Uniform convergence of sine transforms of general monotone functions

A. Debernardi*

Centre de Recerca Matemàtica and Universitat Autònoma de Barcelona

08193, Bellaterra, Barcelona, Spain

AMS 2010 Primary subject classification: 42A38. Secondary: 26A45, 26A48, 40A10.

Abstract

We obtain necessary and sufficient conditions for the uniform convergence of sine integrals

$$\int_0^\infty g(t)\sin\xi t\,dt,$$

where g satisfies general monotonicity conditions. In contrast with the previous results on this topic, here we do not assume $g\geq 0$.

1 Introduction

Throughout this paper we denote by $F(\xi)$ the cosine transform

$$F(\xi)=\int_0^\infty f(t)\cos\xi t\,dt,\quad \xi\in\mathbb{R}, \tag{1}$$

and by $G(\xi)$ the sine transform

$$G(\xi)=\int_0^\infty g(t)\sin\xi t\,dt,\quad \xi\in\mathbb{R}, \tag{2}$$

whenever they exist in the improper sense. We assume that $f,g:\mathbb{R}_+\to\mathbb{C}$ (here $\mathbb{R}_+:=(0,+\infty)$) are locally of bounded variation. Moreover, in order to guarantee the existence of integrals (1) and (2) near the origin, we suppose that $f(t)\in L^1(0,1)$ and $tg(t)\in L^1(0,1)$.

*E-mail: adebernardi@crm.cat

This research was supported by the Centre de Recerca Matemàtica in Barcelona and the MTM2014–59174–P grant.

First, let us summarize some known results related to uniform convergence of trigonometric series, which in part motivates our work. In general, we can only guarantee the uniform convergence of

$$\sum_{n=1}^{\infty} a_n \sin nx, \tag{3}$$

in the trivial case $\sum |a_n| < \infty$. However, imposing restrictions on $\{a_n\}$ has proved to be convenient, not only to study pointwise convergence of (3), but also its uniform convergence, or integrability, among other topics.

The study of uniform convergence of sine series and integrals has recently attracted several authors due to the number of generalizations of sequence and function classes during the last years. However, it was back in 1916 when this problem was first considered: Chaundy and Jolliffe [1], [19, V. I, p. 182] obtained the following characterization for monotone sequences.

Theorem 1.1. *Let $a_n \geq 0$ be monotonically decreasing to zero. Then, the series*

$$\sum_{n=1}^{\infty} a_n \sin nx$$

converges uniformly in $[0, 2\pi]$ if and only if $na_n \to 0$.

In view of this result, one may ask whether the hypothesis of $\{a_n\}$ being monotone can be weakened in order to generalize Theorem 1.1. In [8], Leindler introduced the class of *sequences with rest of bounded variation*, in short $RBVS$. These sequences satisfy

$$\sum_{k=n}^{\infty} |a_k - a_{k+1}| \leq C|a_n| \quad \text{for all } n, \tag{4}$$

where $C > 0$ is independent of n. This idea was generalized by Tikhonov [18], replacing the infinite sum of (4) by $\sum_{k=n}^{2n} |a_k - a_{k+1}|$. Such sequences are called *general monotone sequences*, in short GMS. Later on, the GMS condition was further generalized by considering non-negative sequences β_n instead of $|a_n|$ on the right hand side of (4). A sequence $\{a_n\}$ satisfying

$$\sum_{k=n}^{2n} |a_k - a_{k+1}| \leq C\beta_n \quad \text{for all } n. \tag{5}$$

is called *β-general monotone sequence* (or $\{a_n\} \in GMS(\beta)$, to shorten). Such an extension of monotone sequences, as mentioned earlier, has led to several generalizations of Theorem 1.1 in the past few years (c.f. [4], [17], [18], and [5], among others). Very recently, Feng, Totik, and Zhou proved an analogue of Theorem 1.1 [5, Theorem 3.1] for the $GMS(\beta)$ class defined by

$$\beta_n = \frac{1}{n} \sum_{k=n/\lambda}^{\lambda n} |a_k|, \quad \text{for some } \lambda > 2,$$

without assuming that $a_k \geq 0$ (in fact, the same result with $a_k \geq 0$ was proved earlier in [17]).

We also mention the generalization of $GMS(\beta)$ introduced by Szal [16], where he considered differences $|a_k - a_{k+r}|$ with $r \in \mathbb{N}$ in place of $|a_k - a_{k+1}|$ on the left hand side of (5).

As one could expect, there is an analogue of Theorem 1.1 for sine integrals (see [12]).

