This is a preprint of: "Weak periodic solutions of $x\ddot{x}+1=0$ and the Harmonic balance method", Johanna Denise García Saldaña, Armengol Gasull, *J. Phys. Conf. Ser.*, vol. 811, 012003, 2017.
DOI: [10.1088/1742-6596/811/1/012003]

Weak periodic solutions of $x\ddot{x}+1=0$ and the Harmonic Balance Method

J D García-Saldaña[1,3] and A Gasull [2]

[1,2] Departament de Matemàtiques, Universitat Autònoma de Barcelona
Edifici C. 08193 Bellaterra, Barcelona. Spain.

E-mail: jgarcias@ucsc.cl, gasull@mat.uab.cat

Abstract. We prove that the differential equation $x\ddot{x}+1=0$ has continuous weak periodic solutions and compute their periods. Then, we use the Harmonic Balance Method until order six to approximate these periods and to illustrate how the accuracy of the method increases with the order. Our computations rely on the Gröbner basis approach.

1. Introduction and main results

The nonlinear differential equation

$$x\ddot{x}+1=0 \tag{1}$$

appears in the modeling of certain phenomena in plasma physics. More concretely, studying a certain electron beam injected into a plasma tube, where the magnetic field is cylindrical and increases towards the axis in inverse proportion to the radius, see [1, 8]. In [11] and [13, Sec. 3.2.2], the author calculates the period of its periodic orbits and use the N-th order Harmonic Balance Method (HBM), for $N=1,2$, to obtain approximations of these periodic solutions and of their corresponding periods. Similarly, the same functions are approximated in [8]. However, strictly speaking, it can be seen that neither equation (1), nor its associated planar system

$$\begin{cases} \dot{x}=y, \\ \dot{y}=-\frac{1}{x}, \end{cases} \tag{2}$$

which is singular at $x=0$, have smooth periodic solutions. For system (2) this is so, because it has no critical points and it is well-known ([9]) that periodic orbits of planar autonomous systems must surround a critical point. For equation (1) it is not difficult to see that any periodic solution $x(t)$ must vanish for some $t^*\in\mathbb{R}$, that is $x(t^*)=0$. Then $\lim_{t\to t^*}\ddot{x}(t)=\infty$ and as a consequence this equation cannot have $\mathcal{C}^2$-periodic solutions.

We start giving two different interpretations of the computations of [8, 11] of the period function. The first one states their results in terms of weak (or generalized) solutions, where in this work a weak solution is a function satisfying the differential equation (1) on an open and dense set, but which is only continuous ($\mathcal{C}^0$) at some isolated points.

[3] Current address: Departamento de Matemática y Física Aplicadas, Facultad de Ingeniería, Universidad Católica de la Santísima Concepción. Casilla 297, Concepción, Chile.

The second interpretation shows that, for a given initial condition, this period is the limit when k tends to zero, of the period of an actual periodic solution with the same initial condition, of the smooth planar differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\frac{x}{x^2+k^2}, \quad k \neq 0, \end{cases} \tag{3}$$

which has a global center at the origin.

Next theorem collects our first results. The method used to obtain $T(A)$ is based on standard procedures applied to 1-dimensional oscillatory physical systems, see [2, 12, 13, 15, 16, 19], adapted to this singular setting. Item *(ii)* provides a new result. It is a consequence of applying the Lebesgue's dominated convergence theorem to the integral expression that gives the periods of the periodic orbits of system (3).

Theorem 1.1. *(i) For the initial conditions $x(0) = A$, $\dot{x}(0) = 0$, with $A \neq 0$, the differential equation* (1) *has a weak $\mathcal{C}^0$-periodic solution with period $T(A) = 2\sqrt{2\pi}A$.*

(ii) Let $T(A;k)$ be the period of the periodic orbit of system (3) *with initial conditions $x(0) = A$, $y(0) = 0$. Then*

$$T(A;k) = 4\,A \int_0^1 \frac{ds}{\sqrt{\ln\left(\frac{A^2+k^2}{A^2s^2+k^2}\right)}}$$

and

$$\lim_{k\to 0} T(A;k) = 4A \int_0^1 \frac{1}{\sqrt{-2\ln s}} ds = 2\sqrt{2\pi}A = T(A).$$

Recall that the N-th order HBM consists in approximating the periodic solutions of differential equations by truncated Fourier series with N harmonics and an unknown frequency; see for instance [6, 7, 12, 13] or Section 3 for a short overview of the method. In [13] the author asks for techniques to deal analytically with the N-th order HBM, for $N \geq 3$. In [5] it is shown how resultants can be used when $N = 3$.

