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# Sharp bounds for composition with quasiconformal mappings in Sobolev spaces

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## Abstract

Let  $\phi$  be a quasiconformal mapping, and let  $T_\phi$  be the composition operator which maps  $f$  to  $f \circ \phi$ . Since  $\phi$  may not be bi-Lipschitz, the composition operator need not map Sobolev spaces to themselves. The study begins with the behavior of  $T_\phi$  on  $L^p$  and  $W^{1,p}$  for  $1 < p < \infty$ . This cases are well understood but alternative proofs of some known results are provided. Using interpolation techniques it is seen that compactly supported Bessel potential functions in  $H^{s,p}$  are sent to  $H^{s,q}$  whenever  $0 < s < 1$  for appropriate values of  $q$ . The techniques used lead to sharp results and they can be applied to Besov spaces as well.

**Keywords:** Sobolev spaces, fractional smoothness, quasiconformal mappings, composition operator.

**MSC 2010:** 30C65, 46E35, 47A57.

## 1 Introduction

Given a quasiconformal homeomorphism  $\phi : \Omega \rightarrow \Omega'$  between domains in  $\mathbb{R}^n$ , we consider the composition operator  $T_\phi$  which maps every measurable function  $f : \Omega' \rightarrow \mathbb{R}$  to  $f \circ \phi$ . It is well known that  $\phi$  lies in a certain Sobolev space  $W_{loc}^{1,p}$  with  $p > n$ , that is, the space of locally  $p$ -integrable functions with locally  $p$ -integrable derivatives, and in some Hölder class  $C^s$  with  $0 < s < 1$ , i.e. for any  $K \subset \Omega$  compact  $\phi$  is bounded and continuous in  $K$  with  $|\phi(x) - \phi(y)| \leq C_K |x - y|^s$  for  $x, y \in K$ . The composition operator  $T_\phi$  is a self-map of  $W^{1,n}$ . However, since  $\phi$  may not be bi-Lipschitz, the composition operator does not necessarily map other Sobolev spaces to themselves.

A characterization of homeomorphisms which give rise to bounded composition operators is given in [Ukh93] and [Kle12].

Over the last decade, the study on the stability of the planar Calderón inverse conductivity problem raised questions on the range of  $T_\phi$ . The ground-breaking work of Astala and Päivärinta [AP06] showing uniqueness of the solution to the problem was adapted by Barceló, Faraco and Ruiz in [BFR07] to provide stability in Lipschitz domains with only Hölder a priori conditions on the conductivities. Some years later, Clop, Faraco and Ruiz weakened the a priori assumption on the conductivities to just a fractional Sobolev condition in [CFR10], allowing the method to be applied to non-continuous conductivities, and later on the regularity assumptions on the boundary of the domain where severely reduced in [FR13]. A deeper knowledge of the behavior

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of the composition operator may lead to better numerical methods for the electric impedance tomography (see [AMPS10], for instance).

We study quasiconformal mappings  $\phi$  whose Jacobian determinant  $J_\phi$  satisfies the estimates

$$\left( \int_U J_\phi(x)^a dx \right)^{\frac{1}{a}} \leq C_a, \quad \text{and} \quad \left( \int_U J_\phi(x)^{-b} dx \right)^{\frac{1}{b}} \leq C_b \quad (1.1)$$

for values  $a > 1$  and  $b > 0$ , where  $U \subset \mathbb{R}^n$  is a certain domain (open and connected set). The existence of these values for any domain  $U$  compactly contained in  $\Omega$  can be derived from quasiconformality itself, but we may have better exponents for particular mappings, and this will imply better behavior of the composition operator.

In [HK08, Theorems 1.1 and 1.2], it is shown that, under the first condition in (1.1) the composition operator  $T_\phi$  sends compactly supported  $W^{1,p}$  functions to  $W^{1,q}$  for certain couples  $n \leq q < p$ , that is, some integrability of the function and its gradient is lost. The main result of the present paper shows that this loss of integrability is common to all the Bessel potential spaces  $H^{s,p}$  with  $0 \leq s \leq 1$  and  $1 < p < \infty$  and Besov spaces  $B_{p,r}^s$  with  $0 < s < 1$ ,  $1 < p < \infty$  and  $0 < r \leq \infty$  (see Section 3 for the definitions). Note that  $H^{0,p} = L^p$  and  $H^{1,p} = W^{1,p}$ .

**Theorem 1.1.** *Let  $n \geq 2$ ,  $0 \leq s \leq 1$  and  $1 < p < \infty$ . Given a quasiconformal mapping  $\phi : \Omega \rightarrow \Omega'$  between two domains in  $\mathbb{R}^n$ , a ball  $B$  with  $2B \subset \Omega$ , a ball  $B'$  with  $\phi(2B) \subset B'$  and positive real numbers  $a, b, C_a$  and  $C_b$  satisfying (1.1) for  $U = 2B$ , let  $q$  be defined by  $\frac{1}{q} = \frac{1}{p} + \frac{1}{c} \left| \frac{s}{n} - \frac{1}{p} \right|$ , where we take  $c = a$  if  $sp \geq n$  and  $c = b$  if  $sp < n$ . If  $q > 1$ , then there exists a constant  $C$  such that*

$$\|T_\phi f\|_{H^{s,q}(B)} \leq C \|f\|_{H^{s,p}(B')} \quad (1.2)$$

and, if in addition  $s \notin \{0, 1\}$ , then

$$\|T_\phi f\|_{B_{q,r}^s(B)} \leq C \|f\|_{B_{p,r}^s(B')} \quad (1.3)$$

for every locally integrable function  $f$  and every  $0 < r \leq \infty$ , with constants not depending on  $\phi$ . If  $s \in \{0, 1\}$  and  $p = \infty$ , then (1.2) holds as well.

Previous results ([CFR10, Proposition 4.2] and [HK13, Theorem 1.2]) show that  $T_\phi$  sends compactly supported  $H^{s,q} \cap L^\infty$  functions (with  $0 < s < 1$ ) to  $H^{\beta,q}$  functions for certain  $\beta < s$ , that is, with a loss on the smoothness parameter. More precisely, the statement of the latter theorem settles that question for mappings between diagonal Besov spaces  $B_{q,q}^s \rightarrow B_{q,q}^\beta$ , and it establishes the supremum of the admissible values of  $\beta$  as  $\frac{bs}{b+1-\frac{sq}{n}} = \frac{\frac{s}{q}}{\frac{1}{q} + \frac{1}{b} \left( \frac{1}{q} - \frac{s}{n} \right)}$ .

These results can be recovered from Theorem 1.1. Indeed, by [RS96, Theorem 2.2.5], for  $0 < s < \infty$ ,  $0 < q < \infty$ ,  $0 < r, \ell \leq \infty$  and  $0 < \Theta < 1$ ,

$$\|f\|_{F_{\frac{q}{\Theta},r}^{\Theta s}} \leq C_{s,q,r,\Theta} \|f\|_{F_{q,\ell}^s} \|f\|_{L^\infty} \quad \text{and} \quad \|f\|_{B_{\frac{q}{\Theta},r}^{\Theta s}} \leq C_{s,q,r,\Theta} \|f\|_{B_{q,r}^s} \|f\|_{L^\infty}.$$

For  $sq < n$ , taking  $\beta := \frac{\frac{s}{q}}{\frac{1}{q} + \frac{1}{b} \left( \frac{1}{q} - \frac{s}{n} \right)}$  and  $p$  so that  $\frac{1}{q} = \frac{1}{p} + \frac{1}{b} \left( \frac{1}{p} - \frac{\beta}{n} \right)$  one can check that  $\frac{\beta}{s} = \frac{1/p}{1/q}$ . Thus, choosing  $\Theta = \frac{q}{p}$ ,  $r := q$  and given a function  $f \in B_{q,q}^s \cap L^\infty$  with compact support, we have that  $\|f\|_{B_{p,p}^\beta} \leq C \|f\|_{B_{q,q}^s} \|f\|_{L^\infty}$ . Using Theorem 1.1, it follows that

$$\|T_\phi f\|_{B_{q,p}^\beta(B)} \leq C_{s,p,q} \|f\|_{B_{p,p}^\beta(B)} \leq C \|f\|_{B_{q,q}^s} \|f\|_{L^\infty},$$

recovering [HK13, Theorem 1.2] by elementary embeddings and obtaining the end-point with a change on the secondary integrability index. In Bessel potential spaces the estimate reads as

$$\|T_\phi f\|_{H^{\beta,q}(B)} \leq C_{s,p,q} \|f\|_{H^{\beta,p}(B)} \leq C \|f\|_{H^{s,q}} \|f\|_{L^\infty}.$$