Theorem 1.2. *Let $g : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ be a monotonically decreasing function. Then, integral* (2) *converges uniformly if and only if $xg(x) \to 0$ as $x \to \infty$.*

The concept of general monotonicity for functions was introduced in [11], and it is further discussed together with the known generalizations of Theorem 1.2 in Section 2. We say that a function g belongs to $GM(\beta)$ if

$$\int_x^{2x} |dg(s)| \leq C\beta(x) \quad \text{for all } x > 0,$$

where

$$V_a^b(g) = \int_a^b |dg(s)| := \sup_{\mathcal{P}} \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)|$$

is the total variation of g in $[a,b]$, and $\mathcal{P}$ is the set of all partitions $\{a = x_0 < \cdots < x_n = b\}$ of $[a,b]$. For a survey of general monotone functions and sequences, we refer to [9].

Our main goal is to generalize Theorem 1.2 for functions of a larger GM class containing the ones already considered in [2] or [9]. To this end, we rely on the technique developed in [5] that we mentioned before.

The paper is organized as follows. In Section 2 we review the known generalizations of Theorem 1.2 and introduce a new GM class of functions. In Section 3, we present our main result (Theorem 3.1), and discuss its sharpness. Section 4 is devoted to the proofs of main results. Finally, we give in Section 5 several examples of $GM(\beta)$ classes we deal with, as well as we prove that the new GM class is strictly larger than the previously known ones.

2 A new class of general monotone functions

Let us consider the $GM(\beta_0)$ class of functions defined by

$$\beta_0(x) := \frac{1}{x} \sup_{s \geq x/c} \int_s^{2s} |g(t)|\, dt, \quad c > 1.$$

Such class was introduced by Dyachenko, Liflyand and Tikhonov in [2], where they proved the following criteria.

Theorem 2.1. *[2, Theorem 2] Let $f \in GM(\beta)$. If either*

(i) $f \geq 0$, or

(ii) $\beta(x) = o(1/x)$ *as* $x \to \infty$,

then (1) *converges uniformly on* $\mathbb{R}$ *if and only if* $\int_0^\infty f(t)\,dt < \infty$.

Theorem 2.2. *[2, Theorem 3]*

(i) Let $g \in GM(\beta)$. *If* $\beta(x) = o(1/x)$ *as* $x \to \infty$, *then* (2) *converges uniformly on* $\mathbb{R}$.

(ii) If $g \in GM(\beta_0)$ *is a non-negative function and* (2) *converges uniformly, then* $xg(x) \to 0$ *as* $x \to \infty$.

It is not hard to see that if $\beta_0(x) = o(1/x)$ as $x \to \infty$, then $xg(x) \to 0$ as $x \to \infty$ (it actually follows from Lemma 4.1 below). Combining this fact with Theorem 2.2 we have the following extension of Theorem 1.2.

Corollary 2.3. *[2] Let* $g \in GM(\beta_0)$ *be non-negative. Then, a necessary and sufficient condition for integral* (2) *to converge uniformly is that* $xg(x) \to 0$ *as* $x \to \infty$.

We now proceed to define a class of functions extending $GM(\beta_0)$.

Definition 2.4. Let $\mathcal{M}$ be the space of nonnegative functions defined on $\mathbb{R}_+$. We say that an operator $B : \mathcal{M} \to \mathcal{M}$ is *admissible* if for any $\varphi \in \mathcal{M}$, the function $B(\cdot, \varphi)$ satisfies the following properties:

(i) If φ vanishes at infinity, $B(\cdot, \varphi)$ also vanishes at infinity.

(ii) If φ is bounded at infinity, $B(\cdot, \varphi)$ is also bounded at infinity.

(iii) For every $x > 0$, $\varphi(x) \le B(x, \varphi)$.

(iv) $B(x, \varphi)$ is decreasing on x.

Let us denote, for simplicity, $I(x) := I(x, g) = \int_x^{2x} |g(t)|\,dt$. Then, we can rewrite β_0 as

$$\beta_0(x) = \frac{1}{x} \sup_{s \ge x/c} I(s),$$

and note that $B(x, I) = \sup_{s \ge x/c} I(s)$ is an admissible operator.

The GM class we aim to define is obtained by means of admissible operators.

