.$$

By hypothesis  $P$  and  $Q$  are polynomials and  $f_1$  and  $f_2$  are irreducible polynomials, then we can consider  $K_1 = -\{f_1, f_2\} \lambda_1$  and  $K_2 = \{f_1, f_2\} \lambda_2$ , with  $\lambda_1$  and  $\lambda_2$  arbitrary rational functions. So we obtain  $X = \lambda_1 f_1 H_{f_2} + \lambda_2 f_2 H_{f_1}$ . Note that even  $\lambda_1$  and  $\lambda_2$  as rational functions we can choose them such a way that  $X$  is a polynomial vector field.

However this is not the most general form of vector field  $X$ . Indeed, since the flow of  $X$  preserves  $f_1 = 0$  and  $f_2 = 0$ , we can add  $f_1 f_2 \tilde{X}$ , where  $\tilde{X}$  is an arbitrary planar polynomial vector field, into the expression of  $X$ . Then the most general form of  $X$  is (2).  $\square$

Before introducing the normal form (4), we present the next result which give us a way to construct first integrals using invariant algebraic curves and we can find a proof for it in [3] (see statement (i) of Theorem 8.7).

**Proposition 2.2:** *Suppose that a polynomial differential system (1) of degree  $d$  admits  $p$  invariant algebraic curves  $f_i = 0$  with cofactors  $K_i$ , for  $i = 1, \dots, p$ . There exist  $\mu_i \in \mathbb{R}$  not all zero such that*

$$\sum_{i=1}^p \mu_i K_i = 0$$

*if and only if the function  $H = f_1^{\mu_1} + \dots + f_p^{\mu_p}$  is a first integral of  $X$ .*

**Proposition 2.3:** *Any planar polynomial vector field  $X$  of degree  $d \leq 3$  having two nonconcentric circles  $C_1$  and  $C_2$  as invariant algebraic curves and the function  $H = C_1^\beta C_2^\alpha$ , with  $\alpha$  and  $\beta$  real values, as first integral, after a change of coordinates, can be written as (4).*

*Proof.* After an adequate change of coordinates, we can suppose that the equations of the two nonconcentric invariant circles  $C_1$  and  $C_2$  are given by (3). By Theorem 2.1 we can write vector field  $X$  as (2) taking  $f_i = C_i$ , for

$i = 1, 2$  (remember that  $a \neq 0$  so the Jacobian  $\{C_1, C_2\} \neq 0$ ). Since  $H = C_1^\beta C_2^\alpha$ , with  $\alpha$  and  $\beta$  real values, is a first integral of  $X$ , by Proposition 2.2, we obtain equation

$$\beta K_1 + \alpha K_2 = 0, \quad (8)$$

where  $K_1 = -4ay\lambda_1 + 2C_2(x\tilde{P} + y\tilde{Q})$  and  $K_2 = 4ay\lambda_2 + 2C_1((x-a)\tilde{P} + y\tilde{Q})$ , with  $\tilde{X} = (\tilde{P}, \tilde{Q})$  into (2), are the cofactors of  $C_1 = 0$  and  $C_2 = 0$ , respectively. From (8) we obtain that  $\tilde{P} \equiv \tilde{Q} \equiv 0$  and  $\beta\lambda_1 = \alpha\lambda_2$ . Therefore  $X$  can be written as (4).  $\square$

Proposition 2.3 gives us the normal form of all planar polynomial vector fields of degree  $d \leq 3$  having two nonconcentric circles  $C_1$  and  $C_2$  as invariant algebraic curves and the function  $H = C_1^\beta C_2^\alpha$ , with  $\alpha$  and  $\beta$  real values, as first integral. Now we do a study of the global phase portraits of vector fields (4) when  $\lambda_1 = \alpha$  and  $\lambda_2 = \beta$ , that is the subclass (5). Our first result claims that vector fields (5) have no limit cycles.

**Proposition 2.4:** *The vector fields (5) have no limit cycles.*

*Proof.* We have that  $H = C_1^\beta C_2^\alpha$  is first integral of vector fields (5). More precisely, note that the function  $C_1^{\beta-1} C_2^{\alpha-1}$ , is an integrating factor of the vector field (5), and that this vector field multiplied by that integrating factor becomes a Hamiltonian system with Hamiltonian  $H = C_1^\beta C_2^\alpha$ . Since a Hamiltonian vector field cannot have limit cycles, the proposition follows.  $\square$

### 3. “Quadratic case”

In this section we characterize all the phase portraits on the Poincaré disc of vector fields (5) of degree 2, that is the quadratic ones. These vector fields appear when we consider  $\alpha = -\beta$  into (5).