Next, in this work we will show how to utilize a more powerful tool, the computation of Gröbner basis ([3, Ch. 5]), to obtain higher accuracy in the approximations obtained by using the N-th order HBM, for $N \geq 3$. As an example, we will apply this method to approach the function $T(A)$ introduced in Theorem 1.1. Clearly, it can also be applied to other similar differential equations to approach their periodic orbits and their corresponding periods, like for instance, the ones considered in [2, 14, 15, 16]

Notice that all solutions of equation (1) are also solutions of the family of differential equations

$$x^{m+1}\ddot{x} + x^m = 0, \quad m \in \mathbb{N} \cup \{0\}. \tag{4}$$

In fact, in this new family only the new solution $x = 0$ is added to the solutions of (1).

In the second part of this paper we address the question of the effect of this parameter m and of the integer N, that measures the order of the HBM, on the accuracy of the approaches to the period function,

$$T(A) = 2\sqrt{2\pi}A \approx 5.0132A,$$

by the periods of the trigonometric polynomials obtained from applying the N-th order HBM to (4). Our results are given in next theorem, where $[a]$ denotes the integer part of a. In particular, it can be seen that at first order, the best approximations of $T(A)$ are obtained when $m \in \{1, 2\}$ and not when $m = 0$.

Theorem 1.2. *Let $\mathcal{T}_N(A;m)$ be the period of the truncated Fourier series obtained from applying the N-th order HBM to equation (4). Then there exist constants $C_N(m)$ such that $\mathcal{T}_N(A;m) = C_N(m)A$. Moreover*

(i) For all $m \in \mathbb{N} \cup \{0\}$,

$$C_1(m) = 2\pi\sqrt{\frac{2[\frac{m+1}{2}]+1}{2[\frac{m+1}{2}]+2}}. \tag{5}$$

(ii) For $m = 0$,

$$C_1(0)=\sqrt{2}\,\pi \approx 4.4428, \qquad C_4(0) \approx 5.0455,$$

$$C_2(0)=(\sqrt{218}/9)\,\pi \approx 5.1539, \qquad C_5(0) \approx 4.9841,$$

$$C_3(0)=\frac{13810534\,\pi}{3\sqrt{5494790257313+115642506449\sqrt{715}}} \approx 4.9353, \qquad C_6(0) \approx 5.0260.$$

(iii) For $m = 1$,

$$C_1(1) = \sqrt{3}\,\pi \approx 5.4414, \qquad C_3(1) \approx 5.1476,$$

$$C_2(1) \approx 5.2733, \qquad C_4(1) \approx 5.1186.$$

(iv) For $m = 2$,

$$C_1(2) = \sqrt{3}\,\pi \approx 5.4414, \qquad C_2(2) \approx 5.2724, \qquad C_3(2) \approx 5.1417.$$

Moreover, the approximate values stated above are the roots of certain given polynomials with integer coefficients, and they can be obtained with any desired accuracy using Sturm sequences.

Notice that the values $C_1(m)$, for $m \in \{0,1,2\}$ given in items (*ii*), (*iii*) and (*iv*), respectively, are already computed in item (*i*). We only make them explicit to clarify the reading.

Table 1. Percentage of relative errors $e_N(m)$.

$e_N(m)$	$m=0$	$m=1$	$m=2$
$N=1$	11.38%	8.54%	8.54%
$N=2$	2.80%	5.19%	5.17%
$N=3$	1.55%	2.68%	2.56%
$N=4$	0.64%	2.10%	–
$N=5$	0.58%	–	–
$N=6$	0.25%	–	–

To elucidate which of the approaches given in Theorem 1.2 is better, in Table 1 we give the percentage of the relative errors

$$e_N(m) = 100\left|\frac{\mathcal{T}_N(A;m) - T(A)}{T(A)}\right| = 100\left|\frac{C_N(m) - 2\sqrt{2\pi}}{2\sqrt{2\pi}}\right|.$$

The best approximation that we have found corresponds to $\mathcal{T}_6(A;0)$. Our computers have not been able to get the Gröbner basis needed to fill the gaps of the table.

The paper is organized as follows. Theorem 1.1 is proved in Section 2. In Section 3 we describe the N-th order HBM adapted to our purpose. Finally, in Section 4, we use this method to prove Theorem 1.2.

2. Proof of Theorem 1.1.

(i) We start by proving that the solution of (1) with initial conditions $x(0)=A$, $\dot{x}(0)=0$ and for $t\in\left(-\frac{\sqrt{2\pi}}{2}A, \frac{\sqrt{2\pi}}{2}A\right)$ is

$$x(t)=\phi_0(t):=Ae^{-\left(\operatorname{erf}^{-1}\left(\frac{2\,t}{\sqrt{2\pi}A}\right)\right)^2}, \tag{6}$$

