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ZERO-HOPF BIFURCATION IN A CHUA SYSTEM

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Abstract. A zero-Hopf equilibrium is an isolated equilibrium point whose eigenvalues are $\pm\omega i \neq 0$ and 0. In general for a such equilibrium there is no theory for knowing when from it bifurcates some small–amplitude limit cycle moving the parameters of the system. Here we study the zero-Hopf bifurcation using the averaging theory. We apply this theory to a Chua system depending on 6 parameters, but the way followed for studying the zero-Hopf bifurcation can be applied to any other differential system in dimension 3 or higher.

In this paper first we show that there are three 4-parameter families of Chua systems exhibiting a zero-Hopf equilibrium. After, by using the averaging theory, we provide sufficient conditions for the bifurcation of limit cycles from these families of zero-Hopf equilibria. From one family we can prove that 1 limit cycle bifurcate, and from the other two families we can prove that 1, 2 or 3 limit cycles bifurcate simultaneously.

1. Introduction and statement of the main results

The Chua system is a classical model for electronic circuit and one of the most simplest models presenting chaos. It was presented by Chua, Komuro and Matsumoto [11] in 1986 and exhibit a rich range of dynamical behaviour. Precisely, the Chua circuit is a relaxation oscillator with a cubic nonlinear characteristic. It can be though as a circuit comprising a harmonic oscillator for which the operation is based on a field-effect transistor, coupled to a relaxation oscillator composed of a tunnel diode. The Chua system can be described by the following equations

$$\begin{aligned} \frac{dx}{dt} &= a(z - bx - a_2x^2 - a_1x^3), \\ \frac{dy}{dt} &= -z, \\ \frac{dz}{dt} &= -b_1x + y + b_2z. \end{aligned} \tag{1}$$

Note that it depends on six parameters a, a_1, a_2, b, b_1 and b_2.

In [19] the autors analyze the existence of local and global analytic first integrals in the Chua system. In [21] the authors use techniques of Differential Geometry in order to obtain an analytical expression of the slow manifold equation of Chua system. In [20] was studied the dynamics at infinity of the Chua system for the particular case where b_1 and b_2 are booth one. Besides, we can found some aspects about the Hopf bifurcation in [7] and [2]. In this paper, by using averaging theory, we study the limit cycles that can bifurcate from (1) from zero-Hopf equilibrium

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points of the Chua system. We recall that at these points we cannot apply the classical Hopf bifurcation theory which needs that the real eigenvalue be non-zero.

The Chua system can have at most three equilibria, namely: the origin and the two equilibria

$$p_{\pm} = \left(\frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 b}}{2a_1}, -\frac{a_2 b_1}{2a_1} \pm \frac{b_1 \sqrt{a_2^2 - 4a_1 b}}{2a_1}, 0 \right),$$

if $a_2^2 - 4a_1 b > 0$ and $a_1 \neq 0$. When $a_2^2 - 4a_1 b = 0$ and $a_1 a_2 \neq 0$ the system has only two equilibra, the origin and the equilibrium

$$p = \left(\frac{-a_2}{2a_1}, -\frac{a_2 b_1}{2a_1}, 0 \right).$$

Otherwise the origin is the unique equilibrium of the system.

As far as we know, the study of existence or non-existence of zero-Hopf equilibria and zero-Hopf bifurcation in the Chua system have not been considered in the literature. In this paper we have this objective. The method used here for studying the zero-Hopf bifurcation can be applied to any differential system in $\mathbb{R}^3$. In fact, this method already has been used in order to study the zero-Hopf bifurcations of the Rössler differential system, see [18].

A *zero-Hopf equilibrium* is an equilibrium point of a $3-$dimensional autonomous differential system which has a zero eigenvalue and a pair of purely imaginary eigenvalues. In general the zero-Hopf bifurcation is a $2-$parameter unfolding of a $3-$dimensional autonomous differential system with a zero-Hopf equilibrium. The unfolding has an isolated equilibrium with a zero eigenvalue and a pair of purely imaginary eigenvalue if the two parameters take zero values and the unfolding has different dynamics in a small neighborhood of this isolated equilibrium as the two parameters vary in a small neighborhood of the origin. To read more about zero-Hopf bifurcation, see Guckenheimer, Han, Holmes, Kuznetsov, Marsden and Scheurle in [13], [14], [15], [16] and [23]. Moreover, complex phenomena can occur at an isolated zero-Hopf equilibrium, as bifurcation of complicated invariant sets of the unfolding and a local birth of "chaos", as can be seen in the work of Baldomá and Seara, Broer and Vegter, Champneys and Kirk, Scheurle and Marsden in [3], [4], [8], [10] and [23].