**Proposition 3.1:** *If vector field (5) is quadratic, that is  $d = 2$ , then it can be written as*

$$X = -C_1 H_{C_2} + C_2 H_{C_1} \quad (9)$$

where  $C_i = C_i(x, y)$  are given by (3) and  $H_{C_i}$  denotes the Hamiltonian vector field  $(-\partial C_i / \partial y, \partial C_i / \partial x)$ , for  $i = 1, 2$ .

*Proof.* Considering an adequate change of coordinates, we can suppose that the equations of the two nonconcentric invariant circles are given by (3). Taking  $\alpha = -\beta$  into vector field (5), it becomes quadratic, otherwise it is cubic, and after the rescaling of time  $T = \beta t$ , where  $T$  is the new time, we obtain vector field (9).  $\square$

Note that function  $H = C_1/C_2$  is a first integral of vector fields (9). So the phase portrait of them are arcs of circles. Indeed,  $H = c$  is almost equivalent to  $C_1 - c C_2 = 0$ , which is an equation of a circle. This shows that bifurcation diagram of vector fields (9) (Figure 5) is given by whether the two circles  $C_1$  and  $C_2$  intersect or not: if they do, the phase portrait are circles passing

through the intersection point, if they do not, the plane is foliated by such circles (Figure 6). An analytic proof for this is given by Proposition 3.2.

**Proposition 3.2:** *Let  $A = A(a, r) = (a + 1 - r)(a + 1 + r)(a - 1 - r)(a - 1 + r)$ . The following statements hold for vector field (9).*

- (i) *If  $A < 0$ , then vector field (9) has two singular points, which are an unstable and a stable node, symmetric with respect to the  $x$ -axis. Moreover, these singular points belong to the intersection of the invariant circles  $C_1$  and  $C_2$ .*
- (ii) *If  $A = 0$ , then vector field (9) has only one singular point, which is on the  $x$ -axis and it is a nonelementary singular point, formed by the union of two elliptic sectors, see Figure 4. Moreover, it belongs to the tangency point between the invariant circles  $C_1$  and  $C_2$ .*
- (iii) *If  $A > 0$ , then vector field (9) has two singular points, which are centers belonging to the  $x$ -axis.*

*Proof.* Computing the singular points of vector field (9) we obtain

$$\begin{aligned} M_1 &= \left( \frac{1 + a^2 - r^2}{2a}, \frac{\sqrt{-A}}{2a} \right), & N_1 &= \left( \frac{1 + a^2 - r^2}{2a}, -\frac{\sqrt{-A}}{2a} \right), \\ P_1 &= \left( \frac{1 + a^2 - r^2 + \sqrt{A}}{2a}, 0 \right), & Q_1 &= \left( \frac{1 + a^2 - r^2 - \sqrt{A}}{2a}, 0 \right), \end{aligned} \quad (10)$$

where  $A = A(a, r) = (a + 1 - r)(a + 1 + r)(a - 1 - r)(a - 1 + r)$ . So, we have three cases to consider:  $A < 0$ ,  $A = 0$  and  $A > 0$ .

(i) If  $A < 0$ , then from (10) only  $M_1$  and  $N_1$  are singular points of vector field (9). Note that  $M_1$  and  $N_1$  are symmetric with respect to the  $x$ -axis. The eigenvalues of the Jacobian matrix of vector field (9) at the point  $M_1$  are  $\lambda_1 = \lambda_2 = \sqrt{-A}$ , while the eigenvalues at the point  $N_1$  are  $\lambda_1 = \lambda_2 = -\sqrt{-A}$ . Thus  $M_1$  is an unstable node and  $N_1$  is a stable node. Furthermore, taking  $x = (1 + a^2 - r^2)/(2a)$  and  $y = \sqrt{-A}/(2a)$ , i.e.  $x$  and  $y$  are given by the coordinates of  $M_1$  then  $x$  and  $y$  satisfy simultaneously the equations  $x^2 + y^2 - 1 = 0$  and  $(x - a)^2 + y^2 - r^2 = 0$ . So  $M_1 \in C_1 \cap C_2$ . Analogously, we obtain that  $N_1 \in C_1 \cap C_2$ .