where erf^{-1} is the inverse of the error function

$$\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}\,ds.$$

Notice that $\lim_{t\to\pm\frac{\sqrt{2\pi}}{2}}\phi_0(t)=0$ and $\lim_{t\to\pm\frac{\sqrt{2\pi}}{2}}\phi_0'(t)=\mp\infty$. To obtain (6), observe that from system (2) we arrive at the simple differential equation $\frac{dx}{dy}=-xy$, which has separable variables and can be solved by integration. The particular solution that passes through the point $(x,y)=(A,0)$ is $x=Ae^{-y^2/2}$. Plugging this equation in (2) we get $\frac{dy}{dt}=-\frac{e^{y^2/2}}{A}$, which is again a separable equation. It has the solution

$$y(t)=-\sqrt{2}\,\operatorname{erf}^{-1}\left(\frac{2\,t}{\sqrt{2\pi}A}\right), \tag{7}$$

which is well defined for $t\in\left(-\frac{\sqrt{2\pi}}{2}A, \frac{\sqrt{2\pi}}{2}A\right)$ since $\operatorname{erf}^{-1}(\cdot)$ is defined in $(-1,1)$. Finally, since $x(t)=Ae^{-y^2(t)/2}$, we get (6).

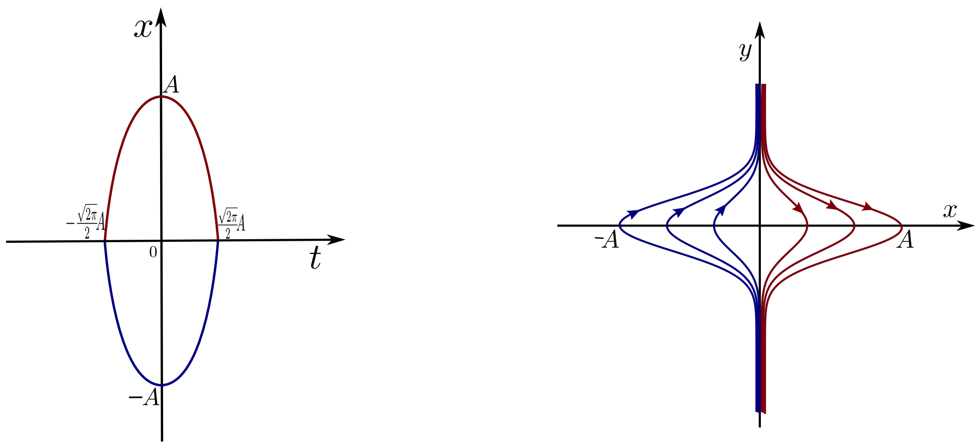

Figure 1. (a) Two solutions of equation (1) (b) Phase-portrait of system (2)

By using $x(t)$ and $y(t)$ given by (6) and (7), respectively, we can draw the phase portrait of (2) which, as we can see in Figure 1.(b), is symmetric with respect to both axes. Notice that its orbits do not cross the y-axis, which is a singular locus for the associated vector field. Moreover,

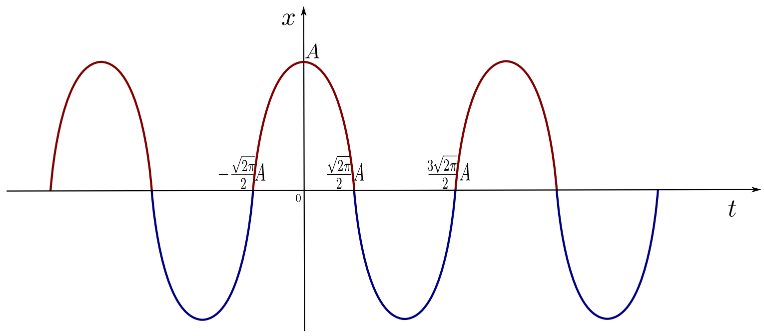

Figure 2. A weak $\mathcal{C}^0$-periodic solution of (1).

the solutions of (1) are not periodic (see Figure 1.(a)), and the transit time of $x(t)$ from $x = A$ to $x = 0$ is $\sqrt{2\pi}\,A/2$.

From (6) we introduce the $\mathcal{C}^0$-function, defined on $\mathbb{R}$, as

$$\phi(t) = \begin{cases} (-1)^n \phi_0(t - n\sqrt{2}\pi), & \text{for} \quad t \in \left(\frac{2n-1}{2}\sqrt{2\pi}, \frac{2n+1}{2}\sqrt{2\pi}\right), \quad n \in \mathbb{Z}, \\ 0, & \text{for} \quad t = \frac{2n+1}{2}\sqrt{2\pi}, \quad n \in \mathbb{Z}, \end{cases}$$

see Figure 2. It is a $\mathcal{C}^0$-periodic function of period $T(A) = 2\sqrt{2\pi}A$ and $x = \phi(t)$ satisfies (1) for all $t \in \mathbb{R} \setminus \cup_{n\in\mathbb{Z}}\{\frac{2n+1}{2}\sqrt{2\pi}\}$. Hence (i) of the theorem follows.

