# Uniform convergence of double sine transforms of general monotone functions

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## Abstract

We consider different classes of functions of two variables that satisfy general monotonicity conditions, and obtain necessary and sufficient conditions for the uniform convergence in the regular sense and in the sense of Pringsheim of double sine transforms of functions of such classes.

## 1 Convergence of double integrals in $\mathbb{R}^2_+$

In this paper we consider the double sine transform

$$\mathcal{F}(u,v)=\int_0^\infty\int_0^\infty f(x,y)\,\sin ux\,\sin vy\,dx\,dy,\quad u,v\in\mathbb{R},\tag{1}$$

whenever it exists in the improper sense, where $f:\mathbb{R}^2_+\to\mathbb{C}$ is a Lebesgue measurable function (here $\mathbb{R}_+:=[0,+\infty)$) that vanishes at infinity (that is, $f(x,y)\to 0$ as $\max\{x,y\}\to\infty$). We also assume that $f$ satisfies the so-called Hardy bounded variation condition on $\mathbb{R}^2_+$ (or on a certain subset of $\mathbb{R}^2_+$), and $xyf(x,y)\in L^1_{loc}(\mathbb{R}^2_+)$.

The convergence of a double integral

$$\int_0^\infty\int_0^\infty g(s,t)\,ds\,dt,\tag{2}$$

has more than one interpretation, which is not the case in one dimension. If $g\in L^1_{loc}(\mathbb{R}^2_+)$, we say that (2) converges in *Pringsheim's sense* if the partial

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integrals

$$I(g;x,y) = \int_0^x \int_0^y g(s,t)\,ds\,dt,$$

converge to a finite limit as $\min\{x,y\} \to \infty$. It is easy to check that the Cauchy convergence criterion for Pringsheim convergence of double integrals holds. In other words, a necessary and sufficient condition for (2) to converge in the sense of Pringsheim is that for every $\varepsilon > 0$, there exists $z = z(\varepsilon)$ such that

$$|I(g;x_1,y_1) - I(g;x_2,y_2)| < \varepsilon, \quad \text{if } \min\{x_1,y_1,x_2,y_2\} > z. \tag{3}$$

On the other hand, we say that (2) converges in the *regular sense* if

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} g(s,t)\,ds\,dt \to 0 \quad \text{as } \max\{x_1,y_1\} \to \infty,\ x_2 > x_1,\ y_2 > y_1.$$

Convergence in the regular sense was first introduced by Hardy in [12] (see also [23]). It is useful to observe [24] that convergence in the sense of Pringsheim is equivalent to the following two conditions:

$$\int_{x_1}^{x_2} \int_0^y g(s,t)\,ds\,dt \to 0 \quad \text{as } x_1 \to \infty,\ x_2 > x_1,\ y \to \infty,$$
$$\int_0^x \int_{y_1}^{y_2} g(s,t)\,ds\,dt \to 0 \quad \text{as } y_1 \to \infty,\ y_2 > y_1,\ x \to \infty,$$

whilst regular convergence is equivalent to

$$\int_{x_1}^{x_2} \int_0^y g(s,t)\,ds\,dt \to 0 \quad \text{as } x_1 \to \infty,\ x_2 > x_1,\ y \in \mathbb{R}_+,$$
$$\int_0^x \int_{y_1}^{y_2} g(s,t)\,ds\,dt \to 0 \quad \text{as } y_1 \to \infty,\ y_2 > y_1,\ x \in \mathbb{R}_+.$$

Thus, it is clear that if a double integral converges in the regular sense, then it converges in the sense of Pringsheim. However, the converse is not true. Indeed, consider the following example from [22]:

$$g(x,y) = \begin{cases} k, & \text{if } (x,y) \in (k+2,k+3] \times (0,1], \\ & \text{or } (0,1] \times (k+2,k+3],\ k \in \mathbb{N}, \\ -k, & \text{if } (x,y) \in (k+2,k+3] \times (1,2], \\ & \text{or } (1,2] \times (k+2,k+3],\ k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for every $x, y \geq 2$, it is easy to verify that $I(g;x,y) = 0$. Hence, in this case, (2) converges to 0 in the sense of Pringsheim, even though $g$ is an unbounded function. However, (2) does not converge in the regular sense, since for any $k \in \mathbb{N}$,

$$\int_0^1 \int_{k+2}^{k+3} g(s,t)\,ds\,dt = k \nrightarrow 0 \quad \text{as } k \to \infty.$$

For a more detailed discussion of these two types of convergence (also defined for double series) and examples, we refer to [22].

For $M, N > 0$, we denote the partial double integral of (1) as

$$S_{M,N}(u,v) = S_{M,N}(f;u,v) := \int_0^M \int_0^N f(x,y) \sin ux \sin vy \, dx \, dy.$$

Then, by (3), we observe that uniform convergence of (1) in the sense of Pringsheim is equivalent to

$$|S_{M,N}(u,v) - S_{M',N'}(u,v)| < \varepsilon, \quad \text{if } \min\left\{M, N, M', N'\right\} > z,$$

uniformly in $u, v$, whilst uniform convergence of (1) in the regular sense is equivalent to

$$\int_M^{M'} \int_N^{N'} f(x,y) \sin ux \sin vy \, dx \, dy \to 0,$$

as $\max\{M,N\} \to \infty$, and $M' > M$, $N' > N$.

# 2 Bounded variation in $\mathbb{R}_+^2$

We recall the concept of bounded variation for functions of one variable. Given $g : \mathbb{R}_+ \to \mathbb{C}$, and $I := [a,b] \subset \mathbb{R}_+$, $g$ is then said to be of bounded variation on $I$ ($g \in BV(I)$) if

$$V_I(g) := \sup \sum_{k=0}^{n-1} |g(x_k) - g(x_{k+1})| < \infty,$$

where the supremum is taken over all partitions of $I$, i.e., all the finite sequences $a = x_0 < x_1 < \cdots < x_n = b$. We define $V_I(g)$ the *variation of $g$ over $I$*. It can be proved that whenever such a variation is finite, it equals the Stieltjes integral

$$\int_a^b |dg(s)| := \lim_{\delta \to 0} \sum_{k=0}^{n-1} |g(x_k) - g(x_{k+1})|,$$

where the $\max_k\{x_{k+1} - x_k\} < \delta$ (cf. [27, pp. 26–28]). Moreover, if $g$ is differentiable, then

$$\int_a^b |dg(s)| = \int_a^b |g'(s)| \, ds.$$

We can also extend the concept of bounded variation to the whole $\mathbb{R}_+$; we define the *variation of $g$ over $\mathbb{R}_+$* as

$$V_{\mathbb{R}_+}(g) := \sup_{[a,b] \subset \mathbb{R}_+} V_{[a,b]}(g).$$

If $V_{\mathbb{R}_+}(g)$ is finite we say that $g$ is of bounded variation on $\mathbb{R}_+$, and in this case one can check that such variation equals the improper Stieltjes integral

$$\int_0^\infty |dg(s)|.$$

For functions of two variables, there are several definitions for the concept of bounded variation (cf. [5]). We will focus on the so-called Hardy-$BV$ condition, but first, we introduce some notation: for any increasing sequences $\{x_n\}, \{y_n\} \subset \mathbb{R}_+$, we define

$$\begin{aligned}
\Delta_{10} f(x_i, y) &:= f(x_i, y) - f(x_{i+1}, y), \\
\Delta_{01} f(x, y_j) &:= f(x, y_j) - f(x, y_{j+1}), \\
\Delta_{11} f(x_i, y_j) &:= \Delta_{01}(\Delta_{10} f(x_i, y_j)) = \Delta_{10}(\Delta_{01} f(x_i, y_j)) \\
&= f(x_i, y_j) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1}).
\end{aligned}$$