However, the result in [HK13, Theorem 1.1] can only be partially recovered. Indeed, condition (1.1) implies that  $\phi^{-1} \in W^{1,n(b+1)}(U)$ , arguing as in (2.2) below. Thus,  $\phi^{-1} \in C^{\frac{b}{b+1}}(U)$ . Therefore,  $\phi$  satisfies that

$$|\phi(B)| \geq C|B|^\alpha \quad (1.4)$$

for every ball  $B \subset U$ , with  $\alpha = \frac{b+1}{b}$ . According to [HK13, Theorem 1.1] this implies that given a function  $f \in B_{q,q}^s(\phi(B))$ , the composition operator maps it to  $f \circ \phi \in B_{q,q}^\beta(B)$ , where  $\beta = \frac{n}{q} - \alpha \left( \frac{n}{q} - s \right)$ . As before, one can check that this statement when  $\alpha = \frac{b+1}{b}$  is a consequence of Theorem 1.1 using the embeddings

$$\|f\|_{F_{p,r}^\beta} \leq C_{s,q,r,\ell} \|f\|_{F_{q,\ell}^s} \quad \text{and} \quad \|f\|_{B_{p,r}^\beta} \leq C_{s,q,r} \|f\|_{B_{q,r}^s},$$

which hold because  $\beta = s - \frac{n}{q} + \frac{n}{p}$  (see [Tri83, Theorem 2.7.1]). As the reader may note, there may be values of  $\alpha < \frac{b+1}{b}$  for which (1.4) holds but with no counterpart in the spirit of (1.1). The planar case illustrates how this result must be taken into account.

Indeed, given a  $K$ -quasiconformal mapping  $\phi : \mathbb{C} \rightarrow \mathbb{C}$ , we may choose

$$-\mathbf{b}_K < -b < 0 < a < \mathbf{a}_K$$

where  $\mathbf{b}_K := \frac{1}{K-1}$  and let  $\mathbf{a}_K := \frac{K}{K-1}$ . In [AIM09, Theorem 13.4.2] it is shown that  $J_\phi^a$  and  $J_\phi^{-b}$  are locally integrable, with

$$\int_B J_\phi^a \leq \frac{C_K}{1 - \frac{a}{\mathbf{a}_K}} \left( \frac{|\phi(B)|}{|B|} \right)^a,$$

and

$$\int_B J_\phi^{-b} \leq \frac{C_K}{1 - \frac{b}{\mathbf{b}_K}} \left( \frac{|B|}{|\phi(B)|} \right)^b.$$

On the other hand, although for  $b = \mathbf{b}_K$  this integral may blow up (see [AIM09, Theorem 13.2.3], for instance), still we have the end-point Hölder regularity

$$|\phi(B)| \geq C|B|^K.$$

Combining Theorem 1.1 and [HK13, Theorem 1.1] we get the following corollary.

**Corollary 1.2.** *Let  $K \geq 1$ . Let  $0 \leq s \leq 1$  and  $1 < p < \infty$ . Given domains  $\Omega, \Omega' \subset \mathbb{C}$ , a  $K$ -quasiconformal mapping  $\phi : \Omega \rightarrow \Omega'$ , a ball  $B$  with  $2B \subset \Omega$ , a ball  $B'$  with  $\phi(2B) \subset B'$ , if  $1 > \frac{1}{q} > \frac{1}{p} + \frac{1}{\mathbf{c}_K} \left| \frac{s}{2} - \frac{1}{p} \right|$  (where we take  $\mathbf{c}_K = \mathbf{a}_K$  if  $sp \geq 2$  and  $\mathbf{c}_K = \mathbf{b}_K$  if  $sp < 2$ ), there exists a constant  $C$  such that*

$$\|T_\phi f\|_{H^{s,q}(B)} \leq C \|f\|_{H^{s,p}(B')}$$

and, if  $s \notin \{0, 1\}$ , then

$$\|T_\phi f\|_{B_{q,r}^s(B)} \leq C \|f\|_{B_{p,r}^s(B')}.$$

Moreover, whenever  $sp < 2$  and  $\beta = s - (K-1) \left( \frac{2}{p} - s \right) > 0$ , we have that

$$\|T_\phi f\|_{B_{p,p}^\beta(B)} \leq C \|f\|_{B_{p,p}^s(B')}.$$

The results in Theorem 1.1 and Corollary 1.2 are sharp concerning the loss of integrability. In the critical setting ( $sp = n$ ), this fact is obvious since the composition is a self-map for these spaces. Both examples presented in [HK13] illustrate the sharpness of the subcritical setting ( $sp < n$ ),

adapting the arguments above. In Section 4 we check that this extends to the Bessel-potential subcritical setting and we adapt one of these examples to the supercritical setting ( $sp > n$ ).

In conclusion, we have seen that better integrability properties of the derivatives of quasiconformal mappings imply less loss of integrability in the composition, while better Hölder regularity implies less loss of smoothness. In general, we expect the integrability properties to be an open condition for higher dimensions as it happens in dimension 2 (see [AIM09, Section 13.4.1]). In higher dimensions, it is conjectured that one can take  $b < \mathbf{b}_K := \frac{1}{K^{(1/n-1)}-1}$  in (1.1). If this is the case, the Hölder condition of the inverse mapping of exponent  $K^{-1/(n-1)}$ , which coincides with the Sobolev embedding of the conjectured endpoint Sobolev space for the inverse mapping, is reached by quasisymmetry (see [Kos]), that is, a closed condition for the loss of smoothness. It remains to see if this extends to the supercritical case and what happens in the fractional spaces between both end-points. We believe that these techniques can be applied to the finite distortion setting described in [Kle12].

The paper is structured as follows. In Section 2 we revisit the proofs of the loss of integrability in the classical Lebesgue and Sobolev spaces. In Section 3, we derive the proof of Theorem 1.1 by a subtle interpolation of those classical results. Finally, in Section 4, we check the sharpness of the main theorem.

## 2 Classical spaces

### 2.1 Composition in Lebesgue spaces

**Definition 2.1.** Let  $1 \leq K < \infty$  and let  $\Omega, \Omega'$  be two domains in  $\mathbb{R}^n$ . We say that a homeomorphism  $\phi : \Omega \rightarrow \Omega'$  is a  $K$ -quasiconformal mapping if  $\phi \in W_{loc}^{1,1}(\Omega)$ , and the distributional Jacobian matrix  $D\phi$  satisfies the distortion inequality

$$|D\phi(x)|^n \leq K |J_\phi(x)| \quad \text{a.e. } x \in \Omega,$$

where  $J_\phi$  stands for the Jacobian determinant of  $\phi$  and  $|A| = \sup_{|h|=1} |A \cdot h|$  stands for the usual operator norm of a matrix  $A$ .

We recall some properties of  $K$ -quasiconformal mappings.

**Theorem 2.2** (see [Kos, Remark 6.1, Theorem 6.3]). Let  $\phi : \Omega \rightarrow \Omega'$  be a  $K$ -quasiconformal mapping between planar domains. Then

1.  $\phi \in W_{loc}^{1,p}$  for some  $p > n$  depending on  $K$ .
2. Either  $J_\phi > 0$  almost everywhere or  $J_\phi < 0$  almost everywhere.
3. For  $E \subset \Omega$  measurable,  $|E| = 0$  if and only if  $|\phi(E)| = 0$ .
4. Given a ball  $B \subset 2B \subset \Omega$  and a measurable set  $E \subset B$ , there are constants  $C$  and  $\alpha$  depending on  $n$  and  $K$  such that

$$\frac{|\phi(E)|}{|\phi(B)|} \leq C \left( \frac{|E|}{|B|} \right)^\alpha.$$

5.  $\phi^{-1}$  is a  $K^{n-1}$ -quasiconformal mapping.
6. Moreover, given  $f \in L^1(\Omega')$ , we have that  $f \circ \phi |J_\phi| \in L^1(\Omega)$  with

$$\int_{\Omega} f(\phi(z)) |J_\phi(z)| \, dm(z) = \int_{\Omega'} f(w) \, dm(w). \quad (2.1)$$

From now on we assume  $J_\phi > 0$  almost everywhere.