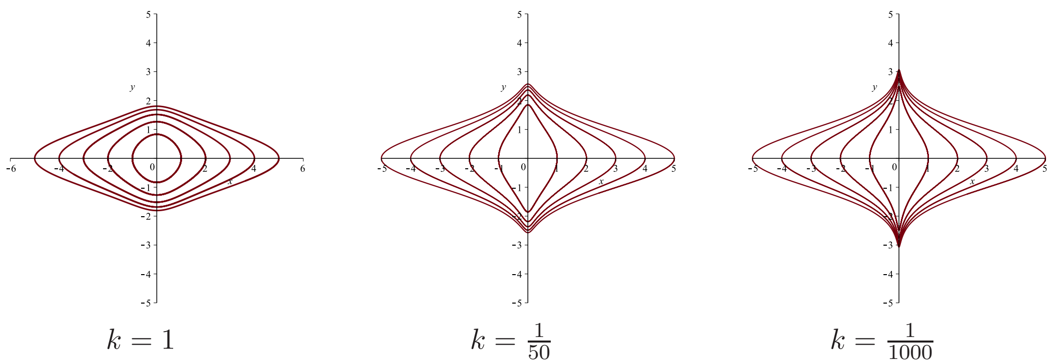

Figure 3. Phase portraits of (3) for different values of k.

(ii) System (3) is Hamiltonian with Hamiltonian function

$$H(x, y) = \frac{y^2}{2} + \frac{\ln(x^2 + k^2)}{2}.$$

Since $\ln(x^2 + k^2)$ has a global minimum at 0 and $\ln(x^2 + k^2)$ tends to infinity when $|x|$ does, system (3) has a global center at the origin. In Figure 3 we can see its phase portrait for some values of k. This figure also illustrates how the periodic orbits of (3) approach to the solutions of system (2).

Its period function is

$$T(A; k) = 2\int_{-A}^{A} \frac{dx}{y(x)} = 2\int_{-A}^{A} \frac{dx}{\sqrt{2h - \ln(x^2 + k^2)}},$$

where $h = \ln(A^2+k^2)/2$ is the energy level of the orbit passing through the point $(A,0)$. Therefore,

$$T(A;k) = 2\int_{-A}^{A} \frac{dx}{\sqrt{\ln\left(\frac{A^2+k^2}{x^2+k^2}\right)}} = 4\,A\int_0^1 \frac{ds}{\sqrt{\ln\left(\frac{A^2+k^2}{A^2s^2+k^2}\right)}},$$

where we have used the change of variable $s = x/A$ and the symmetry with respect to x. Then,

$$\lim_{k\to 0} T(A;k) = \lim_{k\to 0}\int_0^1 \frac{4\,A\,ds}{\sqrt{\ln\left(\frac{A^2+k^2}{A^2s^2+k^2}\right)}}.$$

If we prove that

$$\lim_{k\to 0}\int_0^1 \frac{4\,A\,ds}{\sqrt{\ln\left(\frac{A^2+k^2}{A^2s^2+k^2}\right)}} = \int_0^1 \lim_{k\to 0} \frac{4\,A\,ds}{\sqrt{\ln\left(\frac{A^2+k^2}{A^2s^2+k^2}\right)}}, \tag{8}$$

then

$$\lim_{k\to 0} T(A;k) = 4\,A\int_0^1 \frac{ds}{\sqrt{-2\ln(s)}} = 2\sqrt{2\pi}A = T(A)$$

and the theorem will follow. To show that (8) holds, take any sequence $1/z_n$, with z_n tending monotonically to infinity, and consider the functions $f_n(s) = \left(\ln\left(\frac{(Az_n)^2+1}{(Az_n)^2s^2+1}\right)\right)^{-1/2}$. We have that the sequence $\{f_n(s)\}_{n\in\mathbb{N}}$ is formed by measurable and positive functions defined on the interval $(0,1)$. It is not difficult to prove that it is a decreasing sequence. In particular, $f_n(s) < f_1(s)$ for all $n > 1$. Therefore, if we show that $f_1(s)$ is integrable, then we can apply the Lebesgue's dominated convergence theorem ([17]) and (8) will follow. The function $f_1(s)$ tends to infinity when $s \to 1^-$. Hence, to prove that $\int_0^1 f_1(s)\,ds < \infty$ we need to study $f_1(s)$ near $s = 1$. Set $B = (Az_1)^{-2}$ and $s = 1-w$, with $w > 0$ and $w \sim 0$. Then,

$$\frac{(Az_1)^2+1}{(Az_1)^2s^2+1} = \frac{B^{-1}+1}{B^{-1}s^2+1} = \frac{1+B}{(1-w)^2+B} = \frac{1+B}{1+B-2w+w^2} = \frac{1}{1-\frac{2}{1+B}w+O(w^2)} =$$

$$= 1+\frac{2}{1+B}w+O(w^2) = 1+\frac{2(Az_1)^2}{(Az_1)^2+1}(1-s)+O((1-s)^2).$$

By using that when $x \sim 0$, $\ln(1+x) = x+O(x^2)$, it holds that when $s<1$, $s\sim 1$,

$$f_1(s) = \left(\ln\left(\frac{A^2z_1^2+1}{A^2z_1^2s^2+1}\right)\right)^{-1/2} = \left(\ln\left(1+\frac{2(Az_1)^2}{(Az_1)^2+1}(1-s)+O((1-s)^2)\right)\right)^{-1/2}$$

$$\sim \left(\frac{2(Az_1)^2(1-s)}{(Az_1)^2+1}\right)^{-1/2}.$$

Since this last expression is integrable, the convergence of the integral is consequence of the comparison test for improper integrals.