Definition 2.5. We say that $g \in GM_{\mathrm{adm}}$ if there exists an admissible operator B such that $g \in GM(\beta)$, where

$$\beta(x) = \frac{1}{x} B(x, I).$$

It is clear that $GM(\beta_0) \subset GM_{\mathrm{adm}}$, whilst $GM(\beta_0) \subsetneq GM_{\mathrm{adm}}$ is shown in Proposition 5.2. Also, if we define, for any admissible B,

$$\beta_B(x) = \frac{B(x, I)}{x},$$

then it is clear that

$$GM_{\text{adm}} = \bigcup_{B \text{ admissible}} GM(\beta_B).$$

Remark 2.6. In Definition 2.4, we do not lose generality by assuming conditions (iii)–(iv). Indeed, if φ is non-negative and we define

$$\widetilde{B}(x,\varphi) = \sup_{y\geq x} \max\big\{\varphi(y), B(y,\varphi)\big\},$$

then $\widetilde{B}(x,\varphi)$ satisfies (i)–(iv) whenever $B(x,\varphi)$ satisfies (i)–(ii). Therefore, denoting

$$\beta(x) = \frac{1}{x}B(x,I), \quad \tilde{\beta}(x) = \frac{1}{x}\widetilde{B}(x,I),$$

one has $GM(\beta) \subset GM(\tilde{\beta})$.

In Section 5 the reader can find several examples of admissible B.

3 Main results

3.1 Uniform convergence of sine transforms

The counterpart of Theorem 1.1 for the class GM_{adm} reads as follows.

Theorem 3.1. *Let $g \in GM_{\text{adm}}$, and assume that $I(x)$ is bounded at infinity. Then, a necessary and sufficient condition for* (2) *to converge uniformly on $\mathbb{R}$ is that*

$$x|g(x)| \to 0 \quad \text{as } x \to \infty.$$

As it is noted in Remark 4.2 below, the hypothesis of $I(x)$ being bounded at infinity is not needed if we assume that g is non-negative. The following result is an analogue of [3, Theorem 3], and shows the sharpness of Theorem 3.1 with respect to the aforementioned hypothesis.

Theorem 3.2. *There exists a uniformly converging sine integral $\int_0^\infty g(t)\sin\xi t\,dt$ such that*

$$x|g(x)| \to \infty \quad \text{as } x \to \infty$$

and

$$\int_x^{2x} |g(t)|\,dt \to \infty \quad \text{as } x \to \infty.$$

Note that if g is such that $\int_x^{2x}|g(t)|\,dt$ is not bounded at infinity, then formally $g \in GM(\beta_0)$.

For the sake of completeness, we present an analogue of [5, Theorem 3.1], though we omit its proof since it can be derived easily by combining the proofs of the result we just cited and Theorem 3.1.

Theorem 3.3. *Let $g \in GM(\beta)$ be a real-valued function, where*

$$\beta(x) = \frac{1}{x} \int_{x/\lambda}^{\lambda x} |g(t)|\, dt. \tag{6}$$

Then, a necessary and sufficient condition for the sine integral (2) *to converge uniformly on $\mathbb{R}$ is that*

$$x|g(x)| \to 0 \quad \text{as } x \to \infty.$$

In contrast with Theorem 3.1, we do not need to assume that integrals $\int_{x/\lambda}^{\lambda x} |g(t)|\, dt$ are bounded at infinity in Theorem 3.3. We emphasize that the $GM(\beta)$ class with β as in (6) has proved to be very convenient to replace the class of monotone functions, for instance, to prove Boas' conjecture (cf. [6, 10]), or to prove Theorem 3.3, where we do not even need to assume that $g \geq 0$.

4 Proofs

In order to prove Theorem 3.1, we need the following estimate.

Lemma 4.1. *Let $g \in GM_{\mathrm{adm}}$. Then, for every $x > 0$ and $u \in [x, 2x]$,*

$$|g(u)| \leq C_0 \frac{B(x, I)}{x}.$$

Proof. Let $u, v \in [x, 2x]$. It is clear that

$$|g(u)| - |g(v)| \leq |g(u) - g(v)| \leq \int_x^{2x} |dg(s)| \leq C \frac{B(x, I)}{x}.$$

Integrating both sides with respect to v over $[x, 2x]$, and using property (iii) of $BI(x)$, we get

$$x|g(u)| \leq CB(x, I) + \int_x^{2x} |g(v)|\, dv = CB(x, I) + I(x) \leq C_0 B(x, I),$$

which establishes the estimate. □

Proof of Theorem 3.1. Sufficiency immediately follows from (i) of Theorem 2.2, together with property (i) of $BI(x)$. In order to prove necessity, we adapt the technique developed for trigonometric series in [5, Theorem 3.1] to the context of trigonometric integrals. Our goal is to prove that $I(x) \to 0$ as $x \to \infty$. Once this is done, the result will simply follow by property (i) of $BI(x)$ and Lemma 4.1.

