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# Spectral stability of periodic waves in the generalized reduced Ostrovsky equation

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**Abstract**

We consider stability of periodic travelling waves in the generalized reduced Ostrovsky equation with respect to co-periodic perturbations. Compared to the recent literature, we give a simple argument that proves spectral stability of all smooth periodic travelling waves independently on the nonlinearity power. The argument is based on the energy convexity and does not use coordinate transformations of the reduced Ostrovsky equations to the semi-linear equations of the Klein–Gordon type.

*Key words:* reduced Ostrovsky equations; stability of periodic waves; energy-to-period map; negative index theory

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## 1 Introduction

We address the generalized reduced Ostrovsky equation written in the form

$$(u_t + u^p u_x)_x = u, \tag{1}$$

where $p \in \mathbb{N}$ is the nonlinearity power and $u$ is a real-valued function of $(x, t)$. This equation was derived in the context of long surface and internal gravity waves in a rotating fluid for $p = 1$ [21] and $p = 2$ [10]. These two cases are the only cases, for which the reduced Ostrovsky equation is transformed to integrable semi-linear equations of the Klein–Gordon type by means of a change of coordinates [3, 9].

We consider existence and stability of travelling periodic waves in the generalized reduced Ostrovsky equation (1) for any $p \in \mathbb{N}$. The travelling $2T$-periodic waves are given by $u(x, t) = U(x - ct)$, where $c > 0$ is the wave speed, $U$ is the wave profile satisfying the boundary-value problem

$$\frac{d}{dz}\left[(c - U^p)\frac{dU}{dz}\right] + U(z) = 0, \quad U(-T) = U(T), \quad U'(-T) = U'(T), \tag{2}$$

and $z = x - ct$ is the travelling wave coordinate. We are looking for smooth periodic waves $U \in H^\infty_{\rm per}(-T,T)$ satisfying (2). It is straightforward to check that periodic solutions of the second-order equation (2) correspond to level curves of the first-order invariant,

$$E = \frac{1}{2}(c - U^p)^2 \left(\frac{dU}{dz}\right)^2 + \frac{c}{2}U^2 - \frac{1}{p+2}U^{p+2} = \text{const}. \tag{3}$$

In this work, perturbations to the travelling wave are supposed to be *co-periodic*, that is, to have the same period $2T$. Separating the variables, the spectral stability problem for the perturbation $v$ to $U$ is given by $\lambda v = \partial_z L v$, where

$$L = P_0 \left(\partial_z^{-2} + c - U(z)^p\right) P_0 : \; \dot{L}^2_{\rm per}(-T,T) \to \dot{L}^2_{\rm per}(-T,T), \tag{4}$$

where $\dot{L}^2_{\rm per}(-T,T)$ denotes the space of $2T$-periodic, square-integrable functions with zero mean and $P_0 : L^2_{\rm per}(-T,T) \to \dot{L}^2_{\rm per}(-T,T)$ is the projection operator that removes the mean value of $2T$-periodic functions.

**Definition 1.** We say that the travelling wave is spectrally stable with respect to co-periodic perturbations if the spectral problem $\lambda v = \partial_z L v$ with $v \in \dot{H}^1_{\rm per}(-T,T)$ has no eigenvalues $\lambda \notin i\mathbb{R}$.

Local solutions of the Cauchy problem associated with the generalized reduced Ostrovsky equation (1) exist in the space $\dot{H}^s_{\rm per}(-T,T)$ for $s > \frac{3}{2}$ [24]. For sufficiently large initial data, the local solutions break in finite time, similar to the inviscid Burgers equation [17, 18]. However, if the initial data $u_0$ is small in a suitable norm, then local solutions are continued for all times in the same space, at least in the integrable cases $p = 1$ [11] and $p = 2$ [23].

Travelling periodic waves to the generalized reduced Ostrovsky equation (1) were recently considered in the cases $p = 1$ and $p = 2$. In these cases, travelling waves can be found in the explicit form given by the Jacobi elliptic functions after a change of coordinates to the semi-linear equations of Klein–Gordon type [3, 9]. Exploring this idea further, it was shown in [13, 14, 25] that the spectral stability of travelling periodic waves can be studied with the help of the eigenvalue problem $M\psi = \lambda \partial_z \psi$, where $M$ is a second-order Schrödinger operator.

Independently, by using higher-order conserved quantities which exist in the integrable cases $p = 1$ and $p = 2$, it was shown in [4] that the travelling periodic waves are unconstrained minimizers of energy functions in suitable function spaces with respect to *subharmonic* perturbations, that is, perturbations with a multiple period to the periodic waves. This result yields not only spectral but also nonlinear stability of the travelling wave. The nonlinear stability of periodic waves was established analytically for small-amplitude waves and shown numerically for waves of arbitrary amplitude [4].

In this paper, we give a simple argument that proves spectral stability of all smooth periodic travelling waves to the generalized reduced Ostrovsky equation (1) independently on the nonlinearity power $p$. The spectral stability of periodic waves is defined here with respect to co-periodic perturbations in the sense of Definition 1. The argument is based on convexity of the energy function

$$H(u) = -\frac{1}{2}\|\partial_x^{-1}u\|^2_{L^2_{\rm per}} - \frac{1}{(p+1)(p+2)}\int_{-T}^{T} u^{p+2}dx, \tag{5}$$

at the travelling wave profile $U$ in the energy space with fixed momentum:

$$X_q = \left\{ u \in \dot{L}^2_{\rm per}(-T,T) \cap L^{p+2}_{\rm per}(-T,T) : \quad \|u\|_{L^2_{\rm per}} = q > 0 \right\}. \tag{6}$$

Note that the self-adjoint operator $L$ given by (4) is the Hessian operator of the extended energy function $H(u) + cQ(u)$, where

$$Q(u) = \frac{1}{2}\|u\|^2_{L^2_{\rm per}} \tag{7}$$

is the momentum function.