If $J := [a, b] \times [c, d] \subset \mathbb{R}_+^2$ is a compact rectangle, we define the *Hardy variation of $f$ over $J$* as

$$HV_J(f) := \sup \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\Delta_{11} f(x_i, y_j)|,$$

where the supremum is taken over all partitions of $J$, that is,

$$\mathcal{P}(J) = \big\{\{a = x_0 < x_1 < \cdots < x_m = b\} \times \{c = y_0 < y_1 < \cdots < y_n = d\}\big\}.$$

Similarly, as in the one-dimensional case, we can show that if $HV_J(f)$ is finite, it coincides with the double Stieltjes integral

$$\int_a^b \int_c^d |d_{xy} f(s, t)| := \lim_{\delta \to 0} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\Delta_{11} f(x_i, y_j)|,$$

where $\{x_i\} \times \{y_j\} \in \mathcal{P}(J)$, and

$$\max\big\{\max_i \{x_{i+1} - x_i\}, \max_j \{y_{j+1} - y_j\}\big\} < \delta. \tag{4}$$

**Definition 1.** 1. We say that $f$ is of Hardy bounded variation on $J$ ($f \in HBV(J)$) if $HV_J(f) < \infty$ and, in addition, the functions $f(x_0, \cdot)$ and $f(\cdot, y_0)$ are of bounded variation on $[c, d]$ and $[a, b]$ respectively, for some $x_0 \in [a, b]$ and $y_0 \in [c, d]$.

2. We say that $f$ is of Hardy bounded variation on $\mathbb{R}_+^2$ ($f \in HBV(\mathbb{R}_+^2)$) if $f(x_0, \cdot)$, $f(\cdot, y_0)$ are of bounded variation on $\mathbb{R}_+$ for any $x_0, y_0 \in \mathbb{R}_+$, $HV_J(f) < \infty$ for any compact rectangle $J \subset \mathbb{R}_+^2$, and

$$HV_{\mathbb{R}_+^2}(f) := \sup_{J \subset \mathbb{R}_+^2} HV_J(f) < \infty.$$

If $f \in HBV(\mathbb{R}^2_+)$, it holds that

$$HV_{\mathbb{R}^2_+}(f) = \int_0^\infty \int_0^\infty |d_{xy} f(s,t)|,$$

as an improper Stieltjes integral. Several additional properties of the single and double Stieltjes integral are discussed in [26].

**Remark 1.** Originally, Definition 1 (for compact rectangles $J$) required that the restrictions of $f$ over lines satisfy the following conditions:

$$f_2^{x_0} := f(x_0, \cdot) \in BV\big([c,d]\big), \quad f_1^{y_0} := f(\cdot, y_0) \in BV\big([a,b]\big), \tag{5}$$

for all $x_0 \in [a,b]$ and all $y_0 \in [c,d]$, respectively. However, W. H. Young [30] proved that the original definition was redundant, i.e., it is sufficient that (5) holds for only one $x_0$ and one $y_0$, provided that $HV_J(f)$ is finite, since in this case, it follows that (5) holds for any choice of $x_0, y_0$. For further discussion, see [13, §254].

**Remark 2.** The Hardy bounded variation property on $\mathbb{R}^2_+$ is rather restrictive: in particular, if $f \in HBV(\mathbb{R}^2_+)$, then it follows that $f$ must be bounded (since all the marginal functions are of bounded variation on $\mathbb{R}_+$). Under this definition, we neglect functions such as

$$f(x,y) = \begin{cases} (xy)^{-1} & \text{if } x, y < 1, \\ e^{-(x+y)} & \text{otherwise,} \end{cases} \tag{6}$$

which may have a good behaviour at infinity in the sense that they decrease rapidly. We show in Sections 5 and 6 that we can relax the condition of bounded variation to some subset

$$\mathbb{R}^2_+ \backslash [0,c)^2, \quad c > 0,$$

instead of the whole $\mathbb{R}^2_+$. This condition allows functions to be unbounded near the origin. For simplicity, we will consider functions of Hardy bounded variation on $\mathbb{R}^2_+$, but always keeping in mind that the presented results in Sections 5 and 6 are also valid for functions satisfying the following definition:

**Definition 1'.** Let $c > 0$. We say that $f : \mathbb{R}^2_+ \to \mathbb{C}$ is of Hardy bounded variation on $\mathbb{R}^2_+ \backslash [0,c)^2$ if

$$\sup_{J \subset \mathbb{R}_+ \times [c,+\infty)} HV_J(f), \qquad \sup_{J \subset [c,+\infty) \times \mathbb{R}_+} HV_J(f),$$

are finite, and moreover, the functions $f(x_0, \cdot)$, $f(\cdot, y_0)$ are of bounded variation on $\mathbb{R}_+$ for all $x_0, y_0 \geq c$, respectively.

Under this definition, the functions are required to be bounded only on $\mathbb{R}^2_+ \backslash [0,c)^2$. Thus, functions such as (6) satisfy the latter definition.

In the follow-up to this paper we refer to functions of Hardy bounded variation just as functions of bounded variation.

# 3 Uniform convergence of sine series and integrals: General monotonicity

The uniform convergence of sine series and integral transforms is a problem that has recently attracted several authors. In this section we present some of the known results on uniform convergence of sine series and integrals. We also introduce a new class of functions in order to generalize these results.

## 3.1 Sine series and general monotone sequences

In general it is difficult to prove whether a sine series

$$\sum_{n=1}^{\infty} a_n \sin nx \tag{7}$$

converges uniformly or not, besides the case when $\sum |a_n| < \infty$, which is a trivial sufficient condition to guarantee uniform convergence. However, there are cases for which we can characterize the sequences $\{a_n\}$ whose corresponding series (7) converge uniformly. For example, in 1916, Chaundy and Jolliffe proved their well-known theorem (cf. [3] and [31, V. II, p. 182]):

**Theorem A** (Chaundy and Jolliffe)**.** *Let $\{a_n\}$ be a non-negative sequence monotonically decreasing to zero. Then, the series* (7) *converges uniformly in $x$ if and only if*

$$na_n \to 0 \quad \textit{as } n \to \infty.$$

In view of this result, we note that imposing certain conditions on $\{a_n\}$ can help to characterise sequences whose sine series converge uniformly. Since the monotonicity condition is too restrictive, one may consider if this result is still true if we assume a weaker hypothesis on $\{a_n\}$. Several efforts have been made in order to generalize Chaundy and Jolliffe's theorem for quasi-monotone sequences [28], or sequences with rest of bounded variation [17], among others. Moreover, a new class has recently been introduced: the so-called *$\beta$-general monotone sequences* [19, 29], i.e., those that satisfy

$$\sum_{k=n}^{2n} |a_n - a_{n+1}| \le C\beta_n, \quad \text{for all } n, \tag{8}$$

where $\beta_n$ is a sequence of positive numbers that depend on $\{a_n\}$, and $C$ is a constant independent of $n$. If $\{a_n\}$ satisfies (8) for a certain $\beta := \{\beta_n\}$, then we write $\{a_n\} \in GMS(\beta)$, where the $S$ denotes sequences. Thus, the general monotonicity condition expresses quantitative characteristics for sequences of bounded variation. Some examples of such $\beta$ are

1. $\beta_n^1 := |a_n|$,
2. $\beta_n^2 := \frac{1}{n}\sum_{k=n/\lambda}^{\lambda n} |a_k|$, with $\lambda > 1$,

3. $\beta_n^3 := \frac{1}{n}\sup_{m\geq n/\lambda}\sum_{k=m}^{2m}|a_k|$, with $\lambda > 1$,

4. $\beta_n^4 := \frac{1}{n}\sup_{m\geq b_n}\sum_{k=m}^{2m}|a_k|$, where $b_n$ can be any sequence that increases towards infinity, as for example $b_n = \sqrt{n}$ or $b_n = \log n$.