In the next proposition we characterize the Hopf equilibria of the Chua system.

Proposition 1. *There are three* 4*-parameter families of Chua systems having a zero-Hopf equilibrium point, one for the equilibrium point located at the origin and the other two for each one of the equilibria* $p_{\pm}$ *when they exist. Namely,*

(a) $b = b_2 = 0$ *and* $ab_1 + 1 > 0$ *for the origin; and*
(b) $b = a_2^2/(4a_1)$, $b_2 = 0$, $ab_1 + 1 > 0$, $a_2^2 - 4a_1 b > 0$ *and* $a_1 \neq 0$ *for* $p_{\pm}$.

The next result gives sufficient conditions for the bifurcation of a limit cycle from the origin when it is a zero-Hopf equilibrium.

Theorem 2. *Let*

$$(a, a_1, a_2, b, b_1, b_2) = (\overline{a}_0 + \varepsilon\alpha_0, \overline{a}_1 + \varepsilon\alpha_1, \overline{a}_2 + \varepsilon\alpha_2, \varepsilon\beta_0, \frac{\omega^2 - 1}{a} + \varepsilon\beta_1, \varepsilon\beta_2).$$

If $\overline{a}_0 \overline{a}_2 \neq 0$, $|\omega| \neq 0, 1$ *and*

$$\Gamma = (\overline{a}_0 \beta_0 (1 - \omega^2) + \beta_2 \omega^2)(\overline{a}_0 \beta_0 \omega^2 (1 - \omega^2) + \beta_2 \omega^4) > 0,$$

then for $\varepsilon > 0$ sufficiently small the Chua system has a zero-Hopf bifurcation at the equilibrium point located at the origin of coordinates, and a limit cycle borns at this equilibrium when $\varepsilon = 0$. Moreover, this limit cycle has the same kind of stability or instability than an equilibrium point of a planar differential system with eigenvalues

$$\frac{-\beta_2\omega^5 \pm \sqrt{\omega^6(\beta_2^2\omega^4(3-2\omega^2)+2\overline{a}_0^2\beta_0^2(\omega^2-1)^3)}}{2\omega^6(\omega^2-1)}. \tag{2}$$

The following result provides sufficient conditions for the bifurcation of a limit cycle from the equilibrium p_- when it is zero-Hopf equilibrium.

Theorem 3. *Consider the vector $(a, a_1, a_2, b, b_1, b_2)$ given by*

$$\begin{aligned} a &= \overline{a}_0 + \varepsilon\alpha_0 + \varepsilon^2\xi_0, \\ a_1 &= \overline{a}_1 + \varepsilon\alpha_1 + \varepsilon^2\xi_1, \\ a_2 &= \varepsilon\alpha_2 + \varepsilon^2\xi_2, \\ b &= \frac{a_2^2}{4a_1} + \varepsilon^2\zeta_0, \\ b_1 &= \frac{\omega^2-1}{a} + \varepsilon\beta_1 + \varepsilon^2\zeta_1, \\ b_2 &= \varepsilon^2\zeta_2. \end{aligned} \tag{3}$$

If $\overline{a}_1\omega \neq 0$ and $\overline{a}_1\zeta_0 < 0$ then, for $\varepsilon > 0$ sufficiently small the Chua system has a zero-Hopf bifurcation at the equilibrium point located at p_- and three limit cycles can bifurcate from this equilibrium when $\varepsilon = 0$. Moreover, examples of systems where 1, 2 or 3 limit cycles bifurcate simultaneously are given.

Proposition 1 and Theorems 2 and 3 are proved in section 3. In particular, booth theorems are proved using the averaging method. This method will be briefly summarized in the next section. We note that Theorem 2 is proved using averaging theory of first order, but the proof of Theorem 3 needs averaging of second order.

Also the stability or instability of the bifurcated limit cycles in Theorem 3 can be studied, but the expressions of the eigenvalues which provide such stability or instability are huge and we do not give them here.