Adopting estimates from [6, 7, 8], we prove in Section 2 that the energy-to-period map $E \to 2T$ is strictly monotonically decreasing for the family of smooth periodic solutions satisfying (2) and (3). Then, in Section 3, we use this result to prove that the self-adjoint operator $L$ given by (4) has a simple negative eigenvalue, a one-dimensional kernel, and the rest of its spectrum is bounded from below by a positive number. Finally, in Section 4, we prove that the operator $L$ constrained on the space

$$L^2_c = \left\{ u \in \dot{L}^2_{\rm per}(-T,T) : \quad \langle U, u\rangle_{L^2_{\rm per}} = 0 \right\} \tag{8}$$

is strictly positive except for the one-dimensional kernel induced by the translational symmetry. This gives convexity of $H(u)$ at $U$ in space (6). By using the standard Hamilton–Krein theorem in [15] (see also the review in [22]), this rules out existence of eigenvalues $\lambda \notin i\mathbb{R}$ of the spectral problem $\lambda v = \partial_z L v$ with $v \in \dot{H}^1_{\rm per}(-T,T)$. All together, the existence and spectral stability of smooth periodic travelling waves of the generalized reduced Ostrovsky equation (1) is summarized in the following theorem.

**Theorem 1.** *For every $p \in \mathbb{N}$ and every $c > 0$,*

*(a) there exists a smooth family of periodic solutions $U \in \dot{L}^2_{\rm per}(-T,T) \cap H^\infty_{\rm per}(-T,T)$ of equation (2), parameterized by the energy $E$ given in (3) for $E \in (0, E_c)$, with*

$$E_c := \frac{p}{2(p+2)} c^{\frac{p+2}{p}},$$

*such that the map $E \to T$ is strictly monotonically decreasing;*

*(b) for each point of the family, the operator $L$ given by (4) has a simple negative eigenvalue, a simple zero eigenvalue associated with* $\mathrm{Ker}(L) = \mathrm{span}\{\partial_z U\}$*, and the positive spectrum bounded away from zero;*

*(c) the spectral problem $\lambda v = \partial_z L v$ with $v \in \dot{H}^1_{\rm per}(-T,T)$ admits no eigenvalues $\lambda \notin i\mathbb{R}$.*

*Consequently, periodic waves of the generalized reduced Ostrovsky equation (1) are spectrally stable with respect to co-periodic perturbations in the sense of Definition 1.*

We now compare the main result to the existing literature on spectral and orbital stability of periodic waves with respect to co-periodic perturbations.

First, in comparison with the analysis in [14], the result of Theorem 1 is more general since $p \in \mathbb{N}$ is not restricted to the integrable cases $p = 1$ and $p = 2$. On a technical level, the method of proof of Theorem 1 is simple and robust, so that many unnecessary explicit computations from [14] are avoided. Indeed, in the transformation of the spectral problem $\lambda v = \partial_z L v$ to the spectral problem $M\psi = \lambda \partial_z \psi$, where $M$ is a second-order Schrödinger operator from $H^2_{\rm per}(-T,T) \to L^2_{\rm per}(-T,T)$, the zero-mean constraint is lost[1]. Consequently, the operator $M$ was found in [14] to admit two negative eigenvalues in $L^2_{\rm per}(-T,T)$, which are computed explicitly by using eigenvalues of the Schrödinger operator with finite-gap elliptic potentials. By adding three constraints for the spectral problem $M\psi = \lambda \partial_z \psi$, the authors of [14] showed that the operator $M$ becomes positive on the constrained space, again by means of symbolic computations involving explicit Jacobi elliptic functions. All these technical details become redundant in our simple approach.

Second, we mention another type of improvement of our method compared to the analysis of spectral stability of periodic waves in other nonlinear evolution equations [19, 20]. By establishing first the monotonicity of the energy-to-period map $E \to 2T$ for a smooth family of periodic waves, we give a very precise count on the number of negative eigenvalues of the operator $L$ in $L^2_{\rm per}(-T,T)$ without doing numerical approximations on solutions of the homogeneous equation $Lv = 0$. Indeed, the smooth family of periodic waves has a limit to zero solution, for which eigenvalues of $L$ in $L^2_{\rm per}(-T,T)$ are found from Fourier series. The zero eigenvalue of $L$ is double in this limit and it splits once the amplitude of the periodic wave becomes nonzero. Owing to the monotonicity of the map $E \to 2T$ and continuation arguments, the negative index of the operator $L$ remains invariant along the entire family of the smooth periodic waves. Therefore, the negative index of the operator $L$ is found for the entire family of periodic waves by a simple perturbative argument, again avoiding cumbersome analytical or approximate numerical computations.

Finally, we also mention that the spectral problem $\lambda v = \partial_z L v$ is typically difficult when it is posed in the space $L^2_{\rm per}(-T,T)$ because the mean-zero constraint is needed on $v$ in addition to the orthogonality condition $\langle U, v \rangle_{L^2_{\rm per}} = 0$. The two constraints are taken into account by studying the two-parameter family of smooth periodic waves and working with a 2-by-2 matrix of projections [1, 5]. This complication is avoided for the reduced Ostrovsky equation (1) because the spectral problem $\lambda v = \partial_z L v$ is posed in space $\dot{L}^2_{\rm per}(-T,T)$ and the only orthogonality condition $\langle U, v \rangle_{L^2_{\rm per}} = 0$ is studied with a simple scaling argument.

