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On the Darboux integrability of a three–dimensional forced–damped differential system

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In 2011 Pehlivan proposed a three–dimensional forced–damped autonomous differential system which can display simultaneously unbounded and chaotic solutions. This untypical phenomenon has been studied recently by several authors. In this paper we study the opposite to its chaotic motion, i.e. its integrability, mainly the existence of polynomial, rational and Darboux first integrals through the analysis of its invariant algebraic surfaces and its exponential factors.

Keywords: Invariant algebraic surfaces; exponential factors; first integral; four-scroll chaotic system.

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1. Introduction and statement of the main result

We consider in $\mathbb{R}^3$ the autonomous system of differential equations

$$\begin{aligned} \dot{x} &= -ax + y + yz, \\ \dot{y} &= x - ay + bxz, \\ \dot{z} &= cz - bxy, \end{aligned} \tag{1.1}$$

where a, b, c are real parameters with $b > 0$. This system arise in mechanical, electrical and fluid–dynamical contexts, see for more details the articles of Miyaji, Okamoto and Craik [14, 15] and the references quoted there. This system was proposed and studied by Pehlivan [16]. The system extends a previous study of Craik and Okamoto [1], including linear forcing and damping.

Pehlivan showed that system (1.1) displays simultaneously unbounded and chaotic solutions. This phenomenon has been studied in more depth by Miyaji, Okamoto and Craik who also find that can be accompanied by three distinct period–doubling cascades of periodic orbits to chaos.

Chaotic systems are nonlinear deterministic systems which exhibits a complex and unpredictable behavior, hence it is a very interesting phenomenon in nonlinear dynamical systems and it has been intensively studied starting with the Lorenz system. The majority of the known chaotic system have one or more quadratic non-linearities.

Pehlivan system as the Lorenz system are two polynomial differential systems in $\mathbb{R}^3$, with very different dynamics. For the Lorenz system their invariant algebraic surfaces defined by their Darboux polynomials are very well known, see [12]. In this paper we provide the invariant algebraic surfaces of the Pehlivan system. Moreover for the Lorenz system also their polynomial, rational and Darboux first integrals were studied in [17, 18]. Here we also provide the polynomial, rational and Darboux first integrals of the Pehlivan system.

As far as we know this rich dynamical system (1.1) has never been investigated from the integrability point of view. The main goal of this paper is to characterize the polynomial and rational first integrals of system (1.1). For doing this we need to provide a complete characterization of the invariant algebraic surfaces of system (1.1) depending on its parameters. In order to obtain such invariant algebraic surfaces we shall use the Darboux theory of integrability which gives a link between the algebraic geometry of the system and its first integrals, see for more details about this theory [5–7, 9–11, 13].

It is well known that the existence of a first integral for three–differential system allows to reduce the study of its dynamics in one dimension, and that the existence of two independent first integrals allows to describe completely the dynamics of the system. These arguments justify the study of the integrability of a differential system. The Darboux theory of integrability is classical. The Darboux integrability essentially captures the elementary first integrals, i.e. the first integrals given by elementary functions, which are the ones that roughly speaking can be obtained by composition of exponential, trigonometric, logarithmic and polynomial functions, see for more details about the Darboux integrability the Chapter 8 of [5], and the references quoted there. The Darboux integrability in dimension three is based in the existence of invariant algebraic surfaces $f(x,y,z) = 0$, where $f(x,y,z)$ is a polynomial, called a Darboux polynomial. A sufficient number of such polynomials taking into account their multiplicity (through the so–called exponential factors) force the existence of first integrals.

Historically, the theory received mainly contributions from Darboux [3] who gave a link between the algebraic geometry and the search of first integrals and showed how to construct a first integral of a polynomial differential system in the plane having sufficient number of invariant algebraic curves. Poincaré noticed the difficulty in obtaining an algorithm to compute Darboux first integrals and Singer proved the relation for polynomial differential system in the plane to have a Liouvillian first integral in terms of a Darbouxian integrating factor.

Let U be an open and dense subset of $\mathbb{R}^3$. A nonconstant function $H\colon U \to \mathbb{R}$ is called a *first integral* of system (1.1) on U if $H(x(t),y(t),z(t))$ is constant for all of the values of t for which $(x(t),y(t),z(t))$ is a solution of system (1.1) contained in U. So H is a first integral of system (1.1) if and only if

$$(-ax+y+yz)\frac{\partial H}{\partial x}+(x-ay+bxz)\frac{\partial H}{\partial y}+(cz-bxy)\frac{\partial H}{\partial z}=0,$$

for all $(x,y,z)\in U$. If H is a polynomial (respectively a rational function) we say that H is a *polynomial* (respectively *rational) first integral*.

Let $\mathbb{R}[x,y,z]$ be the ring of the polynomials in the variables x, y and z with coefficients in the field $\mathbb{R}$.

Given $g \in [x,y,z]$ the surface $g(x,y,z)=0$ is called an *invariant algebraic surface* of system (1.1) and g is called a *Darboux polynomial* if there exists $k \in [x,y,z]$ such that

$$(-ax+y+yz)\frac{\partial g}{\partial x}+(x-ay+bxz)\frac{\partial g}{\partial y}+(cz-bxy)\frac{\partial g}{\partial z}=kg. \tag{1.2}$$

The polynomial k satisfying (1.2) is called the *cofactor* of the invariant surface $g(x,y,z)=0$ and it has degree at most 1. The name of invariant algebraic surface comes from the fact that if a solution of system (1.1) has a point on the such surface the whole solution is contained in it.

Let U be an open and dense subset of $\mathbb{R}^3$. We recall that two functions $f,g : U \to \mathbb{R}^3$ are *functionally independent* or simply *independent* if their gradients are linearly independent at all points of U except perhaps in a zero Lebesgue set. Differential system (1.1) is *completely integrable* if it has two first integrals which are functionally independent.

The aim of this paper is to study the existence of first integrals of system (1.1) that can be described by functions of Darboux type (see (1.3)). In general, for a given differential system it is difficult to determine the existence or nonexistence of first integrals.

An *exponential factor* $F(x,y,z)$ of system (1.1) is an exponential function of the form $F = \exp(g/h)$ with $g,h \in \mathbb{C}[x,y,z]$ coprime, denoted by $(g,h)=1$, and satisfying

$$(yz-ax+y)\frac{\partial F}{\partial x}+(bxz+x-ay)\frac{\partial F}{\partial y}+(-bxy+cz)\frac{\partial F}{\partial z}=\ell F$$

for some $\ell(x,y,z)\in\mathbb{C}[x,y,z]$ a polynomial of degree at most one, which is called the cofactor of F.

A first integral H of system (1.1) is called a *generalized Darboux first integral* or here simply a *Darboux first integral* if it has the form

$$G_= f_1^{\lambda_1}\cdots f_p^{\lambda_p}F_1^{\mu_1}\cdots F_q^{\mu_q}, \tag{1.3}$$

where $f_1,\ldots,f_p$ are Darboux polynomials and $F_1,\ldots,F_q$ are exponential factors and $\lambda_j,\mu_k\in\mathbb{C}$ for $j=1,\ldots,p$ and $k=1,\ldots,q$.

Note that polynomial first integrals and rational first integrals are Darboux first integrals.

The main results of this paper are the following five theorems.

Theorem 1.1. *If $c = a = 0$ and $b = 1$, then system* (1.1) *is completely integrable with the two independent first integrals $H_1(x,y,z) = 2z+z^2+x^2$ and $H_2(x,y,z) = x^2-y^2$.*

Theorem 1.2. *Assume $c^2+a^2 \neq 0$ and $b > 0$. System* (1.1) *has an invariant algebraic surface if and only if $a+c = 0$ or $b = 1$. The irreducible invariant algebraic surfaces are described in Table 1 with their corresponding cofactors.*

Table 1. The invariant algebraic surfaces of system (1.1) with its corresponding cofactors.

Parameters	Irreducible invariant algebraic surface	Cofactor
$a+c=0$	$b(x^2-y^2-z^2)+z^2=0$	$-2a$
$b=1$	$x+y=0$	$1-a+z$
$b=1$	$x-y=0$	$-1-a-z$

Theorem 1.3. *Assume $c^2+a^2 \neq 0$ and $b > 0$. System* (1.1) *has a polynomial first integral if and only if $a = 0$ and $b = 1$. This first integral is x^2-y^2.*

Theorem 1.4. *Assume $c^2+a^2 \neq 0$ and $b > 0$. System* (1.1) *has no rational first integrals which are not polynomial.*

Theorem 1.5. *For all $a,c \in \mathbb{R}$ and $b > 0$, except when $a = c = 0$, or $a = 0$ and $b = 1$, system* (1.1) *has no Darboux first integrals.*

The proofs of these theorems are given in Sections 3 and 4. Similar results on the integrability of a polynomial Lotka–Volterra differential system in $\mathbb{R}^3$ can be found in [8].

2. Preliminary results

Before to proof the main results of this paper we will introduce some well-known results. The first was proved in [5].

Lemma 2.1. *Let f be a polynomial and $f = \partial rod_{j=1}^{s} f_j^{\alpha_j}$ its decomposition into irreducible factors in $\mathbb{C}[x,y,z]$. Then f is a Darboux polynomial if and only if all the f_j are Darboux polynomials. Moreover, if k and k_j are the cofactors of f and f_j, then $k = \sum_{j=1}^{s} \alpha_j k_j$.*

The second result whose proof and geometrical meaning is given in [2] is the following.