**Corollary 2.3.** *There exist  $a = a_K > 1$  and  $b = b_K > 0$  such that for every  $K$ -quasiconformal mapping  $\phi : \Omega \rightarrow \Omega'$  and every measurable set  $U$  contained in a ball  $B$  with  $2B \subset \Omega$ , there exist constants  $C_a, C_b$  depending on  $K, a$  (resp.  $b$ ),  $|B|$  and  $|\phi(B)|$ , such that (1.1) holds.*

*Proof.* Take  $a_K = \frac{p}{n}$  from the first property in the theorem above.

$$\int_B J_\phi(x)^{a_K} dm(x) \leq \|D\phi\|_{L^p(B)} \leq C_a.$$

Take  $b_K = a_{K^{n-1}} - 1$  and the last item above:

$$\int_B J_\phi(x)^{-b_K} dm(x) = \int_B J_\phi(x) J_\phi(x)^{-b_K-1} dm(x) = \int_{\phi(B)} J_{\phi^{-1}}(x)^{b_K+1} dm(x) \leq C_b. \quad (2.2)$$

By the fifth property, the corollary follows.  $\square$

Next we see how does interact the composition operator with Lebesgue spaces.

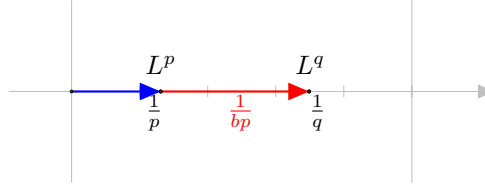


Figure 2.1: Action of the composition operator described in Lemma 2.4. The gap (loss of integrability) is proportional to  $\frac{1}{p}$ .

**Lemma 2.4.** *Let  $n \geq 2$ ,  $K \geq 1$  and  $0 < p \leq \infty$ . Given a  $K$ -quasiconformal mapping  $\phi : \Omega \rightarrow \Omega'$  between two domains in  $\mathbb{R}^n$  and a function  $f \in L^p(\Omega')$ , let  $b$  and  $C_b$  satisfy (1.1) for  $U = \Omega$  and let  $q$  be defined as  $\frac{1}{q} = \frac{1}{p} + \frac{1}{b} \frac{1}{p}$ . We have that*

$$\|f \circ \phi\|_{L^q(\Omega)} \leq C_b^{\frac{1}{p}} \|f\|_{L^p(\Omega')}.$$

*Proof.* The case  $p = \infty$  being trivial, let us assume that  $p < \infty$ . By the Hölder inequality

$$\begin{aligned} \|f \circ \phi\|_{L^q(\Omega)} &= \left\| f \circ \phi J_\phi^{\frac{1}{p}} J_\phi^{-\frac{1}{p}} \right\|_{L^q(\Omega)} \\ &\leq \left( \int_\Omega |f \circ \phi(x)|^p J_\phi(x) dm(x) \right)^{\frac{1}{p}} \left( \int_\Omega J_\phi(x)^{-b} dm(x) \right)^{\frac{1}{b} \frac{1}{p}}. \end{aligned}$$

Note that  $\phi$  acts as a change of variables by Theorem 2.2, so

$$\|f \circ \phi\|_{L^q(\Omega)} \leq C_b^{\frac{1}{p}} \|f\|_{L^p(\phi(\Omega))}.$$

$\square$

Let us write  $B_1$  for the ball with radius 1 centered at the origin. There exists a bump function  $\varphi \in C_c^\infty(2B_1)$ , such that  $\chi_{B_1} \leq \varphi \leq \chi_{2B_1}$ . For every ball  $B$  we define  $\varphi_B$  by precomposing  $\varphi$  with an appropriate affine mapping, so that  $\chi_B \leq \varphi_B \leq \chi_{2B}$ .

**Definition 2.5.** Let  $\phi : \Omega \rightarrow \Omega'$  be a  $K$ -quasiconformal mapping. Then, the composition operator  $T_\phi : L_{loc}^{1+\frac{1}{bK}}(\Omega') \rightarrow L_{loc}^1(\Omega)$  is defined as

$$T_\phi f := f \circ \phi.$$

Moreover, for every ball  $B$  with  $2B \subset \Omega$ , we define

$$T_\phi^B f := \varphi_B \cdot f \circ \phi.$$

**Corollary 2.6.** Let  $K \geq 1$  and  $0 < p \leq \infty$ . Suppose that  $\phi : \Omega \rightarrow \Omega'$  is a  $K$ -quasiconformal mapping and  $f \in L^p(\Omega')$ . Let  $B$  be a ball with  $2B \subset \Omega$ , and let  $b$  and  $C_b$  satisfy (1.1) for  $U = 2B$ . If  $\frac{1}{q} = \frac{1}{p} + \frac{1}{pb}$ , then

$$\|T_\phi^B\|_{L^p \rightarrow L^q} \leq C_b^{\frac{1}{p}}.$$

## 2.2 Composition in classical Sobolev spaces

In this section we study the behavior of the composition operator in Sobolev spaces  $W^{1,p}$  with  $1 \leq p \leq \infty$ . The following is partially contained in [HK08, Theorems 1.1 and 1.2]. In that case only the critical and supercritical cases are covered.

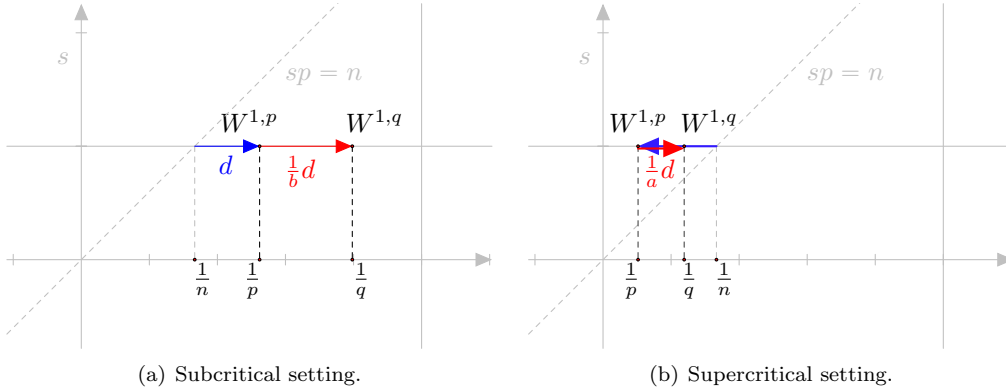


Figure 2.2: Action of the composition operator described in Theorem 2.7. The gap (loss of integrability) is proportional to  $d$ , i.e. to the horizontal distance to the critical line (which is the homogeneity of the space seminorm under rescaling divided by  $n$ ).

Next we present a particular case of [Kle12, Theorem 1.3], which we prove for the sake of completeness and to keep control on the constants.

**Theorem 2.7.** Let  $n \geq 2$ ,  $K \geq 1$  and  $1 < p \leq \infty$ . Given a  $K$ -quasiconformal mapping  $\phi : \Omega \rightarrow \Omega'$  between two domains in  $\mathbb{R}^n$  and a function  $f \in W^{1,p}(\Omega')$ , let  $a$ ,  $b$ ,  $C_a$  and  $C_b$  be real numbers satisfying (1.1) for  $U = \Omega$  and let  $q$  be defined as follows:

- If  $n < p \leq \infty$ ,  $\frac{1}{q} = \frac{1}{p} + \frac{1}{a} \left( \frac{1}{n} - \frac{1}{p} \right)$ .
- If  $p = n$ ,  $\frac{1}{q} = \frac{1}{p}$ .
- If  $1 + \frac{n-1}{nb+1} \leq p < n$ ,  $\frac{1}{q} = \frac{1}{p} + \frac{1}{b} \left( \frac{1}{p} - \frac{1}{n} \right)$

(see Figure 2.2). We have that

$$\|\nabla(f \circ \phi)\|_{L^q(\Omega)} \leq K^{\frac{1}{n}} C_c^{\left|\frac{1}{n} - \frac{1}{p}\right|} \|\nabla f\|_{L^p(\Omega')}, \quad (2.3)$$

where  $c$  stands for  $a$  (resp.  $b$  or  $1$  with  $C_1 = 1$ ) if we are in the supercritical (resp. subcritical or critical) case.

Note that  $1 \geq \frac{1}{q}$  is granted in the supercritical and the critical cases, while in the subcritical it is equivalent to  $p \geq 1 + \frac{n-1}{nb+1}$ .