3. The Harmonic Balance Method

This section gives a brief description of the HBM applied to second order differential equations

$$\mathcal{F} := \mathcal{F}(x(t), \ddot{x}(t)) = 0, \tag{9}$$

with $\mathcal{F}:\mathbb{R}^2\to\mathbb{R}$ a smooth function satisfying $\mathcal{F}(-u,-v)=\mathcal{F}(u,v)$, because of our interests. Notice that if $x(t)$ is a solution of (9) then $x(-t)$ is also a solution.

Suppose that equation (9) has a T-periodic solution $x(t)$ with initial conditions $x(0)=A$, $\dot{x}(0)=0$ and period $T=T(A)$. If $x(t)$ satisfies $x(t)=x(-t)$ it is clear that its Fourier series writes as

$$\sum_{k=1}^{\infty} a_k\cos(k\omega t), \quad \text{with} \quad \sum_{k=1}^{\infty} a_k = A \quad \text{and} \quad \omega=\frac{2\pi}{T}.$$

As we have seen in the previous section, the weak periodic solutions of equation (1) which we want to approximate satisfy the above property. Moreover, $x(T/4)=0$ and $\dot{x}(T/4)$ does not exist. In any case, smooth approximations to $x(t)$, should also satisfy $\dot{x}(t)\ddot{x}(t)+x(t)\dddot{x}(t)=0$, and hence $\dot{x}(T/4)=0$. For this reason, in this work we will consider Fourier series in cosine, not including the even terms $\cos(2j\omega t)$, $j\in\mathbb{N}\cup\{0\}$, which do not satisfy this property. This type of a priori simplifications are similar to the ones introduced in [10] for other problems.

Hence, in our setting, the HBM of order N follows the next five steps:

1. Consider a trigonometric polynomial

$$x_N(t)=\sum_{j=1}^{N} a_{2j-1}\cos((2j-1)\omega_N t) \quad \text{with} \quad \sum_{j=1}^{N} a_{2j-1}=A. \tag{10}$$

2. Compute the $2\pi/\omega_N$-periodic function $\mathcal{F}_N := \mathcal{F}(x_N(t),\ddot{x}_N(t))$, which also has an associated Fourier series,

$$\mathcal{F}_N(t)=\sum_{j\geq 0}\mathcal{A}_j\cos(j\,\omega_N t),$$

where $\mathcal{A}_j=\mathcal{A}_j(\mathbf{a},\omega_N,A)$ $j\geq 0$, with $\mathbf{a}=(a_1,a_3,\ldots,a_{2N-1})$.

3. Find all values $\mathbf{a}$ and ω_N such that

$$\mathcal{A}_j(\mathbf{a},\omega_N,A)=0 \qquad \text{for} \quad 1\leq j\leq j_N, \tag{11}$$

where j_N is the value such that (11) consists exactly of N non trivial equations. Notice also that each equation $\mathcal{A}_j(\mathbf{a},\omega_N,A)=0$ is equivalent to

$$\int_0^{2\pi/\omega_N}\cos(j\omega_N t)\mathcal{F}_N(t)\,dt=0. \tag{12}$$

4. Then the expression (10), with the values of $\mathbf{a}=\mathbf{a}(A)$ and $\omega_N=\omega_N(A)$ obtained in point *3*, provides candidates for the approximations of the actual periodic solutions of the initial differential equation. In particular, the functions $\mathcal{T}_N=\mathcal{T}_N(A)=2\pi/\omega_N$ give approximations of the periods of the corresponding periodic orbits.

5. The final approximation is chosen to be the one associated to the solution that minimizes the norm

$$||\mathcal{F}_N(t)||=\int_0^{\mathcal{T}_N}\mathcal{F}_N^2(t)\,dt,$$

because an actual solution of our differential equation would satisfy $||\mathcal{F}(x(t),\ddot{x}(t))||=0$ and $\mathcal{F}_N(t)=\mathcal{F}(x_N(t),\ddot{x}_N(t))$ must approach $\mathcal{F}(x(t),\ddot{x}(t))$.