(ii) If  $A = 0$ , then from (10) the unique singular point of vector field (9) is

$$R_1 = \left( \frac{1 + a^2 - r^2}{2a}, 0 \right),$$

which is on the  $x$ -axis. The Jacobian matrix of vector field (9) at the point  $R_1$  is the null matrix. So  $R_1$  is a nonelementary singular point. In order to study this singularity we use a polar blowing-up, considering the change of coordinates

$$x = \rho \cos \theta + \frac{1 + a^2 - r^2}{2a}, \quad y = \rho \sin \theta$$

into vector field (9) and doing the rescaling  $T = \rho t$ , where  $T$  is the new time, from which we obtain

$$\dot{\rho} = \rho a \sin \theta, \quad \dot{\theta} = -a \cos \theta. \quad (11)$$

The zeros of system (11) on  $\{\rho = 0\}$  are located at  $\theta = \pi/2$ ,  $\theta = 3\pi/2$ . It is easy to



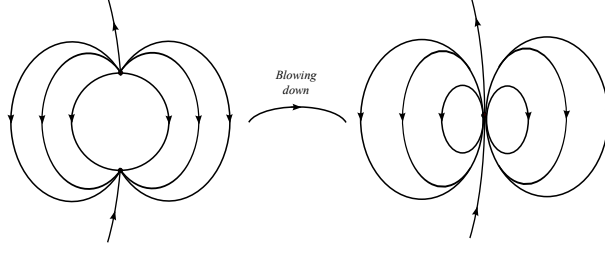


Figure 4. Polar blow-up of the singular point  $R_1$ .

verify that  $\theta = \pi/2$  is an unstable node and  $\theta = 3\pi/2$  is a stable node. Lastly doing a blowing down we obtain the behavior of solutions near the singular point  $R_1$ . Figure 4 illustrates this process. Moreover,  $R_1$  belongs to the intersection of the invariant circles  $C_1$  and  $C_2$ . Indeed if we take  $x = (1 + a^2 - r^2)/(2a)$  and  $y = 0$ , then  $x^2 + y^2 - 1 = 0$  and  $(x - a)^2 + y^2 - r^2 = 0$ .

(iii) If  $A > 0$ , then from (10) only  $P_1$  and  $Q_1$  are singular points of vector field (9). Note that  $P_1$  and  $Q_1$  are on the  $x$ -axis. The eigenvalues of the Jacobian matrix of vector field (9) at  $P_1$  are  $\lambda_{1,2} = \pm i\sqrt{A}$ , and the eigenvalues at the point  $Q_1$  are also  $\lambda_{1,2} = \pm i\sqrt{A}$ . So these points can be centers or weak foci of vector field (9). As the function

$$H = \frac{(x - a)^2 + y^2 - r^2}{x^2 + y^2 - 1}$$

is a first integral of vector field (9), so it is integrable, hence we can conclude that  $P_1$  and  $Q_1$  are centers.  $\square$

**Note that the symmetry  $y \rightarrow -y$  with respect to the  $x$ -axis reverses the time of the flow of vector fields (5), this, in particular, explains why in Proposition 3.2 the nodes are symmetric with respect to the  $x$ -axis and they are of opposite types. The same remark can be done to Proposition 4.2.**

**Proposition 3.3:** *Vector field (9) has two infinite singular points, which are saddles.*

*Proof.* In order to study the infinite singular points of vector field (9), we consider the change of coordinates

$$x = \frac{1}{\rho} \cos \theta, \quad y = \frac{1}{\rho} \sin \theta, \quad (12)$$

and doing the rescaling  $T = (1/\rho)t$ , where  $T$  is the new time, we obtain the system

$$\dot{\rho} = f_0(\theta)\rho + f_1(\theta)\rho^3, \quad \dot{\theta} = g_0(\theta) + g_1(\theta)\rho + g_2(\theta)\rho^2, \quad (13)$$

where  $f_i$  ( $i = 0, 1$ ) and  $g_j$  ( $j = 0, 1, 2$ ) are polynomials in  $\cos \theta$  and  $\sin \theta$ . The singular points at infinity are given by  $\rho = 0$  and  $g_0(\theta) = 0$ , where  $g_0(\theta) = a \cos \theta$ , hence we obtain the pair of singular points  $\theta = \pi/2$  and  $\theta = 3\pi/2$ . The eigenvalues of the Jacobian matrix of system (13) at  $\theta = \pi/2$  and at  $\theta = 3\pi/2$  are  $\lambda_{1,2} = \pm a$ . Thus, these infinite singular points are saddles.  $\square$

The results obtained about vector field (9) are summarized in Figures 5 and 6, which show respectively the bifurcation diagram and the phase portraits on the Poincaré disc