*Proof.* By the chain rule (use that  $\frac{|D\phi|^n}{J_\phi} \in L^\infty$  and apply [Kle12, Theorem 1.3] for instance) we have that

$$\|\nabla(f \circ \phi)\|_{L^q(\Omega)} \leq \left( \int_{\Omega} |\nabla f(\phi(z))|^q |D\phi(z)|^q \right)^{\frac{1}{q}} \leq K^{\frac{1}{n}} \left( \int_{\Omega} |\nabla f(\phi(z))|^q |J_\phi(z)|^{\frac{q}{n}} \right)^{\frac{1}{q}}.$$

In case  $p = q = n$ , by (2.1) we have shown that

$$\|\nabla(f \circ \phi)\|_{L^p(\Omega)} \leq K^{\frac{1}{n}} \|\nabla f\|_{L^p(\phi(\Omega))}.$$

We need to study the supercritical and the subcritical cases. Let  $1 < p < \infty$  with  $p \neq n$  (the case  $p = \infty$  follows by an analogous reasoning). The Hölder inequality implies that

$$\begin{aligned} \|\nabla(f \circ \phi)\|_{L^q(\Omega)} &\leq K^{\frac{1}{n}} \left( \int_{\Omega} |\nabla f(\phi(z))|^p |J_\phi(z)| \right)^{\frac{1}{p}} \left( \int_{\Omega} J_\phi(z)^{\left(\frac{q}{n} - \frac{q}{p}\right) \frac{p}{p-q}} \right)^{\frac{p-q}{pq}} \\ &\leq K^{\frac{1}{n}} \left( \int_{\Omega} J_\phi(z)^{\frac{p-n}{n} \frac{q}{p-q}} \right)^{\frac{p-q}{pq}} \|\nabla f\|_{L^p(\phi(\Omega))}, \end{aligned}$$

and it remains to control the integral of the Jacobian. For (1.1) to apply we need that  $\frac{p-n}{n} \frac{q}{p-q}$  equals  $a$  or  $-b$  depending on the setting.

First we study the supercritical case,  $p > n$ . Since  $p > q$ , we have that  $\frac{p-n}{n} \frac{q}{p-q} > 0$ , and we only need to check that

$$\frac{p-n}{n} \frac{q}{p-q} = a,$$

that is,  $\frac{p-q}{pq} = \frac{p-n}{nap}$ , which is equivalent to our first assumption

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{a} \left( \frac{1}{n} - \frac{1}{p} \right).$$

Next we study the case  $p < n$  and  $p > q$ . Here  $\frac{p-n}{n} \frac{q}{p-q} < 0$ , and we will check the other equality, that is  $\frac{p-n}{n} \frac{q}{p-q} = -b$ , or, changing signs,

$$\frac{n-p}{n} \frac{q}{p-q} = b,$$

which is equivalent to  $\frac{p-q}{pq} = \frac{n-p}{nbp}$  and finally, to

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{b} \left( \frac{1}{p} - \frac{1}{n} \right).$$

□



**Corollary 2.8.** *Let  $K \geq 1$  and  $1 < p \leq \infty$ . Given a  $K$ -quasiconformal mapping  $\phi : \Omega \rightarrow \Omega'$ , let  $B$  be a ball with  $2B \subset \Omega$ , let  $B'$  a ball with  $\phi(2B) \subset B'$ , and let  $b > 0$  and  $C_b$  satisfy (1.1) for  $U = 2B$ . Let  $q$  be defined as before. If  $p > n$ , then*

$$\|T_\phi^B\|_{W^{1,p} \rightarrow W^{1,q}} \leq C_{K,n,p,q,|B|,|B'|} \left(1 + C_a^{\frac{1}{n}-\frac{1}{p}}\right).$$

If  $p = n$ , then

$$\|T_\phi^B\|_{W^{1,n} \rightarrow W^{1,n}} \leq C_{K,n,b,C_b,|B|,|B'|}.$$

If  $1 + \frac{n-1}{nb+1} \leq p < n$ , then

$$\|T_\phi^B\|_{W^{1,p} \rightarrow W^{1,q}} \leq C_{K,n,p,q,|B|,|B'|} C_b^{\frac{1}{p}-\frac{1}{n}}.$$

*Proof.* We will assume that  $B = B' = B_1$ , which can be achieved by translation and rescaling. Let  $f \in W^{1,p}$ . First take  $n > p > q$ , and let  $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$ . By Leibniz' rule and Hölder's inequality,

$$\begin{aligned} \|\nabla(\varphi \cdot f \circ \phi)\|_{L^q(2B_1)} &\leq \|\nabla \varphi \cdot f \circ \phi\|_{L^q(2B_1)} + \|\varphi \cdot \nabla(f \circ \phi)\|_{L^q(2B_1)} \\ &\leq \|\nabla \varphi\|_{L^n} \|f \circ \phi\|_{L^{q^*}(2B_1)} + \|\varphi\|_{L^\infty} \|\nabla(f \circ \phi)\|_{L^q(2B_1)}. \end{aligned} \quad (2.4)$$

Since  $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n} = \frac{1}{p^*} + \frac{1}{b} \frac{1}{p^*}$ , Lemma 2.4 and the Gagliardo-Nirenberg-Sobolev inequality (see [Eva98, Theorem 5.6.1/2], for instance) give

$$\|f \circ \phi\|_{L^{q^*}(2B_1)} \leq C_b^{\frac{1}{p^*}} \|f\|_{L^{p^*}(\phi(2B_1))} \leq C_b^{\frac{1}{p}-\frac{1}{n}} \|f\|_{L^{p^*}(B_1)} \leq C_{p,n} C_b^{\frac{1}{p}-\frac{1}{n}} \|f\|_{W^{1,p}(B_1)}.$$

Theorem 2.7 gives

$$\|\nabla(f \circ \phi)\|_{L^q(2B_1)} \leq K^{\frac{1}{n}} C_b^{\frac{1}{p}-\frac{1}{n}} \|\nabla f\|_{L^p(\phi(2B_1))}.$$

Back to (2.4), we have shown that

$$\|\nabla(\varphi_B \cdot f \circ \phi)\|_{L^q(2B_1)} \leq C_{K,n,p} C_b^{\frac{1}{p}-\frac{1}{n}} \|f\|_{W^{1,p}}.$$

By the Poincaré inequality,

$$\|\varphi \cdot f \circ \phi\|_{W^{1,q}} \leq C \|\nabla(\varphi \cdot f \circ \phi)\|_{L^q(2B_1)} \leq C_{K,n,p} C_b^{\frac{1}{p}-\frac{1}{n}} \|f\|_{W^{1,p}}.$$

The critical case,  $p = q = n$ , instead of (2.4) we use

$$\|\nabla(\varphi \cdot f \circ \phi)\|_{L^n(2B_1)} \leq \|\nabla \varphi\|_{L^\infty} \|f \circ \phi\|_{L^n(2B_1)} + \|\varphi\|_{L^\infty} \|\nabla(f \circ \phi)\|_{L^n(2B_1)}.$$

Theorem 2.7 gives

$$\|\nabla(f \circ \phi)\|_{L^n(2B_1)} \leq K^{\frac{1}{n}} \|\nabla f\|_{L^n(\phi(2B_1))} \leq K^{\frac{1}{n}} \|\nabla f\|_{L^n}$$

and using Lemma 2.4 and the continuous embedding  $W^{1,n}(B_1) \subset L^{\frac{(b+1)n}{b}}(B_1)$  we obtain

$$\|f \circ \phi\|_{L^n(2B_1)} \leq C_b^{\frac{b}{b+1} \frac{1}{n}} \|f\|_{L^{\frac{(b+1)n}{b}}(\phi(2B_1))} \leq C_b^{\frac{b}{b+1} \frac{1}{n}} \|f\|_{L^{\frac{(b+1)n}{b}}(B_1)} \leq C_{b,n,C_b} \|f\|_{W^{1,n}}.$$

Using again the Poincaré inequality we end the proof.

In the supercritical case, we can use the algebra structure of  $W^{1,q}(2B_1)$  (see [RS96, Section 4.6.4], for instance) to get

$$\|\varphi \cdot f \circ \phi\|_{W^{1,q}} \leq C_{n,q} \|\varphi\|_{W^{1,q}(2B_1)} \|f \circ \phi\|_{W^{1,q}(2B_1)} = C_{n,q} \left( \|f \circ \phi\|_{L^q(2B_1)} + \|\nabla(f \circ \phi)\|_{L^q(2B_1)} \right).$$

By Theorem 2.7 we obtain that

$$\begin{aligned} \|\varphi \cdot f \circ \phi\|_{W^{1,q}} &\leq C_{n,q} \left( \|f\|_{L^\infty(B_1)} + K^{\frac{1}{n}} C_a^{\frac{1}{n} - \frac{1}{p}} \|\nabla f\|_{L^p(\phi(2B_1))} \right) \\ &\leq C_{K,n,p,q} \left( 1 + C_a^{\frac{1}{n} - \frac{1}{p}} \right) \|f\|_{W^{1,p}}. \end{aligned}$$