Remark 1. *For the equilibrium point p_+ we have analogous results to the ones of Theorem 3 for p_-. For this reason, we omit the statement of the result for the equilibrium p_+ and its proof.*

2. Limit cycles via averaging theory

The averaging method is a classical tool in nonlinear analysis and dynamical systems. The procedure of averaging can be found already in the work of Lagrange and Laplace who provided an intuitive justification of the method. After them, Poincaré considered the determination of periodic solutions by series expansion with respect to a small parameter and around 1930 we see the start of precise statements and proofs in averaging theory. After this time many new results in the theory of averaging have been obtained. The main contribution in direction to the formalization of the method is due to Fatou [12]. The work of Krylov and Bogoliubov [6] and Bogoliubov [5] also provide important practical and theoretical improvements in the theory.

Now we present the basic results on the averaging theory of first and second order. The averaging of first order for studying periodic orbits can be found in [22], see Theorems 11.5 and 11.6. It can be summarized as follows.

Theorem 4. *We consider the following two initial value problems*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon), \qquad x(0) = x_0, \tag{4}$$

and

$$\dot{y} = \varepsilon f^0(y), \qquad y(0) = x_0. \tag{5}$$

where $x, y, x_0 \in \Omega$ *an open subset of* $\mathbb{R}^n$, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, f *and* g *are periodic of period* T *in the variable* t, *and* $f^0(y)$ *is the averaged function of* $f(t,x)$ *with respect to* t, *i.e.,*

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt. \tag{6}$$

Suppose:

(i) f, *its Jacobian* $\dfrac{\partial f}{\partial x}$, *its Hessian* $\dfrac{\partial^2 f}{\partial x^2}$, g *and its Jacobian* $\dfrac{\partial g}{\partial x}$ *are defined, continuous and bounded by a constant independent on* ε *in* $[0,\infty) \times \Omega$ *and* $\varepsilon \in (0,\varepsilon_0]$;

(ii) T *is a constant independent of* ε; *and*

(iii) $y(t)$ *belongs to* Ω *on the interval of time* $[0, 1/\varepsilon]$. *Then the following statements hold.*

  (a) *On the time scale* $1/\varepsilon$ *we have that* $x(t) - y(t) = O(\varepsilon)$, *as* $\varepsilon \to 0$.

  (b) *If* p *is a singular point of the averaged system* (5) *such that the determinant of the Jacobian matrix*

$$\left.\frac{\partial f^0}{\partial y}\right|_{y=p} \tag{7}$$

  is not zero, then there exists a limit cycle $\phi(t,\varepsilon)$ *of period* T *for system* (4) *which is close to* p *and such that* $\phi(0,\varepsilon) \to p$ *as* $\varepsilon \to 0$.

  (c) *The stability or instability of the limit cycle* $\phi(t,\varepsilon)$ *is given by the stability or instability of the singular point* p *of the averaged system* (5). *In fact, the singular point* p *has the stability behaviour of the Poincaré map associated to the limit cycle* $\phi(t,\varepsilon)$.

The next result present the second order averaging method of a periodic differential system. For a proof see Theorem 3.5.1 of Sanders and Verhulst in [22], see also [9].

Theorem 5. *We consider the following two initial value problems*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 R(t,x,\varepsilon), \quad x(0) = x_0 \tag{8}$$

and

$$\dot{y} = \varepsilon f^0(y) + \varepsilon^2(f^{10}(y) + g^0(y)), \quad y(0) = x_0, \tag{9}$$

with $f, g : [0,\infty) \times \Omega \to \mathbb{R}^n$, $R : [0,\infty) \times \Omega \times (0,\varepsilon_0] \to \mathbb{R}^n$, Ω *an open subset of* $\mathbb{R}^n$, f, g *and* R *periodic of period* T *in the variable* t,

$$f^1(t,x) = \frac{\partial f}{\partial x} y^1(t,x), \quad \textit{where} \quad y^1(t,x) = \int_0^t f(s,x)ds.$$

Of course, f^0, f^{10} and g^0 denote the averaged functions of f, f^1 and g, respectively, defined as in (6)*. Suppose:*