Proposition 2.1. *The following statements hold.*

- *(a) If $E = \exp(g_0/g)$ is an exponential factor for the polynomial system* (1.1) *and g is not a constant polynomial, then $g = 0$ is an invariant algebraic hypersurface.*
- *(b) Eventually e^{g_0} can be an exponential factor, coming from the multiplicity of the infinite invariant hyperplane.*

The proof of the third and fourth results is given in [5].

Theorem 2.1. *If system* (1.1) *has a rational first integral then either it has a polynomial first integral or two Darboux polynomials with the same nonzero cofactor.*

Theorem 2.2. *Suppose that system* (1.1) *admits p Darboux polynomials with cofactors k_i and q exponential factors F_j with cofactors ℓ_j. Then there exists $\lambda_j, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{q} \lambda_k k_i + \sum_{i=1}^{q} \mu_i \ell_i = 0$$

if and only if the function G given in (1.3) *(called of Darboux type) is a first integral of system* (1.1).

Since system (1.1) is real, we claim that if it has a Darboux first integral, this can be chosen to be real. Indeed, it is well-known that if a complex Darboux polynomial or exponential factor appears then its conjugate must appear simultaneously. If among the Darboux polynomials of system (1.1) a complex conjugate pair f, $\bar{f}$ occurs the first integral (1.3) has a real factor of the form $f^{\lambda} \bar{f}^{\bar{\lambda}}$, which is the multi-valued real function

$$[(\mathrm{Re} f)^2 + (\mathrm{Im} f)^2]^{\mathrm{Re}\lambda} \exp\left(-2\mathrm{Im}\lambda \arctan\left(\frac{\mathrm{Im} f}{\mathrm{Re} f}\right)\right),$$

if $\mathrm{Im} f \mathrm{Im}\lambda \neq 0$. If among the exponential factors of system (1.1) a complex conjugate pair $F = \exp(h/g)$ and $\bar{F} = \exp(\bar{h}/\bar{g})$ occurs the first integral (1.3) has a real factor of the form

$$\left(\exp\left(\frac{h}{g}\right)\right)^{\mu} \left(\exp\left(\frac{\bar{h}}{\bar{g}}\right)\right)^{\bar{\mu}} = \exp\left(2\mathrm{Re}\left(\mu \frac{h}{g}\right)\right).$$

So the claim is proved.

We introduce the change of variables

$$X = \sqrt{b}x, \quad Y = y, \quad Z = z \tag{2.1}$$

and the rescaling of time $t = \tau/\sqrt{b}$. In these new variables system (1.1) is written as

$$\begin{aligned} \dot{X} &= -a_1 X + Y + YZ, \\ \dot{Y} &= \frac{1}{b} X - a_1 Y + XZ, \\ \dot{Z} &= c_1 Z - XY, \end{aligned} \tag{2.2}$$

where $a_1 = a/\sqrt{b}$ and $c_1 = c/\sqrt{b}$.

Now consider the linear operator

$$L = YZ\frac{\partial}{\partial X} + XZ\frac{\partial}{\partial Y} - XY\frac{\partial}{\partial Z} \tag{2.3}$$

The characteristic equation associated to L is

$$\frac{dX}{dZ} = -\frac{YZ}{XY}, \quad \frac{dY}{dZ} = -\frac{XZ}{XY}.$$

It general solution is

$$X^2 + Z^2 = d_1, \quad Y^2 + Z^2 = d_2$$

where d_1, d_2 are arbitrary constants. We make the change of variables

$$u = X^2 + Z^2, \quad v = Y^2 + Z^2, \quad w = Z. \tag{2.4}$$

Its inverse change is

$$X = \pm\sqrt{u - w^2}, \quad Y = \pm\sqrt{v - w^2}, \quad Z = w. \tag{2.5}$$

In the paper we only use the positive case. The negative one gives the same results.

We also introduce the linear operator

$$D_{a_1,b,c,s_1} = \big(a_1X - Y\big)\frac{\partial}{\partial X} - \big(\frac{1}{b}X - a_1Y\big)\frac{\partial}{\partial Y} - c_1Z\frac{\partial}{\partial Z} + s_1. \tag{2.6}$$

3. Proofs of Theorems 1.1 to 1.4

The proofs of the theorems will be divided into several propositions.

Let $\tau\colon \mathbb{C}[x,y,z] \to \mathbb{C}[x,y,z]$ be the automorphism $\tau(x,y,z) = (-x,-y,z)$. We recall that an irreducible Darboux polynomial is a polynomial irreducible in $\mathbb{C}[x,y,z]$.

Proposition 3.1. *If $g(x,y,z)$ is an irreducible Darboux polynomial for system* (1.1) *with cofactor $K = px + qy + rz + s$ then $f = g \cdot \tau g$ is a Darboux polynomial invariant by τ with cofactor $k = 2rz + 2s$.*

Proof. Since system (1.1) is invariant under τ, then τg is also a Darboux polynomial of system (1.1) with cofactor $\tau(K)$. Moreover, by Lemma 2.1, $g \cdot \tau g$ is also a Darboux polynomial with cofactor $k = K + \tau(K) = 2rz + 2s$, as we wanted to prove. □

From Proposition 3.1 we shall consider the Darboux polynomials invariant by τ and also two cases in the cofactor, the case where the cofactor k is written as $k = rz + s$, where $r \neq 0$ and the case $k = s$. Let $f(x,y,z)$ be a Darboux polynomial invariant by τ. In these new variables (X,Y,Z) introduced in (2.1) if $\bar{f}(X,Y,Z) = f(x,y,z)$ then we have that f is an invariant algebraic surface of system (1.1) invariant by τ with cofactor $k = rz + s$ if and only if $\bar{f}$ is an invariant algebraic surface of system (2.2) invariant by τ with cofactor $k = r_1z + s_1$ where $r_1 = r/\sqrt{b}$ and $s_1 = s/\sqrt{b}$. So from now on we will study the invariant algebraic surfaces of system (2.2) and in the proofs we are concerned with characterizing polynomials $f \in [X,Y,Z]$ such that

$$(-a_1X + Y + YZ)\frac{\partial f}{\partial X} + (\frac{1}{b}X - a_1Y + XZ)\frac{\partial f}{\partial X} + (c_1Z - XY)\frac{\partial f}{\partial Z} = (r_1Z + s_1)f. \tag{3.1}$$

Note that for simplicity now we shall write f instead of $\bar{f}$.

We first consider the case $r = 0$ (i.e., $r_1 = 0$).

Proposition 3.2. *Assume $c_1^2 + a_1^2 \neq 0$ and $b > 0$. Let $f = f(x,y,z) = 0$ be an invariant algebraic surface of system* (2.2) *of degree $n \geq 1$ with cofactor $k = s_1$. Then n is even, $s_1 = -an/\sqrt{b}$, and its invariant algebraic surfaces are described in Table 2.*

Table 2. Invariant algebraic surfaces of system (2.2) with its corresponding cofactors.

Parameters	Invariant algebraic surface	Cofactor
$a_1 + c_1 = 0$	$X^2 + Z^2 - b(Y^2 + Z^2) = 0$	$-a/\sqrt{b}$
$b = 1$	$X^2 - Y^2 = 0$	$-a/\sqrt{b}$

Proof. Assume that $k = s_1$ is the cofactor of the invariant algebraic surface $f = 0$ of degree n. We write $f = \sum_{i=1}^{n} f_i$ where f_i are homogeneous polynomials of degree i, $f_n \neq 0$, and f_n satisfies

$$YZ\frac{\partial f_n}{\partial X} + XZ\frac{\partial f_n}{\partial Y} - XY\frac{\partial f_n}{\partial Z} = 0.$$

The solutions of this linear partial differential equation are of the form $F(X^2+Z^2, Y^2+Z^2)$ where F is an arbitrary C^1 function. Since f_n must be a homogeneous polynomial it follows that F is a homogeneous polynomial in the variables X^2+Z^2 and Y^2+Z^2. So n must be even, i.e. $n = 2m$, where m is a positive integer.

From (3.1) the following partial differential equations

$$L[f_{2m}] = 0, \qquad L[f_{2i-1}] = D_{a_1,b,c_1,s_1}[f_{2i}] \tag{3.2}$$

for $i = m, \ldots, 1$, and $s_1 f_0 = 0$.