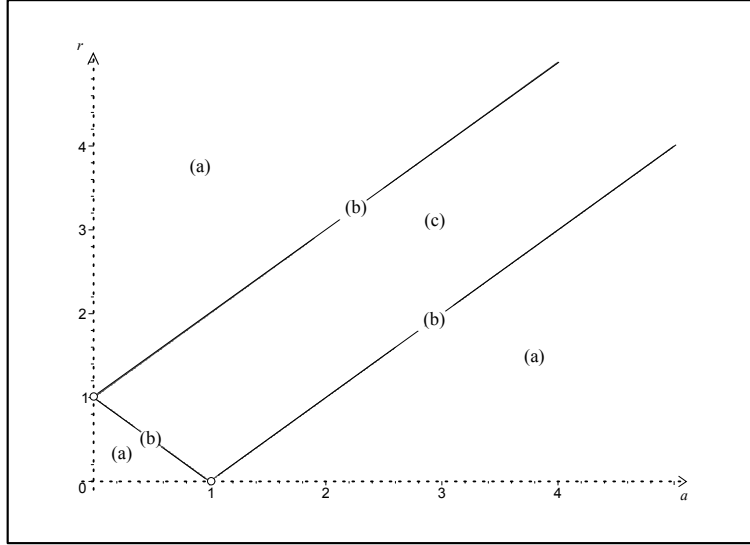


Figure 5. Bifurcation diagram of vector field (9).

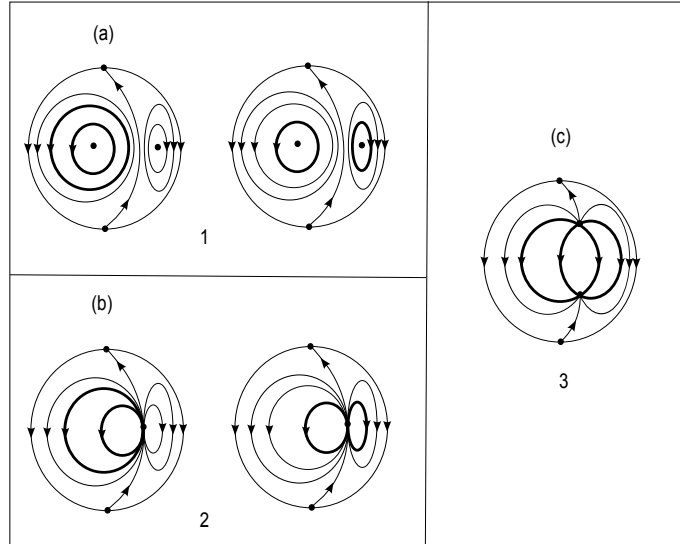


Figure 6. Phase portraits of vector field (9) on the Poincaré disc related to each region described in Figure 5, corresponding to the phase portraits 1, 2 and 3 of Figure 3. The bold curves represent the circles  $C_1 = 0$  and  $C_2 = 0$ .

related to each region of the bifurcation diagram. In short, this proves Theorem 1.1 when  $d = 2$  providing in this cases the phase portraits 1, 2 and 3 of Figure 3.

#### 4. “Cubic case”

In this section we consider the case of vector fields (5) of degree 3, that is the cubic ones. We shall consider three cases: case (i)  $\alpha = 0$  and  $\beta \neq 0$ ; case (ii)  $\beta = 0$  and  $\alpha \neq 0$ ; and case (iii)  $\alpha\beta \neq 0$  and  $\alpha \neq -\beta$ , since the case  $\alpha = -\beta$  was studied in section 3.

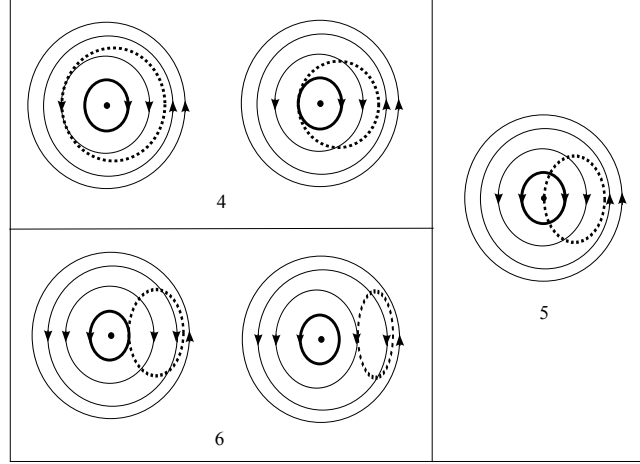


Figure 7. Phase portraits of vector field (14) on the Poincaré disc. The dotted curve represents the circle  $C_2$ , which is filled by singular points.