(i) *$\partial f/\partial x$ is Lipschitz in x, g and R are Lipschitz in x and all these functions are continuous on their domain of definition;*
(ii) *$|R(t,x,\varepsilon)|$ is bounded by a constant uniformly in $[0,L/\varepsilon)\times\Omega\times(0,\varepsilon_0]$;*
(iii) *T is a constant independent of ε; and*
(iv) *$y(t)$ belongs to Ω on the interval of time $[0,1/\varepsilon]$. Then the following statements hold.*
   (a) *In the time scale $1/\varepsilon$ we have that $x(t)=y(t)+\varepsilon y^1(t,y(t))=O(\varepsilon^2)$.*
   (b) *If $f^0(y)\equiv 0$ and p is a singular point of averaged system* (9) *such that*
$$\left.\frac{\partial(f^{10}+g^0)(y)}{\partial y}\right|_{y=p}$$
   is not zero, then there exist a limit cycle $\phi(t,\varepsilon)$ of period T for system (8) *which is close to p and such that $\phi(0,\varepsilon)\to p$ as $\varepsilon\to 0$.*
   (c) *The stability or instability of the limit cycle $\phi(t,\varepsilon)$ is given by the stability os instability of the singular point p of the averaged system* (9)*. In fact, the singular point p has the stability behaviour of the Poincaré map associated to the limit cycle $\phi(t,\varepsilon)$.*

3. Proofs

In this section we give the proofs of the results presented in section 1.

Proof of Proposition 1. The characteristic polynomial of the linear part of the Chua system at the origin is
$$p(\lambda)=-\lambda^3+(b_2-ab)\lambda^2+(b_2ab-ab_1-1)\lambda-ab.$$
Imposing that $p(\lambda)=-\lambda(\lambda^2+\omega^2)$, we obtain $b=b_2=0$ and $b_1=(\omega^2-1)/a$. So statement (a) follows.

The characteristic polynomial of the linear part of the Chua system at p_- is given by
$$p(\lambda)=-\frac{(1-b_2\lambda+\lambda^2)[2a_1\lambda+a(a_2^2+a_2\sqrt{a_2^2-4a_1b}-4a_1b)]+2a_1b_1a\lambda}{2a_1}$$
The proposition follows imposing that $p(\lambda)=-\lambda(\lambda^2+\omega^2)$, and that the equilibrium point p_- exists. □

Proof of Theorem 2. If we consider
$$(a,a_1,a_2,b,b_1,b_2)=(\overline{a}_0+\varepsilon\alpha_0,\overline{a}_1+\varepsilon\alpha_1,\overline{a}_2+\varepsilon\alpha_2,\varepsilon\beta_0,\frac{\omega^2-1}{a}+\varepsilon\beta_1,\varepsilon\beta_2).$$
with $\varepsilon>0$ a sufficiently small parameter, then the Chua system becomes
$$\begin{aligned}\dot{x}&=(\overline{a}_0+\varepsilon\alpha_0)(\varepsilon\beta_0x+z-(\overline{a}_2+\varepsilon\alpha_2)x^2-(\overline{a}_1+\varepsilon\alpha_1)x^3),\\ \dot{y}&=-z,\\ \dot{z}&=-\left(\varepsilon\beta_1+\frac{\omega^2-1}{\overline{a}_0+\varepsilon\alpha_0}\right)x+y+\varepsilon\beta_2z.\end{aligned}\tag{10}$$

By the rescaling of variables $(x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$, system (10) becomes

$$
\begin{aligned}
\dot{X} &= (\overline{a}_0 + \varepsilon\alpha_0)(\varepsilon\beta_0 X + Z - \varepsilon(\overline{a}_2 + \varepsilon\alpha_2)X^2 - \varepsilon^2(\overline{a}_1 + \varepsilon\alpha_1)X^3), \\
\dot{Y} &= -Z, \\
\dot{Z} &= -\left(\varepsilon\beta_1 + \frac{\omega^2 - 1}{\overline{a}_0 + \varepsilon\alpha_0}\right) X + Y + \varepsilon\beta_2 Z.
\end{aligned} \tag{11}
$$

Now we shall write the linear part at the origin of (11) into its real Jordan normal form

$$
\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{12}
$$

when $\varepsilon = 0$. For doing that we do the linear change of variables $(X, Y, Z) \to (u, v, w)$ given by

$$
\begin{aligned}
X &= \frac{\overline{a}_0(w + \omega v)}{\omega^2}, \\
Y &= w - \frac{w}{\omega^2} - \frac{v}{\omega}, \\
Z &= u.
\end{aligned} \tag{13}
$$