As a limitation of the results of Theorem 1 we mention that the nonlinear orbital stability of travelling periodic waves cannot be established for the reduced Ostrovsky equations (1) by using the energy function (5) in space (6). This is because the local solution is defined in $\dot{H}^s_{\rm per}(-T,T)$ for $s > \frac{3}{2}$ [24], whereas the energy function is defined in

[1]Note that this transformation reflects the change of coordinates owing to which the reduced Ostrovsky equations are reduced to the semi-linear equations of the Klein–Gordon type. This transformation also changes the period of the travelling periodic wave.

$\dot{L}^2_{\rm per}(-T,T) \cap L^{p+2}_{\rm per}(-T,T)$. As a result, coercivity of $H(u)$ in the space of fixed momentum (6) only controls the $L^2$ norm of time-dependent perturbations. Local well-posedness in such spaces of low regularity is questionable and so is the proof of orbital stability of the travelling periodic waves in the time evolution of the reduced Ostrovsky equations (1).

## 2 Monotonicity of the energy-to-period map

Traveling wave solutions of the reduced Ostrovsky equation (1) are solutions of the second-order differential equation (2) with $c > 0$ and $p \in \mathbb{N}$. The following lemma establishes a correspondence between the smooth periodic solutions of the second-order equation (2) and the periodic orbits around the center of an associated planar system, see Figure 1. For lighter notation we replace $U(z)$ by $u(z)$.

**Lemma 1.** *For every $c > 0$ and $p \in \mathbb{N}$ the following holds:*

- $(i)$ *A function $u$ is a smooth periodic solution of equation* (2) *if and only if* $(u, v) = (u, u')$ *is a periodic orbit of the planar differential system*

$$\begin{cases} u' = v, \\ v' = \dfrac{-u + pu^{p-1}v^2}{c - u^p}. \end{cases} \tag{9}$$

- $(ii)$ *The system (9) has a first integral given by* (3), *which we write as*

$$E(u, v) = A(u) + B(u)v^2, \tag{10}$$

*with* $A(u) = \frac{c}{2}u^2 - \frac{1}{p+2}u^{p+2}$, $B(u) = \frac{1}{2}(c - u^p)^2$.

- $(iii)$ *Every periodic orbit of system* (9) *belongs to the period annulus*[2] *of the center at the origin of* (9).

*Proof.* The assertion in $(ii)$ is proved with a straightforward calculation. To prove $(iii)$, notice that system (9) has no limit cycles in view of the existence of a first integral, and hence the periodic orbits form period annuli. A periodic orbit must surround at least one critical point. The unique critical point of system (9) is a center at the origin, corresponding to the energy level $E = 0$. In view of the presence of the singular lines

$$\{u = \pm c^{1/p}, \quad v \in \mathbb{R}\} \subset \mathbb{R}^2$$

we may conclude, applying the Poincaré-Bendixon Theorem, that the set of periodic orbits forms a punctured neighbourhood of the center, and that no other period annulus is possible.

It remains to show $(i)$. It is clear that $z \mapsto (u, v) = (u, u')$ is a solution of the differential system (9) if and only if $u$ is a smooth solution of the second-order equation (2) satisfying

[2]Recall that the largest punctured neighbourhood of a center which consists entirely of periodic orbits is called *period annulus*, see [2].

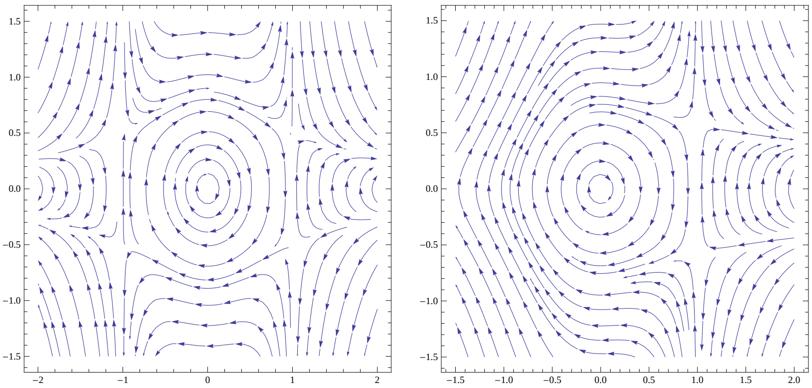


Figure 1: Phase portraits of system (9) for $p = 2$ (left) and $p = 1$ (right).

$c \neq u(z)^p$ for all $z$. We claim that $c \neq u(z)^p$ for all $z \in \mathbb{R}$ for smooth periodic solutions $u$. Indeed, let $p$ be odd for simplicity and recall that every periodic orbit in a planar system has exactly two turning points $(u, u') = (u_\pm, 0)$ per fundamental period. The turning points correspond to the maximum and minimum of the periodic solution $u$ and satisfy the equation $A(u_\pm) = E$. The graph of $A(u)$ on $\mathbb{R}^+$ has a global maximum at $u = c^{1/p}$ with

$$E_c := A(c^{1/p}) = \frac{p}{2(p+2)} c^{\frac{p+2}{p}}. \tag{11}$$

The equation $A(u) = E$ has exactly two positive solutions for $E \in (0, E_c)$, where $u = u_+$ corresponds to the smaller one inside the period annulus. At $E = E_c$, the equation $A(u) = E$ has only one positive solution given by $u_+ = c^{1/p}$. Now assume that for a smooth periodic solution $u$ there exists $z_1$ such that $u(z_1) = c^{1/p}$. Then, equation (2) implies that $u'(z_1) = \pm p^{-1/2} c^{-\frac{p-2}{2p}}$, hence the solution $(u, u')(z)$ to system (9) tends to the points $p_\pm = (c^{1/p}, \pm p^{-1/2} c^{-\frac{p-2}{2p}})$ as $z \to z_1$. Since $E(p_\pm) = E_c$ and by continuity of the first integral, this orbit lies inside the $E_c$-level set. For such an orbit, we have seen that its turning point is located at $u_+ = c^{1/p} = u(z_1)$. However, since $u'(z_1) \neq 0$, this cannot be a turning point, which leads to a contradiction. $\square$