For any $x > 0$, let

$$A(x) := \{ y \in [x, 2x] : |g(y)| \geq I(x)/2x \}.$$

By definition of $A(x)$ and Lemma 4.1, we have the following estimate for $I(x)$:

$$I(x) = \left(\int_{[x,2x]\setminus A(x)} |g(t)|\,dt + \int_{A(x)} |g(t)|\,dt \right) \le \frac{I(x)}{2} + \int_{A(x)} |g(t)|\,dt$$
$$\le \frac{I(x)}{2} + C_0|A(x)|\frac{B(x,I)}{x},$$

where $|A(x)|$ denotes the measure of the set $A(x)$. It follows from the latter estimate that

$$|A(x)| \ge \frac{x}{2C_0} \cdot \frac{I(x)}{B(x,I)},$$

and consequently,

$$\int_{A(x)} |g(t)|\,dt \ge |A(x)|\frac{I(x)}{2x} \ge \frac{1}{4C_0} \cdot \frac{I(x)^2}{B(x,I)}. \tag{7}$$

Since the integral $\int_0^\infty g(t)\sin\xi t\,dt$ converges uniformly, for a fixed $\varepsilon > 0$ there exists $y > 0$ such that

$$\left| \int_{y_1}^{y_2} g(t)\sin\xi t\,dt \right| < \varepsilon, \quad \text{if } y \le y_1 \le y_2, \quad \xi \in \mathbb{R}. \tag{8}$$

Now we can choose $x \ge y$ such that $I(x) > 0$. Indeed, if such x did not exist, it would mean that $I(x) = 0$ for all $x \ge y$, and our assertion would be trivial. Notice that $g(x)$ is bounded at infinity; this follows from Lemma 4.1, along with the fact that $I(x)$ is bounded at infinity and property (ii) of $BI(x)$. Thus, there exists $\delta = \delta(\varepsilon, x)$ such that

$$\int_w^{w+\delta} |g(t)|\,dt \le \varepsilon, \text{ for all } w \ge x. \tag{9}$$

For example, take $\delta = \min\{\delta', x\}$, where $\delta' = \varepsilon / \sup_{t\ge x} |g(t)|$.

Our next goal is, roughly speaking, to "cover" the set $A(x)$ by almost disjoint intervals S_j. More precisely, we look for a collection $\{S_j\}_{j=1}^n$ (here and from now on $n = n(x)$) such that $|S_j \cap S_k| = 0$ whenever $j \ne k$ and $\left|A(x)\setminus\left(S_1 \cup \ldots \cup S_n\right)\right| = 0$, or in other words,

$$A(x) \subset \left(\bigcup_{j=1}^n S_j \right) \cup E(x),$$

where $|E(x)| = 0$. For such a collection, one has

$$\int_{A(x)} |g(t)|\,dt \le \sum_{j=1}^n \int_{S_j} |g(t)|\,dt.$$

We proceed to construct the intervals $S_j = [u_j, \nu_j]$ as follows: let $u_1 = \inf A(x)$.

(1) If there exists $u_1 < v_1 \leq 2x$ such that g has constant sign[1] in $(u_1, v_1]$, and $|g(z)| > I(x)/4x$ for every $z \in (u_1, v_1)$, while $|g(v_1)| \leq I(x)/4x$, we define $\nu_1 = v_1 + \delta$, with δ as above.

(2) If there is no $v_1 \in (u_1, 2x]$ satisfying all the properties described in (1), let

$$z_1 = \inf\{w \in [u_1, 2x] : g(u_1)g(w) \leq 0\}.$$

If such z_1 exists, we define $\nu_1 = z_1 + \delta$.

(3) If neither v_1 nor z_1 exist, let $\nu_1 = 2x$.

We set $S_1 = [u_1, \nu_1]$, and if $A(x)\backslash S_1$ has positive measure, we define $u_2 = \inf A(x)\backslash S_1$. By the same procedure, we find ν_2 and define $S_2 = [u_2, \nu_2]$, and so on until we reach n such that

$$\big|A(x)\backslash\big(S_1 \cup \ldots \cup S_n\big)\big| = 0.$$

Let $1 \leq j < n$. We now prove that

$$\int_{u_j}^{\nu_j} |dg(s)| \geq \frac{I(x)}{4x},$$

which will allow us to obtain an upper estimate for n.