#### 4.1. Case (i)

Let  $\alpha = 0$  and  $\beta \neq 0$  into (5). After the rescaling of time  $T = \beta t$ , where  $T$  is the new time, vector field (5) reduces to

$$X = C_2 H_{C_1} \quad (14)$$

which has the origin as an isolated singular point and the invariant circle  $C_2$  is filled by singular points. **Note that vector field (14) is obtained from Hamiltonian vector field  $H_{C_1} = (-2y, 2x)$  by “slowing down” the multiplication by  $C_2$  and hence its trajectories are the same as those of  $H_{C_1}$  or their pieces to which they are cut by  $\{C_2 = 0\}$ .** The phase portrait of vector field (14) on the Poincaré disc is one of those shown in Figure 7, providing the phase portraits 4, 5 and 6 of Figure 3. Therefore this part of Theorem 1.1 is proved.

#### 4.2. Case (ii)

Let  $\beta = 0$  and  $\alpha \neq 0$  into (5). After the rescaling of time  $T = \alpha t$ , where  $T$  is the new time, vector field (5) reduces to

$$X = C_1 H_{C_2} \quad (15)$$

which has the point  $(a, 0)$  as an isolated singular point and the invariant circle  $C_1$  is filled by singular points. **The phase portrait of vector field (15) on the Poincaré disc is topologically equivalent to one of those shown in Figure 7, changing the dotted curve with the bold one.**

#### 4.3. Case (iii)

Let  $\alpha\beta \neq 0$  and  $\alpha \neq -\beta$  into (5). After the rescaling of time  $T = \alpha t$ , where  $T$  is the new time, vector field (5) can be written as

$$X = C_1 H_{C_2} + \gamma C_2 H_{C_1} \quad (16)$$

where  $\gamma = \beta/\alpha$  is a real parameter, with  $\gamma \neq 0$  and  $\gamma \neq -1$ . In what follows we state the following results about vector field (16): Proposition 4.1 ensures that vector field (16) has at least one and at most five singular points; Propositions 4.2 and 4.3 provide a classification for these singular points; Proposition 4.4 ensures that vector field (16) has no infinite singular points.

**Proposition 4.1:** *Vector field (16) has at least one and at most five singular points. At least one and at most three of them are on the  $x$ -axis and the other two, when they exist, are symmetric with respect to the  $x$ -axis.*

*Proof.* From vector field (16), we obtain the differential system

$$\begin{aligned}\dot{x} &= -2y(C_1 + \gamma C_2), \\ \dot{y} &= 2x(C_1 + \gamma C_2) - 2aC_1.\end{aligned}\tag{17}$$

Note that  $\dot{x} = -y \cdot f(x, y)$ . So, if  $\dot{x} = 0$ , then either  $y = 0$  or  $f(x, y) = 0$ .

If we consider  $f(x, y) = 0$ , then system (17) has the singular points

$$M_2 = \left( \frac{a^2 - r^2 + 1}{2a}, \frac{\sqrt{-A}}{2a} \right), \quad N_2 = \left( \frac{a^2 - r^2 + 1}{2a}, -\frac{\sqrt{-A}}{2a} \right),\tag{18}$$

where  $A = A(a, r) = (a + 1 - r)(a + 1 + r)(a - 1 - r)(a - 1 + r)$ . Note that  $M_2$  and  $N_2$  are symmetric with respect to the  $x$ -axis. We must consider three cases:

- (a) if  $A < 0$ , then  $M_2$  and  $N_2$  are distinct singular points;
- (b) if  $A = 0$ , then  $M_2$  and  $N_2$  are coincident singular points;
- (c) if  $A > 0$ , then  $M_2$  and  $N_2$  are not singular points.