In these new variables, system (11) is written as follow

$$
\begin{aligned}
\dot{u} = \;& -v\omega + \varepsilon \frac{-(\alpha_0(1-\omega^2) + \overline{a}_0^2\beta_1)(w + \omega v) + \overline{a}_0 u \beta_2 \omega^2}{\overline{a}_0 \omega^2} \\
& -\varepsilon^2 \frac{\alpha_0^2(\omega^2 - 1)(w + \omega v)}{\overline{a}_0^2 \omega^2}, \\
\dot{v} = \;& u\omega - \varepsilon \frac{(\omega^2 - 1)(-u\alpha_0\omega^4 + \overline{a}_0^2(w + \omega v)(\beta_0\omega^2 + \overline{a}_0\overline{a}_2(w + \omega v)))}{\overline{a}_0\omega^5} \\
& -\varepsilon^2 \frac{1}{\omega^7}(\omega^2 - 1)(w + \omega v)(\alpha_0\beta_0\omega^4 + \overline{a}_0(w + \omega v)(\overline{a}_2\alpha_0\omega^2 \\
& + \overline{a}_0\alpha_2\omega^2 + \overline{a}_0^2\alpha_1(w + \omega v))), \\
\dot{w} = \;& -\varepsilon \frac{-u\alpha_0\omega^4 + \overline{a}_0^2(w + \omega v)(\beta_0\omega^2 + \overline{a}_0\overline{a}_2(w + \omega v))}{\overline{a}_0\omega^4} \\
& -\varepsilon^2 \frac{(w + \omega v)(\alpha_0\beta_0\omega^4 + \overline{a}_0(w + \omega v)(\overline{a}_2\alpha_0\omega^2 + \overline{a}_0\alpha_2\omega^2))}{\omega^6}.
\end{aligned} \tag{14}
$$

Writing the differential system (14) in cylindrical coordinates (r, θ, w) by $u = r\cos\theta$, $v = r\sin\theta$ and $w = w$ we have

$$
\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon\left(\frac{r\beta_2\cos^2\theta}{\omega} - \frac{\overline{a}_0(\omega^2-1)\sin\theta(w+r\omega\sin\theta)(\overline{a}_0\overline{a}_2 w+\beta_0\omega^2}{\omega^6}\right. \\
&\quad \frac{+\overline{a}_0\overline{a}_2 rw\sin\theta)}{\omega^6} - \frac{\cos\theta(w(\alpha_0+\overline{a}_0^2\beta_1-\alpha_0\omega^2)+rw(\overline{a}_0\beta_1}{\overline{a}_0\omega^3} \\
&\quad \left.\frac{-2\alpha_0(\omega^2-1))\sin\theta)}{\overline{a}_0\omega^3}\right) + O(\varepsilon^2), \qquad (15) \\
\frac{dw}{d\theta} &= \varepsilon\frac{r\alpha_0\omega^4\cos\theta - \overline{a}_0^2(w+r\omega\sin\theta)(\overline{a}_0\overline{a}_2 w+\beta_0\omega^2+\overline{a}_0\overline{a}_2 r\omega\sin\theta)}{\overline{a}_0\omega^5} \\
&\quad +O(\varepsilon^2).
\end{aligned}
$$

Now we apply the first order averaging theory as described in Theorem 4 of section 2. In order to do this, we note that (15) satisfies all the assumptions of Theorem 4, where we identify $t = \theta$, $T = 2\pi$, $x = (r, w)^T$, $F(\theta, r, w) = (F_1(\theta, r, w), F_2(\theta, r, w))$ and $f(r, w) = (f_1(r, w), f_2(r, w))$.

By calculating f_1 and f_2, we get

$$
\begin{aligned}
f_1(r,w) &= \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,r,w)d\theta \\
&= \frac{r(\beta_2\omega^4 - 2\overline{a}_0^2\overline{a}_2 w(\omega^2-1) - \overline{a}_0\beta_0\omega^2(\omega^2-1))}{2\omega^5}, \\
f_2(r,w) &= \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta,r,w)d\theta \\
&= -\frac{\overline{a}_0(2w\beta_0\omega^2+\overline{a}_0\overline{a}_2(2w^2+r^2\omega^2))}{2\omega^5}.
\end{aligned}
$$

There is only one solution (r^*, w^*) for $f_1(r, w) = f_2(r, w) = 0$ satisfying $r^* > 0$ and this solution is

$$
\begin{aligned}
r^* &= \sqrt{\frac{\Gamma}{2\overline{a}_0^4\overline{a}_2^2(\omega^2-1)^2}}, \\
w^* &= \frac{\overline{a}_0\beta_0\omega^2(1-\omega^2)+\beta_2\omega^4}{2\overline{a}_0^2\overline{a}_2(\omega^2-1)},
\end{aligned}
$$

since $\overline{a}_0\overline{a}_2 \neq 0$, $|\omega| \neq 1$ and $\Gamma > 0$.