By Lemma 1, every smooth periodic solution $u$ of the differential equation (2) corresponds to a periodic orbit $(u, v) = (u, u')$ inside the period annulus of the differential system (9). Since $E$ is a first integral of (9), this orbit lies inside some energy level curve of $E$, where $E \in (0, E_c)$. We denote this orbit by $\gamma_E$. The period of this orbit is given by

$$2T(E) = \int_{\gamma_E} \frac{du}{v}, \tag{12}$$

since $\frac{du}{dz} = v$ in view of (9). The energy levels of the first integral $E$ parameterize the set of periodic orbits inside the period annulus, and therefore this set forms a smooth family $\{\gamma_E\}_{E \in (0, E_c)}$. In view of Lemma 1, we can therefore assert that the set of smooth periodic solutions of (2) forms a smooth family $\{u_E\}_{E \in (0, E_c)}$, which is parameterized by $E$ as well.

Moreover, it ensures that the period $2T(E)$ of the periodic orbit $\gamma_E$ is equal to the period of the corresponding smooth periodic solution $u_E$ of the second-order equation (2). The main result of this section is the following proposition, from which we conclude that the energy-to-period map $E \to 2T$ for the smooth periodic solutions of equation (2) is strictly monotonically decreasing. This proves statement (a) of Theorem 1.

**Proposition 1.** *For every $c > 0$ and $p \in \mathbb{N}$ the function*

$$T : (0, E_c) \longrightarrow \mathbb{R}^+, \quad E \longmapsto T(E) = \frac{1}{2} \int_{\gamma_E} \frac{du}{v},$$

*is strictly monotonically decreasing and satisfies*

$$T'(E) = -\frac{p}{4(2+p)E} \int_{\gamma_E} \frac{u^p}{(c-u^p)} \frac{du}{v} < 0. \tag{13}$$

*Proof.* Since $A(u) + B(u)v^2 = E$ is constant along an orbit $\gamma_E$, we find that

$$2E\,T(E) = \int_{\gamma_E} B(u)v du + \int_{\gamma_E} A(u) \frac{du}{v}. \tag{14}$$

To compute the derivative of $T$ with respect to $E$, we first resolve the singularity in the second integral in equation (14). To this end, recall that the orbit $\gamma_E$ belongs to the level curve $\{A(u) + B(u)v^2 = E\}$ and therefore

$$\frac{dv}{du} = -\frac{A'(u) + B'(u)v^2}{2B(u)v} \tag{15}$$

along the orbit. Note that $B(u)$ is different from zero for $E \in (0, E_c)$. Furthermore, $BA/A'$ is bounded on $\gamma_E$. Using the fact that the integral of a total differential $d$ over the closed orbit $\gamma_E$ yields zero, we find that

$$\begin{aligned}
0 &= \int_{\gamma_E} d\left[\left(\frac{2BA}{A'}\right)(u)\,v\right] \\
&= \int_{\gamma_E} \left(\frac{2BA}{A'}\right)'(u)\,v\,du + \left(\frac{2BA}{A'}\right)(u)\,dv \\
&= \int_{\gamma_E} \left(\frac{2BA}{A'}\right)'(u)\,v\,du - \left(\frac{2BA}{A'}\frac{A'}{2B}\right)(u)\frac{du}{v} - \left(\frac{2BA}{A'}\frac{B'}{2B}\right)(u)\,v\,du \\
&= \int_{\gamma_E} \left[\left(\frac{2BA}{A'}\right)'(u) - \left(\frac{AB'}{A'}\right)(u)\right] v\,du - A(u)\frac{du}{v},
\end{aligned}$$

where we have used relation (15) in the third equality. Denoting

$$G = \left(\frac{2BA}{A'}\right)' - \frac{AB'}{A'}, \tag{16}$$

this ensures that

$$2ET(E) = \int_{\gamma_E} [B(u) + G(u)]\,v du, \tag{17}$$

where the integrand is no longer singular at the turning points, where the orbit $\gamma_E$ intersects with the horizontal axis $v = 0$[3]. Taking now the derivative of equation (17) with respect to $E$ we obtain that

$$2T(E) + 2E\,T'(E) = \int_{\gamma_E} \frac{B(u) + G(u)}{2B(u)v} du, \tag{18}$$

where we have used that

$$\frac{\partial v}{\partial E} = \frac{1}{2B(u)v}$$

in view of (10)[4]. From (18), we conclude that

$$\begin{aligned} 2T'(E) &= \frac{1}{E}\int_{\gamma_E} \left(\frac{B+G}{2B}\right)(u)\frac{du}{v} - \frac{1}{E}\int_{\gamma_E} \frac{du}{v} \\ &= \frac{1}{E}\int_{\gamma_E} \frac{1}{2B}\left(\left(\frac{2AB}{A'}\right)' - \frac{(AB)'}{A'}\right)(u)\frac{du}{v}. \end{aligned}$$