(1) Assume first ν_j was chosen by case (1). Note that there exists[2] $y \in [u_j, v_j)$ such that $|g(y)| \geq I(x)/2x$, whilst $|g(v_j)| \leq I(x)/4x$. Thus,

$$\int_{u_j}^{\nu_j} |dg(s)| \geq |g(y) - g(v_j)| = |g(y)| - |g(v_j)| \geq \frac{I(x)}{4x}.$$

(2) Assume ν_j was chosen by case (2). Similarly as in case (1), there exists $y \in [u_j, u_j + \delta)$ such that $|g(y)| \geq I(x)/2x$. Since

$$z_j = \inf\{w \in [u_j, 2x] : g(u_j)g(w) \leq 0\},$$

there must exist $z \in [u_j, z_j + \delta)$ such that $g(z)g(y) \leq 0$. Indeed, if this z does not exist, then $g(y)g(z) > 0$ for all $z \in [u_j, z_j + \delta)$, and in particular, $g(y)g(u_j) > 0$. But this implies that $g(u_j)g(z) > 0$ for all $z \in [u_j, z_j + \delta)$, or in other words, g has constant sign in the latter interval. Hence,

$$\inf\{w \in [u_j, 2x] : g(u_j)g(w) \leq 0\} > z_j,$$

which is a contradiction. Therefore, we conclude

$$\int_{u_j}^{\nu_j} |dg(s)| \geq |g(y) - g(z)| \geq \frac{I(x)}{2x} \geq \frac{I(x)}{4x}.$$

[1]We will always consider that g has constant sign in a set X if and only if $g(x_1)g(x_2) > 0$ for all $x_1, x_2 \in X$.

[2]For any set $A \subset \mathbb{R}$, we have that $m = \inf A$ if and only if (a) m is a lower bound of A, and (b) for every $m' > m$ there exists $x \in A$ such that $x < m'$. In our case, we can find $y \geq u_j$ with $y \in A(x)$, i.e., $|g(y)| \geq I(x)/2x$.

Finally, it is only left to remark that if ν_j is chosen by case (3), then $j = n$. We can now proceed estimating n from above (when $n > 1$). By the GM property, property (iv) of $BI(x)$, and the fact that $\delta \leq x$, we have

$$\begin{aligned} 2\frac{C}{x}B(x,I) &\geq \frac{C}{2x}B(2x,I) + \frac{C}{x}B(x,I) \geq \int_{2x}^{4x} |dg(s)| + \int_{x}^{2x} |dg(s)| \\ &\geq \int_{x}^{2x+\delta} |dg(s)| \geq \sum_{j=1}^{n-1} \int_{u_j}^{\nu_j} |dg(s)| \geq (n-1)\frac{I(x)}{4x}. \end{aligned}$$

Thus,

$$n \leq \frac{8Cx}{x}\frac{B(x,I)}{I(x)} + 1 \leq 9C\frac{B(x,I)}{I(x)}. \tag{10}$$

We note that if $n = 1$, inequality (10) is trivially true.

Let now $\xi = \pi/8x$. Then, it holds that $\xi t \leq \pi/2$ for all $t \in [x, 4x]$, so that $\sin \xi t \geq 1/4$ on the latter interval. By (9) and the fact that for any $1 \leq j \leq n$, g has constant sign in $(u_j, \nu_j - \delta)$ (by construction), it follows from (8) and (9) that

$$\begin{aligned} \frac{1}{4}\int_{u_j}^{\nu_j} |g(t)|\, dt &= \frac{1}{4}\left(\int_{u_j}^{\nu_j-\delta} |g(t)|\, dt + \int_{\nu_j-\delta}^{\nu_j} |g(t)|\, dt\right) \\ &\leq \left|\int_{u_j}^{\nu_j-\delta} g(t)\sin \xi t\, dt\right| + \frac{\varepsilon}{4} < \varepsilon + \frac{\varepsilon}{4}. \end{aligned}$$

Therefore, for any $1 \leq j \leq n$,

$$\int_{u_j}^{\nu_j} |g(t)|\, dt < 5\varepsilon. \tag{11}$$

Since $\big|A(x)\backslash\big(S_1 \cup \ldots \cup S_n\big)\big| = 0$, summing up on j the integrals of (11), it follows together with (10) that

$$\int_{A(x)} |g(t)|\, dt \leq \sum_{j=1}^{n} \int_{u_j}^{\nu_j} |g(t)|\, dt < 5n\varepsilon \leq 45C\frac{B(x,I)}{I(x)}\varepsilon. \tag{12}$$

Finally, combining (7) and (12), we obtain

$$\frac{1}{4C_0}\cdot\frac{I(x)^2}{B(x,I)} \leq 45C\frac{B(x,I)}{I(x)}\varepsilon; \qquad \frac{I(x)^3}{B(x,I)^2} \leq 180CC_0\varepsilon,$$

so that the left-hand side tends to zero as x tends to infinity. Moreover, since $BI(x)$ is bounded for x large enough (by property (ii) of $B(x,I)$), then $I(x)^3 \to 0$ as $x \to \infty$, as required. $\square$

Remark 4.2. Notice that if the function g satisfies $g(x) \geq 0$ for all $x > x_0$, then we do not need to assume the boundedness of $I(x)$: if $\xi = \pi/4x$, it follows from (8) that

$$\varepsilon > \left| \int_x^{2x} g(t) \sin \frac{\pi}{4x} t \, dt \right| \geq \frac{1}{2} \left| \int_x^{2x} g(t) \, dt \right| = \frac{1}{2} \int_x^{2x} |g(t)| \, dt = \frac{I(x)}{2},$$

which, moreover, trivially proves that $I(x) \to 0$ as $x \to \infty$. Thus, it is easier to prove Theorem 3.1 if we assume $g \geq 0$. On the other hand, the sufficiency condition of Theorem 2.2 does not require any assumption on the sign of g (in this case g can even be complex-valued).