If we take  $y = 0$  into system (17), as

$$\dot{y} = Q(x, y) = x[x^2 + y^2 - 1 + \gamma((x - a)^2 + y^2 - r^2)] - a(x^2 + y^2 - 1),$$

then

$$Q(x, 0) = Q(x) = (1 + \gamma)x^3 - a(2\gamma + 1)x^2 + (\gamma(a^2 - r^2) - 1)x + a.$$

Doing  $Q(x) = 0$  we get a third degree equation, whose discriminant is given by the function

$$\begin{aligned}D = D(a, r, \gamma) &= \frac{1}{4} \left( \frac{a}{1 + \gamma} + \frac{a(2\gamma + 1)(\gamma(a^2 - r^2) - 1)}{3(1 + \gamma)^2} - \frac{2a^3(2\gamma + 1)^3}{27(1 + \gamma^3)} \right)^2 \\ &\quad + \frac{1}{27} \left( \frac{\gamma(a^2 - r^2) - 1}{1 + \gamma} - \frac{a^2(2\gamma + 1)^2}{3(1 + \gamma)^2} \right)^3.\end{aligned}$$

So we must consider three cases:

(d) if  $D < 0$ , then equation  $Q(x) = 0$  has three real roots, so system (17) has three singular points on the  $x$ -axis;

(e) if  $D = 0$ , then equation  $Q(x) = 0$  has two real roots, so system (17) has two singular points on the  $x$ -axis;

(f) if  $D > 0$ , then equation  $Q(x) = 0$  has one real root, so system (17) has only one singular point on the  $x$ -axis.  $\square$

In the next result we give a classification for the singular points of vector field (16) symmetric with respect to the  $x$ -axis, that is the singular points  $M_2$  and  $N_2$  given in (18). For the definitions of hyperbolic and semi-hyperbolic singular points, see for instance chapter 2 of [3].

**Proposition 4.2:** *Let  $A = A(a, r) = (a + 1 - r)(a + 1 + r)(a - 1 - r)(a - 1 + r)$ . The following statements hold for vector field (16).*

- (i) *If  $A < 0$ , then the singular points  $M_2$  and  $N_2$  are both either saddles or nodes, in this last case they have opposite stability.*
- (ii) *If  $A = 0$ , then  $M_2$  and  $N_2$  are coincident and it is a nonelementary singular point, formed by the union of two elliptic sectors, as shown in Figure 4.*
- (iii) *If  $A > 0$ , then  $M_2$  and  $N_2$  are not singular points.*

Moreover when  $M_2$  and  $N_2$  are singular points they are on the intersection of the invariant circles  $C_1$  and  $C_2$ .

*Proof.* According to the proof of Proposition 4.1, we have that the singular points of vector field (16)  $M_2$  and  $N_2$  are given by (18). Calculating the Jacobian matrix of vector field (16) at  $M_2$  we obtain that its eigenvalues are  $\lambda_1 = -\sqrt{-A}$  and  $\lambda_2 = \gamma\sqrt{-A}$ . Analogously, calculating the Jacobian matrix of vector field (16) at  $N_2$  we have that its eigenvalues are  $\lambda_1 = \sqrt{-A}$  and  $\lambda_2 = -\gamma\sqrt{-A}$ .

(i) If  $A < 0$ , then  $M_2$  and  $N_2$  are distinct singular points and in this case we have that  $\lambda_1$  and  $\lambda_2$  are real eigenvalues. Hence, if  $\gamma < 0$ , then  $M_2$  is a stable node and  $N_2$  is an unstable node; while for  $\gamma > 0$ ,  $M_2$  and  $N_2$  are both saddles.

(ii) If  $A = 0$ , then  $M_2$  and  $N_2$  are coincident singular points and  $\lambda_1 = \lambda_2 = 0$ . So  $M_2$  and  $N_2$  reduce to a nonelementary singular point, which is given by

$$R_1 = \left( \frac{1 + a^2 - r^2}{2a}, 0 \right).$$

Note that  $R_1$  is the same singular point of the proof of item (ii) of Proposition 3.2, hence with a polar blow-up we show that  $R_1$  is formed by the union of two elliptic sectors, as in the proof of Proposition 3.2 (see Figure 4).

(iii) If  $A > 0$ , then by Proposition 4.1,  $M_2$  and  $N_2$  are not singular points.