**Remark 3.** By writing $a := \{a_n\} \in GMS(\beta)$ we encounter a problematic notational issue because of the dependence of $\{\beta_n\}$ on $\{a_n\}$. One way to solve this problem is to write $(a, \beta) \in GMS$ instead of $a \in GMS(\beta)$, as in [19, Def. 1.1]. For simplicity, we will stick to the notation $\{a_n\} \in GMS(\beta)$.

Let $M$ denote the class of monotone sequences. It is known [9, 14, 19, 29] that

$$M \subsetneq GMS(\beta^1) \subsetneq GMS(\beta^2) \subsetneq GMS(\beta^3) \subsetneq GMS(\beta^4).$$

Furthermore, Theorem A is also true for positive sequences that belong to the class $GMS(\beta^4)$ (see [14]). On the other hand, in the definition of $\beta^1$ we already consider sequences that are not only non-negative, contrarily to the case of monotone sequences. Thus, when studying the uniform convergence of (7) we can also consider real sequences $\{a_n\}$ that vary their sign, or even complex sequences. In the case of real sequences, Theorem A is still true for the class $GMS(\beta^2)$ (cf. [10]), but if we try to extend the proof of the latter fact to the class $GMS(\beta^3)$, we need an extra assumption, namely

$$\tilde{a}_n := \sum_{k=n}^{2n}|a_k|$$

is bounded. Moreover, this hypothesis is essential, i.e., there exists a sequence $\{a_n\} \in GMS(\beta^3)$ such that $\tilde{a}_n$ is unbounded, but its corresponding sine series (7) converges uniformly even though $na_n \nrightarrow 0$ (see [8]).

General monotonicity is a useful property, not only to study uniform convergence of sine series and transforms, but also in other topics of Fourier analysis such as summability [2] or pointwise convergence [11] of trigonometric series. With respect to functions, general monotonicity has been used to obtain weighted Fourier inequalities [7, 20], among other applications.

## 3.2 Sine integrals and general monotone functions

The concept of general monotonicity for functions is defined in a similar way as for sequences [20, 21]: we say that $g : \mathbb{R}_+ \to \mathbb{C}$ is a *$\beta$-general monotone function* if

$$\int_x^{2x} |dg(s)| \leq C\beta(x), \quad \text{for all } x > 0,$$

where $\beta : (0, +\infty) \to \mathbb{R}_+$, $C$ is a constant independent of $x$, and the majorant $\beta$ depends on $g$. In this case we say that $g \in GMF(\beta)$, where $F$ denotes functions. We observe that the class of monotone functions is a subclass of $GMF(\beta)$, with $\beta(x) = g(x)$ and $C = 1$ (this also occurs in the case of $GMS$).

As one may expect, there is a version of Chaundy and Jolliffe's theorem for sine transforms of monotone functions [25]:

**Theorem B.** *Let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be monotonically decreasing and such that* $xf(x) \in L^1(0,1)$*. Then, the sine transform*

$$\int_0^\infty f(x) \sin ux \, dx, \tag{9}$$

*converges uniformly in* $u$ *if and only if*

$$xf(x) \to 0 \quad \text{as } x \to \infty.$$

We note that there is a slight difference between the latter result and Chaundy and Jolliffe's theorem: we need to assume $xf(x) \in L^1(0,1)$ in order to ensure the existence of the improper integral (9) near the origin.

Similarly as in the case of sine series, we can obtain a number of generalizations of Theorem B by considering wider classes of general monotone functions (cf. [9, 25]). For each one of the classes $GMS(\beta^i)$, $i = 1, 2, 3$, appearing in Subsection 3.1, we can define a converse class of functions $GMF(\beta_i)$ in a natural way, that is, replacing sequences by functions and sums by integrals. We can also define a similar class $GMF(\beta_4)$, but we would need some conditions that do not appear in the definition of $GMS(\beta^4)$, due to the fact that we work with functions instead of sequences. We will focus on the class $GMF(\beta_3)$ introduced in [7], where

$$\beta_3(x) = \frac{1}{x} \sup_{s \geq x/\lambda} \int_s^{2s} |f(t)| \, dt, \quad \lambda > 1. \tag{10}$$

If we denote $I(x) := I(x, f) = \int_x^{2x} |f(t)| \, dt$, then the expression

$$B(x) := B\big(x, I(\cdot)\big) = \sup_{s \geq x/\lambda} I(s),$$

satisfies the following properties:

(a) If $I(x)$ vanishes at infinity, $B(x)$ also vanishes.

(b) If $I(x)$ is bounded at infinity[1], $B(x)$ is also bounded.

(c) For every $x > 0$, $I(x) \leq B(x)$.

(d) $B(x)$ is decreasing on $x$.

We call any operator $B(x) = B(x, I(\cdot))$ *admissible* if properties (a)–(d) hold. Also, we define the following class of functions: if $f : \mathbb{R}_+ \to \mathbb{C}$ and there exists an admissible operator $B$ such that $f \in GMF(\beta)$, with

$$\beta(x) = \frac{B(x)}{x},$$

then we say that $f \in GMF_{adm}$. It is clear from the definition that $GMF(\beta_3) \subset GMF_{adm}$, and the inclusion is proper. Moreover, we have the following result [6]:

[1] We say that a function $\varphi$ is bounded at infinity if there exist $C, x_0 > 0$ such that $|\varphi(x)| \leq C$ for all $x > x_0$.

**Theorem C.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be such that $xf(x) \in L^1(0,1)$, and assume that $f \in GMF_{adm}$. Furthermore, suppose that $I(x)$ is bounded at infinity. Then, the sine transform* (9) *converges uniformly if and only if*

$$xf(x) \to 0 \quad \text{as } x \to \infty.$$

It is important to note that if we assume that $f$ is non-negative, then we do not need the hypothesis of $I(x)$ being bounded at infinity. Moreover, as in the case of sequences discussed in the previous subsection, the result is sharp with respect to the hypothesis "$I(x)$ bounded at infinity", i.e., there exists a function $g$ such that

$$\int_x^{2x} |g(s)|\, ds,$$

is not bounded at infinity, but its corresponding sine integral

$$\int_0^{\infty} g(x) \sin ux\, dx,$$

converges uniformly although $x|g(x)| \to \infty$ as $x \to \infty$ (cf. [6]).

We can also define a class of sequences $GMS_{adm}$, similarly as we defined $GMF_{adm}$. In this case there is also a re-statement of Theorem C for sequences [8], and as mentioned previously, it is also sharp with respect to the hypothesis $\sum_{k=n}^{2n} |a_k|$ bounded.

# 4 Double sine integrals and general monotone functions of two variables

We can extend the theory of general monotone sequences and functions to the two-dimensional framework.