In order to prove Theorem 3.2, we make use of the *Rudin-Shapiro sequence* [7, 13, 15]. The following is a well-known result [13, Theorem 1], also referred to as Rudin-Shapiro's lemma:

Lemma 4.3. *There exists a sequence* $\{\varepsilon_n\}$, $\varepsilon_n = \pm 1$, *such that*

$$\left| \sum_{n=0}^{N} \varepsilon_n e^{in\xi} \right| < 5\sqrt{N+1} \tag{13}$$

for all $\xi \in [0, 2\pi]$ *and* $N \in \mathbb{N}$.

The sequence $\{\varepsilon_n\}$ from the previous lemma is the aforementioned Rudin-Shapiro sequence, and the trigonometric polynomials on the left hand side of (13) are known as the *Rudin-Shapiro polynomials*. We need the following counterpart of Lemma 4.3 for trigonometric integrals.

Lemma 4.4 (Rudin-Shapiro's lemma for Fourier integrals)**.** *There exists a function* $h : [0, +\infty) \to \{-1, 1\}$ *such that*

$$\left| \int_0^M h(x) e^{ix\xi} \, dx \right| < 6\sqrt{M} \tag{14}$$

for every $\xi \in \mathbb{R}$ *and* $M > 0$.

Proof. The function h we are looking for is obtained by means of the Rudin-Shapiro sequence: for $n \in \mathbb{N} \cup \{0\}$, we define

$$h(x) = \varepsilon_n, \quad \text{if } x \in [n, n+1).$$

Our assertion is trivial for $M < 1$, so we can assume that $M \geq 1$. Also, observe that we only need to consider the case $\xi \geq 0$. Let $N = [M]$, where $[\cdot]$ denotes the floor function. If $\xi = 0$, it follows from Lemma 4.3 that

$$\left| \int_0^M h(x) \, dx \right| \leq \left| \sum_{n=0}^{N-1} \varepsilon_n \right| + \left| \int_N^M h(x) \, dx \right| < 6\sqrt{M}.$$

On the other hand, if $\xi > 0$, we have the following identity.

$$\int_0^N h(x)e^{ix\xi}\,dx = \sum_{n=0}^{N-1}\int_n^{n+1} h(x)e^{ix\xi}\,dx = \frac{e^{i\xi}-1}{i\xi}\sum_{n=0}^{N-1}\varepsilon_n e^{in\xi}.$$

Thus, by Lemma 4.3 we have

$$\left|\int_0^N h(x)e^{ix\xi}\,dx\right| < 5\sqrt{N}\left|\frac{e^{i\xi}-1}{\xi}\right| \le 5\sqrt{N},$$

Finally,

$$\left|\int_0^M h(x)e^{ix\xi}\,dx\right| \le \left|\int_0^N h(x)e^{ix\xi}\,dx\right| + \left|\int_N^M h(x)e^{ix\xi}\,dx\right| < 6\sqrt{M},$$

which completes the proof. □

Remark 4.5. Note that estimate (14) is not optimal; we could improve it, for instance, by considering sharper estimates of (13). It is known that if we write $C\sqrt{N+1}$ on the right hand side of (13), then the optimal C lies between $\sqrt{6}$ and $(2+\sqrt{2})\sqrt{3/5}$ (see [14], as well as the references therein).

Proof of Theorem 3.2. Let us define $c_n = n^{-2}2^{-n/2}$, and let h be the *Rudin-Shapiro function* (i.e., the function we defined in Lemma 4.4). We prove that the Fourier integral given by

$$\int_0^1 h(t)e^{it\xi}\,dt + \sum_{n=1}^{\infty} c_n \int_{2^{n-1}}^{2^n} h(t)e^{it\xi}\,dt = \int_0^\infty g(t)e^{it\xi}\,dt, \tag{15}$$

converges uniformly, where $g(t) = c_n h(t)$ for all $t \in [2^{n-1}, 2^n)$ with $n \ge 1$, and $g(t) = h(t)$ for $t \in [0,1)$. Fix $n \ge 1$ and let $2^{n-1} \le z_1 < z_2 \le 2^n$. Then, by Lemma 4.4,

$$\begin{aligned}\left|\int_{z_1}^{z_2} g(t)e^{it\xi}\,dt\right| &\le c_n\left(\left|\int_0^{z_1} h(t)e^{it\xi}\,dt\right| + \left|\int_0^{z_2} h(t)e^{it\xi}\,dt\right|\right)\\ &\lesssim n^{-2}2^{-n/2}\cdot 2^{n/2} = n^{-2}.\end{aligned}$$