Finally, if we consider  $x$  and  $y$  given by the coordinates of  $M_2$  and  $N_2$ , that is  $x = (a^2 - r^2 + 1)/(2a)$  and  $y = \pm\sqrt{-A}/(2a)$ , we have that they satisfy simultaneously the equations  $x^2 + y^2 - 1 = 0$  and  $(x - a)^2 + y^2 - r^2 = 0$ . So  $M_2, N_2 \in C_1 \cap C_2$ .  $\square$

**Proposition 4.3:** *The singular points of vector field (16) on the  $x$ -axis are saddles, centers or nonelementary singular points, whose local phase portraits are described in Figure 8.*

*Proof.* By Proposition 4.1, we know that vector field (16) has at least one and at most three singular points on the  $x$ -axis. If we consider the change of coordinates  $x = \rho \cos \theta$ ,  $y = \rho \sin \theta$  into (16), then we obtain the system

$$\begin{aligned} \dot{\rho} &= -a(\rho^2 - 1) \sin \theta, \\ \dot{\theta} &= (1 + \gamma)\rho^2 + a \cos \theta(2\gamma - 1)\rho + \gamma(a^2 - r^2) + 1 + \frac{1}{\rho} \cos \theta. \end{aligned} \tag{19}$$

In order to study the behavior of the solutions near the singular points of vector field

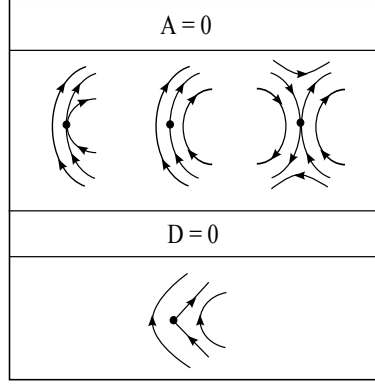


Figure 8. Nonelementary singular points of vector field (16).

(16) on the  $x$ -axis, we calculate the Jacobian matrix of system (19) and take  $\theta = 0$ , which have the eigenvalues  $\lambda_{1,2} = \pm\sqrt{B}/\rho$ , where  $B = B(a, \gamma, \rho) = a(\rho^2 - 1)(-2(1 + \gamma)\rho^3 + a(2\gamma + 1)\rho^2 + a)$ . So we have.

If  $B < 0$ , then the singular point on the  $x$ -axis can be a center or a weak focus. As the function  $H(x, y) = C_1^\gamma C_2$  is a first integral of vector field (16), then it is integrable and these singular points are centers.

If  $B = 0$ , then the singular point on the  $x$ -axis is a nonelementary singular point, since the eigenvalues are both zero. After the desingularization of this singular point using the tools of Chapter 2 and 3 of [3], we have that its local phase portrait is topologically equivalent to one of those described in 8.

If  $B > 0$ , then the singular point on the  $x$ -axis is a saddle.

Therefore, the singular points of vector field (16) on the  $x$ -axis are saddles, centers or nonelementary singular points (see Figure 8), depending on the value of  $B$ .  $\square$

**Proposition 4.4:** *Vector field (16) has no infinite singular points.*

*Proof.* In order to study the infinite singular points of vector field (16) we consider the change of coordinates (12) and the rescaling of time  $T = (1/\rho^2)t$ , where  $T$  is the new time, from what we obtain the system

$$\dot{\rho} = f_0(\theta)\rho^2 + f_1(\theta)\rho^4, \quad \dot{\theta} = g_0(\theta) + g_1(\theta)\rho + g_2(\theta)\rho^2 + g_3(\theta)\rho^3.$$

The infinite singular points are given by  $\rho = 0$  and  $g_0(\theta) = 0$ . Since  $g_0(\theta) = 1$ , for all  $\theta \in \mathbb{R}$ , then vector field (16) has no infinite singular points.  $\square$

We can summarize the results about Case (iii) and classify the phase portraits of vector field (16) as follows. First we present in Figure 9 the cases in which vector field (16) has only elementary singular points, which correspond to the phase portraits 7 to 13 of Figure 3. Then we present in Figures 10, 11 and 12 the other cases, when vector field (16) has nonelementary singular points, which correspond to the phase portraits 14 to 22 of Figure 3. In short Figures 9, 10, 11 and 12 provide the phase portraits 7 to 22 of Figure 3, which end the proof of Theorem 1.1.