We note that the Jacobian (7) at (r^*, w^*) takes the value

$$
\frac{\beta_2^2\omega^4 - \overline{a}_0^2\beta_0^2(\omega^2-1)^2}{2\omega^6(\omega^2-1)}
$$

and the eigenvalues of the Jacobian matrix

$$
\left.\frac{\partial(f_1,f_2)}{\partial(r,w)}\right|_{(r,w)=(r^*,w^*)} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}\omega^5}\sqrt{\overline{a}_0(\omega^2-1)\Gamma} \\ -\frac{1}{\omega^3}\sqrt{\frac{\overline{a}_0\Gamma}{2(\omega^2-1)^3}} & \frac{\beta_2}{\omega(1-\omega^2)} \end{pmatrix}
$$

are the ones given in (2).

In short, from Theorem 4 we conclude the proof once we show that periodic solutions corresponding to (r^*, w^*) provides a periodic solution bifurcating from the origin of coordinates of the differential system (10) when $\varepsilon = 0$. Theorem 4 guarantees for $\varepsilon > 0$ sufficiently small the existence of a periodic solution corresponding to the point (r^*, w^*) of the form $(r(\theta,\varepsilon), w(\theta,\varepsilon))$ such that $(r(0,\varepsilon), w(0,\varepsilon)) \to (r^*, w^*)$ when $\varepsilon \to 0$. So system (14) has a periodic solution

$$(u(\theta,\varepsilon) = r(\theta,\varepsilon)\cos\theta, v(\theta,\varepsilon) = r(\theta,\varepsilon)\sin\theta, w(\theta,\varepsilon)) \tag{16}$$

for $\varepsilon > 0$ sufficiently small. Consequently, from relation (16) through the linear change of variables (13) system (11) has a periodic solution $(X(\theta), Y(\theta), Z(\theta))$. Finally, for $\varepsilon > 0$ sufficiently small system (10) has a periodic solution $(x(\theta), y(\theta), z(\theta)) = (\varepsilon X(\theta), \varepsilon Y(\theta), \varepsilon Z(\theta))$ which tends to the origin of coordinates when $\varepsilon \to 0$. Thus, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when $\varepsilon = 0$. This completes the proof of theorem. □

Since the proof of Theorem 3 is very similar to the of Theorem 2, then we will omit some steps in order to avoid some long expressions.

Proof of Theorem 3. Suppose that we have the conditions given in (3) on the parameters of Chua system (1). Then, by a translation of the equilibrium point p_- at the origin of coordinates, and a rescaling of variables given by $(x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$ the Chua system becomes

$$\begin{aligned} \dot{X} &= A_1 X + (\overline{a}_0 + \alpha_0\varepsilon + \varepsilon^2\xi)Z + A_2X^2 + A_3X^3, \\ \dot{Y} &= -Z, \\ \dot{Z} &= A_4X + Y + \varepsilon^2\zeta_2 Z, \end{aligned} \tag{17}$$

where

$$\begin{aligned} A_1 &= \varepsilon^2(1/2\overline{a}_1^2)(\overline{a}_0(-4\overline{a}_1^2\zeta_0 - \alpha_1\alpha_2\varepsilon\sqrt{-\overline{a}_1\zeta_0} + 2\overline{a}_1\sqrt{-\overline{a}_1\zeta_0}(\alpha_2 + \varepsilon\xi_2) \\ &\quad +2\overline{a}_1\alpha_0\varepsilon(-2\overline{a}_1\zeta_0 + \alpha_2\sqrt{-\overline{a}_1\zeta_0})), \\ A_2 &= \varepsilon^2(1/2\overline{a}_1)(\overline{a}_0(3\alpha_1\varepsilon\sqrt{-\overline{a}_1\zeta_0} + \overline{a}_1(\alpha_2 + 6\sqrt{-\overline{a}_1\zeta_0} + \varepsilon\xi_2)) \\ &\quad +\overline{a}_1\alpha_0\varepsilon(\alpha_2 + 6\sqrt{-\overline{a}_1\zeta_0})), \\ A_3 &= \varepsilon^2(\overline{a}_1\alpha_0\varepsilon + \overline{a}_0(\overline{a}_1 + \alpha_1\varepsilon)), \\ A_4 &= (\omega^2 - 1)[\overline{a}_0^3 - \alpha_0^3\varepsilon^3 - \overline{a}_0^2\varepsilon(\alpha_0 + \varepsilon\xi_0 + \overline{a}_0\alpha_0\varepsilon^2(\alpha_0 + 2\varepsilon\xi_0))] \\ &\quad +\overline{a}_0^4\varepsilon(\beta_1 + \varepsilon\zeta_1). \end{aligned}$$