□

**Remark 2.9.** *The proof given above gives constants which blow up when  $p \rightarrow n$ , because they depend on Gagliardo-Nirenberg and Morrey inequalities. However, one can show using the (exact) real interpolation functor with the critical case (see [BL76, Theorems 4.1.2 (5) and 3.1.2]) that there exists a constant  $C = C_{K,n,p,b,C_b,a,C_a,|B|,|B'|}$  which depends continuously on  $p \in \left[ \frac{nb+n}{nb+1}, \infty \right)$  such that*

$$\|T_\phi^B\|_{W^{1,p} \rightarrow W^{1,q}} \leq C.$$

### 3 Fractional Spaces

#### 3.1 A note on interpolation

When dealing with fractional smoothness there are two families which are studied the most, the Besov and the Triebel-Lizorkin scale, which include a number of classical function spaces, as the Hölder-Zygmund class, the Bessel potentials, the Sobolev traces... (see [Tri83, Section 2.3.5]). We recall their definition:

**Definition 3.1.** *Let  $\Phi(\mathbb{R}^n)$  be the collection of all the families  $\Psi = \{\psi_j\}_{j=0}^\infty \subset C_c^\infty(\mathbb{R}^n)$  such that*

$$\begin{cases} \text{supp } \psi_0 \subset \mathbb{D}(0, 2), \\ \text{supp } \psi_j \subset \mathbb{D}(0, 2^{j+1}) \setminus \mathbb{D}(0, 2^{j-1}) \quad \text{if } j \geq 1, \end{cases}$$

*for all multiindex  $\alpha \in \mathbb{N}^n$  there exists a constant  $c_\alpha$  such that*

$$\|D^\alpha \psi_j\|_\infty \leq \frac{c_\alpha}{2^{j|\alpha|}} \quad \text{for every } j \geq 0$$

*and*

$$\sum_{j=0}^\infty \psi_j(x) = 1 \quad \text{for every } x \in \mathbb{R}^n.$$

**Definition 3.2.** *Given any Schwartz function  $\psi \in \mathcal{S}(\mathbb{R}^n)$ , its Fourier transform is*

$$F\psi(\zeta) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \zeta} \psi(x) dm(x).$$

*This notion extends to the tempered distributions  $\mathcal{S}(\mathbb{R}^n)'$  by duality (see [Gra08, Definition 2.3.7]).*

**Definition 3.3.** Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$ ,  $1 \leq q \leq \infty$  and  $\Psi \in \Phi(\mathbb{R}^n)$ . For any tempered distribution  $f \in \mathcal{S}'(\mathbb{R}^n)$  we define its non-homogeneous Besov norm

$$\|f\|_{B_{p,q}^s}^\Psi = \left\| \left\{ 2^{sj} \left\| (\psi_j \hat{f})^\sim \right\|_{L^p} \right\} \right\|_{l^q},$$

and we call  $B_{p,q}^s \subset \mathcal{S}'$  to the set of tempered distributions such that this norm is finite.

Let  $s \in \mathbb{R}$ ,  $1 \leq p < \infty$ ,  $1 \leq q \leq \infty$  and  $\Psi \in \Phi(\mathbb{R}^n)$ . For any tempered distribution  $f \in \mathcal{S}'(\mathbb{R}^n)$  we define its non-homogeneous Triebel-Lizorkin norm

$$\|f\|_{F_{p,q}^s}^\Psi = \left\| \left\| \left\{ 2^{sj} \left\| (\psi_j \hat{f})^\sim \right\|_{L^p} \right\} \right\|_{l^q} \right\|_{L^p},$$

and we call  $F_{p,q}^s \subset \mathcal{S}'$  to the set of tempered distributions such that this norm is finite.

These norms are equivalent for different choices of  $\Psi$ . Usually one works with radial  $\psi_j$  and such that  $\psi_{j+1}(x) = \psi_j(x/2)$  for  $j \geq 1$ . Of course we will omit  $\Psi$  in our notation since it plays no role (see [Tri83, Section 2.3]).

We will work with the so-called Bessel potential spaces

$$H^{s,p} := F_{p,2}^s.$$

For  $s \in \mathbb{N}$  and  $1 < p < \infty$  we have that  $W^{s,p} = H^{s,p}$  and  $L^p = H^{0,p}$ .

We recall a result from interpolation theory.

**Theorem 3.4** (see [BL76, Theorem 6.4.5 - (7)]). Let  $0 < s < 1$ , let  $1 < p_0, p_1 < \infty$ . Let

$$\frac{1}{p} = \frac{1-s}{p_0} + \frac{s}{p_1}.$$

Then

$$(L^{p_0}, W^{1,p_1})_{[s]} = H^{s,p},$$

where the interpolation space  $(\cdot, \cdot)_{[s]}$  is defined as in [BL76, Chapter 4].

Next, the interpolation property.

**Theorem 3.5** (See [BL76, Theorem 4.1.2]). Let  $0 < s < 1$ , let  $1 < p_0, p_1, q_0, q_1 < \infty$ . Let

$$\frac{1}{p} = \frac{1-s}{p_0} + \frac{s}{p_1}, \quad \frac{1}{q} = \frac{1-s}{q_0} + \frac{s}{q_1}.$$

Then, given a linear operator  $T : L^{p_0} + W^{1,p_1} \rightarrow L^{q_0} + W^{1,q_1}$ , it satisfies that

$$\|T\|_{H^{s,p}, H^{s,q}} \leq \|T\|_{L^{p_0}, L^{q_0}}^{1-s} \|T\|_{W^{1,p_1}, W^{1,q_1}}^s.$$

### 3.2 Composition in Bessel potential spaces

**Proposition 3.6.** Let  $n \geq 2$ ,  $K \geq 1$ ,  $0 \leq s \leq 1$  and  $1 + \frac{n-1}{nb+1} \leq p < \frac{n}{s}$ . Given a  $K$ -quasiconformal mapping  $\phi : \Omega \rightarrow \Omega'$  between two domains in  $\mathbb{R}^n$ , with  $2B_1 \subset \Omega$  and  $\phi(2B_1) \subset B_1$ , and real numbers  $b$  and  $C_b$  satisfying (1.1) for  $U = 2B$ , if  $\frac{1}{q} := \frac{1}{p} + \frac{1}{b} \left( \frac{1}{p} - \frac{s}{n} \right) < 1$ , then there exists a constant  $C = C_{K,n,s,p,q}$  such that

$$\left\| T_\phi^{B_1} \right\|_{H^{s,p} \rightarrow H^{s,q}} \leq CC_b^{\frac{1}{p} - \frac{s}{n}}. \quad (3.1)$$

*Proof.* Since  $p > q$ , we will argue as follows: we will find indices  $p_\lambda, q_\lambda$  for  $\lambda \in \{0, 1\}$ , so that

$$\frac{1}{p} = \frac{1-s}{p_0} + \frac{s}{p_1} \quad (3.2)$$

and

$$\frac{1}{q} = \frac{1-s}{q_0} + \frac{s}{q_1}. \quad (3.3)$$

Moreover, we will need that  $(p_0, q_0)$  satisfy the condition in Corollary 2.6 and  $(p_1, q_1)$  satisfy the condition in Corollary 2.8. Then we will use interpolation.

We define  $p_0, p_1, q_0$  and  $q_1$  by the following relations:

$$\frac{1 - \frac{1}{p}}{n-s} = \frac{1 - \frac{1}{p_\lambda}}{n-\lambda} \quad \text{and} \quad \frac{1 - \frac{1}{q}}{n-s} = \frac{1 - \frac{1}{q_\lambda}}{n-\lambda} \quad (3.4)$$

for  $\lambda \in \{0, 1\}$  (see Figure 3.1). It follows immediately that  $\frac{1}{p_\lambda} < \frac{1}{q_\lambda} < 1$ . It is a routine to check that (3.3) and (3.2) are satisfied. Indeed, using (3.4) we have that

$$1 - \frac{1}{p} = (1-s) \left(1 - \frac{1}{p}\right) + s \left(1 - \frac{1}{p}\right) = (1-s) \left(1 - \frac{s}{n}\right) \left(1 - \frac{1}{p_0}\right) + s \left(1 + \frac{1-s}{n-1}\right) \left(1 - \frac{1}{p_1}\right).$$

Rearranging and using (3.4) again, we get

$$\begin{aligned} 1 - \frac{1}{p} &= (1-s) \left(1 - \frac{1}{p_0}\right) + s \left(1 - \frac{1}{p_1}\right) + s(1-s) \left(-\frac{1 - \frac{1}{p_0}}{n} + \frac{1 - \frac{1}{p_1}}{n-1}\right) \\ &= (1-s) \left(1 - \frac{1}{p_0}\right) + s \left(1 - \frac{1}{p_1}\right), \end{aligned}$$

showing (3.2). The counterpart (3.3) is proven using the same reasoning. Next we see that