Thus, for arbitrary $y_1 < y_2$, we have

$$\left|\int_{y_1}^{y_2} g(t)e^{it\xi}\,dt\right| \le \sum_{k=n_1}^{n_2}\frac{1}{k^2} \to 0 \quad \text{as } y_2 > y_1 \to \infty,$$

where $n_1 = \max\{n \in \mathbb{N} : 2^n \le y_1\}$ and $n_2 = \min\{n \in \mathbb{N} : 2^n \ge y_2\}$. Thus, the uniform convergence of (15) follows. However, the integrals

$$\int_x^{2x} |g(t)|\,dt$$

are not bounded at infinity. Indeed, fix $x > 1$ and let $n \in \mathbb{N} \cup \{0\}$ be such that $2^n \leq x < 2^{n+1}$. Then,

$$\int_x^{2x} |g(t)|\,dt \geq x c_{n+1} \gtrsim n^{-2} 2^{n/2} \to \infty \quad \text{as } n \to \infty.$$

Furthermore, with n and x as above, we have

$$x|g(x)| \geq 2^n c_n = n^{-2} 2^{n/2} \to \infty \quad \text{as } n \to \infty.$$

To conclude the proof we only have to note that since (15) converges uniformly, the integral

$$\int_0^\infty g(t) \sin \xi t\,dt$$

also converges uniformly, as desired. □

5 Examples

Let us present some examples of admissible operators (cf. Definition 2.4). Note that for the operators B_j appearing here we only require that the functions $B_j(\cdot, \varphi)$, $\varphi \in \mathcal{M}$, satisfy properties (i)–(ii) (see Remark 2.6).

(1) $B_1(x, \varphi) = \varphi(x)$,

(2) $B_2(x, \varphi) = \varphi(x)^\alpha$, with $\alpha > 0$,

(3) $B_3(x, \varphi) = \int_{x/\lambda}^{\lambda x} \varphi(t)/t\,dt$, where $\lambda > 1$,

(4) $B_4(x, \varphi) = x^\alpha \int_{x/\lambda}^{\infty} \varphi(t)/t^{\alpha+1}$, where $\lambda > 1$ and $\alpha > 0$,

(5) $B_5(x, \varphi) = \sup_{s \geq x/\lambda} \varphi(s)$, where $\lambda > 1$,

(6) $B_6(x, \varphi) = \sup_{s \geq \log(x+1)} \varphi(s)$.

(7) The composition of two admissible operators is an admissible operator, i.e., if C and D are admissible, then the function

$$B_7(x, \varphi) = C(x, D(\cdot, \varphi))$$

satisfies properties (i)–(iv).

Remark 5.1. We cannot allow $\alpha = 0$, in B_4, since the operator would not be admissible. Indeed, if

$$\varphi(x) = \begin{cases} 0, & \text{if } x < 2 \\ \frac{1}{\log x}, & \text{otherwise,} \end{cases}$$

then φ clearly vanishes at infinity, but for any $x > 2\lambda$, one has

$$\int_{x/\lambda}^{\infty} \frac{\varphi(t)}{t}\,dt = \int_{x/\lambda}^{\infty} \frac{1}{t \log t}\,dt = \infty,$$

and therefore $B_4\varphi$ does not satisfy condition (ii) whenever $\alpha = 0$.

To conclude, we show that the class GM_{adm} is strictly larger than $GM(\beta_0)$.

Proposition 5.2. $GM(\beta_0) \subsetneq GM_{adm}$.