Remark 3.1. *(i) Notice that going from order N to order $N+1$ in the method implies to compute again all the coefficients of the Fourier polynomial, because in general the coefficients of the Fourier series of $x_N(t)$ and $x_{N+1}(t)$ do not coincide.*

(ii) The above set of equations (11) is a system of polynomial equations which usually is not easy to solve. For this reason only the values of $N=1,2$ are considered in many works, see

for instance [11, 13] *and the references therein. To solve system* (11) *for* $N \geq 3$ *we use the Gröbner basis approach* ([3])*. In general this method is faster that using successive resultants and moreover it does not give spurious solutions.*

(iii) As far as we know, the test proposed in point 5 to select the best approach is not commonly used. We propose it following the definition of accuracy of an approximated solution used in [4] *and inspired in the classical works* [18, 20].

4. Application of the HBM

We start proving a lemma that allows to reduce our computations to the case $A = 1$.

Lemma 4.1. *Let* $\mathcal{T}_N(A;m)$ *be the period of the truncated Fourier series obtained from applying the* N*-th order HBM to equation* (4)*. Then there exist constants* $C_N(m)$ *such that* $\mathcal{T}_N(A;m) = C_N(m)A$*.*

Proof. Consider $\mathcal{F}_N = x_N^{m+1}\ddot{x}_N + x_N^m = 0$, with x_N given in (10). We have to solve the set of $N+1$ non-trivial equations

$$\int_0^{2\pi/\omega_N} \cos(j\omega_N t)\mathcal{F}_N(t)\,dt = 0 \qquad 1 \leq j \leq j_N, \qquad \sum_{j=1}^{N} a_{2j-1} = A, \tag{13}$$

with $N+1$ unknowns $a_1, a_3, \ldots, a_{2N-1}$ and ω_N and $A \neq 0$. The lemma clearly follows if we prove the following assertion: $\tilde{a}_1, \tilde{a}_3, \ldots, \tilde{a}_{2N-1}$ and $\tilde{\omega}_N$ is a solution of (13) with $A = 1$ if and only if $A\tilde{a}_1, A\tilde{a}_3, \ldots, A\tilde{a}_{2N-1}$ and $\tilde{\omega}_N/A$ is a solution of (13). This equivalence is a consequence of the fact that the change of variables $s = At$ transforms the integral equation (13) into

$$\frac{1}{A}\int_0^{2\pi A/\omega_N} \cos\left(j\frac{\omega_N}{A}s\right)\mathcal{F}_N\left(\frac{s}{A}\right)\,ds = 0$$

and of the structure of the right-hand side equation of (13).

Hence, $\mathcal{T}_N(A;m) = \mathcal{T}_N(1;m)A =: C_N(m)A$, as we wanted to prove. □

Proof of Theorem 1.2. By Lemma 4.1 we know that $\mathcal{T}_N(A;m) = C_N(m)A$ for some unknown constants $C_N(m)$. Therefore we can restrict our attention to the case $A = 1$.

(i) Following section 3, we consider $x_1(t) = \cos(\omega_1 t)$ as the first approximation to the actual solution of the functional equation $\mathcal{F}(x(t), \ddot{x}(t)) = x^{m+1}\ddot{x} + x^m = 0$. Then

$$\mathcal{F}_1(t) = -\omega_1^2 \cos^{m+2}(\omega_1 t) + \cos^m(\omega_1 t).$$

When $m = 2k$ the above expression writes as

$$\mathcal{F}_1(t) = -\omega_1^2 \cos^{2k+2}(\omega_1 t) + \cos^{2k}(\omega_1 t) = 0.$$

Using (12) for $j = 0$ we get

$$\int_0^{2\pi/\omega_1} \mathcal{F}_1(t)\,dt = -\omega_1^2 I_{2k+2} + I_{2k} = 0, \tag{14}$$

where $I_{2\ell} = \int_0^{2\pi/\omega_1} \cos^{2\ell}(\omega_1 t)dt$. Using integration by parts we prove that $(2k+2)I_{2k+2} = (2k+1)I_{2k}$. Combining this equality and (14) we obtain

$$\omega_1 = \sqrt{\frac{2k+2}{2k+1}},$$

or equivalently,

$$C_1(m) = 2\pi\sqrt{\frac{2k+1}{2k+2}},$$

which in terms of m coincides with (5). The case m odd follows similarly. The only difference is that to find $\mathcal{T}_1(A;m)$, instead of condition (14) we have to impose that

$$\int_0^{2\pi/\omega_1} \cos(\omega_1 t)\mathcal{F}_1(t)\,dt = 0,$$

because $\int_0^{2\pi/\omega_1} \mathcal{F}_1(t)\,dt \equiv 0$.

(ii) Case $m = 0$. Consider the functional equation $\mathcal{F}(x(t), \ddot{x}(t)) = x(t)\ddot{x}(t) + 1 = 0$.