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{b} \left(\frac{1}{p} - \frac{s}{n}\right) \implies \frac{1}{q_\lambda} - \frac{1}{p_\lambda} = \frac{1}{b} \left(\frac{1}{p_\lambda} - \frac{\lambda}{n}\right).$$

So we assume the condition and plugging (3.4) we get

$$\begin{aligned} \frac{1}{q_\lambda} - \frac{1}{p_\lambda} &= \left(1 - \frac{1}{p_\lambda}\right) - \left(1 - \frac{1}{q_\lambda}\right) = \frac{n-\lambda}{n-s} \left(\frac{1}{q} - \frac{1}{p}\right) \\ &= \frac{n-\lambda}{n-s} \frac{1}{b} \left(\frac{1}{p} - \frac{s}{n}\right) = \frac{1}{b} \left(\frac{n-\lambda}{n-s} \left(\frac{1}{p} - 1\right) + \frac{n-\lambda}{n-s} - \frac{(n-\lambda)s}{(n-s)n}\right). \end{aligned}$$

Again by (3.4) we obtain

$$\begin{aligned} \frac{1}{q_\lambda} - \frac{1}{p_\lambda} &= \frac{1}{b} \left(\frac{1}{p_\lambda} - 1 + \frac{n-\lambda}{n-s} - \frac{(n-\lambda)s}{(n-s)n}\right) = \frac{1}{b} \left(\frac{1}{p_\lambda} + \frac{-n^2 + ns + n^2 - n\lambda - ns + \lambda s}{(n-s)n}\right) \\ &= \frac{1}{b} \left(\frac{1}{p_\lambda} - \frac{\lambda}{n}\right). \end{aligned}$$

Thus, we can use again both Corollary 2.6 to establish

$$\left\|T_\phi^{B_1}\right\|_{L^{p_0} \rightarrow L^{q_0}} \leq C_b^{\frac{1}{p_0}} \quad (3.5)$$

and Corollary 2.8 to find that

$$\left\| T_\phi^{B_1} \right\|_{W^{1,p_1} \rightarrow W^{1,q_1}} \leq C_{K,n,p,q} C_b^{\frac{1}{p_1} - \frac{1}{n}}. \quad (3.6)$$

Estimate (3.1) follows by the interpolation property in Theorem 3.5. Namely,

$$\left\| T_\phi^{B_1} \right\|_{H^{s,p} \rightarrow H^{s,q}} \leq \left\| T_\phi^{B_1} \right\|_{L^{p_0} \rightarrow L^{q_0}}^{1-s} \left\| T_\phi^{B_1} \right\|_{W^{1,p_1} \rightarrow W^{1,q_1}}^s \leq C_{K,n,p_1,q_1}^s C_b^{\frac{1}{p_0}(1-s) + \left(\frac{1}{p_1} - \frac{1}{n}\right)s}.$$

□

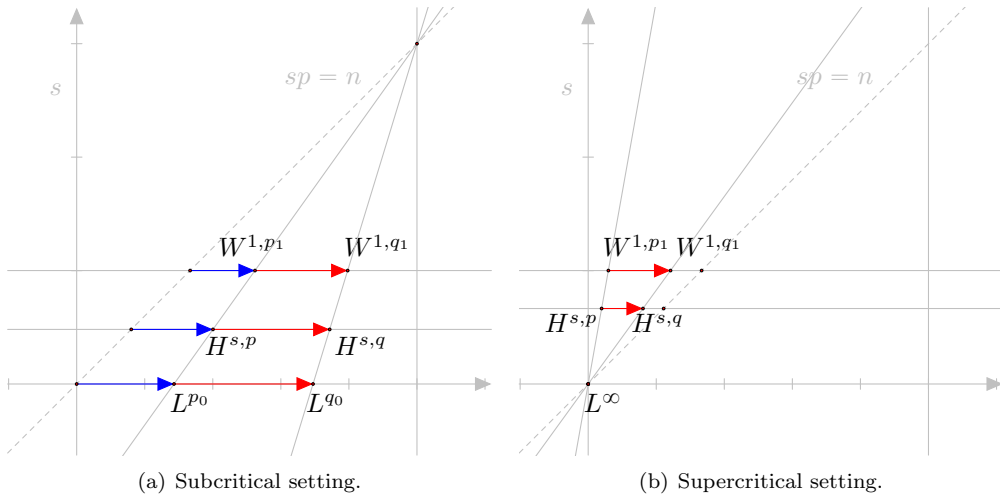


Figure 3.1: Interpolation in the proofs of Propositions 3.6 and 3.9.

Proposition 3.6 is proven by an interpolation argument, which is summarized in Figure 3.1 (a). The idea behind is just the Thales Theorem: horizontal lines correspond to a given smoothness parameter  $s$ . The homogeneity in the critical line is 0, and therefore, any pair of straight lines which intersect at the critical line will be suitable for interpolating the results in Lemma 2.4 and Theorem 2.7. When using lines intersecting at  $(1, n)$ , we make sure that  $q_0 > 1$  and  $q_1 > 1$ , which is necessary for the interpolation to make sense.

Next we study the critical case. This case was studied previously in [KYZ11, Theorem 1.1], and we provide an alternative proof by means of interpolation. Here the interpolation limits according to the previous argument should be  $L^\infty$  and  $W^{1,n}$ . But this is not a complex interpolation couple, so we will get uniform bounds by an interpolation of  $L^{q_0}$  and  $W^{1,n}$  and then we will use Lemma 3.7 below to end the proof.

**Lemma 3.7.** *Let  $0 < s < \infty$  and let  $1 < p_0 < \infty$ . Let  $f \in \mathcal{S}'$  satisfy that for every  $\varepsilon$ , there exists  $p$  with  $|p - p_0| < \varepsilon$  such that*

$$\|f\|_{H^{s,p}} \leq 1.$$

*Then,*

$$\|f\|_{H^{s,p_0}} \leq C,$$

*with  $C$  depending on  $s$  and  $p_0$ .*

*Proof.* By the lifting property (see [Tri83, Theorem 2.3.8]), the linear map  $I_s = F^{-1}(1 + |x|^2)^{\frac{s}{2}}F$  induces an isomorphism between  $H^{s,r}$  and  $L^r$ . By interpolation, the norm of this map varies continuously on  $1 < r < \infty$ . Let  $\varepsilon > 0$  and let  $p$  with  $|p - p_0| < \varepsilon$  such that

$$\|f\|_{H^{s,p}} \leq 1.$$

Thus,

$$\|f\|_{H^{s,p_0}} \leq C_{s,p_0} \|I_s f\|_{L^{p_0}} = C_{s,p_0} (\|I_s f\|_{L^{p_0}} - \|I_s f\|_{L^p}) + C_{s,p_0} \|I_s f\|_{L^p}.$$

The last term in the right-hand side above is uniformly bounded by the continuity of  $I_s$  and our hypothesis. Namely, for  $\varepsilon$  small enough, we can grant

$$\|I_s f\|_{L^p} \leq C_{s,p_0} \|f\|_{H^{s,p}} \leq C_{s,p_0}.$$

On the other hand, by the monotone convergence theorem

$$\int |I_s f|^p = \int_{|I_s f| \geq 1} |I_s f|^p + \int_{|I_s f| < 1} |I_s f|^p \xrightarrow{p \rightarrow p_0} \int_{|I_s f| \geq 1} |I_s f|^{p_0} + \int_{|I_s f| < 1} |I_s f|^{p_0} = \int |I_s f|^{p_0}.$$

□

**Proposition 3.8.** *Let  $n \geq 2$ ,  $K \geq 1$ ,  $0 \leq s \leq 1$ . Given a  $K$ -quasiconformal mapping  $\phi : \Omega \rightarrow \Omega'$  between two domains in  $\mathbb{R}^n$ ,  $\phi : \Omega \rightarrow \Omega'$  with  $2B_1 \subset \Omega$  and  $\phi(2B_1) \subset B_1$ , and real numbers  $b$  and  $C_b$  satisfying (1.1) for  $U = 2B$ , there exists a constant  $C = C_{K,n,s,b,C_b}$  such that*

$$\left\| T_\phi^{B_1} \right\|_{H^{s, \frac{n}{s}} \rightarrow H^{s, \frac{n}{s}}} \leq C_{K,n,s,b,C_b}. \quad (3.7)$$

*Proof.* Let  $q$  be such that  $0 < \frac{1}{q} - \frac{s}{n} < \varepsilon_0$  with  $\varepsilon_0$  small enough. Let us define  $p \in (q, \frac{n}{s})$  by the relation

$$\frac{1}{q} =: \frac{1}{p} + \frac{1}{b} \left( \frac{1}{p} - \frac{s}{n} \right). \quad (3.8)$$