Our definition of general monotonicity for functions of two variables is the following:

**Definition 2.** We say that a function $f : \mathbb{R}_+^2 \to \mathbb{C}$ belongs to $GMF^2(\beta)$, where $\beta : (0, +\infty)^2 \to \mathbb{R}_+$, if there exists $C > 0$ such that for any $x, y > 0$,

$$\int_x^{2x} \int_y^{2y} |d_{xy} f(s,t)| \le C\beta(x,y). \tag{11}$$

Here the superscript 2 represents the dimension.

Comparing Definition 2 to the previous definitions for general monotone functions and sequences in two dimensions (see, for example, [16] and [18, Definition 4]), we also require the existence of $\beta_1$ and $\beta_2$ such that

$$\int_x^{2x} |d_x f(s,y)| \le C\beta_1(x,y), \quad \int_y^{2y} |d_y f(x,t)| \le C\beta_2(x,y). \tag{12}$$

Instead of choosing these $\beta_1, \beta_2$ arbitrarily, we use the following intrinsic expressions that can be obtained from (11): since

$$\int_x^{2x} |d_x f(s,y)| = \int_x^{2x} \left| \int_y^{\infty} d_{yx} f(s,t) \right|, \quad \int_y^{2y} |d_y f(x,t)| = \int_y^{2y} \left| \int_x^{\infty} d_{xy} f(s,t) \right|,$$

we can define

$$\beta_1(x,y) := \sum_{j=0}^{\infty} \beta(x, 2^j y), \quad \beta_2(x,y) := \sum_{i=0}^{\infty} \beta(2^i x, y),$$

and (12) follows.

The widest class of general monotone functions that has been introduced so far is given by

$$\beta(x,y) = \frac{1}{xy} \int_{x/\lambda}^{\lambda x} \int_{y/\lambda}^{\lambda y} |f(s,t)|\, ds\, dt, \quad \lambda > 1. \tag{13}$$

In [15, 16], the authors consider locally absolutely continuous functions from this class (see [1]), and they obtain a two-dimensional version of Chaundy and Jolliffe's theorem, among others:

**Theorem D.** *Let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$, $f \in AC_{loc}(\mathbb{R}_+^2)$, be such that $f \in GMF^2(\beta)$, where $\beta$ is defined by (13). Then, the double sine transform (1) converges uniformly in the regular sense if and only if*

$$xyf(x,y) \to 0 \quad \text{as } \max\{x,y\} \to \infty.$$

Thus, our goal is not only to extend Theorem D for a wider class containing $GMF^2(\beta)$, where $\beta$ is defined by (13), but also to prove it for functions that are only of bounded variation, instead of absolutely continuous.

Motivated by the $GMF_{adm}$ class we have mentioned before, we proceed to define a similar class of functions of two variables, namely $GMF^2_{adm}$.

**Definition 3.** Let $h : (0,+\infty)^2 \to \mathbb{R}_+$. We say that an operator $B = B(h) : (0,+\infty)^2 \to \mathbb{R}_+$ is admissible if the following hold:

(a) If $h(x_0, y_0) \to 0$ as $\max\{x_0, y_0\} \to \infty$, then $B\big(x, y, h(\cdot,\cdot)\big) \to 0$ as $\max\{x,y\} \to \infty$.

(b) For all $x, y > 0$, $h(x,y) \le B\big(x, y, h(\cdot,\cdot)\big)$.

Noteworthy is the fact that property (b) does not make us lose generality: Indeed, if $B$ is an operator that satisfies (a), then we can define a new operator $\tilde{B}$ as

$$\tilde{B}\big(x, y, h(\cdot,\cdot)\big) = \max\big\{h(x,y), B\big(x,y,h(\cdot,\cdot)\big)\big\},$$

which is clearly admissible. Also, we could require that $B$ is monotone in each variable; we would not lose generality neither, and it would simplify the computations. For instance, equation (25) would read as

$$|f(u,v)| \le \frac{C}{xy} B\big(x, y, I_{12}(f,\cdot,\cdot)\big)$$

under the monotonicity assumption of $B$. Thus, broadly speaking, monotonicity on $B$ is optional here (and we will not use it). The essential property that $B$ should satisfy is (a) of Definition 3. Here we list some operators that satisfy both properties (a) and (b) of Definition 3:

(i) $B_1\big(x, y, h(\cdot,\cdot)\big) = h(x, y)$,

(ii) $B_2\big(x, y, h(\cdot,\cdot)\big) = h(x, y)^\alpha$, with $\alpha > 0$,

(iii) $B_3\big(x, y, h(\cdot,\cdot)\big) = \int_{x/\lambda}^{\lambda x}\int_{y/\lambda}^{\lambda y} h(s, t)/(st)\, ds\, dt \asymp (xy)^{-1}\int_{x/\lambda}^{\lambda x}\int_{y/\lambda}^{\lambda y} h(s, t)\, ds\, dt$, where $\lambda > 1$,

(iv) $B_4\big(x, y, h(\cdot,\cdot)\big) = (xy)^\alpha \int_{x/\lambda}^{\infty}\int_{y/\lambda}^{\infty} h(s, t)/(st)^{\alpha+1}\, ds\, dt$, where $\lambda > 1$ and $\alpha > 0$,

(v) $B_5\big(x, y, h(\cdot,\cdot)\big) = \sup_{s+t\geq (x+y)/\lambda} h(s, t)$, where $\lambda > 1$,

(vi) $B_6\big(x, y, h(\cdot,\cdot)\big) = \sup_{s+t\geq \log(x+y+1)} h(s, t)$.

(vii) The composition of admissible operators is admissible. That is, if $C$ and $D$ are admissible operators, then

$$B_7\big(x, y, h(\cdot,\cdot)\big) = (C \circ D)\big(x, y, h(\cdot,\cdot)\big) = C\big(x, y, D\big(\cdot,\cdot, h(\cdot,\cdot)\big)\big)$$

is admissible.

**Remark 4.** We cannot allow $\alpha = 0$ in $B_4$, since the operator would not be admissible. Indeed, if we consider

$$h(x, y) = \begin{cases} 0, & \text{if } x \text{ or } y < 2, \\ (\log x \log y)^{-1}, & \text{otherwise,} \end{cases}$$

then $h(x, y) \to 0$ as $\max\{x, y\} \to \infty$, but

$$\int_{x/c}^{\infty}\int_{y/c}^{\infty} \frac{h(s, t)}{st}\, ds\, dt = \int_{x/c}^{\infty} \frac{1}{s\log s}\, ds \int_{y/c}^{\infty} \frac{1}{t\log t}\, dt = +\infty,$$

for any $x, y > 2c$, so that condition (a) of Definition 3 fails to be true.

Now we introduce the new class of functions we will deal with:

**Definition 4.** We say that $f : \mathbb{R}_+^2 \to \mathbb{C}$ belongs to the class $GMF^2_{adm}$ if $f \in GMF^2(\beta)$, with

$$\beta(x, y) = \frac{1}{xy} B\big(x, y, I_{12}(f,\cdot,\cdot)\big), \tag{14}$$

where $B$ is admissible, and

$$I_{12}(f, x, y) = I_{21}(f, x, y) := \int_x^{2x}\int_y^{2y} |f(s, t)|\, dt\, ds. \tag{15}$$

To conclude this section, we will rewrite the functions $\beta(x,y)$ from certain known $GMF^2(\beta)$ classes in terms of admissible operators. In Definition 4, the operator $B$ is acting on the function $I_{12}$, but we can also choose a different one, such as

(i) $h_1(f,x,y) = xy|f(x,y)|$,

(ii) $h_2(f,x,y) = \int_{x/\lambda}^{\lambda x}\int_{y/\lambda}^{\lambda y} |f(s,t)|\,ds\,dt$, where $\lambda > 1$.