It follows from section 2 that all solutions of $L[f_{2m}] = 0$ can be written as

$$f_{2m} = \sum_{i=0}^{m} a_i^m \left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^i,$$

where a_i^m is a constant for $i = 0, 1, \ldots, m$. Introducing f_{2m} into the second equation of (3.2) we have

$$\begin{aligned}
L[f_{2m-1}] &= D_{a_1,b,c_1,s_1}[f_{2m}] \\
&= 2X(a_1X - Y)\sum_{i=0}^{m}(m-i)a_i^m \left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^i \\
&\quad + 2Y(a_1Y - \frac{1}{b}X)\sum_{i=0}^{m} i a_i^m \left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^{i-1} \\
&\quad - 2c_1Z^2\sum_{i=0}^{m}(m-i)a_i^m \left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^i \\
&\quad - 2c_1Z^2\sum_{i=0}^{m} i a_i^m \left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^{i-1} \\
&\quad + s_1\sum_{i=0}^{m} a_i^m \left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^i.
\end{aligned} \tag{3.3}$$

Now writing $X^2 = X^2+Z^2-Z^2$ and $Y^2 = Y^2+Z^2-Z^2$, we get

$$\begin{aligned}
L[f_{2m-1}] = &-2XY\left[\sum_{i=0}^{m}(m-i)a_i^m \left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^i + \sum_{i=0}^{m}\frac{ia_i^m}{b}\left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^{i-1}\right] \\
&+ 2a_1\sum_{i=0}^{m}(m-i)a_i^m \left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^i - 2a_1Z^2\sum_{i=0}^{m}(m-i)a_i^m \left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^i \\
&+ 2a_1\sum_{i=0}^{m} ia_i^m \left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^i - 2a_1Z^2\sum_{i=0}^{m} ia_i^m \left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^{i-1} \\
&- 2c_1Z^2\left[\sum_{i=0}^{m}(m-i)a_i^m \left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^i + \sum_{i=0}^{m} ia_i^m \left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^{i-1}\right] \\
&+ s_1\sum_{i=0}^{m} a_i^m \left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^i.
\end{aligned}$$

Then, making the change $i \to j-1$ in some sums and joint the sums conveniently we get

$$
\begin{aligned}
L[f_{2m-1}] = & -2XY \sum_{j=1}^{m} ((m-j+1)a_{j-1}^m + \frac{j}{b} a_j^m) \left(X^2+Z^2\right)^{m-j} (Y^2+Z^2)^{j-1} \\
& + \sum_{j=0}^{m} (2a_1 m + s_1) a_j^m \left(X^2+Z^2\right)^{m-j} (Y^2+Z^2)^{j} \\
& - 2Z^2 \sum_{j=1}^{m} (a_1+c_1) \left((m-j+1)a_{j-1}^m + j a_j^m\right) \left(X^2+Z^2\right)^{m-j} (Y^2+Z^2)^{j-1}.
\end{aligned} \tag{3.4}
$$

Using u, v, w, introduced in section 2, we obtain the ordinary differential equation

$$
\begin{aligned}
\frac{d\overline{f}_{2m-1}}{dw} = & -\frac{1}{\sqrt{u-w^2}\sqrt{v-w^2}} \sum_{j=0}^{m} (2a_1 m + s_1) a_j^m u^{m-j} v^j \\
& + \frac{w^2}{\sqrt{u-w^2}\sqrt{v-w^2}} \sum_{j=1}^{m} 2(a_1+c_1) \left((m-j+1)a_{j-1}^m + j a_j^m\right) u^{m-j} v^{j-1} \\
& + 2 \sum_{j=1}^{m} ((m-j+1)a_{j-1}^m + \frac{j a_j^m}{b}) u^{m-j} v^{j-1}.
\end{aligned}
$$

By solving it, we get

$$
\begin{aligned}
\overline{f}_{2m-1} = & \left(\sum_{j=0}^{m} (2a_1 m + s_1) a_j^m u^{m-j} v^j \right) \int \frac{dw}{\sqrt{u-w^2}\sqrt{v-w^2}} \\
& -2(a_1+c_1) \left(\sum_{j=1}^{m} \left((m-j+1)a_{j-1}^m + j a_j^m\right) u^{m-j} v^{j-1} \right) \int \frac{w^2 dw}{\sqrt{u-w^2}\sqrt{v-w^2}} \\
& +2w \sum_{j=1}^{m} \left((m-j+1)a_{j-1}^m + \frac{j}{b} a_j^m \right) u^{m-j} v^{j-1} + B_{2m-1}(u,v),
\end{aligned}
$$

where B_{2m-1} is an arbitrary function in the variables u and v.

Since

$$
\int \frac{w^2 dw}{\sqrt{u-w^2}\sqrt{v-w^2}} = -\int \frac{\sqrt{u-w^2}}{\sqrt{v-w^2}} dw + u \int \frac{dw}{\sqrt{u-w^2}\sqrt{v-w^2}}, \tag{3.5}
$$

the two integrals which appear in the expression of the polynomial $\overline{f}_{2m-1}$ are reduced to the integrals

$$
\int \frac{dw}{\sqrt{u-w^2}\sqrt{v-w^2}} \quad \text{and} \quad \int \frac{\sqrt{u-w^2}}{\sqrt{v-w^2}} dw. \tag{3.6}
$$

Since these are elliptic integrals of the second and first kinds, respectively (note that these integrals cannot produce a polynomial, fact that can be verified considering their expansions in Taylor series), and f_{2m-1} is a homogeneous polynomial of degree $2m-1$, we must have

$$
\begin{aligned}
& B_{2m-1}\left(X^2+Z^2, Y^2+Z^2\right) = 0, \\
& (2a_1 m + s_1) a_j^m = 0, \qquad j = 0, 1, \ldots, m, \\
& (a_1+c_1)\left((m-j+1)a_{j-1}^m + j a_j^m\right) = 0, \qquad j = 1, \ldots, m.
\end{aligned} \tag{3.7}
$$

Therefore, writing $b_j^m = 2\left((m-j+1)a_{j-1}^m + \dfrac{ja_j^m}{b}\right)$ we have

$$f_{2m-1} = \sum_{j=1}^{m} b_j^m \left(X^2+Z^2\right)^{m-j} (Y^2+Z^2)^{j-1} Z.$$

If $a_j^m = 0$, for $j = 0,1,..,m$, we should have that $f_{2m} = 0$, consequently we obtain that $s_1 = -2a_1 m = -a_1 n$. By the third equation in (3.7) we get either $a_1 + c_1 = 0$, or $a_1 + c_1 \neq 0$ and $(m-j+1)a_{j-1}^m + ja_j^m = 0$, for $j = 1,..,m$.

Now, we split the proof in two cases.

Case 1: $a_1 + c_1 = 0$. As $a_1^2 + c_1^2 \neq 0$, $k = s_1 \neq 0$ and $s_1 = -2a_1 m$ it follows that $s_1 = 2mc_1$. Introducing f_{2m}, f_{2m-1} into the second equation of (3.2) with $i = 2m-1$ and doing similar computations as the ones for passing from (3.3) to (3.4) we obtain

$$\begin{aligned} L[f_{2m-2}] &= D_{a_1,b,c_1,2mc_1}[f_{2m-1}] \\ &= -2XYZ \sum_{i=2}^{m} \left((m-i+1)b_{i-1}^m + \frac{(i-1)}{b} b_i^m\right) \left(X^2+Z^2\right)^{m-i} (Y^2+Z^2)^{i-2} \\ &\quad - 2c_1 Z \sum_{i=1}^{m} b_i^m \left(X^2+Z^2\right)^{m-i} (Y^2+Z^2)^{i-1}. \end{aligned} \tag{3.8}$$

Again, using u, v, w we get

$$\begin{aligned} \frac{d\overline{f}_{2m-2}}{dw} &= 2w \sum_{i=2}^{m} \left((m-i+1)b_{i-1}^m + \frac{(i-1)}{b} b_i^m\right) u^{m-i} v^{i-2} \\ &+ 2c_1 \frac{w}{\sqrt{u-w^2}\sqrt{v-w^2}} \sum_{i=1}^{m} b_i^m u^{m-i} v^{i-1} + B_{2m-2}(u,v). \end{aligned}$$

Since

$$\int \frac{wdw}{\sqrt{u-w^2}\sqrt{v-w^2}} = \log|\sqrt{w^2-u} + \sqrt{w^2-v}|, \tag{3.9}$$

where $f_{2m-2}(x,y,z) = \overline{f}_{2m-2}$ is a homogeneous polynomial in the variables x, y and z, we must have either $c_1 = 0$ or $b_i^m = 0$, for all $i = 1,2,...,m$. But $a_1^2 + c_1^2 \neq 0$ and $a_1 = -c_1$ so $b_i^m = 2\Big((m-i+1)a_{i-1}^m + \dfrac{i}{b} a_i^m\Big) = 0$, consequently $f_{2m-1} = 0$, and

$$a_i^m = (-1)^i b^i \binom{m}{i} a_0^m.$$

Consequently,

$$f_{2m-2} = B_{2m-2}\left(X^2+Z^2, Y^2+Z^2\right) \quad \text{and} \quad f_{2m} = a_0^m \left(X^2+Z^2-bY^2-bZ^2\right)^m.$$

Repeating the same steps done for f_{2m} we conclude that

$$f_{2m-2} = \sum_{i=0}^{m-1} a_i^{m-1} \left(X^2+Z^2\right)^{m-i-1} (Y^2+Z^2)^i$$

$$f_{2m-3}=0$$

and

$$(2a_1(m-1)+s_1)a_i^{m-1}=0, \qquad i=0,1,...,m-1.$$

Since $s_1=-2a_1m$ we have $a_i^{m-1}=0$ for $i=0,1,...,m-1$. Hence $f_{2m-2}=0$.

Finally following this recursive method we conclude that

$$f=f_{2m}=a_0^m\left(X^2+Z^2-bY^2-bZ^2\right)^m.$$

In short $f=0$ is an invariant algebraic surface with cofactor $k=nc/\sqrt{b}$ in the case $a_1+c_1=0$.