The linear part of (17) at p_- in the real Jordan normal form when $\varepsilon = 0$ is given by (12), and doing also the linear change of variables $(X, Y, Z) \to (u, v, w)$ given by (13) we write the linear part of system (17) in its real Jordan normal form when $\varepsilon = 0$, we obtain the system

$$\begin{aligned} \dot{u} &= -\omega v + \varepsilon(B_1 v + B_2 w) + \varepsilon^2\zeta_2 u, \\ \dot{v} &= \omega u + \frac{\varepsilon\alpha_0(\omega^2 - 1)u}{\overline{a}_0\omega} + \varepsilon\frac{\omega^2 - 1}{\overline{a}_0\omega}B_3, \\ \dot{w} &= \frac{\varepsilon\alpha_0}{\overline{a}_0}u + \frac{\varepsilon^2}{\overline{a}_0}B_3, \end{aligned} \tag{18}$$

where

$$\begin{aligned} B_1 &= -\frac{\overline{a}_0^3(\beta_1+\zeta_1)-\overline{a}_0(\alpha_0+\varepsilon\xi_0)(\omega^2-1)-\varepsilon\alpha_0^2(\omega^2-1)}{\overline{a}_0^2\omega},\\ B_2 &= -\frac{\overline{a}_0^3(\beta_1+\zeta_1)-\overline{a}_0(\alpha_0+\varepsilon\xi_0)(\omega^2-1)+\varepsilon\alpha_2^2(\omega^2-1)}{\overline{a}_0^2\omega^2},\\ B_3 &= \xi_0 u-\frac{\overline{a}_0^2(w+\omega v)(2(-2\overline{a}_1\zeta_0+\alpha_2\sqrt{\overline{a}_1\zeta_0})\omega^4-\overline{a}_0\overline{a}_1(\alpha_2}{2\overline{a}_1\omega^6}\\ &\quad \frac{+6\omega^2(w+\omega v)\sqrt{\overline{a}_1\zeta_0})+2\overline{a}_0^2\overline{a}_1^2(w+\omega v)^2)}{2\overline{a}_1\omega^6}. \end{aligned}$$

If we write system (18) in cylindrical coordinates (r,θ,w) defined by $u=r\cos\theta$, $v=r\sin\theta$ and $w=w$, after we take as new independent variable the angle θ, and we apply to the system $dr/d\theta$ and $dw/d\theta$ that we obtain the second order averaging method described in Theorem 5, we get that the function $f=(f_1,f_2)$ is identically zero, and that the function $g=(g_1,g_2)$ is

$$\begin{aligned} g_1(r,w) &= \frac{\pi r}{4\omega}\left(4\zeta_2+\frac{\overline{a}_0(\omega^2-1)(4\overline{a}_0\overline{a}_1 w(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0})\omega^2}{\overline{a}_1\omega^6}\right.\\ &\quad \left.+\frac{4(2\overline{a}_1\zeta_0-\alpha_2\sqrt{-\overline{a}_1\zeta_0})\omega^4-3\overline{a}_0^2\overline{a}_1^2(4w^2+3r^2\omega^2))}{\overline{a}_1\omega^6}\right),\\ g_2(r,w) &= \frac{\overline{a}_0\pi}{2\overline{a}_1\omega^7}\left(4w(2\overline{a}_1\zeta_0-\alpha_2\sqrt{-\overline{a}_1\zeta_0})\omega^4-2\overline{a}_0^2\overline{a}_1^2 w(2w^2+3r^2\omega^2)\right.\\ &\quad \left.+\overline{a}_0\overline{a}_1(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0})\omega^2(2w^2+r^2\omega^2)\right). \end{aligned}$$

In order to find solutions (r^*,w^*) of $g=0$ we compute a Gröbner basis $\{b_k(r,w), k=1,\dots,20\}$ in the variables r and w for the set of polynomials $\{\overline{g}_1(r,w),\overline{g}_2(r,w)\}$ where $\overline{g}_1=4(\overline{a}_1\omega^7/\pi r)g_1$ and $\overline{g}_2=(2\overline{a}_1\omega^7/\overline{a}_0\pi)g_2$ and then we will look for roots of b_1 and b_2. It is a known fact that the solutions of a Gröbner basis of $\{\overline{g}_1(r,w),\overline{g}_2(r,w)\}$ are the solutions of $\overline{g}_1=0$ and $\overline{g}_2=0$, consequently solutions of $g_1=0$ and $g_2=0$ as well. For more information about Gröbner basis see [1] and [17].