For instance, a two-dimensional version for functions of the original class of $GM$ sequences, introduced in [29], is given by $\beta(x,y) = |f(x,y)|$. In terms of the previous admissible operators, this can be written as

$$\beta(x,y) = \frac{1}{xy}B_1\big(x,y,h_1(f,\cdot,\cdot)\big).$$

Furthermore, the $GMF^2(\beta)$ class introduced in [16], whose $\beta$ is defined by (13), can be written as

$$\beta(x,y) = \frac{1}{xy}B_1\big(x,y,h_2(f,\cdot,\cdot)\big),$$

and finally, the two-dimensional version of the $GMF(\beta_3)$ class (cf. (10)), introduced in [7], can also be expressed by means of an admissible operator:

$$\beta(x,y) = \frac{1}{xy}B_5\big(x,y,I_{12}(f,\cdot,\cdot)\big).$$

# 5 Main results

Here we present necessary and sufficient conditions for the uniform convergence of the double sine transform (1).

The following result is a two-dimensional version of [7, Theorem 3, (1)]. The counterpart for a double sine series can be found in [18, Theorem 1].

**Theorem 1.** *Let* $f:\mathbb{R}_+^2\to\mathbb{C}$, $f\in GMF^2(\beta)$. *If*

$$\beta(x,y) = o\big((xy)^{-1}\big) \quad \text{as } \max\{x,y\}\to\infty. \tag{16}$$

*Then,* (1) *converges uniformly in the regular sense, and moreover,*

$$\|\mathcal{F}(u,v) - S_{M,N}(u,v)\|_\infty \le 9(\varepsilon_{M,0} + \varepsilon_{0,N} + \varepsilon_{M,N}), \tag{17}$$

*where*

$$\varepsilon_{\mu,\nu} = \sup_{\substack{\mu'\ge\mu\\ \nu'\ge\nu}} \mu'\nu'\int_{\mu'}^\infty\int_{\nu'}^\infty |d_{xy}f(s,t)|. \tag{18}$$

**Remark 5.** Condition (16) can be rewritten as

$$\int_x^\infty \int_y^\infty |d_{yx} f(s,t)| = o\big((xy)^{-1}\big) \quad \text{as } \max\{x,y\} \to \infty,$$

since they are equivalent. Indeed, if the latter holds, then (16) is obviously true. For the other implication, note that since

$$\int_x^{2x} \int_y^{2y} |d_{yx} f(s,t)| = o\big((xy)^{-1}\big) \quad \text{as } \max\{x,y\} \to \infty,$$

then

$$\begin{aligned}
\int_x^\infty \int_y^\infty |d_{yx} f(s,t)| &= \sum_{m=0}^\infty \sum_{n=0}^\infty \int_{2^m x}^{2^{m+1}x} \int_{2^n y}^{2^{n+1}y} |d_{yx} f(s,t)| \\
&\le C \sum_{m=0}^\infty \sum_{n=0}^\infty \beta(2^m x, 2^n y) = C \sum_{m=0}^\infty \sum_{n=0}^\infty o\bigg(\frac{1}{2^m 2^n xy}\bigg) \\
&= o\bigg(\frac{1}{xy}\bigg) \quad \text{as } \max\{x,y\} \to \infty.
\end{aligned}$$

The next result of this section is a corollary of Theorem 1, which establishes sufficient conditions for the uniform convergence of (1) whenever $f$ belongs to the $GMF^2_{adm}$ class.

**Corollary 2.** *Let $f \in GMF^2_{adm}$ be such that*

$$xy|f(x,y)| \to 0 \quad \textit{as } \max\{x,y\} \to \infty.$$

*Then,* (1) *converges uniformly in the regular sense, and moreover,*

$$\|\mathcal{F}(u,v) - S_{M,N}(u,v)\|_\infty \le 9C(\delta_{M,0} + \delta_{0,N} + \delta_{M,N}), \tag{19}$$

*where*[2]

$$\delta_{\mu,\nu} = \sup_{\substack{\mu' \ge \mu \\ \nu' \ge \nu}} B\big(\mu', \nu', I_{12}(f,\cdot,\cdot)\big),$$

*and $C$ is the constant from the GM condition given by* (11).

Note that the condition $xy|f(x,y)| \to 0$ as $\max\{x,y\} \to \infty$ always implies $\delta_{\mu,\nu} \to 0$ as $\max\{\mu,\nu\} \to \infty$ whenever $f \in GMF^2_{adm}$.

Finally, we obtain necessary conditions for the uniform convergence in the regular sense, and in the sense of Pringsheim. However, in this case we have to restrict ourselves to the case of $f$ being non-negative.

**Theorem 3.** *Let $f \in GMF^2_{adm}$ be non-negative. A necessary condition for the uniform regular convergence of* (1) *is that*

$$xyf(x,y) \to 0 \quad \textit{as } \max\{x,y\} \to \infty.$$

[2]Since $B\big(x,y,I_{12}(f,\cdot,\cdot)\big)$ is not defined for $x = 0$ or $y = 0$, then we define $\delta_{\mu,\nu}$ as the supremum over $\mu' > 0$ or $\nu' > 0$, whenever $\mu = 0$ or $\nu = 0$.

**Theorem 4.** *Let $f \in GMF^2_{adm}$ be non-negative, and suppose that the operator $B$ satisfies, instead of property (a) of Definition 3, the following weaker condition: if $I_{12}(f,x,y) \to 0$ as $\min\{x,y\} \to \infty$, then $B\big(x,y,I_{12}(f,\cdot,\cdot)\big) \to 0$ as $\min\{x,y\} \to \infty$. Then, a necessary condition for the uniform Pringsheim convergence of* (1) *is that*

$$xyf(x,y) \to 0 \quad \text{as } \min\{x,y\} \to \infty.$$

Combining Corollary 2 and Theorem 3, we obtain a version of Chaundy and Jolliffe's criterion for functions of two variables:

**Theorem 5.** *Let $f \in GMF^2_{adm}$ be non-negative. Then, the double sine integral* (1) *converges uniformly in the regular sense if and only if*

$$xyf(x,y) \to 0 \quad \text{as } \max\{x,y\} \to \infty.$$

Summarizing, we have presented necessary and sufficient conditions for the double sine transform (1) to converge uniformly in the regular sense, and also necessary conditions that follow from the uniform convergence in the sense of Pringsheim. However, it is likely that the necessary condition stated in Theorem 4 should not be sufficient in this case.

**Open Problem.** Find sufficient conditions to guarantee the uniform convergence in the sense of Pringsheim of the double sine transform (1), besides the trivial sufficient condition that ensures uniform convergence in the regular sense (cf. Corollary 2).