Case 2: $a_1+c_1\neq 0$ *and* $(m-j+1)a_{j-1}^m+ja_j^m=0$, *for* $j=1,..,m$. In this case working in a similar way to the previous case we get

$$a_j^m=(-1)^j\binom{m}{j}a_0^m \quad \text{and} \quad f_{2m-1}=\sum_{j=1}^m b_j^m(X^2+Z^2)^{m-j}(Y^2+Z^2)^{j-1}Z,$$

where

$$b_j^m=2\left((m-j+1)a_{j-1}^m+\frac{j}{b}a_j^m\right). \tag{3.10}$$

So $b_j^m=2\left(-ja_j^m+\dfrac{j}{b}a_j^m\right)=\dfrac{2}{b}(1-b)ja_j^m$.

Proceeding as in Case 1 we have that the second equation of (3.2) for $i=2m-2$ can be written as

$$\begin{aligned}
L[f_{2m-2}]&=D_{a_1,b,c_1,-2a_1m}[f_{2m-1}]\\
&=-2XYZ\sum_{i=2}^m\left((m-i+1)b_{i-1}^m+\frac{(i-1)}{b}b_i^m\right)\left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^{i-1}\\
&\quad-2(c_1+2a_1)Z\sum_{i=1}^m b_i^m\left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^{i-1}\\
&\quad-2(a_1+c_1)Z^3\sum_{i=2}^m\left((m-i+1)b_{i-1}^m+b_i^m(i-1)\right)\left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^{i-2}.
\end{aligned} \tag{3.11}$$

Therefore, using u,v,w we get

$$\begin{aligned}
\frac{d\overline{f}_{2m-2}}{dw}=&-2w\sum_{i=2}^m\left((m-i+1)b_{i-1}^m+\frac{(i-1)b_i^m}{b}\right)u^{m-i+1}v^{i-1}\\
&-2(c_1+2a_1)\frac{w}{\sqrt{u-w^2}\sqrt{v-w^2}}\sum_{i=1}^m b_i^m u^{m-i}v^{i-1}\\
&-2(a_1+c_1)\frac{w^3}{\sqrt{u-w^2}\sqrt{v-w^2}}\sum_{i=2}^m\left((m-i+1)b_{i-1}^m+(i-1)b_i^m\right)u^{m-i}v^{i-2}.
\end{aligned} \tag{3.12}$$

By equation (3.9), since

$$\int\frac{w^3dw}{\sqrt{u-w^2}\sqrt{v-w^2}}=\frac{1}{2}\sqrt{u-w^2}\sqrt{w^2-v}+(u+v)\log|\sqrt{w^2-u}+\sqrt{w^2-v}|, \tag{3.13}$$

and f_{2m-2} is a homogeneous polynomial in the variables x,y and z of degree $2m-2$ we must have

$$\begin{aligned} &(c_1+2a_1)b_i^m = 0, \qquad \text{for } i=1,\dots,m,\\ &(m-i+1)b_{i-1}^m+(i-1)b_i^m = 0, \qquad \text{for } i=2,\dots,m. \end{aligned} \tag{3.14}$$

From

$$(m-i+1)a_{i-1}^m+ia_i^m = 0,$$

we have $a_j^m = -\dfrac{1}{j}(m-j+1)a_{j-1}^m$. Then $b_j^m = \dfrac{2}{b}\,(b-1)\,(m-j+1)a_{j-1}^m$. Hence

$$(m-i+1)b_{i-1}^m+(i-1)b_i^m = \frac{2}{b}\,(b-1)\,(m-j+1)\left((m-j+2)a_{j-2}^m+\frac{(j-1)}{b}a_{j-1}^m\right) = 0.$$

Then we only need to consider two subcases, $c_1+2a_1 = 0$ and $b_j^m = 0$, for $j = 1,2,..,n$.

Subcase 2.1: $b_j^m = (m-j+1)a_{j-1}^m+\dfrac{j}{b}a_j^m = 0$ for $j = 1,\dots,m$. From the hypothesis of Case 2 we get $\dfrac{1}{b}ja_j^m(1-b) = 0$ for $j = 1,\dots,m$. So $a_j^m = 0$ or $b = 1$. But if $a_j^m = 0$, for $j = 1,\dots,m$ we have $f_{2m} = 0$, a contradiction. If $b = 1$ then $a_j^m = (-1)^j\binom{m}{j}a_0^m$ and $f_{2m} = a_0^m(x^2-y^2)^m$, $f_{2m-1} = 0$ and

$$f_{2m-2} = B_{2m-2}\left(X^2+Z^2,Y^2+Z^2\right) = \sum_{i=0}^{m-1} a_i^{m-1}\left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^i.$$

Repeating the same steps for passing from f_{2m} to f_{2m-2}, and so on as we have done in Case 1, we get that $f_k = 0$ for $k = 0,1,\dots,2m-1$. Consequently $f = f_{2m} = a_0^m(X^2-Y^2)$ and $g = a_0^m(x^2-y^2)^m$ with $b = 1$.

Subcase 2.2: $c_1+2a_1 = 0$. In this case solving the differential equation (3.12) we have

$$\begin{aligned} f_{2m-2} &= \sum_{i=2}^{m} c_j^m\left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^{i-1}Z^2+B_{2m-2}\left(X^2+Z^2,Y^2+Z^2\right)\\ &= \sum_{i=2}^{m} c_j^m\left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^{i-1}Z^2+\sum_{j=0}^{m-1} d_j^{m-1}\left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^i, \end{aligned}$$

where

$$c_j^m = ((m-j+1)b_{j-1}^m+\frac{(j-1)}{b}b_j^m) = \frac{2}{b^2}(1-b)^2j(j-1)a_j^m. \tag{3.15}$$

Taking in equation (3.2) $i=m-2$ we obtain

$$\begin{aligned}
L[f_{2m-3}] &= D_{a_1,b,-2a_1,-2a_1m}[f_{2m-2}]\\
&= -2XYZ^2\sum_{i=3}^{m}\left((m-i)c_{i-1}^m+\frac{(i-1)c_i^m}{b}\right)(X^2+Z^2)^{m-i-1}(Y^2+Z^2)^{i-2}\\
&\quad -2Z^4(c_1+a_1)\sum_{i=2}^{m}\left((m-i)c_{i-1}^m+(i-1)c_j^m\right)\left(X^2+Z^2\right)^{m-i-1}(Y^2+Z^2)^{i-2}\\
&\quad -2XY\sum_{i=1}^{m-1}((m-i+1)d_{i-1}^m+\frac{i}{b}d_i^m)\left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^{i-1}\\
&\quad -2Z^2\sum_{i=1}^{m-1}(a_1+c_1)\left((m-i+1)d_{i-1}^m+id_i^m\right)\left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^{i-1}.
\end{aligned} \tag{3.16}$$

Passing to the variables u, v and w, we have the ordinary differential equation

$$\begin{aligned}
\frac{d\overline{f}_{2m-3}}{dw} &= -2w^2\sum_{i=3}^{m}\left((m-i)c_{i-1}^m+\frac{(i-1)c_i^m}{b}\right)u^{m-i-1}v^{i-2}\\
&\quad -2\frac{w^4}{\sqrt{u-w^2}\sqrt{v-w^2}}(c_1+a_1)\sum_{i=2}^{m}\left((m-i)c_{i-1}^m+(i-1)c_j^m\right)u^{m-i-1}v^{i-2}\\
&\quad -2\sum_{i=1}^{m-1}((m-i+1)d_{i-1}^m+\frac{i}{b}d_i^m)u^{m-i}v^{i-1}\\
&\quad -2\frac{w^2}{\sqrt{u-w^2}\sqrt{v-w^2}}(c_1+a_1)\sum_{i=1}^{m-1}\left((m-i+1)d_{i-1}^m+id_i^m\right)u^{m-i}v^{i-1}.
\end{aligned} \tag{3.17}$$

Again the expression of f_{2m-3} depends on elliptic integrals and logarithmic functions and they force that $(m-i)c_{i-1}^m+\dfrac{i-1}{b}c_j^m=0$ for $i=2,3,..,m$, and $(m-i+1)d_{i-1}^m+\dfrac{i}{b}d_j^m=0$ for for $i=1,2,3,..,m-1$, because $a+c\neq 0$. Since $(m-i)c_{i-1}^m+\dfrac{i-1}{b}c_j^m=0$ and we are in Case 2, we obtain

$$\frac{4}{b^2}(1-b)^2 i(i-1)a_i^m=0.$$

If some of the a_i^m is zero then all the a_i^m's are zero, because $d_j^m=(-1)^j b^j\begin{pmatrix} m\\ j\end{pmatrix}a_0^m$. But this is a contradiction because then $f_{2m}=0$. Therefore $b=1$, and consequently from (3.15) all the c_i^m's are zero. Hence $f_{2m-2}=\sum_{j=0}^{m-1}d_j^{m-1}\left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^i$. And as in the Case 1 with $b=1$ we obtain $f=f_{2m}=a_0^m(X^2-Y^2)^m$. This complete the proof of the proposition. ☐

Proposition 3.3. *Assume $c_1^2+a_1^2\neq 0$. Let $g=g(X,Y,Z)=0$ be an irreducible invariant algebraic surface of system* (2.2) *of degree n. Then $b=1$ and the algebraic invariant surfaces are $g(X,Y)=X+Y$ with cofactor $k=Z+1-a$ and $g(X,Y)=X-Y$ with cofactor $k=-Z+1-a$*

Proof. If g is invariant by τ, then its cofactor is also invariant by τ and thus $k=r_1Z+s_1$. On the other hand, if g is not invariant by τ then by Proposition 3.1 we can assume that $f=g\cdot\tau g$ is a Darboux polynomial invariant by τ with degree $2n$ and cofactor $k=2r_1Z+2s_1$.