In view of the expressions for $A$ and $B$ defined in Lemma 1, further calculations show that

$$T'(E) = -\frac{p}{4(2+p)E}\int_{\gamma_E} \frac{u^p}{(c-u^p)}\frac{du}{v}. \tag{19}$$

We now need to show that $T'(E) < 0$ for every $E \in (0, E_c)$. In view of the symmetry of the vector field with respect to the horizontal axis and taking into account (10), we write (19) in the form

$$\begin{aligned} T'(E) &= -\frac{p}{2(2+p)E}\int_{u_-}^{u_+} \frac{u^p}{(c-u^p)}\sqrt{\frac{B(u)}{E-A(u)}}du \\ &= -\frac{p}{2\sqrt{2}(2+p)E}\int_{u_-}^{u_+} \frac{u^p}{\sqrt{E-A(u)}}du, \end{aligned}$$

where $u_\pm$ denote the turning points of the orbit $\gamma_E$ with $E = A(u_\pm)$, i.e. the intersections of the orbit $\gamma_E$ with the horizontal axis $v = 0$. Therefore, we find that $T'(E) < 0$ if $p$ is even. Now we show that the same property also holds when $p$ is odd. Denote

$$I_1(E) := \int_{u_-}^{0} \frac{u^p}{\sqrt{E-A(u)}}du, \quad I_2(E) := \int_{0}^{u_+} \frac{u^p}{\sqrt{E-A(u)}}du, \tag{20}$$

then

$$T'(E) = -\frac{p}{2\sqrt{2}(2+p)E}\big[I_1(E) + I_2(E)\big]. \tag{21}$$

[3] The idea for this approach of resolving the singularity is taken from [8, Lemma 4.1], where the authors prove a more general result for polynomial systems having first integrals of the form (10).

[4] Note that (18) also follows by applying Gelfand-Leray derivatives in (17) (see Theorem 26.32 on p. 526 in [16]).

We perform the change of variables $u = u_+ x$ and find that

$$\begin{aligned} I_2(E) &= \int_0^{u_+} \frac{u^p}{\sqrt{A(u_+) - A(u)}} du \\ &= \int_0^1 \frac{u_+^p x^p}{\sqrt{A(u_+) - A(u_+ x)}} u_+ dx \\ &= \sqrt{2} u_+^p \int_0^1 \frac{x^p}{\sqrt{c(1-x^2) - \frac{2u_+^p}{p+2}(1 - x^{p+2})}} dx. \end{aligned}$$

To rewrite the first integral we change variables according to $u = -|u_-|x$ and obtain

$$\begin{aligned} I_1(E) &= \int_{-|u_-|}^{0} \frac{u^p}{\sqrt{A(-|u_-|) - A(u)}} du \\ &= \int_1^0 \frac{-|u_-|^p x^p}{\sqrt{A(-|u_-|) - A(u_- x)}} (-|u_-|) dx \\ &= -\sqrt{2} |u_-|^p \int_0^1 \frac{x^p}{\sqrt{c(1-x^2) + \frac{2|u_-|^p}{p+2}(1 - x^{p+2})}} dx. \end{aligned}$$

We claim that $-u_- < u_+$. Indeed, we have that $A(u) < A(-u)$ on $(0, c^{1/p})$, since

$$A(u) - A(-u) = u^2 \left( \frac{c}{2} - \frac{1}{p+2} u^p \right) - u^2 \left( \frac{c}{2} + \frac{1}{p+2} u^p \right) = -\frac{2}{p+2} u^{p+2} < 0.$$

Moreover, $A$ is monotone on $(0, c^{1/p})$. Assuming to the contrary that $-u_- \geq u_+$, we would have that $A(-u_-) \geq A(u_+)$ and hence $A(u_+) \leq A(-u_-) < A(u_-)$, which contradicts the fact that $A(u_+) = A(u_-)$. Hence $0 < |u_-| < u_+ < c^{1/p}$, which implies that $|I_1(E)| < I_2(E)$, and therefore, $T'(E) < 0$ also in the case when $p$ is odd. The proof of Proposition 1 is complete. □

# 3 Negative index of operator $L$

We start with the limit of zero energy for the family of periodic waves, when $U = 0$ at $E = 0$. As $E \to 0$, $T(E) \to T(0) = \pi c^{1/2}$, which can be established from (12). In this limit, the operator $L$ given by (4) becomes an integral operator with constant coefficients, therefore, its spectrum can be found explicitly by the Fourier series:

$$U = 0, \quad T(0) = \pi c^{1/2}: \quad \sigma(L) = \left\{ c(1 - n^{-2}), \quad n \in \mathbb{Z} \backslash \{0\} \right\}. \tag{22}$$

For every $c > 0$, the spectrum is purely discrete and consists of double eigenvalues accumulating to the point $c$. All double eigenvalues are strictly positive except for the lowest eigenvalue, which is located at the origin. As is shown in [4] with the perturbation argument, the spectrum of $L$ for $E$ near 0 includes a simple negative eigenvalue, a simple zero eigenvalue, and the positive spectrum is bounded away from zero. We will show that this conclusion remains true for the entire family of smooth periodic waves.

Let us first prove the following.