# 6 Auxiliary results

We can represent any function $f : \mathbb{R}^2_+ \to \mathbb{C}$ of bounded variation that vanishes at infinity as an improper Stieltjes integral (defined in the usual way), as follows (see [4]):

$$\begin{aligned} f(x,y) &= \int_x^\infty d_x f(s,y) = \int_y^\infty d_y f(x,t) = \int_x^\infty \int_y^\infty d_y(d_x f(s,t)) \\ &= \int_y^\infty \int_x^\infty d_x(d_y f(s,t)) =: \int_y^\infty \int_x^\infty d_{xy} f(s,t), \end{aligned}$$

with

$$\begin{aligned} \int_x^{x'} d_x f(s,y) &= \lim_{\delta\to 0} \sum_{j=0}^{n-1} \Delta_{10} f(x_j,y) = f(x,y) - f(x',y), \\ \int_y^{y'} d_y f(x,t) &= \lim_{\delta\to 0} \sum_{k=0}^{m-1} \Delta_{01} f(x,y_k) = f(x,y) - f(x,y'), \\ \int_x^{x'} \int_y^{y'} d_{xy} f(s,t) &= \lim_{\delta\to 0} \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \Delta_{11} f(x_j,y_k) \\ &= f(x,y) - f(x',y) - f(x,y') + f(x',y'), \end{aligned}$$

where $\{x_k\}$ and $\{y_k\}$ are partitions of $[x, x']$ and $[y, y']$ respectively, and satisfying (4). Moreover, all of these integrals are absolutely convergent due to the bounded variation condition (cf. Section 2).

**Lemma 6.** *Let $M, N \geq 0$ and let $f : \mathbb{R}^2_+ \to \mathbb{C}$ be of bounded variation on $\mathbb{R}^2_+$. Then,*

$$\int_N^\infty \int_M^\infty f(x,y) \sin ux \sin vy \, dx \, dy = \int_M^\infty \int_N^\infty d_{yx} f(s,t) \int_M^s \sin ux \, dx \int_N^t \sin vy \, dy.$$

*Proof.* Since $f$ is of bounded variation,

$$\int_x^\infty \int_y^\infty |d_{xy} f(s,t)| = o(1) \quad \text{as } \max\{x,y\} \to \infty.$$

Now let $M' > M$, $N' > N$, and $u, v > 0$. Also, let us denote

$$\phi_u(s) = \int_M^s \sin ux \, dx, \quad \psi_v(t) = \int_N^t \sin vy \, dy.$$

Notice that for every $s, t \in \mathbb{R}_+$, $|\phi_u(s)| \leq 2/u$, $|\psi_v(t)| \leq 2/v$. Rewriting $f$ as an improper Stieltjes integral, we have the following equality:

$$\begin{aligned} &\int_N^{N'} \int_M^{M'} f(x,y) \sin ux \sin vy \, dx \, dy \\ &= \int_N^{N'} \int_M^{M'} \left( \int_x^\infty d_x f(s,y) \right) \sin ux \sin vy \, dx \, dy. \end{aligned} \tag{20}$$

The latter integral converges absolutely for all values of $M' > M$ and $N' > N$. Thus, we can switch the order of integration, and obtain that (20) is equal to

$$\begin{aligned} &\int_N^{N'} \int_M^{M'} d_x f(s,y) \phi_u(s) \sin vy \, dy + \phi_u(M') \int_N^{N'} \int_{M'}^\infty d_x f(s,y) \sin vy \, dy \\ &= \int_M^{M'} \phi_u(s) \int_N^{N'} d_x f(s,y) \sin vy \, dy \; + \phi_u(M') \int_{M'}^\infty \int_N^{N'} d_x f(s,y) \sin vy \, dy \\ &= \int_M^{M'} \phi_u(s) \int_N^{N'} \left( \int_y^\infty d_{yx} f(s,t) \right) \sin vy \, dy \\ &\quad + \phi_u(M') \int_{M'}^\infty \int_N^{N'} \left( \int_y^\infty d_{yx} f(s,t) \right) \sin vy \, dy \\ &= \int_M^{M'} \int_N^{N'} \phi_u(s) \psi_v(t) \, d_{yx} f(s,t) + \psi_v(N') \int_M^{M'} \phi_u(s) \int_{N'}^\infty d_{yx} f(s,t) \\ &\quad + \phi_u(M') \int_{M'}^\infty \int_N^{N'} \psi_v(t) \, d_{yx} f(s,t) + \phi_u(M') \psi_v(N') \int_{M'}^\infty \int_{N'}^\infty d_{yx} f(s,t). \end{aligned} \tag{21}$$

Now it is clear that the first term of (21) tends to

$$\int_M^\infty \int_N^\infty \phi_u(s)\psi_v(t) d_{yx} f(s,t) = \int_M^\infty \int_N^\infty d_{yx} f(s,t) \int_N^t \sin vy\, dy \int_M^s \sin ux\, dx$$

as $M', N' \to \infty$. Therefore, the statement of Lemma 6 follows if we prove that the three remaining terms of (21) vanish as $M', N' \to \infty$. For the second term, we have

$$\begin{aligned} \left| \psi_v(N') \int_M^{M'} \phi_u(s) \int_{N'}^\infty d_{xy} f(s,t) \right| &\le \frac{2}{v} \int_M^{M'} \int_{N'}^\infty |\phi_u(s)\, d_{xy} f(s,t)| \\ &\le \frac{4}{uv} \int_M^\infty \int_{N'}^\infty |d_{yx} f(s,t)| = \frac{1}{uv} o(1), \end{aligned}$$

as $M + N' \to \infty$. The third term of (21) is estimated similarly:

$$\begin{aligned} \left| \phi_u(M') \int_{M'}^\infty \int_N^{N'} \psi_v(t)\, d_{yx} f(s,t) \right| &\le \frac{2}{u} \int_{M'}^\infty \int_N^{N'} |\psi_v(t)\, d_{yx} f(s,t)| \\ &\le \frac{4}{uv} \int_M^\infty \int_{N'}^\infty |d_{yx} f(s,t)| = \frac{1}{uv} o(1), \end{aligned}$$

as $M' + N \to \infty$. Finally, the last term of (21) is estimated as follows:

$$\left| \phi_u(M')\psi_v(N') \int_{M'}^\infty \int_{N'}^\infty d_{yx} f(s,t) \right| \le \frac{4}{uv} \int_{M'}^\infty \int_{N'}^\infty |d_{yx} f(s,t)| = \frac{1}{uv} o(1),$$

as $M' + N' \to \infty$, which concludes the proof. □

**Lemma 7.** *Let $f : \mathbb{R}_+^2 \to \mathbb{C}$ be of bounded variation on $\mathbb{R}_+^2$. Then, for $M, N \ge 0$,*

$$\left\| \int_M^\infty \int_N^\infty f(x,y) \sin ux \sin vy\, dx\, dy \right\|_\infty \le 9\varepsilon_{M,N},$$

*where $\varepsilon_{\mu,\nu}$ is defined by* (18).