In short we can write $f(X,Y,Z)$ a Darboux polynomial of system (2.2) invariant by τ with degree n_1 and cofactor $k=r_2Z+s_2$ where $n_1=n$, $r_2=r_1,s_2=s_1$ if $f=g$ (in the case in which g is itself), or $n_1 = 2n$, $r_2 = 2r_1$, $s_2 = 2s_1$ if $f \neq g$ (which corresponds to the case in which g is not invariant by τ).

Assume $r_2 > 0$, the case r_2 negative can be proved in the same way interchanging $X+Y$ by $X-Y$. Note that if f is an algebraic invariant surface of degree n_1 with cofactor $k = r_2 Z + s_2$ then f can be written as $f = \sum_{i=1}^{n_1} f_i$, where f_i are homogeneous polynomials of degree i and f_{n_1} satisfies

$$YZ\frac{\partial f_{n_1}}{\partial X} + XZ\frac{\partial f_{n_1}}{\partial y} - XY\frac{\partial f_{n_1}}{\partial z} = f_{n_1} r_2 Z.$$

The solutions of this linear partial differential equations are of the form

$$(Y+X)^{\pm r_2} G(X^2+Z^2, Y^2+Z^2),$$

where G is an arbitrary C^1 function. Since f_{n_1} must be a homogeneous polynomial of degree n_1 it follows that r_2 must be a non-negative integer and G must be a homogeneous polynomial in the variables X^2+Z^2 and Y^2+Z^2. Since $r_2 > 0$ we can write

$$f_{n_1} = (X+Y)^{r_2} \sum_{i=0}^{m} a_i^m (X^2+Z^2)^{m-i} (Y^2+Z^2)^i,$$

where $n_1 = 2m + r_2$, or equivalently, $r_2 = n_1 - 2m$. (Note that if $r_2 < 0$ then we would have $f_{n_1} = (X-Y)^{r_2} \sum_{i=0}^{m} a_i^m (X^2+Z^2)^{m-i} (Y^2+Z^2)^i$,).

Then, from (3.1) we get the following partial differential equations

$$L[f_{n_1}] = (n_1 - 2m) Z f_{n_1} \qquad L[f_i] = D_{a_1,b,c_1,s_1}[f_{i+1}] + (n_1 - 2m) Z f_i, \tag{3.18}$$

for $i = n_1 - 1, \ldots, 1$ and

$$D_{a_1,b_1,c_1,s_1}[f_1] + (n-2m) Z f_0 = 0.$$

Introducing f_{n_1} in the second equation of (3.18) with $i = n_1 - 1$ and writing $X^2 = X^2 + Z^2 - Z^2$, $Y^2 = Y^2 + Z^2 - Z^2$ and $Y = X + Y - X$ or doing $i = j-1$ if necessary, we get the following differential equation

$$\begin{aligned} L[f_{n_1-1}] = {} & (n-2m) Z f_{n_1-1} - 2XY(X+Y)^{n_1-2m} \sum_{i=1}^{m} \left((m-i+1) a_{i-1}^m + \frac{i a_i^m}{b} \right) \left(X^2+Z^2\right)^{m-i} (Y^2+Z^2)^{i-1} \\ & + [(a_1 - 1 + 2a_1 m)(n-2m) + s_1](X+Y)^{n_1-2m} \sum_{i=0}^{m} a_i^m \left(X^2+Z^2\right)^{m-i} (Y^2+Z^2)^i \\ & - 2(a_1 + c_1) Z^2 (X+Y)^{n_1-2m} \sum_{i=1}^{m} ((m-i+1) a_{i-1} + i a_i^m) \left(X^2+Z^2\right)^{m-i} (Y^2+Z^2)^{i-1} \\ & + \left(1 - \frac{1}{b}\right)(n-2m) X (X+Y)^{n_1-2m-1} \sum_{i=0}^{m} a_i^m \left(X^2+Z^2\right)^{m-i} (Y^2+Z^2)^i. \end{aligned}$$

Passing to the variables u,v,w from the above equation we obtain

$$\begin{aligned}
&\sqrt{u-w^2}\sqrt{v-w^2}\frac{d\overline{f}_{n_1-1}}{dw}=-(n_1-2m)w\overline{f}_{n_1-1}\\
&-2\sqrt{u-w^2}\sqrt{v-w^2}(\sqrt{u-w^2}+\sqrt{v-w^2})^{n_1-2m}\sum_{i=1}^{m}\left((m-i+1)a_{i-1}^m+\frac{ia_i^m}{b}\right)u^{m-i}v^{i-1}\\
&+[(a_1-1+2a_1m)(n_1-2m)+s_2](\sqrt{u-w^2}+\sqrt{v-w^2})^{n_1-2m}\sum_{i=0}^{m}a_i^m u^{m-i}v^i\\
&-2(a_1+c_1)w^2(\sqrt{u-w^2}+\sqrt{v-w^2})^{n_1-2m}\sum_{i=1}^{m}((m-i+1)a_{i-1}+ia_i^m)\,u^{m-i}v^{i-1}\\
&+\left(1-\frac{1}{b}\right)(n_1-2m)\sqrt{u-w^2}(\sqrt{u-w^2}+\sqrt{v-w^2})^{n_1-2m-1}\sum_{i=0}^{m}a_i^m u^{m-i}v^i.
\end{aligned}$$

This is a linear ordinary differential equation in $\overline{f}_{n_1-1}$, its corresponding homogeneous differential equation is

$$\sqrt{u-w^2}\sqrt{v-w^2}\frac{d\overline{f}_{n_1-1}}{dw}=-(n_1-2m)w\overline{f}_{n_1-1},$$

Its general solution is

$$\overline{f}_{n_1-1}=E_{n_1-1}(u,v)\left(\sqrt{u-w^2}+\sqrt{v-w^2}\right)^{n_1-2m},$$

where E_{n_1-1} is any C^1 function in the variables u and v. Hence, the general solution of the non–homogeneous linear differential equation for $\overline{f}_{n_1-1}$ is

$$\begin{aligned}
\overline{f}_{n_1-1}=&E_{n_1-1}(u,v)\left(\sqrt{u-w^2}+\sqrt{v-w^2}\right)^{n_1-2m}+\left(\sqrt{u-w^2}+\sqrt{v-w^2}\right)^{n_1-2m}\\
&\left(-2\sum_{i=1}^{m}\left((m-i+1)a_{i-1}^m+\frac{ia_i^m}{b}\right)u^{m-i}v^{i-1}\int dw\right.\\
&+[(a_1-1+2a_1m)(n_1-2m)+s_2]\sum_{i=0}^{m}a_i^m u^{m-i}v^i\int\frac{1}{\sqrt{u-w^2}\sqrt{v-w^2}}\,dw\\
&-2(a_1+c_1)\sum_{i=1}^{m}\left((m-i+1)a_{i-1}^m+ia_i^m\right)u^{m-i}v^{i-1}\int\frac{w^2}{\sqrt{u-w^2}\sqrt{v-w^2}}\,dw\\
&\left.+\left(1-\frac{1}{b}\right)(n_1-2m)\sum_{i=0}^{m}a_i^m u^{m-i}v^i\int\frac{(\sqrt{u-w^2}+\sqrt{v-w^2})^{-1}}{\sqrt{v-w^2}}\,dw\right).
\end{aligned}$$

Solving this integral, proceeding as above taking into account that $r_2>0$ we must have

$$\begin{aligned}
&[(a_1-1+2a_1m)(n_1-2m)+s_1]a_i^m=0,\\
&(a_1+c_1)((m-i+1)a_{i-1}+ia_i^m)=0,\\
&\left(1-\frac{1}{b}\right)(n_1-2m)a_i^m=0.
\end{aligned}$$

Since $n_1>2m$, it follows from the last identity above that either $b=1$ or $a_i^m=0$, for $i=1,2,\dots,m$. But if $a_i^m=0$ then f_{2m+r_2} is zero (a contraction), so $b=1$. Moreover, $(a_1-1+2a_1m)(n_1-2m)+s_2=0$ and either $a_1+c_1=0$ or $a_{i-1}(m-i+1)+ia_i^m=0$.

Assuming $b=1$ we have

$$f_{2m+r_2-1}=(X+Y)^{n_1-2m}\sum_{i=1}^{m} b_i^m (X^2+Z^2)^{m-i}(Y^2+Z^2)^{i-1}Z,$$

where $b_i^m = 2((m-i+1)a_{i-1}^m + ia_i^m)$.

We consider two cases.