**Lemma 2.** *For every $c > 0$, $p \in \mathbb{N}$, and $E \in (0, E_c)$, the operator $L$ given by (4) is self-adjoint and its spectrum includes a countable set of isolated eigenvalues below*

$$C_-(E) := \inf_{z \in [-T(E), T(E)]} (c - U(z)^p) > 0. \tag{23}$$

*Proof.* The self-adjoint properties of $L$ are obvious. For every $E \in (0, E_c)$, there are positive constants $C_\pm(E)$ such that

$$C_-(E) \leq c - U(z)^p \leq C_+(E) \quad \text{for every } \; z \in [-T(E), T(E)]. \tag{24}$$

The eigenvalue equation $(L - \lambda I)v = 0$ for $v \in \dot{L}^2_{\rm per}(-T, T)$ is equivalent to the spectral problem

$$P_0(c - U^p - \lambda)P_0 v = -P_0 \partial_z^{-2} P_0 v. \tag{25}$$

Under the condition $\lambda < C_-(E)$, we have $c - U^p - \lambda \geq C_-(E) - \lambda > 0$. Setting

$$w := (c - U^p - \lambda)^{1/2} P_0 v \in L^2_{\rm per}(-T, T), \quad \lambda < C_-(E), \tag{26}$$

we find that $\lambda$ is an eigenvalue of the spectral problem (25) if and only if 1 is an eigenvalue of the self-adjoint operator

$$K(\lambda) = -(c - U^p - \lambda)^{-1/2} P_0 \partial_z^{-2} P_0 (c - U^p - \lambda)^{-1/2} : L^2_{\rm per}(-T, T) \to L^2_{\rm per}(-T, T), \tag{27}$$

that is, $w = K(\lambda)w$[5]. The operator $K(\lambda)$ for every $\lambda < C_-(E)$ is a compact (Hilbert–Schmidt) operator thanks to the bounds (24) and the compactness of $P_0 \partial_z^{-2} P_0$. Consequently, the spectrum of $K(\lambda)$ in $L^2_{\rm per}(-T, T)$ for every $\lambda < C_-(E)$ is purely discrete and consists of isolated eigenvalues. Moreover, these eigenvalues are positive thanks to the positivity of $K(\lambda)$, as follows:

$$\langle K(\lambda) w, w \rangle_{L^2_{\rm per}} = \| P_0 \partial_z^{-1} P_0 (c - U^p - \lambda)^{-1/2} w \|^2_{L^2_{\rm per}} \geq 0, \quad \forall w \in L^2_{\rm per}(-T, T). \tag{28}$$

We note that

(a) $K(\lambda) \to 0^+$ as $\lambda \to -\infty$,

(b) $K'(\lambda) > 0$ for every $\lambda < C_-(E)$.

Claim (a) follows from (28) via spectral calculus:

$$\langle K(\lambda) w, w \rangle_{L^2_{\rm per}} \sim |\lambda|^{-1} \| P_0 \partial_z^{-1} P_0 w \|^2_{L^2} \quad \text{as} \quad \lambda \to -\infty.$$

Claim (b) follows from the differentiation of $K(\lambda)$,

$$\begin{aligned} \langle K'(\lambda) w, w \rangle_{L^2_{\rm per}} \;&=\; -\frac{1}{2} \langle (c - U^p - \lambda)^{-3/2} P_0 \partial_z^{-2} P_0 (c - U^p - \lambda)^{-1/2} w, w \rangle_{L^2_{\rm per}} \\ &\quad -\frac{1}{2} \langle (c - U^p - \lambda)^{-1/2} P_0 \partial_z^{-2} P_0 (c - U^p - \lambda)^{-3/2} w, w \rangle_{L^2_{\rm per}}. \end{aligned}$$

[5] This reformulation can be viewed as an adjoint version of the Birmann–Schwinger principle used in analysis of isolated eigenvalues of Schrödinger operators with rapidly decaying potentials [12].

Each term in the representation above is equivalent to the quadratic form for the positive operator $K(\lambda)$ defined in the weighted space with the strictly positive and uniformly bounded weight function $\rho(\lambda) := (c - U^p - \lambda)^{-1}$, thanks to (24). Therefore, each term in the representation above is positive.

As follows from claims (a) and (b), positive isolated eigenvalues of $K(\lambda)$ in $L^2_{\rm per}(-T,T)$ are monotonically increasing functions of $\lambda$ from the zero level as $\lambda \to -\infty$. The location and number of crossings of these eigenvalues with the unit level gives the location and number of eigenvalues $\lambda$ in the spectral problem (25). It follows from compactness of $K(\lambda)$ for $\lambda < C_-(E)$ that there exists a countable (finite or infinite) set of isolated eigenvalues of $L$ below $C_-(E)$. ☐

Next, we inspect analytical properties of eigenvectors for isolated eigenvalues below $C_-(E) > 0$ given by (23).

**Lemma 3.** *Under the condition of Lemma 2, let $\lambda_0 < C_-(E)$ be an eigenvalue of the operator $L$ given by (4). Then, $\lambda_0$ is at most double and the eigenvector $v_0$ belongs to $\dot{L}^2_{\rm per}(-T,T) \cap H^\infty_{\rm per}(-T,T)$.*

*Proof.* The eigenvector $v_0 \in \dot{L}^2_{\rm per}(-T,T)$ for the eigenvalue $\lambda_0 < C_-(E)$ satisfies the spectral problem (25) written as the integral equation

$$P_0 \partial_z^{-2} v_0 + P_0 (c - U^p - \lambda_0) v_0 = 0. \tag{29}$$

Since $U \in H^\infty_{\rm per}(-T,T)$ and $c - U^p - \lambda_0 \geq C_-(E) - \lambda_0 > 0$, we obtain by bootstrapping arguments that

$$v_0 \in L^2_{\rm per}(-T,T) \quad \Rightarrow \quad v_0 \in H^2_{\rm per}(-T,T) \quad \Rightarrow \quad v_0 \in H^4_{\rm per}(-T,T) \quad \text{and so on,}$$

which yields $v_0 \in H^\infty_{\rm per}(-T,T)$. Applying two derivatives to the integral equation (29), we obtain the equivalent differential equation for the eigenvector $v_0 \in \dot{L}^2_{\rm per}(-T,T) \cap H^\infty_{\rm per}(-T,T)$ and the eigenvalue $\lambda_0 < C_-(E)$:

$$v_0 + \frac{d^2}{dz^2} \left[ (c - U^p - \lambda_0) v_0 \right] = 0. \tag{30}$$