When $N = 2$, we take as approximation $x_2(t) = a_1 \cos(\omega_2 t) + a_3 \cos(3\omega_2 t)$. The vanishing of the coefficients of 1 and $\cos(2\omega_2 t)$ in the Fourier series of $\mathcal{F}_2$ provides the nonlinear system

$$\begin{aligned} 1 - \frac{1}{2}(a_1^2 + 9a_3^2)\omega_2^2 &= 0, \\ a_1 + 10a_3 &= 0, \\ a_1 + a_3 - 1 &= 0. \end{aligned}$$

By solving it and applying point *5* of the HBM we get that $\omega_2 = 18/\sqrt{218}$. Therefore,

$$C_2(0) = \frac{\sqrt{218}}{9}\pi \approx 5.1539,$$

as we wanted to prove.

For the third-order HBM we use $x_3(t) = a_1 \cos(\omega_3 t) + a_3 \cos(3\omega_3 t) + a_5 \cos(5\omega_3 t)$ as the approximate solution. Imposing that the coefficients of 1, $\cos(2\omega_3 t)$, and $\cos(4\omega_3 t)$ in $\mathcal{F}_3$ vanish we arrive at the system

$$\begin{aligned} P &= 2 - \left(a_1^2 + 9a_3^2 + 25a_5^2\right)\omega_3^2 = 0, \\ Q &= a_1^2 + 10a_1a_3 + 34a_3a_5 = 0, \\ R &= 5a_3 + 13a_5 = 0, \\ S &= a_1 + a_3 + a_5 - 1 = 0. \end{aligned}$$

Since all the equations are polynomial, the solutions can be found by using the Gröbner basis approach, see [3]. Recall that the idea of this approach consists in finding a new system of generators, say $G_1, G_2, \ldots, G_\ell$, of the ideal of $\mathbb{R}[a_1, a_3, a_5, \omega_3]$ generated by P, Q, R and S. Hence, solving $P = Q = R = S = 0$ is equivalent to solving $G_i = 0$, $i = 1, \ldots, \ell$. In general, choosing the lexicographic order in the Gröbner basis approach, we get that the polynomials of the equivalent system have triangular structure with respect to the variables and it can be easily solved.

Now, by computing the Gröbner basis of $\{P, Q, R, S\}$ with respect to the lexicographic order $[a_1, a_3, a_5, \omega_3]$ we obtain a new basis with four polynomials ($\ell = 4$), one of them being,

$$G_1(\omega_3) = 1553685075\omega_3^8 - 3692301106\omega_3^6 + 2143547654\omega_3^4 - 402413472\omega_3^2 + 20301192.$$

Solving $G_1(\omega_3) = 0$ and using again point *5* of our approach to HBM we get that the solution which yields the best approximation gives

$$\omega_3 = \frac{3\sqrt{5494790257313 + 115642506449\sqrt{715}}}{6905267}.$$

Hence, the expression $C_3(0) = 2\pi/\omega_3$ of the statement follows.

When $N = 4$ we consider $x_4(t) = a_1\cos(\omega_4 t) + a_3\cos(3\omega_4 t) + a_5\cos(5\omega_4 t) + a_7\cos(7\omega_4 t)$, and we arrive at the system

$$\begin{aligned}
P &= 2 - \left(a_1^2 + 9a_3^2 + 25a_5^2 + 49a_7^2\right)\,\omega_4^2 = 0,\\
Q &= a_1^2 + 10a_1a_3 + 34a_3a_5 + 74a_5a_7 = 0,\\
R &= 5a_1a_3 + 13a_1a_5 + 29a_3a_7 = 0,\\
S &= 9a_3^2 + 50a_1a_7 + 26a_1a_5 = 0,\\
U &= a_1 + a_3 + a_5 + a_7 - 1 = 0.
\end{aligned}$$

The Gröbner basis of $\{P, Q, R, S, U\}$ with respect to the order $[a_1, a_3, a_5, a_7, \omega_4]$ is a new basis with five polynomials, one of them being an even polynomial in ω_4 of degree 16 with integers coefficients. Solving it we obtain that the best approximation is $\omega_4 \approx 1.2453$, which gives $C_4(0) \approx 5.0455$.

For $N = 5$ and $N = 6$ we have done similar computations. In the case $N = 5$ one of the generators of the Gröbner basis is an even polynomial in ω_5 with integers coefficients and degree 32. When $N = 6$ the same happens but with a polynomial of degree 64 in ω_6. Solving the corresponding polynomials we get that $\omega_5 \approx 1.2606$ and $\omega_6 \approx 1.2501$, and consequently, $C_5(0) \approx 4.9843$, and $C_6(0) \approx 5.0260$.

(iii) Case $m = 1$. We apply the HBM to $\mathcal{F}(x(t), \ddot{x}(t)) = x^2(t)\ddot{x}(t) + x(t) = 0$.