Case 1: $a_1+c_1\neq 0$. It follows from the explanation above that $(m-i+1)a_{i-1}+ia_i^m=0$. Then, $f_{2m+r_2-1}=0$ and, by recurrence, $a_i^m=(-1)^m\begin{pmatrix} m\\ i\end{pmatrix}a_0^m$ which yields $f_{2m+r_2}=a_0^m(X+Y)^{n_1-2m}(X^2-Y^2)^m$. Substituting the expression of f_{2m+r_2-1} into (3.18) we get

$$\sqrt{u-w^2}\sqrt{v-w^2}\frac{d\overline{f}_{2m+r_2-2}}{dw}=(n_1-2m)w\overline{f}_{2m+r_2-2}.$$

Solving it, and taking into account that f_{2m+r_2-2} is a homogeneous polynomial of degree $2m+r_2-2$ we get $f_{2m+r_2-2}=f_{n_1-2}=(X+Y)^{n_1-2m}\sum_{i=0}^{m-1} b_i^m (X^2+Z^2)^{m-i}(Y^2+Z^2)^{i-1}$. Substituting the expression of f_{2m+r_2-2} into (3.18) and solving for f_{2m+r_2-3} we get that $f_{2m+r_2-3}=0$ and $f_{2m+r_2-2}=b_0^m(X+Y)^{n_1-2m}(X^2-Y^2)^{m-1}$ for some constant b_0^m. Proceeding inductively we conclude that $f=(X+Y)^{n_1-2m}P(X^2-Y^2)$, being P a polynomial in the variables X^2-Y^2. If $f=g$, that is, g is irreducible, then $n_1=n=1$, $m=0$ and $g=X+Y$, which is not possible because in this case g is invariant by τ. Else, $f=g\cdot\tau g$ with g being irreducible. So $n_2=2n=2$, $m=0$, $f=(X+Y)^2$ which yields $g=X+Y$. The cofactor is $1-a+Z$.

Case 2: $a_1+c_1=0$. In this case if $a_{i-1}(m-i+1)+ia_i^m=0$ then proceeding as in Case 1 we conclude that the irreducible polynomial is $g=X+Y$ with cofactor $1-a+Z$. If $a_{i-1}(m-i+1)+ia_i^m\neq 0$ then substituting the expression of f_{2m+r_2-1} into (3.18) we get

$$\begin{aligned}
&\sqrt{u-w^2}\sqrt{v-w^2}\frac{d\overline{f}_{2m+r_2-2}}{dw}=(n_1-2m)w\overline{f}_{2m+r_2-2}\\
&-2\sqrt{u-w^2}\sqrt{v-w^2}(\sqrt{u-w^2}+\sqrt{v-w^2})^r\sum_{i=2}^{m}((m-i+1)b_{i-1}^m+(i-1)b_i^m)u^{m-i}v^{i-1}w\\
&-c_1(\sqrt{u-w^2}+\sqrt{v-w^2})^{n_1-2m}\sum_{i=1}^{m} b_i^m u^{m-i}v^i.
\end{aligned}$$

Solving this linear equation, using that f_{2m+r_2-2} is a homogeneous polynomial in the variables X,Y and Z we must have $c_1=0$. But then $a_1=0$ in contradiction with the fact that $a_1^2+c_1^2\neq 0$. Hence, this case is not possible and the proposition is proved for $r_2>0$. Note that if $r_2<0$ proceeding as above and repeating the same arguments we conclude that $g=X-Y$ and the cofactor is $k=1-a-Z$. This concludes the proof of the proposition. □

Proof of Theorems 1.1 and 1.2. Theorem 1.1 can be verified by simple computation, and Theorem 1.2 follows from Proposition 3.2 going back through the change of variables given in (2.1). □

Proof of Theorems 1.3 and 1.4. Theorems 1.3 and 1.4 follow directly from Propositions 3.1, 3.2 and 3.3 going back through the change of variables given in (2.1). □

4. Proof of Theorems 1.5

We separate the proof of Theorem 1.5 into a lemma and two propositions.

Lemma 4.1. *If $a+c\neq 0$ or $b\neq 1$ then system* (1.1) *has no Darboux first integrals.*

Proof. In view of Theorems 1.2, 1.3 and 1.4 system (1.1) has no Darboux polynomials. Then in view of Proposition 2.1 if it has an exponential factor F then it must be of the form $F=\exp(f)$ with $f\in\mathbb{C}[x,y,z]\setminus\mathbb{C}$. Finally, from Theorem 2.2 we conclude that if G is a Darboux first integral then it must be of the form $G=F_1^{\mu_1}\cdots F_q^{\mu_q}$ with $F_i=\exp(h_i)$, $h_i\in\mathbb{C}[x,y,z]$ and $\sum_{i=1}^q \mu_i\ell_i=0$. Take $g=\sum_{i=1}^q h_i$ and consider $G=\exp(g)$. Then $g\in\mathbb{C}[x,y,z]\setminus\mathbb{C}$ and G is an exponential factor with cofactor $L=\sum_{i=1}^q \mu_i\ell_i=0$. So, g satisfies, after simplifying by G,

$$(yz-ax+y)\frac{\partial g}{\partial x}+(bxz+x-ay)\frac{\partial g}{\partial y}+(-bxy+cz)\frac{\partial g}{\partial z}=\sum_{i=1}^q \mu_i\ell_i=0.$$

In particular g must be a polynomial first integral. However, in view of Theorems 1.2, 1.3 and 1.4, system (1.1) with either $b\neq 1$ or $a^2+c^2\neq 0$ has no polynomial first integrals. This completes the proof. $\square$

Guided by section 2 instead of working with system (1.1) we will work with system (2.2) and all the results that we will obtain for system (2.2) follow clearly for system (1.1).

Proposition 4.1. *If $b=1$, system (2.2) has a Darboux first integral if and only if $a=0$. In this case the first integral is $H=x^2-y^2$.*

Proof. Let $F=\exp(h/g)$ be an exponential factor of system (1.1) with $b=1$. In view of Proposition 2.1, F can be of the form $F=\exp(h/(f_1^{n_1}f_2^{n_2}))$ with $h\in\mathbb{C}[x,y,z]$ and $n_1,n_2\in\mathbb{N}$, $f_1=x+y$, $f_2=x-y$ with and $(h,f_1)=1$ (coprime) if $n_1>0$ and $(h,f_2)=1$ (coprime) if $n_2>0$.
Case 1*:* $n_1=n_2=0$. In this case $F=\exp(h)$ and h satisfies

$$(-ax+y+yz)\frac{\partial h}{\partial x}+(x-ay+xz)\frac{\partial h}{\partial y}+(cz-xy)\frac{\partial h}{\partial z}=k_0+k_1x+k_2y+k_3z, \tag{4.1}$$

with $k_i\in\mathbb{C}$. Evaluating the above equation on $x=y=z=0$ we obtain that $k_0=0$. Now we write $h=\sum_{i=0}^n h_i$ where each h_i is a homogeneous polynomial in its variables. Without loss of generality we can assume that $h_n\neq 0$ and $n\geq 1$. If $n\leq 2$, i.e., h has degree less than or equal to two, there is a solution if and only if $a=0$ and in this case $h=\alpha(x^2-y^2)$ with $\alpha\in\mathbb{C}$ and $k_0=k_1=k_2=k_3=0$. So, $n\geq 3$.

We use the notation in the proof of Proposition 3.2 (since $b=1$, $X=x, Y=y$ and $Z=z$, $a_1=a$ and $c_1=c$). The terms of degree $n+1$ satisty $L[h_n]=0$ and so $n=2m$ and

$$h_n=\sum_{i=0}^m a_i^m(x^2+z^2)^{m-i}(y^2+z^2)^i.$$

Computing the terms of degree n in (4.1), we get (see (2.6))

$$L[h_{2m-1}]=D_{a,1,c,0}[h_{2m}].$$

Proceeding as in Proposition 3.2 we get that either $a_i^m=0$, for $i=0,1,\ldots,m$ or $a=0$. In the first case $h_{2m}=0$ which is not possible. So $a=0$, $c\neq -a$ (otherwise $c=0$ which is a case not considered here) and $h_n=a_0^m(x^2-y^2)^m$, $a_0^m\in\mathbb{C}$. Moreover $h_{n-1}=h_{2m-1}=0$ because h_{n-1} must be a homogeneous polynomial of degree $n-1$. Note that the terms of degree $2m-i$ for $i=2,\dots,2m-1$ satisfy

$$L[h_{2m-i}] = D_{0,1,c,0}[h_{2m-i+1}], \quad i=1,\dots,2m-1,$$

and

$$0 = L[h_0] = D_{0,1,c,0}[h_1] = (k_1x+k_2y+k_3z). \tag{4.2}$$

Computing the term of degree $n-1$ that is, solving $L[h_{2m-2}] = D_{0,1,c,0}[h_{2m-1}]$ we get $h_{2m-1}=0$ and $h_{2m-2} = a_0^{m-1}(x^2-y^2)^{m-1}$. Proceeding inductively, we get $h_{2k+1}=0$ for $k=0,1,\dots,m-1$ and $h_{2k} = a_0^{2k}(x^2-y^2)^k$ for $k=1,\dots,m$. So, from (4.2) we get $0=k_1x+k_2y+k_3z$, i.e., $k_1=k_2=k3=0$ and so $k_i=0$, for $i=0,1,2,3$. This implies that there are no exponential factors of the form $F=\exp(h)$ for $a\neq 0$ and for $a=0$ the unique exponential factors of the form $F=\exp(h)$ satisfy $h=h(x^2-y^2)$ being h a polynomial of degree n and $k_i=0$, for $i=0,1,2,3$.
Case 2: $n_1>n_2$ or $n_2>n_1$. In this case h is coprime with $f_1=x+y$ (when $n_1\geq 0$) and with $f_2=x-y$ (when $n_2\geq 0$) and satisfies