*Proof.* Assume that $u, v \ne 0$, and let $M, N \ge 0$. By Lemma 6, we can write

$$\int_N^\infty \int_M^\infty f(x,y) \sin ux \sin vy\, dx\, dy = \int_M^\infty \int_N^\infty d_{yx} f(s,t) \int_N^t \sin vy\, dy \int_M^s \sin ux\, dx.$$

Now we distinguish four cases:

1. If $1/u \le M$, and $1/v \le N$, then

$$\begin{aligned} &\left| \int_M^\infty \int_N^\infty d_{yx} f(s,t) \int_N^t \sin vy\, dy \int_M^s \sin ux\, dx \right| \le \\ &\qquad \frac{4}{uv} \int_M^\infty \int_N^\infty |d_{yx} f(s,t)| \le 4\varepsilon_{M,N}. \end{aligned}$$

2. If $1/u \le M$, and $1/v > N$, then

$$
\begin{aligned}
&\left| \int_N^\infty \int_M^\infty f(x,y) \sin ux \sin vy \, dx \, dy \right| \\
=&\left| \int_N^{1/v} \int_M^\infty f(x,y) \sin ux \sin vy \, dx \, dy + \int_{1/v}^\infty \int_M^\infty f(x,y) \sin ux \sin vy \, dx \, dy \right| \\
\le&\left| \int_N^{1/v} \int_M^\infty \Big( - \int_x^\infty d_x f(s,y) \Big) \sin ux \sin vy \, dx \, dy \right| \qquad (22) \\
&+ \left| \int_{1/v}^\infty \int_M^\infty f(x,y) \sin ux \sin vy \, dx \, dy \right|. \qquad (23)
\end{aligned}
$$

We note that integral (23) is covered by Case 1, and therefore it is bounded above by $4\varepsilon_{M,N}$. An argument similar to the one of Lemma 6 shows us that (22) can be written as

$$
\left| \int_N^{1/v} \int_M^\infty d_x f(s,y) \int_M^s \sin ux \, dx \, \sin vy \, dy \right|,
$$

which can be estimated above by

$$
\begin{aligned}
\frac{2}{u} \int_N^{1/v} \int_M^\infty |d_x f(s,y) \sin vy| \, dy &\le 2Mv \int_N^{1/v} y \int_M^\infty \int_y^\infty |d_{yx} f(s,y)| \\
&\le 2\varepsilon_{M,N}.
\end{aligned}
$$

Collecting these estimates, we obtain

$$
\left\| \int_M^\infty \int_N^\infty f(x,y) \sin ux \sin vy \, dx \, dy \right\|_\infty \le 6\varepsilon_{M,N}. \qquad (24)
$$

3. If $1/u > M$ and $1/v \le N$, a similar argument as in Case 2 yields (24) again.

4. If $1/u > M$ and $1/v > N$:

$$
\begin{aligned}
&\left| \int_N^\infty \int_M^\infty f(x,y) \sin ux \sin vy \, dx \, dy \right| \\
=&\left| \int_N^{1/v} \int_M^{1/u} + \int_N^{1/v} \int_{1/u}^\infty + \int_{1/v}^\infty \int_M^{1/u} + \int_{1/v}^\infty \int_{1/u}^\infty f(x,y) \sin ux \sin vy \, dx \, dy \right| \\
=&\left| I_1(u,v) + I_2(u,v) + I_3(u,v) + I_4(u,v) \right|.
\end{aligned}
$$

We will estimate each of these four integrals separately. First of all, for $I_1$, we have

$$
\begin{aligned}
|I_1(u,v)| &\le \int_N^{1/v} \int_M^{1/u} |f(x,y)| ux \, vy \, dx \, dy \\
&\le uv \int_N^{1/v} \int_M^{1/u} xy \int_x^\infty \int_y^\infty |d_{yx}(s,t)| \, dx \, dy \le \varepsilon_{M,N}.
\end{aligned}
$$

For $I_2$, the estimate is obtained similarly as in Case 2:

$$|I_2(u,v)| = \left| \int_N^{1/v} \int_{1/u}^{\infty} d_x f(s,y) \int_{1/u}^{s} \sin ux \sin vy \, dx \, dy \right|$$
$$\leq \frac{2v}{u} \int_N^{1/v} y \int_{1/u}^{\infty} \int_y^{\infty} |d_{yx}(s,t)| \, dy \leq 2\varepsilon_{M,N}.$$

The estimate for $I_3$ is computed in a similar way as the one for $I_2$, yielding

$$|I_3(u,v)| \leq 2\varepsilon_{M,N}.$$

Finally, by Lemma 6, we can write

$$|I_4(u,v)| = \left| \int_{1/u}^{\infty} \int_{1/v}^{\infty} d_{yx} f(s,t) \int_{1/v}^{t} \sin vy \, dy \int_{1/u}^{s} \sin ux \, dx \right|$$
$$\leq \frac{4}{uv} \int_{1/u}^{\infty} \int_{1/v}^{\infty} |d_{yx} f(s,t)| \leq 4\varepsilon_{M,N}.$$

Collecting all the estimates for the integrals $I_j(u,v)$, $j = 1, \ldots, 4$, we conclude that in this case

$$\left\| \int_M^{\infty} \int_N^{\infty} f(x,y) \sin ux \sin vy \, dx \, dy \right\|_{\infty} \leq 9\varepsilon_{M,N}.$$

□

**Remark 6.** As it was already mentioned in Section 2, we emphasize that Lemmas 6 and 7 have their corresponding reformulation for functions in $HBV\big(\mathbb{R}_+^2 \backslash [0,c)^2\big)$ (cf. Definition 1'). In other words, if $f \in HBV\big(\mathbb{R}_+^2 \backslash [0,c)^2\big)$, these results also hold under the restriction $\max\{M,N\} \geq c$.

The following lemma provides an estimate for functions of the class $GMF^2_{adm}$.

**Lemma 8.** *Let $f \in GMF^2_{adm}$, and $x, y > 0$. Then, for any $u \in [x, 2x]$ and $v \in [y, 2y]$,*

$$|f(u,v)| \leq \frac{C}{xy} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^i 2^j} B\big(2^i x, 2^j y, I_{12}(f,\cdot,\cdot)\big), \tag{25}$$

*where $I_{12}$ is defined in* (15).

*Proof.* Let us consider $u_1, u_2 \in [x, 2x]$ and $v_1, v_2 \in [y, 2y]$. Then,

$$|f(u_1,v_1)| - |f(u_2,v_1)| - |f(u_1,v_2)| - |f(u_2,v_2)| \leq |\Delta_{11} f(u_1,v_1)|$$
$$\leq \int_x^{2x} \int_y^{2y} |d_{xy} f(s,t)|.$$

Hence,

$$|f(u_1, v_1)| \le \frac{C}{xy} B\big(x, y, I_{12}(f, \cdot, \cdot)\big) + |f(u_2, v_1)| + |f(u_1, v_2)| + |f(u_2, v_2)|.$$

Integrating both sides with respect to $u_2$ over $[x, 2x]$, and $v_2$ over $[y, 2y]$, and using property (b) of the operator $B$, we obtain

$$xy|f(u_1, v_1)| \le CB\big(x, y, I_{12}(f, \cdot, \cdot)\big) + xI_2(f, y; u_1) + yI_1(f, x; v_1) + I_{12}(f, x, y);$$

$$|f(u_1, v_1)| \le \frac{C}{xy} B\big(x, y, I_{12}(f, \cdot, \cdot)\big) + \frac{1}{y} I_2(f, y; u_1) + \frac{1}{x} I_1(f, x; v_1), \tag{26}$$

where

$$I_1(f, x; v) = \int_x^{2x} |f(s, v)|\, ds, \quad I_2(f, y; u) = \int_y^{2y} |f(u, t)|\, dt.$$