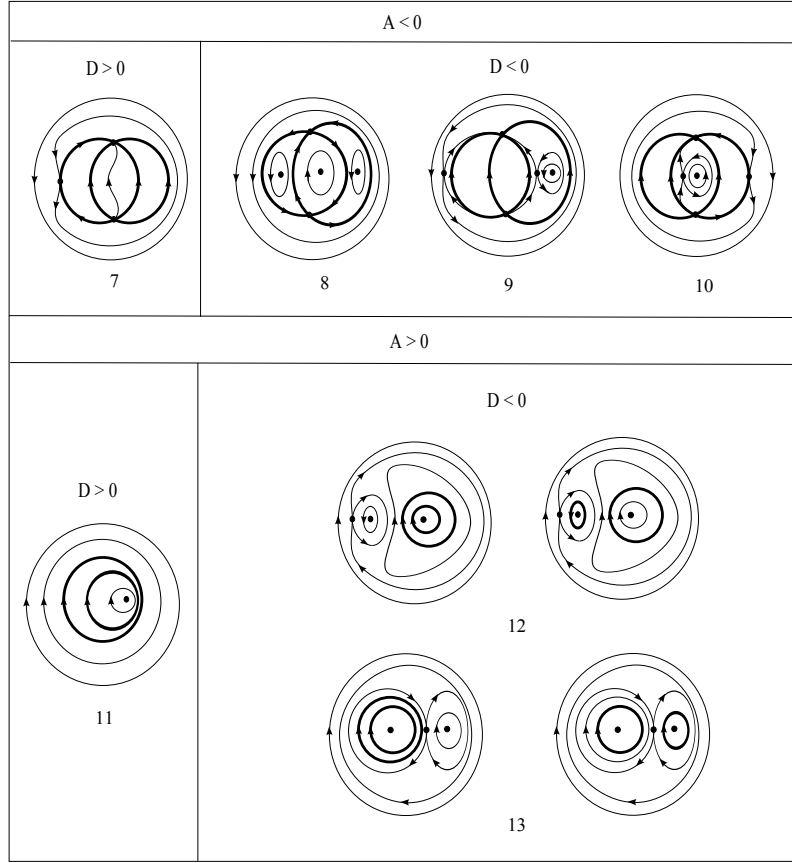


Figure 9. Phase portraits of vector field (16) on the Poincaré disc having only elementary singular points.

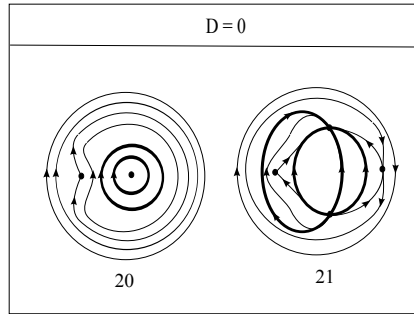


Figure 10. Phase portraits of vector field (16) on the Poincaré disc having nonelementary singular points ( $D = 0$  and  $A \neq 0$ ).

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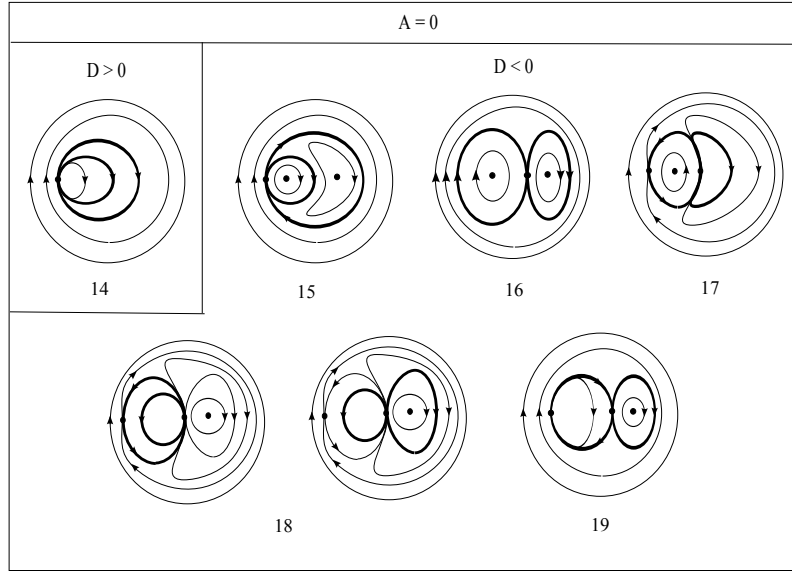


Figure 11. Phase portraits of vector field (16) on the Poincaré disc having nonelementary singular points ( $A = 0$  and  $D \neq 0$ ).

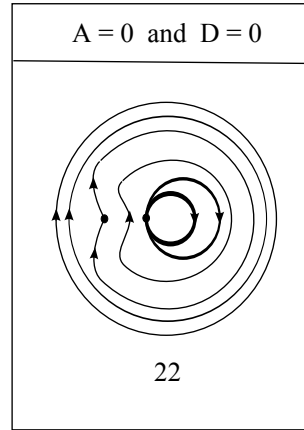


Figure 12. Phase portraits of vector field (16) on the Poincaré disc having nonelementary singular points ( $A = 0$  and  $D = 0$ ).

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