$$\begin{aligned}(-ax+y+yz)\frac{\partial h}{\partial x}+(x-ay+xz)\frac{\partial h}{\partial y}+(cz-xy)\frac{\partial h}{\partial z}\\ -(n_1(1-a+z)+n_2(-1-a-z))h = k f_1^{n_1} f_2^{n_2},\end{aligned} \tag{4.3}$$

where $k=k_0+k_1x+k_2y+k_3z$ with $k_i\in\mathbb{C}$. We consider the case $n_1>n_2$ (i.e, $n_1\geq 1$). The case $n_1<n_2$ can be done in a similar manner and so we do not do it here. Assume that $h=c\in\mathbb{C}$. Then from equation (4.3) we have

$$-c(n_1(1-a+z)+n_2(-1-a-z)) = k(x+y)^{n_1}(x-y)^{n_2}$$

Since $n_1\geq 1$ and the left-hand side of the above equation is not divisible by $x+y$ we get a contradiction. So, h is not constant.

Now we introduce the new variables $(\hat X,\hat Y,z)$ where $\hat X=f_1=x+y$ and $\hat Y=f_2=x-y$. In these new variables we set $h(x,y,z)=g(\hat X,\hat Y,z)$ and so $g\in\mathbb{C}[\hat X,\hat Y,z]$. From (4.3) we obtain that g satisfies

$$\begin{aligned}(1-a+z)\hat X\frac{\partial g}{\partial \hat X}+(-1-a-z)\hat Y\frac{\partial g}{\partial \hat Y}+\left(cz-\frac{\hat X^2-\hat Y^2}{4}\right)\frac{\partial g}{\partial z}\\ -(n_1(1-a+z)+n_2(-1-a-z))g = k\hat X^{n_1}\hat Y^{n_2}.\end{aligned} \tag{4.4}$$

We assume $n_1<n_2$, the case $n_1>n_2$ is done in a similar way. In this case, if we denote by $\bar g$ the restriction of g to $\hat X=0$, i.e. $\bar g=\bar g(y,z)=g(-y,y,z)$, and we restrict (4.4) to $\hat X=0$ (i.e., $x=-y$) we get that $\bar g$ is a Darboux polynomial of system

$$\dot y = -y(1+a+z), \quad \dot z = cz+y^2 \tag{4.5}$$

with cofactor $n_1(1-a+z)+n_2(-1-a-z)$, so it satisfies

$$-y(1+a+z)\frac{\partial \bar g}{\partial y}+(cz+y^2)\frac{\partial \bar g}{\partial z} = (n_1(1-a+z)+n_2(-1-a-z))\bar g. \tag{4.6}$$

We consider two cases.

Case 2.1*:* $c=0$. In this case solving (4.6) we get

$$\bar{g}=K_0(y^2+z(2+2a+z))y^{-n_1+n_2+2an_1/\sqrt{y^2+(1+a+z)^2}}\big(y^2+(1+a+z)^2 \\ +|1+a+z|\sqrt{y^2+(1+a+z)^2}\big)^{2an_1/\sqrt{y^2+(1+a+z)^2}}.$$

Since $n_1\neq 0$ and $\bar{g}$ must be a polynomial we get $\bar{g}=0$, in contradiction with the fact that g is not divisible by $\hat{X}$. So, there are no exponential factors of this form in this case.
Case 2.2*:* $c\neq 0$. We consider two different subcases.
Subcase 2.2.1*: $\bar{g}$ is not divisible by y.* Setting $y=0$ and denoting $\tilde{g}=\tilde{g}(z)=\bar{g}(0,z)$ we get that $\tilde{g}\neq 0$ and satisfies

$$cz\frac{d\tilde{g}}{dz}=(n_1(1-a+z)+n_2(-1-a-z))\tilde{g}.$$

Solving it we obtain

$$\tilde{g}=c_0e^{(n_1-n_2)z/c}z^{((a-1)n_1+(1+a)n_2)/c}, \quad c_0\in .$$

Since $n_1>n_2$ and $\tilde{g}$ is a polynomial we must have $c_0=0$ and so $\tilde{g}=0$, which is not possible.
Subcase 2.2.2*: $\bar{g}$ is divisible by y.* We write $\bar{g}=y^j\bar{g}_1$ where $j\geq 1$ and $\bar{g}_1\neq 0$. Moreover, it follows from (4.6) that $\bar{g}_1$ satisfies

$$-y(1+a+z)\frac{\partial\bar{g}_1}{\partial y}+(cz+y^2)\frac{\partial\bar{g}_1}{\partial z}=(n_1(1-a+z)+(n_2-j)(-1-a-z))\bar{g}_1.$$

Setting $y=0$ and denoting $\tilde{g}_1=\tilde{g}_1(z)=\bar{g}_1(0,z)$ we get that $\tilde{g}_1\neq 0$ and satisfies

$$cz\frac{d\tilde{g}_1}{dz}=(n_1(1-a+z)+(n_2-j)(-1-a-z))\tilde{g}_1.$$

Solving it we get

$$\tilde{g}_1=c_1e^{(n_1-n_2+j)z/c}z^{((a-1)n_1+(1+a)(n_2-j))/c}, \quad c_0\in .$$

Since $n_1>n_2$ and $\tilde{g}_1$ is a polynomial we must have $c_1=0$ and so $\tilde{g}_1=0$, which is not possible.

This means that $\bar{g}=0$ in contradiction with the fact that g is not divisible by $\hat{X}$. Hence, there are no exponential factors of this form in this case.
Case 3*:* $n_1=n_2\geq 1$. Working in a similar way to the proof of Case 1 in Proposition 4.1 we get that the unique possibility is $a=0$ and that $h=h(x^2-y^2)$ with $k_i=0$, for $i=0,1,2,3$. So, in this case there are exponential factors only when $a=0$ and the exponential factors are of the form $F=\exp(h/(x^2-y^2)^{n_1})$ with $h=h(x^2-y^2)$ and $k_i=0$, for $i=0,1,2,3$.

If $a\neq 0$, since there are no exponential factors for system (2.2) when $b=1$ and $a\neq 0$, by Theorem 2.2 we conclude that if G is a Darboux first integral then it must be of the form $G=f_1^{\mu_1}f_2^{\mu_2}$ with $\mu_1,\mu_2\in\mathbb{C}$ being the cofactor $K=(1-a+z)\mu_1-(1+a+z)\mu_2$. Since the cofactor must be zero and $a\neq 0$ we must have $\mu_1=\mu_2=0$ but then G is constant, which is not possible. Hence, there are no Darboux first integrals in this case.

If $a=0$, since the unique exponential factors are of the form $F=\exp(h/(x^2-y^2)^n)$ with $h=h(x^2-y^2)$ and the cofactor $k=0$, in view of (1.3) we get that the unique Darboux first integrals are Darboux functions of the polynomial first integral x^2-y^2. This concludes the proof of the proposition. $\square$

Proposition 4.2. *If $a+c=0$ with $a\neq 0$, system* (2.2) *has no Darboux first integrals.*

Proof. Let $F=\exp(h/g)$ be an exponential factor of system (2.2) with $a_1+c_1=0$ and $a_1\neq 0$. In view of Proposition 2.1, F can be of the form $F=\exp(h/f_3^{n_3})$ with $h\in\mathbb{C}[X,Y,Z]$ and $n_3\in\mathbb{N}$, $f_3=X^2+Z^2-b(Y^2+Z^2)$ and $(h,f_3)=1$ (coprime) if $n_3>0$. We will first compute the exponential factors, showing that there are none.

Case 1*:* $n_3=0$. In this case h satisfies

$$(-a_1X+Y+YZ)\frac{\partial h}{\partial X}+(\frac{1}{b}X-a_1Y+XZ)\frac{\partial h}{\partial Y}+(c_1Z-XY)\frac{\partial h}{\partial Z}=k_0+k_1X+k_2Y+k_3Z, \quad (4.7)$$

with $k_i\in\mathbb{C}$. Evaluating the above equation on $X=Y=Z=0$ we obtain that $k_0=0$. Now we write $h=\sum_{i=0}^n h_i$ where each h_i is a homogeneous polynomial in its variables. Without loss of generality we can assume that $h_n\neq 0$ and $n\geq 1$. The terms of degree $n+1$ satisfy

$$[h_n]=0$$

Proceeding as in the proof of Proposition 3.2 we get that $n=2m$ and

$$h_n=\sum_{i=0}^m a_i^m(X^2+Z^2)^{m-i}(Y^2+Z^2)^i.$$

where a_i^m is a constant for $i=0,1,\dots,m$. Computing the terms of degree n we obtain

$$L[h_{2m-1}]=D_{a_1,b,-a_1,0}[h_{2m}].$$

Proceeding as in the proof of Case 1 of Proposition 3.2 with $s_1=0$ we conclude that $h_{2m}=h_{2m-1}=0$ which is not possible. Hence there are no exponential factors of the form $\exp(h)$, with $h\in\mathbb{C}[X,Y,Z]\setminus\mathbb{C}$.