Finally, we estimate the terms $I_2(f, y; u_1)/y$ and $I_1(f, x; v_1)/x$:

$$\begin{aligned}
\frac{1}{y} I_2(f, y; u_1) &= \frac{1}{y} \int_y^{2y} |f(u_1, w)|\, dw = \frac{1}{y} \int_y^{2y} \left| \int_{u_1}^{\infty} d_x f(s, w) \right| dw \\
&\le \frac{1}{y} \int_y^{2y} \left( \int_x^{\infty} |d_x f(s, w)| \right) dw = \frac{1}{y} \int_y^{2y} \left( \int_x^{\infty} \left| \int_w^{\infty} d_{yx} f(s, t) \right| \right) dw \\
&\le \frac{1}{y} \int_y^{2y} \left( \int_x^{\infty} \int_y^{\infty} |d_{xy} f(s, t)| \right) dw = \int_x^{\infty} \int_y^{\infty} |d_{xy} f(s, t)| \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_{2^i x}^{2^{i+1} x} \int_{2^j y}^{2^{j+1} y} |d_{xy} f(s, t)| \le C \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \beta(2^i x, 2^j y) \\
&= \frac{C}{xy} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^i 2^j} B\big(2^i x, 2^j y, I_{12}(f, \cdot, \cdot)\big).
\end{aligned} \tag{27}$$

Similarly, we have the same bound for the term $I_1(f, x; v_1)/x$. Therefore, combining (26) and (27), we obtain (25). □

# 7 Proofs

*Proof of Theorem 1.* To prove the uniform regular convergence of (1), we must verify that

$$\int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y) \sin ux \sin vy\, dx\, dy \to 0 \quad \text{as } \max\{x_0, y_0\} \to \infty,\ x_0 < x_1,\ y_0 < y_1.$$

Let us denote, for $a, b > 0$, the residual rectangular integral as

$$R_{uv}(f; a, b) := \int_a^{\infty} \int_b^{\infty} f(x, y) \sin ux \sin vy\, dx\, dy.$$

Then,

$$\begin{aligned}
&\left|\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\sin ux \sin vy\,dx\,dy\right| \\
&= \left|R_{uv}(f;x_0,y_0) - R_{uv}(f;x_0,y_1) - R_{uv}(f;x_1,y_0) + R_{uv}(f;x_1,y_1)\right| \\
&\le \left|R_{uv}(f;x_0,y_0)\right| + \left|R_{uv}(f;x_0,y_1)\right| + \left|R_{uv}(f;x_1,y_0)\right| + \left|R_{uv}(f;x_1,y_1)\right|. \quad (28)
\end{aligned}$$

By Lemma 7, we can estimate (28) above by $4\cdot 9\varepsilon_{x_0,y_0}$. By (16) and Remark 5, $\varepsilon_{x_0,y_0}$ tends to zero uniformly in $u,v$ as $\max\{x_0,y_0\}\to\infty$, and thus the uniform convergence of (1) in the regular sense follows. Finally, it remains to estimate the $L^\infty$ error when approximating $\mathcal{F}(u,v)$ by $S_{M,N}(u,v)$:

$$\begin{aligned}
\left\|\mathcal{F}(u,v) - S_{M,N}(u,v)\right\|_\infty =&\left\|R_{uv}(f;0,N) + R_{uv}(f;M,0) - R_{uv}(f;M,N)\right\|_\infty \\
\le&\left\|R_{uv}(f;0,N)\right\|_\infty + \left\|R_{uv}(f;M,0)\right\|_\infty \\
&+ \left\|R_{uv}(f;M,N)\right\|_\infty \\
\le&9(\varepsilon_{0,N} + \varepsilon_{M,0} + \varepsilon_{M,N}),
\end{aligned}$$

where the last inequality follows from Lemma 7. □

*Proof of Corollary 2.* Since $xy|f(x,y)|\to 0$ as $\max\{x,y\}\to\infty$, we have that

$$\begin{aligned}
I_{12}(f,x,y) &= \int_x^{2x}\int_y^{2y} |f(s,t)|\,ds\,dt \\
&\le xy \sup_{(s,t)\in[x,2x]\times[y,2y]} |f(s,t)| \to 0 \quad \text{as } \max\{x,y\}\to\infty. \qquad (29)
\end{aligned}$$

Since the operator $B$ is admissible, by property (a) of Definition 3 and (29), we have that

$$B\big(x,y,I_{12}(f,\cdot,\cdot)\big)\to 0 \quad \text{as } \max\{x,y\}\to\infty.$$

The latter condition precisely means that

$$\beta(x,y) = o\left(\frac{1}{xy}\right) \quad \text{as } \max\{x,y\}\to\infty.$$

Hence, we are under the conditions of Theorem 1, and the uniform convergence of (1) in the regular sense follows, and moreover the estimate (17) holds. Finally, since $f\in GMF^2_{adm}$, combining (14) and (18), we have that for any $M,N>0$

$$\varepsilon_{M,N} \le C \sup_{\substack{M'\ge M\\ N'\ge N}} \frac{M'N'}{M'N'} B\big(M',N',I_{12}(f,\cdot,\cdot)\big) = C\delta_{M,N}.$$

Thus, substituting the latter inequality into (17), we obtain (19). □

*Proof of Theorem 3.* Let $\varepsilon>0$. From the uniform regular convergence of (1), we have that there exists $z>0$ such that

$$\left|\int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x,y)\sin ux\sin vy\,dy\,dx\right| < \varepsilon, \quad \forall u,v\in\mathbb{R},$$

whenever $\max\{x_0, y_0\} > z$, and $x_0 < x_1$, $y_0 < y_1$. If $x_0, y_0 > 0$, setting $x_1 = 2x_0$, $y_1 = 2y_0$, and choosing $u = \pi/4x_0$, $v = \pi/4y_0$, we obtain

$$\begin{aligned}
\varepsilon &> \left| \int_{x_0}^{2x_0} \int_{y_0}^{2y_0} f(x,y) \sin ux \sin vy \, dy \, dx \right| \\
&\geq \frac{4}{\pi^2} \int_{x_0}^{2x_0} \int_{y_0}^{2y_0} f(x,y) \frac{\pi x}{4x_0} \frac{\pi y}{4y_0} \, dx \, dy \geq \frac{1}{4} \int_{x_0}^{2x_0} \int_{y_0}^{2y_0} f(x,y) \, dx \, dy \\
&= \frac{1}{4} I_{12}(f, x_0, y_0).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $I_{12}(f, x, y) \to 0$ as $\max\{x, y\} \to \infty$, and consequently, by property (a) of the operator $B$,

$$B\big(x, y, I_{12}(f, \cdot, \cdot)\big) \to 0 \quad \text{as } \max\{x, y\} \to \infty.$$

Finally, the result follows by Lemma 8, since

$$\begin{aligned}
xy|f(x,y)| &\leq C \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^i 2^j} B\big(2^i x, 2^j y, I_{12}(f, \cdot, \cdot)\big) \\
&\leq C \sup_{\substack{i \geq 0 \\ j \geq 0}} B\big(2^i x, 2^j y, I_{12}(f, \cdot, \cdot)\big) \to 0 \quad \text{as } \max\{x, y\} \to \infty.
\end{aligned}$$

□

*Proof of Theorem 4.* From the uniform Pringsheim convergence of (1), applying the Cauchy criterion, it follows that for every $\varepsilon > 0$, there exists $z > 0$ such that

$$\left| \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x,y) \sin ux \sin vy \, dy \, dx \right| < \varepsilon, \quad \forall u, v \in \mathbb{R},$$

whenever $\min\{x_0, y_0\} > z$, and $x_0 < x_1$, $y_0 < y_1$. The rest of the proof is the same as for Theorem 3, replacing $\max\{x, y\}$ for $\min\{x, y\}$, and using the weaker property of $B$ stated in Theorem 4, instead of property (a) of Definition 3. □

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