Proof. The inclusion is clear. Thus, we only need to find a function $f \in GM_{\text{adm}} \backslash GM(\beta_0)$. Let $0 < \alpha < 1$, and $n_j = 4^j$. We define the function

$$f(x) := \begin{cases} n_j^{-\frac{1}{1-\alpha}}, & \text{if } n_j \le x \le n_j + 1, \quad j \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$$

and the admissible operator $B_\alpha(x, \varphi) := \sup_{s \ge x} \varphi(s)^\alpha$. For any $x \in (n_{j-1} + 1, n_j + 1]$ it holds that

$$\int_x^{2x} |df(s)| \le 2n_j^{-\frac{1}{1-\alpha}}.$$

Moreover, since $n_{j+1} = 4n_j$, we have, for $I(x) = \int_x^{2x} |f(t)|\, dt$ (and j large enough),

$$\frac{1}{x} B_\alpha(x, I) \ge \frac{1}{n_j + 1} B_\alpha(x, I) \asymp n_j^{-1-\frac{\alpha}{1-\alpha}} = n_j^{-\frac{1}{1-\alpha}} \gtrsim \int_x^{2x} |df(s)|,$$

so that $f \in GM(\beta_\alpha)$, where $\beta_\alpha(x) = x^{-1} B_\alpha(x, I)$, and therefore $f \in GM_{\text{adm}}$. On the other hand,

$$\frac{1}{n_j} \sup_{s \ge n_j/c} \int_s^{2s} |f(t)|\, dt \asymp n_j^{-1-\frac{1}{1-\alpha}} = n_j^{-\frac{2-\alpha}{1-\alpha}}.$$

However,

$$\int_{n_j}^{2n_j} |df(s)| = 2n_j^{-\frac{1}{1-\alpha}} \lesssim n_j^{-\frac{2-\alpha}{1-\alpha}}$$

implies that $2 - \alpha \le 1$, which is a contradiction since $0 < \alpha < 1$. Thus, $f \notin GM(\beta_0)$, and our claim follows. □

References

[1] T. W. Chaundy, A. E. Jolliffe, *The Uniform Convergence of a Certain Class of Trigonometrical Series* , Proc. London Math. Soc. **S2-15** (1) (1916), 214–216.

[2] M. Dyachenko, E. Liflyand, S. Tikhonov, *Uniform convergence and integrability of Fourier integrals*, J. Math. Anal. Appl. **372** (2010), 328–338.

[3] M. Dyachenko, A. Mukanov and S. Tikhonov, *Uniform convergence of trigonometric series with general monotone coefficients* (Preprint).

[4] M. Dyachenko, S. Tikhonov, *Integrability and continuity of functions represented by trigonometric series: coefficients criteria*, Studia Math. **193** (3) (2009), 285–306.

[5] L. Feng, V. Totik, S. P. Zhou, *Trigonometric series with a generalized monotonicity condition*, Acta Math. Sin. (Engl. Ser.) **30** (8) (2014), 1289–1296.

[6] D. Gorbachev, E. Liflyand and S. Tikhonov, *Weighted Fourier inequalities: Boas' conjecture in* $\mathbb{R}^n$, J. Anal. Math. **114** (2011), 99–120.

[7] J.-P. Kahane, *Séries de Fourier Absolument Convergentes*, Springer-Verlag, Berlin-New York, 1970.

[8] L. Leindler, *Embedding results regarding strong approximation of sine series*, Acta Sci. Math. (Szeged) **71** (1-2) (2005), 91–103.

[9] E. Liflyand and S. Tikhonov, *A concept of general monotonicity and applications*, Math. Nachr. **284** (8-9) (2011), 1083–1098.

[10] E. Liflyand and S. Tikhonov, *Extended solution of Boas' conjecture on Fourier transforms*, C. R. Math. Acad. Sci. Paris, **346** (21-22) (2008), 1137–1142.

[11] E. Liflyand, S. Tikhonov, *The Fourier transforms of general monotone functions*, Analysis and mathematical physics, Trends Math., Birkhäuser, 2009, 377–395.

[12] F. Móricz, *On the uniform convergence of sine integrals*, J. Math. Anal. Appl. **354** (2009), 213–219.

[13] W. Rudin, *Some theorems on Fourier coefficients*, Proc. Amer. Math. Soc. **10** (1959), 855–859.

[14] B. Saffari, *Une fonction extrémale liée à la suite de Rudin-Shapiro*, C. R. Acad. Sci. Paris Sér. I Math., **303** (4) (1986), 97–100.

[15] H. S. Shapiro, *Extremal problems for polynomials and power series*, Thesis for S. M. Degree, Massachusetts Institute of Technology, 1951.

[16] B. Szal, *A new class of numerical sequences and its applications to uniform convergence of sine series*, Math. Nachr. **284** (14-15) (2011), 1985–2002.

[17] S. Tikhonov, *Best approximation and moduli of smoothness: Computation and equivalence theorems*, J. Approx. Theory **153** (2008), 19–39.

[18] S. Tikhonov, *Trigonometric series with general monotone coefficients*, J. Math. Anal. Appl. **326** (2007), 721–735.

[19] A. Zygmund, *Trigonometric series*, vols. I, II, third ed., Cambridge University Press, Cambridge, 2002.