Case 2*:* $n_3\geq 1$. In this case h satisfies

$$\begin{aligned}&\big(-a_1X+Y+YZ\big)\frac{\partial h}{\partial X}+\Big(\frac{1}{b}X-a_1Y+XZ\Big)\frac{\partial h}{\partial Y}+\big(-c_1Z-XY\big)\frac{\partial h}{\partial Z}\\ &=2n_3a_1h+\big(X^2+Z^2-b(Y^2+Z^2)\big)^{n_3}(k_0+k_1X+k_2Y+k_3Z),\end{aligned} \quad (4.8)$$

with $k_i\in\mathbb{C}$. We claim that $n\geq 2n_3+1$. Otherwise, in what follows we can prove that $k_i=0$, for $i=0,1,2,3$. So h is a Darboux polynomial with cofactor $-2an_3$ and hence from Theorem 1.2, $h=\alpha(X^2+Z^2-b(Y^2+Z^2))^{n_3}=\alpha f_3^{n_3}$ with α an arbitrary constant. But this is not possible because h and f_3 are coprime.

We first prove the claim. If $n-2n_3-1<-2$, from (4.8) and taking in account the degree of equation (4.8), it is easy to see that $k_0=k_1=k_2=k_3=0$, which is not possible.

If $n-2n_3-1=-2$ then proceeding as before we get that $k_1=k_2=k_3=0$ and $L[h_n]=k_0f_3^{n_3}$ (see (2.3)). Applying the method of characteristic curves to this equation, we obtain that

$$h_n=\tilde{h}_n(u,v,w)=k_0\sum_{i=0}^{n_3}\binom{n_3}{i}b^i(-1)^iu^{n_3-i}v^i\int\frac{dw}{\sqrt{u-w^2}\sqrt{v-w^2}}.$$

Since f_n must be a homogeneous polynomial of degree n and using the expression of the integral, given in (3.6), we conclude that $k_0=0$ which it is not possible.

If $n-2n_3-1=-1$, we get $L[h_n]=(k_1X+k_2Y+k_3Z)\,f_3^{n_3}$ or in other words

$$h_n=\sum_{i=0}^{n_3}\binom{n_3}{b}^i(-1)^iu^{n_3-i}v^i\left(k_1\int\frac{dw}{\sqrt{v-w^2}}+k_2\int\frac{dw}{\sqrt{u-w^2}}+k_3\int\frac{wdw}{\sqrt{u-w^2}\sqrt{v-w^2}}\right)+\hat{f}_n(u,v). \tag{4.9}$$

Using (3.9) and that

$$\int\frac{dw}{\sqrt{v-w^2}}=\arctan\left(\frac{w}{\sqrt{v-w^2}}\right),\qquad \int\frac{dw}{\sqrt{u-w^2}}=\arctan\left(\frac{w}{\sqrt{u-w^2}}\right),$$

together with the fact that h_n must be a homogeneous polynomial of degree n we conclude that $k_1=k_2=k_3=0$ and $n=2m$. So $h_n=h_{2m}=\sum_{i=0}^m a_i^m\left(X^2+Z^2\right)^{m-i}(Y^2+Z^2)^i$, with $a_i^m\in\mathbb{C}$. Computing the terms of degree $n=2m$ in (4.8), we must solve

$$L[h_{n-1}]=D_{a_1,b,-a_1,0}[h_n]+k_0\,f_3^{n_3}+2n_3a_1h_n.$$

Using $h_n,\ f_3$, the changes in (2.4) and (2.5) and proceeding as in the proof of Proposition 3.2 we get

$$\begin{aligned}\frac{d\tilde{h}_{n-1}}{dw}&=2a_1\frac{n_3-m}{\sqrt{u-w^2}\sqrt{v-w^2}}\sum_{i=0}^m a_i^mu^{m-i}v^i,\\ &\quad+2w\sum_{i=1}^m(a_{i-1}^m(n-i+1)+\frac{ia_i^m}{b})u^{m-i}v^i\\ &\quad+\frac{k_0}{\sqrt{u-w^2}\sqrt{v-w^2}}\sum_{i=0}^{n_3}\binom{n_3}{i}b^i(-1)^iu^{n_3-i}v^i.\end{aligned} \tag{4.10}$$

Note that now $n=n_3$. So using the integrating formula (3.6) together with the fact that h_{n-1} is a homogeneous polynomial of degree $n-1$ we get $k_0=0$. So, $k_i=0$, for $i=0,1,2,3$ which is not possible. This proves the claim.

We thus have $n=2n_3+1+\zeta$ for some $\zeta\in\mathbb{N}\cup\{0\}$. Then from (4.8) we obtain

$$\begin{aligned}L[h_{n-i}]&=D_{a_1,-a_1,0}[h_{n-i+1}],\quad i=1,\dots,\zeta,\\ L[h_{n-\zeta-1}]&=D_{a_1,-a_1,0}[h_{n-\zeta}]+(k_1x+k_2y+k_3z)\,f_3^{n_3},\\ L[h_{n-\zeta-2}]&=D_{a_1,-a_1,0}[h_{n-\zeta-1}]+k_0\,f_3^{n_3},\\ L[h_{n-\zeta-j}]&=D_{a_1,-a_1,0}[h_{n-\zeta-j+1}],\quad j=1,\dots,n-\zeta-1,\end{aligned} \tag{4.11}$$

where $h_i=0$ for $i<0$ or $i>2n_3+1+\zeta$. Since the operators $D_{a_1,-a_1,0}$ and L are linear we separate h_i in the following way $h_i=h_{i,0}+h_{i,1}$ where

$$L[h_{i,0}]=D_{a_1,-a_1,0}[h_{i-1,0}],\quad i=0,1,\dots,2n_3+\zeta+2, \tag{4.12}$$

$$L[h_{n-i,1}]=0\quad i=1,\dots,\zeta, \tag{4.13}$$

$$L[h_{n-\zeta-1,1}]=(k_1x+k_2y+k_3z)\,f_3^{n_3}, \tag{4.14}$$

$$L[h_{n-\zeta-2,1}]=D_{a_1,-a_1,0}[h_{n-\zeta-1,1}]+k_0\,f_3^{n_3}+2a_1n_3h_{n-\zeta-2,1}, \tag{4.15}$$

$$L[h_{n-\zeta-j,1}] = D_{a_1,-a_1,0}[h_{n-\zeta-j+1,1}], \quad j = 1,\dots,n-\zeta-1.$$

Moreover, we require that in the process to solve $h_{i,l}$ for $i = 0,\dots,n$ and $l = 0,1$ the polynomials $h_{i,1}$ do not contain integrating constants.

From (4.12) working as in Proposition 3.2 we obtain that $h_0 = \sum_{i=0}^{n} h_{i,0}$ is a Darboux polynomial of system (2.2) with cofactor $-2a_1n_3$. So, by Theorem 1.2 we must have $h_0 = \alpha(X^2+Z^2-b(Y^2+Z^2)^{n_3}$ with $\alpha \in \mathbb{C}$.

Under the assumptions on $h_{i,1}$ we obtain that equation (4.13) have the unique solutions $h_{n-i,1} = 0$ for $i = 1,\dots,\zeta$. From equation (4.14) we get

$$\begin{aligned} h_{n-\zeta-1}(x,y,z) = \sum_{i=1}^{n_3} \binom{n_3}{i} b^i(-1)^i u^i v^{n_3-i} \Bigg(k_1 \int \frac{dw}{\sqrt{v-w^2}} + k_2 \int \frac{dw}{\sqrt{u-w^2}} \\ + k_3 \int \frac{wdw}{\sqrt{u-w^2}\sqrt{v-w^2}}\Bigg) + \hat{h}_{n-\zeta-1}(u,v), \end{aligned}$$

which is equation (4.9). Hence, $k_1 = k_2 = k_3 = 0$ and $h_{n-\zeta-1} = 0$. Moreover, equation (4.15) yields

$$\frac{d\tilde{h}_{n-\zeta-2}}{dw} = \frac{k_0}{\sqrt{u-w^2}\sqrt{v-w^2}} \sum_{i=0}^{n_3} (-1)^i b^i u^{n_3-i} v^i.$$

From (3.6) and using that $h_{n-\zeta-2}$ is a homogeneous polynomial we must have $k_0 = 0$. Then $k_i = 0$ for $i = 0,1,2,3$, which is not possible. This shows that there are no exponential factors for system (2.2) and so, there are no exponential factors for system (1.1) in this case.

Since there are no exponential factors for system (2.2) when $a+c=0$ with $a,c \neq 0$, by Theorem 2.2 we conclude that if G is a Darboux first integral then it must be of the form $G = f_3^{\mu_3}$ with $\mu_3 \in \mathbb{C}$ being the cofactor $k = -2a\mu_3$. Since $a \neq 0$ and the cofactor must be zero we must have $\mu_3 = 0$ but then G is constant, which is not possible. Hence, there are no Darboux first integrals in this case. This concludes the proof of the proposition. □

Proof of Theorems 1.5. Theorem 1.5 follows directly from Theorem 1.1 and Lemma 4.1 and Propositions 4.1 and 4.2. □

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