Let us define  $p_0$  and  $q_0$  by the following relations:

$$\left( \frac{1}{n} - \frac{1}{p} \right) \frac{1}{1-s} = \frac{1}{n} - \frac{1}{p_0} \quad \text{and} \quad \left( \frac{1}{n} - \frac{1}{q} \right) \frac{1}{1-s} = \frac{1}{n} - \frac{1}{q_0}. \quad (3.9)$$

It follows immediately that  $p_0 > q_0 > n > 1$ . Let  $p_1 = q_1 = n$ . Note that (3.2) is equivalent to

$$\frac{1}{n} - \frac{1}{p} = (1-s) \left( \frac{1}{n} - \frac{1}{p_0} \right) + s \left( \frac{1}{n} - \frac{1}{p_1} \right),$$

and this holds trivially. The same happens with (3.3). Let us show that  $T_\phi^B : L^{p_0} \rightarrow L^{q_0}$ . Using identities (3.8) and (3.9) we get

$$\begin{aligned} \frac{1}{n} - \frac{1}{q_0} &= \frac{1}{1-s} \left( \frac{1}{n} - \frac{1}{q} \right) = \frac{1}{1-s} \left( \frac{1}{n} - \frac{1}{p} - \frac{1}{b} \left( \frac{1}{p} - \frac{s}{n} + \frac{1}{n} - \frac{s}{n} \right) \right) \\ &= \frac{1}{n} - \frac{1}{p_0} - \frac{1}{b} \left( \frac{1}{p_0} - \frac{1}{n} \right) - \frac{1}{1-s} \left( \frac{1}{b} \left( \frac{1-s}{n} \right) \right) = \frac{1}{n} - \frac{1}{p_0} - \frac{1}{b} \frac{1}{p_0}, \end{aligned}$$

so  $\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{b} \frac{1}{p_0}$  with  $q_0 > 1$ . By Corollary 2.6, we get

$$\left\| T_\phi^{B_1} \right\|_{L^{p_0} \rightarrow L^{q_0}} \leq C_b^{\frac{1}{p_0}}.$$

By Corollary 2.8 we also have that

$$\|T_\phi^{B_1}\|_{W^{1,p_1} \rightarrow W^{1,q_1}} = \|T_\phi^{B_1}\|_{W^{1,n} \rightarrow W^{1,n}} \leq C_{K,n,b,C_b}.$$

Since (3.2) and (3.3) hold, using Theorem 3.5,

$$\|T_\phi^{B_1}\|_{H^{s,p} \rightarrow H^{s,q}} \leq \|T_\phi^{B_1}\|_{L^{p_0} \rightarrow L^{q_0}}^{1-s} \|T_\phi^{B_1}\|_{W^{1,p_1} \rightarrow W^{1,q_1}}^s \leq C_b^{\frac{1-s}{p_0}} C_{K,n,b,C_b}^s.$$

Note that  $\frac{1}{p_0} < \frac{1}{q_0} = \frac{1}{n} - \frac{1}{(1-s)n} + \frac{1}{q(1-s)} = \frac{1}{1-s} \left( \frac{1}{q} - \frac{s}{n} \right) < \frac{\varepsilon_0}{1-s}$ . Thus,  $C_b^{\frac{1-s}{p_0}} \leq 2$  for  $\varepsilon_0 < \frac{\log(2)}{\log(C_b)}$ .

Take  $\varphi_{B_1}$  and let  $\tilde{f} = \varphi_{B_1} f$ . Then  $T_\phi^{B_1} f = T_\phi^{B_1} \tilde{f}$ . Thus, writing  $c_0 = 2C_{K,n,b,C_b}^s$ , we get

$$\begin{aligned} \|T_\phi^{B_1} f\|_{H^{s,q}} &= \|T_\phi^{B_1} \tilde{f}\|_{H^{s,q}} \leq c_0 \|\tilde{f}\|_{H^{s,p}} \leq c_0 C_{n,s} \|\tilde{f}\|_{H^{s,\frac{n}{s}}} \\ &\leq c_0 C_{n,s} \|\varphi_{B_1} \cdot\|_{H^{s,\frac{n}{s}} \rightarrow H^{s,\frac{n}{s}}} \|f\|_{H^{s,\frac{n}{s}}}, \end{aligned}$$

where  $\varphi_{B_1} \cdot : f \mapsto \varphi_{B_1} f$  stands for the pointwise multiplication operator, which has norm one in every  $L^p$ . Thus, by the interpolation property, it can be uniformly bounded by a constant  $C$ , so

$$\|T_\phi^{B_1}\|_{H^{s,\frac{n}{s}} \rightarrow H^{s,q}} \leq C_{K,n,b,C_b}^s C_{n,s}.$$

Lemma 3.7 implies that this uniform bound applies to the limit case, modulo constants depending on  $n$  and  $s$ .  $\square$

The supercritical case follows the same pattern. The interpolation limits according to the previous argument should be  $L^\infty$  for the 0-indices and  $W^{1,sp}$  or  $W^{1,sq}$  for the 1-indices (see Figure 3.1 (b)). Again, this is not a complex interpolation couple, so we will follow the approximation procedure above. Moreover, in this context we need to use the parameter  $a$  for the classical Sobolev spaces, while we are forced to use the parameter  $b$  in the Lebesgue spaces. When taking limits, however, the parameter  $b$  will vanish.

**Proposition 3.9.** *Let  $n \geq 2$ ,  $K \geq 1$ ,  $0 \leq s \leq 1$  and  $\frac{n}{s} < p < \infty$ . Given a  $K$ -quasiconformal mapping  $\phi : \Omega \rightarrow \Omega'$  between two domains in  $\mathbb{R}^n$ , with  $2B_1 \subset \Omega$  and  $\phi(2B_1) \subset B_1$ , and positive real numbers  $a$ ,  $b$ ,  $C_a$  and  $C_b$  satisfying (1.1) for  $U = 2B$ , if  $\frac{1}{q} := \frac{1}{p} + \frac{1}{a} \left( \frac{s}{n} - \frac{1}{p} \right) < 1$ , then there exists a constant  $C = C_{K,n,s,p,a}$  such that*

$$\|T_\phi^{B_1}\|_{H^{s,p} \rightarrow H^{s,q}} \leq C \left( 1 + C_a^{\frac{s}{n} - \frac{1}{p}} \right). \quad (3.10)$$

*Proof.* Let  $0 < \frac{1}{q_0} < \varepsilon_0 < \frac{1}{p}$  with  $\varepsilon_0$  small enough. Let us define  $p_0$  by the relation

$$\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{b} \frac{1}{p_0}.$$

Note that  $\frac{1}{p_0} < \frac{1}{q_0} < \frac{1}{p}$ . Next, we define  $p_1$  by imposing (3.2), that is,  $\frac{1}{p} = \frac{1-s}{p_0} + \frac{s}{p_1}$ . In particular,  $\frac{1}{p_1} = \frac{1}{sp} - \frac{1-s}{sp_0} < \frac{1}{n}$ . Thus, we are in the supercritical range and  $\left| \frac{1}{p_1} - \frac{1}{sp} \right| < \frac{(1-s)\varepsilon_0}{s}$ . Let us define  $q_1$  such that  $T_\phi^{B_1}$  maps  $W^{1,p_1}$  to  $W^{1,q_1}$ , that is,

$$\frac{1}{q_1} := \frac{1}{p_1} + \frac{1}{a} \left( \frac{1}{n} - \frac{1}{p_1} \right).$$

Using the definitions we get

$$\left| \frac{1}{q_1} - \frac{1}{sq} \right| = \left| \frac{1}{p_1} \left( 1 - \frac{1}{a} \right) + \frac{1}{an} - \frac{1}{sp} \left( 1 - \frac{1}{a} \right) - \frac{s}{san} \right| < \frac{(1-s)\varepsilon_0}{s} \frac{a-1}{a}.$$

Finally, we define  $\tilde{q}$  via (3.3), that is

$$\frac{1}{\tilde{q}} = \frac{1-s}{q_0} + \frac{s}{q_1}. \quad (3.11)$$

Note that

$$\left| \frac{1}{\tilde{q}} - \frac{1}{q} \right| = \left| \frac{1-s}{q_0} + \frac{s}{q_1} - \frac{1}{q} \right| \leq \frac{1-s}{q_0} + s \left| \frac{1}{q_1} - \frac{1}{sq} \right| \leq \varepsilon_0(1-s) \frac{2a-1}{a}.$$