The Gröbner basis for the polynomials $\{\overline{g}_1(r,w),\overline{g}_2(r,w)\}$ in the variables r and w is formed by twenty polynomials. We only use two polynomials of this basis, namely,

$$\begin{aligned} G_1(r,w) &= 30(\omega^2-1)\overline{a}_0^4\overline{a}_1^3 w^3-15(\omega^2-1)\omega^2\overline{a}_1^2\overline{a}_0^3(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0})w^2\\ &\quad +2\overline{a}_0\overline{a}_1\omega^4(-6\overline{a}_1\zeta_2\omega^2+\overline{a}_0(\alpha_2^2-42\overline{a}_1\zeta_0\\ &\quad +15\alpha_2\sqrt{-\overline{a}_1\zeta_0}(\omega^2-1)))w+2\omega^6(\overline{a}_1(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0}\zeta_2\omega^2)\\ &\quad +\overline{a}_0(8\overline{a}_1\alpha_2\zeta_0-\alpha_2^2\sqrt{-\overline{a}_1\zeta_0}-12(-\overline{a}_1\zeta_0)\sqrt{-\overline{a}_1\zeta_0}(\omega^2-1))) \end{aligned}$$

and

$$\begin{aligned} G_2(r,w) &= \overline{a}_0\overline{a}_1\omega^2(6\overline{a}_0\overline{a}_1 w-\alpha_2\omega^2-6\omega^2\sqrt{-\overline{a}_1\zeta_0})r^2+2w(2\overline{a}_0^2\overline{a}_1^2 w^2\\ &\quad -\overline{a}_0\overline{a}_1 w(\alpha_2+6\sqrt{-\overline{a}_1\zeta_0})\omega^2+2(-2\overline{a}_1\zeta_0+\alpha_2\sqrt{-\overline{a}_1\zeta_0})\omega^4). \end{aligned}$$

Since $G_1(r,w)=G_1(w)$ is a polynomial of degree 3 in the variable w, it is clear that we can have at most three real solutions for w depending on the parameters of the zero-Hopf family. Replacing these three values of w in the second polynomial

G_2 we have six solutions for r of the form $\pm r_i^*$ for $i = 1, 2, 3$, because $G_2(r, w)$ is of the form $P_1(w)r^2 + P_2(w)$. However, since r must be positive, we have at most three good solutions for $G_1 = 0$ and $G_2 = 0$. Consequently, we have at most three good solutions for $g = (g_1, g_2) = 0$ and then, by Theorem 5 and using the same arguments that in the proof of Theorem 2 when we go back through the changes of coordinates, we can have at most three limit cycles bifurcating from the equilibrium point p_-.

Moreover, if we consider $\alpha_2 = -6\sqrt{-\overline{a}_1\zeta_0}$, then the relations $(g_1(r, w), g_2(r, w)) = (0, 0)$ provide three solutions given by

$$(r^*, w_\pm^*) = \left(\frac{2\omega}{\sqrt{15}} \sqrt{\frac{8\overline{a}_0\zeta_0(1-\omega^2) - \zeta_2}{\overline{a}_0^3\overline{a}_1(\omega^2 - 1)}}, \pm \frac{\omega^2}{\overline{a}_0\sqrt{\overline{a}_1}} \sqrt{-\frac{4\overline{a}_0\zeta_0(1-\omega^2) + 2\zeta_2\omega^2}{5\overline{a}_0(\omega^2 - 1)}} \right)$$

and

$$(r^0, w^0) = \left(\frac{2\omega}{\sqrt{3}} \sqrt{\frac{4\overline{a}_0\zeta_0(1-\omega^2) + \zeta_2}{\overline{a}_0^3\overline{a}_1(\omega^2 - 1)}}, 0 \right).$$

as long as the expressions in the square roots are positives. This shows that three limit cycles can bifurcate simultaneous from the equilibrium p_-. In a similar way we can produce examples with one, or two limit cycles bifurcating from p_-. This completes the proof of the theorem. □

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