The second-order differential equation (30) admits at most two linearly independent solutions in $\dot{L}^2_{\rm per}(-T,T)$ and so does the integral equation (29). Since $L$ is self-adjoint, the eigenvalues of $L$ are not defective[6], and hence the multiplicity of $\lambda_0$ is at most two. ☐

We are now ready to prove the main result of this section given by the following proposition. This proposition gives part (b) of Theorem 1.

**Proposition 2.** *For every $c > 0$, $p \in \mathbb{N}$, and $E \in (0, E_c)$, the operator $L$ given by (4) has a simple negative eigenvalue, a simple zero eigenvalue, and a positive spectrum bounded away from zero.*

[6]Recall that the eigenvalue is called defective if its algebraic multiplicity exceeds its geometric multiplicity.

*Proof.* Thanks to Lemma 2, we only need to inspect the multiplicity of negative and zero eigenvalues of $L$. By Lemma 3, the zero eigenvalue $\lambda_0 = 0 < C_-(E)$ can be at most double. The first eigenvector $v_0 = \frac{dU}{dz} \in \dot{L}^2_{\rm per}(-T,T) \cap H^\infty_{\rm per}(-T,T)$ for $\lambda_0 = 0$ follows by the translational symmetry. Indeed, differentiating (2) with respect to $z$, we verify that $v_0$ satisfies the differential equation (30) with $\lambda_0 = 0$ and, equivalently, the integral equation (29) with $\lambda_0 = 0$.

Another linearly independent solution $v_1 = \partial_E U$ of the same equation (30) is obtained by differentiating (2) with respect to $E$. Here we understand the family $U(z;E)$ of smooth $2T(E)$-periodic solutions constructed in Lemma 1, where the period $2T(E)$ is given by (12). Now, we show that the second solution $v_1$ is not $2T(E)$-periodic under the condition $T'(E) < 0$ established in Proposition 1. Consequently, the zero eigenvalue $\lambda_0 = 0$ is simple. For simplicity, we assume that the family $U(z;E)$ satisfies the condition

$$U(\pm T(E);E) = 0 \tag{31}$$

at the end points, which can be easily fixed by translational symmetry. By differentiating the first boundary condition in (2) with respect to $E$, we obtain

$$\partial_E U(-T(E);E) - T'(E)\partial_z U(-T(E);E) = \partial_E U(T(E);E) + T'(E)\partial_z U(T(E);E).$$

The solution $v_1 = \partial_E U$ is not $2T(E)$-periodic if $T'(E) \neq 0$ because $\partial_z U(\pm T(E);E) \neq 0$[7]. Since $T'(E) \neq 0$ by Proposition 1, the zero eigenvalue $\lambda_0 = 0$ is simple for the entire family of smooth $T(E)$-periodic solutions.

Next, we show that the spectrum of $L$ includes at least one negative eigenvalue. Indeed, it follows from the integral version of the differential equation (2),

$$P_0\left(c - \frac{1}{p+1}U^p\right)U + P_0\partial_z^{-2}U = 0,$$

that

$$LU = -\frac{p}{p+1}P_0 U^{p+1} \quad \Rightarrow \quad \langle LU, U\rangle_{L^2_{\rm per}} = -\frac{p}{p+1}\int_{-T}^{T} U^{p+2}dz < 0. \tag{32}$$

The last inequality is obvious for even $p$. For odd $p$, the inequality follows from a general estimate, which relies on Pohozhaev–type equalities for solutions of elliptic problems, as is given by Lemma 4 below. In both cases, we have proved that $L$ has at least one negative eigenvalue for every $E \in (0, E_c)$.

Finally, the spectrum of $L$ includes at most one simple negative eigenvalue. Indeed, the family of $2T(E)$-periodic solutions is smooth with respect to the parameter $E \in (0, E_c)$ and it reduces to the zero solution as $E \to 0$. It follows from the spectrum (22) at the zero solution and the preservation of the simple zero eigenvalue with the eigenvector $\frac{dU}{dz}$ for every $E \in (0, E_c)$ that the splitting of a double zero eigenvalue for $E \neq 0$ results in appearance of at most one negative eigenvalue of $L$. Thus, there exists exactly one simple negative eigenvalue of $L$ for every $E \in (0, E_c)$. □

[7] If $\partial_z U(\pm T(E);E) = 0$ in addition to (31), the periodic solution is identically zero, which only corresponds to $E = 0$.

Here we prove the last inequality in (32).