From Corollary 2.6 we have that

$$\left\| T_\phi^{B_1} \right\|_{L^{p_0} \rightarrow L^{q_0}} \leq C_b^{\frac{1}{p_0}},$$

and from Corollary 2.8 we have that

$$\left\| T_\phi^{B_1} \right\|_{W^{1,p_1} \rightarrow W^{1,q_1}} \leq C_{K,n,p_1,q_1} \left( 1 + C_a^{\frac{1}{n} - \frac{1}{p_1}} \right).$$

Using Theorem 3.5,

$$\left\| T_\phi^{B_1} \right\|_{H^{s,p} \rightarrow H^{s,\tilde{q}}} \leq \left\| T_\phi^{B_1} \right\|_{L^{p_0} \rightarrow L^{q_0}}^{1-s} \left\| T_\phi^{B_1} \right\|_{H^{s,p_1} \rightarrow H^{s,q_1}}^s \leq C_b^{\frac{1-s}{p_0}} C_{K,n,p_1,a}^s \left( 1 + C_a^{\frac{s}{n} - \frac{s}{p_1}} \right).$$

On one hand, we have that  $C_b^{\frac{1-s}{p_0}} \leq 2$  for  $\varepsilon_0 < \log(2)/\log(C_b^{1-s})$ . On the other hand, from Remark 2.9, the constant  $C_{K,n,p_1,a}^s$  appearing in Corollary 2.8 is uniformly bounded in  $p_1$  for  $\left| \frac{1}{p_1} - \frac{1}{sp} \right| < \frac{(1-s)\varepsilon_0}{s} < \frac{1}{2} \left| \frac{1}{sp} - \frac{1}{n} \right|$ . Thus

$$\left\| T_\phi^{B_1} \right\|_{H^{s,p} \rightarrow H^{s,\tilde{q}}} \leq C_{K,n,s,p,a} \left( 1 + C_a^{\frac{s}{n} - \frac{1}{p}} \right).$$

Lemma 3.7 implies that this uniform bound applies to the limit case, modulo constants depending on  $n$  and  $s$ .  $\square$

*Proof of (1.2).* Propositions 3.6, 3.8 and 3.9 show that (1.2) holds for  $B = B' = B_1$ . Otherwise, estimate (1.2) follows by composing  $\phi$  with convenient translations and homotheties.  $\square$

### 3.3 Other fractional spaces

**Theorem 3.10** (see [Tri83, Theorems 2.4.2, 2.4.3, 2.4.7 and remark 2.4.7/2]). *Let  $-\infty < s_0, s_1 < \infty$ , let  $0 < r \leq \infty$ ,  $0 < p < \infty$  and  $0 < \Theta < 1$ . Let*

$$s = (1 - \Theta)s_0 + \Theta s_1.$$

*If  $s_0 \neq s_1$ , then*

$$(H^{s_0,p}, H^{s_1,p})_{\Theta,r} = B_{p,r}^s, \quad (3.12)$$

*where  $(\cdot, \cdot)_{\Theta,r}$  stands for the real interpolation functor described in [BL76].*



*Proof of (1.3).* The functor  $(\cdot, \cdot)_{\Theta, r}$  is an exact interpolation functor of exponent  $\Theta$ , i.e. it satisfies the interpolation property described in Theorem 3.5. Thus, when  $0 < s < 1$  and the exponents  $p$  and  $q$  satisfy the hypothesis of Theorem 1.1, using (3.12), estimate (1.2) on neighboring  $H^{s, p}$  spaces, and a limiting argument with a Besov version of Lemma 3.7 we can argue as in the previous section, to get

$$\|T_\phi^B\|_{B_{p, r}^s \rightarrow B_{q, r}^s} \leq C$$

for any  $0 < r \leq \infty$ . □

## 4 Sharpness

In this section we provide examples that show the sharpness of Theorem 1.1.

For  $k > 0$ , let  $\phi_k(x) := x|x|^{k-1}$ . We have that

$$D\phi(x) = |x|^{k-1} \left( (\alpha - 1) \frac{xx^t}{|x|^2} + Id \right).$$

Hence, we obtain  $J_{\phi_k}(x) = k|x|^{n(k-1)}$ , and, therefore,  $\phi_k$  is  $k^{n-1}$ -quasiconformal if  $k \geq 1$  and  $\frac{(2-k)^n}{k}$ -quasiconformal if  $k < 1$ . It also satisfies (1.1) for  $a < \frac{1}{1-k}$  and  $b > 0$  if  $k < 1$  and for  $a > 1$  and  $0 < b < \frac{1}{k-1}$  if  $k > 1$  regardless of the chosen domain  $U$ .

Next we recover the example given in [HK13] to show the sharpness on the subcritical setting (that is, when  $sp < n$ ) for the composition of an unbounded  $B_{p, p}^s$  function with a quasiconformal mapping. Let

$$f_\rho(x) := \max\{|x|^{-\rho} - 1, 0\} \text{ with } \rho > 0.$$

It is known that  $f_\rho$  belongs to the space  $H^{s, p}$  if and only if  $0 < \rho < \frac{n}{p} - s$  (see [RS96, Lemma 2.3.1]). Let  $b > 0$  and  $q \geq 1$  such that  $\frac{1}{q} < \frac{1}{p} + \frac{1}{b} \left( \frac{1}{p} - \frac{s}{n} \right)$ , let

$$\varepsilon := \frac{1}{p} + \frac{1}{b} \left( \frac{1}{p} - \frac{s}{n} \right) - \frac{1}{q} > 0 \tag{4.1}$$

and  $\delta > 0$  such that

$$(1 - \delta)^2 \left( \frac{n}{q} - s + n\varepsilon \right) > \frac{n}{q} - s.$$

Define

$$k := (1 - \delta) \left( \frac{1}{b} + 1 \right) \tag{4.2}$$

and  $\rho$  such that

$$(1 - \delta) \left( \frac{n}{p} - s \right) < \rho < \frac{n}{p} - s. \tag{4.3}$$

Then,  $\phi_k$  satisfies (1.1) for  $b$ , and  $f_\rho \in H^{s, p}$ . On the other hand we have  $f_\rho \circ \phi_k = f_{k\rho}$  and combining (4.2) and (4.3) with (4.1), we get

$$\begin{aligned} k\rho - \left( \frac{n}{q} - s \right) &> (1 - \delta)^2 \left( \frac{1}{b} + 1 \right) \left( \frac{n}{p} - s \right) - \left( \frac{n}{q} - s \right) \\ &= (1 - \delta)^2 \left( \frac{n}{q} - s + n\varepsilon \right) - \left( \frac{n}{q} - s \right) > 0. \end{aligned}$$

Therefore,  $f_\rho \circ \phi_k \notin H^{s,q}$ . The interested reader may find in [HK13, Lemma 4.2] an example of a bounded function satisfying the same.

For the supercritical case, i.e.  $sp > n$ , we define

$$g_\rho(x) := \max\{1 - |x|^\rho, 0\} \text{ with } \rho > 0.$$

Then, using again [RS96, Lemma 2.3.1] we have that  $g_\rho$  belongs to the Triebel-Lizorkin space  $H^{s,p}$  if and only if  $s - \frac{n}{p} < \rho$ . Let  $a > 1$  and  $q \geq 1$  such that  $\frac{1}{q} < \frac{1}{p} + \frac{1}{a} \left( \frac{s}{n} - \frac{1}{p} \right)$ , let

$$\varepsilon := \frac{1}{p} + \frac{1}{a} \left( \frac{s}{n} - \frac{1}{p} \right) - \frac{1}{q}$$

and  $\delta > 0$  such that

$$(1 + \delta)^2 \left( s - \frac{n}{q} - n\varepsilon \right) < s - \frac{n}{q}.$$

Define

$$k := (1 + \delta) \left( 1 - \frac{1}{a} \right)$$

with sufficient small  $\delta$  to have  $k < 1$ , and  $\rho$  such that

$$s - \frac{n}{p} < \rho < (1 + \delta) \left( s - \frac{n}{p} \right).$$

Then,  $\phi_k$  satisfies (1.1) for  $a$ , and  $g_\rho \in H^{s,p}$ . On the other hand we have  $g_\rho \circ \phi_k = g_{k\rho}$  and arguing as before we obtain

$$\begin{aligned} s - \frac{n}{q} - k\rho &> s - \frac{n}{q} - (1 + \delta)^2 \left( 1 - \frac{1}{a} \right) \left( s - \frac{n}{p} \right) \\ &= s - \frac{n}{q} - (1 + \delta)^2 \left( s - \frac{n}{q} - n\varepsilon \right) > 0. \end{aligned}$$

Therefore,  $f_\rho \circ \phi_k \notin H^{s,q}$ .

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