**Lemma 4.** *For every $c > 0$, $p \in \mathbb{N}$, and $E \in (0, E_c)$, the periodic solution $U$ of (2) satisfies*

$$\int_{-T}^{T} U^{p+2} dz = \frac{2(p+1)(p+2)}{p} \|\partial_z^{-1} U\|_{L^2}^2 > 0. \tag{33}$$

*Proof.* We recall that the periodic wave $u = U$ is a critical point of $H(u)$ given by (5) in space (6). Indeed, critical points of $H(u) + cQ(u)$ in $X_q$, where $c$ is a Lagrange multiplier, satisfy the integral equation

$$P_0 \left( \partial_z^{-2} + c - \frac{1}{p+1} u^{p+1} \right) P_0 u = 0, \tag{34}$$

which corresponds to the double integral of the second-order differential equation (2) subject to the mean-zero constraints. For simplicity, let $U$ be a periodic solution of the integral equation (34) satisfying the end-point conditions (31), and the mean-zero constraint. Let us consider a scaling transformation

$$U(z) = \alpha \tilde{U}(\tilde{z}), \quad z = \alpha^{-2} \tilde{z}, \quad \alpha > 0 \tag{35}$$

which leaves $Q(U) = Q(\tilde{U})$ invariant. Note that the period of $\tilde{U}$ is now $\tilde{T} = \alpha^2 T$. Under the transformation (35), we obtain

$$H(U) = -\frac{1}{2\alpha^4} \|\partial_{\tilde{z}}^{-1} \tilde{U}\|_{L^2_{\rm per}}^2 - \frac{\alpha^p}{(p+1)(p+2)} \int_{-\tilde{T}}^{\tilde{T}} \tilde{U}^{p+2} d\tilde{z}.$$

Since $\alpha = 1$ yields the identity transformation in (35) and thus $u = U$ is a critical point of $H(u)$ at fixed $Q(u)$, the derivative of $H(U)$ with respect to $\alpha$ vanishes at $\alpha = 1$. Thanks to the end-point conditions and the mean-zero constraint, the derivative of $H(U)$ in $\alpha$ is given by

$$\frac{d}{d\alpha} H(U) = \frac{2}{\alpha^5} \|\partial_{\tilde{z}}^{-1} \tilde{U}\|_{L^2_{\rm per}}^2 - \frac{p\alpha^{p-1}}{(p+1)(p+2)} \int_{-\tilde{T}}^{\tilde{T}} \tilde{U}^{p+2} d\tilde{z}.$$

Equaling it to zero for $\alpha = 1$ yields the equality (33). □

# 4 Applications of the Hamilton–Krein theorem

Since $L$ has a simple zero eigenvalue in $\dot{L}^2_{\rm per}(-T, T)$ by Proposition 2, eigenvectors $v \in \dot{H}^1_{\rm per}(-T, T)$ of the spectral problem $\lambda v = \partial_z L v$ for nonzero eigenvalues $\lambda$ satisfy the constraint $\langle U, v \rangle_{L^2_{\rm per}} = 0$, see (8) for the definition of the constrained space $L^2_c$. Since $\partial_z$ is invertible in space $\dot{L}^2_{\rm per}(-T, T)$ and the inverse operator is bounded from $\dot{L}^2_{\rm per}(-T, T)$ to itself, we can rewrite the spectral problem $\lambda v = \partial_z L v$ in the equivalent form

$$\lambda P_0 \partial_z^{-1} P_0 v = L v, \quad v \in \dot{L}^2_{\rm per}(-T, T). \tag{36}$$

In this form, the Hamilton–Krein theorem from [15] applies directly in $L^2_c$. According to this theorem, the number of unstable eigenvalues with $\lambda \notin i\mathbb{R}$ is bounded by the number of negative eigenvalues of $L$ in the constrained space $L^2_c$. Therefore, we only need to show that the operator $L$ is positive in $L^2_c$ with only a simple zero eigenvalue due to the translational invariance in order to prove part (c) of Theorem 1. The corresponding result is given by the following proposition.

**Proposition 3.** *For every $c > 0$, $p \in \mathbb{N}$, and $E \in (0, E_c)$, the operator $L|_{L^2_c} : L^2_c \to L^2_c$, where $L$ is given by (4), has a simple zero eigenvalue and a positive spectrum bounded away from zero.*

*Proof.* According to the well-known criterion (e.g., see Lemma 1 in [14]), under the condition on $L$ proved in Proposition 2, the assertion is true if

$$\langle L^{-1}U, U\rangle_{L^2_{\text{per}}} < 0,$$

where $L^{-1}$ is defined on $U$ thanks to the Fredholm constraint $\langle v_0, U\rangle_{L^2_{\text{per}}} = 0$ since $\text{Ker}(L) = \{v_0\}$ and $v_0 = \frac{dU}{dz}$. Differentiating (2) with respect to $c$ and integrating it twice with zero mean, we obtain

$$L\partial_c U = -U.$$

Note that the family of periodic waves $U$ is smooth with respect to $c$. Therefore, we only need to prove that

$$0 < -\langle L^{-1}U, U\rangle_{L^2_{\text{per}}} = \langle \partial_c U, U\rangle_{L^2_{\text{per}}} = \frac{1}{2}\frac{d}{dc}\|U\|^2_{L^2_{\text{per}}}. \tag{37}$$

The desired result comes from a simple scaling transformation of solutions of the boundary-value problem (2):

$$U(z) = c^{1/p}\tilde{U}(\tilde{z}), \quad z = c^{1/2}\tilde{z}, \quad T = c^{1/2}\tilde{T}, \tag{38}$$

where $\tilde{U}$ is $2\tilde{T}$-periodic solution of the same boundary-value problem (2) with $c$ normalized to 1. Under the transformation (38), we obtain

$$\|U\|^2_{L^2_{\text{per}}} = c^{\frac{2}{p}+\frac{1}{2}}\|\tilde{U}\|^2_{L^2_{\text{per}}}. \tag{39}$$

Since

$$\frac{2}{p} + \frac{1}{2} > 0$$

and $\|\tilde{U}\|_{L^2_{\text{per}}}$ is $c$-independent, the representation (39) implies positivity in (37). Thus, the assertion of the proposition is proved. □

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