When $N = 2$, doing similar computations as in item *(ii)*, we arrive at

$$\begin{aligned}
P &= 4 - \left(3a_1^2 + 11a_1a_3 + 38a_3^2\right)\,\omega_2^2 = 0,\\
Q &= 4a_3 - \left(a_1^3 + 22a_1^2a_3 + 27a_3^3\right)\omega_2^2 = 0,\\
R &= a_1 + a_3 - 1 = 0.
\end{aligned}$$

Again, by computing the Gröbner basis of $\{P, Q, R\}$ with respect to the lexicographic order $[a_1, a_3, \omega_2]$ we obtain a new basis with three polynomials, one of them being

$$G_1(\omega_2) = 7635411\omega_2^8 - 14625556\omega_2^6 + 5833600\omega_2^4 - 661376\omega_2^2 + 13824.$$

Notice that the equation $G_1(\omega_2) = 0$ can be algebraically solved. Nevertheless, for the sake of shortness, we do not give the exact roots. Following again step *5* of our approach we get that the best solution is $\omega_2 \approx 1.1915$, or equivalently that $C_2(1) \approx 5.2733$.

The HBM when $N = 3$ produces the system

$$\begin{aligned}
P &= 4a_1 - \left(3a_1^3 + 11a_1^2a_3 + 38a_1a_3^2 + 70a_1a_3a_5 + 102a_1a_5^2 + 43a_3^2a_5\right)\omega_3^2 = 0,\\
Q &= 4a_3 - \left(a_1^3 + 22a_1^2a_3 + 27a_1^2a_5 + 70a_1a_3a_5 + 27a_3^3\right)\omega_3^2 = 0,\\
R &= 4a_5 - \left(11a_1^2a_3 + 54a_1^2a_5 + 19a_1a_3^2 + 86a_3^2a_5 + 75a_5^3\right)\omega_3^2 = 0,\\
S &= a_1 + a_3 + a_5 - 1 = 0.
\end{aligned}$$

Computing the Gröbner basis of $\{P, Q, R, S\}$ with respect to the lexicographic order $[a_1, a_3, a_5, \omega_3]$ we get that one of the polynomials of the new basis is an even polynomial in ω_3 of degree 26 with integer coefficients. By solving it we obtain that the best approximation is $\omega_3 \approx 1.2206$, which produces the value $C_3(1)$ of the statement.

When $N = 4$ we arrive at five polynomial equations, which we omit. Once more, using the Gröbner basis approach we obtain a polynomial condition in ω_4 of degree 80. Finally, $\omega_4 \approx 1.2275$ and $C_4(1) \approx 5.1186$.

(*iv*) When $m = 2$ we have to approximate the solutions of $\mathcal{F}(x(t), \ddot{x}(t)) = x^3(t)\ddot{x}(t) + x^2(t) = 0$. We do not give the details of the proof because the results are obtained by using exactly the same type of computations as above.

□

Remark 4.2. *For each N and m our computations also provide a trigonometric polynomial that approximates the continuous weak periodic solution $\phi(t)$ given in the proof of Theorem 1.1.*

Acknowledgments

The first author is supported by FONDECYT Postdoctoral fellowship 3150131/2015 and was supported by the grant FPU AP2009-1189. The second author is supported by a MINECO/FEDER grant number MTM2016-77278-P and by a CIRIT grant number 2014SGR568.

References

[1] Acton J R and Squire P T 1985 *Solving Equations with Physical Understanding* ed Adam Hilger (Bristol and Boston) 219 pp

[2] Beléndez A, Gimeno E, Álvarez M L, Gallego S, Ortuño M and Méndez D I 2009 A novel rational harmonic balance approach for periodic solutions of conservative nonlinear oscillators *International Journal of Nonlinear Sciences & Numerical Simulation* **10** 13–26.

[3] Cox D, Little J and O'Shea D 2007 *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra* Third edition Undergraduate Texts in Mathematics (New York: Springer) 551 pp

[4] García-Saldaña J D and Gasull A 2013 A theoretical basis for the harmonic balance method *J. Differ Equations* **254** 67–80

[5] García-Saldaña J D and Gasull A 2015 The period function and the harmonic balance method *Bull. Sci. Math.* **139** 33–60

[6] Giné J 2013 On the determination of the limit cycles using the harmonic balance method *J. Math. Phys.* **54** 103510

[7] Guo Z and Leung A Y T 2010 The iterative homotopy harmonic balance method for conservative Helmholtz-Duffing oscillators *Appl. Math. Comput.* **215** 3163–69

[8] He J-H 2001 An analytic solution to certain nonlinear equations in plasma physics *Ukr. J. Phys.* **46** 567–568

[9] Lefschetz S 1977 *Differential Equations: Geometric Theory* reprinting of the second edition (New York: Dover Publications Inc.) 390 pp

[10] Mickens R E 2002 Fourier representations for periodic solutions of odd-parity systems *J. Sound Vibration* **258** 398–401

[11] Mickens R E 2007 Harmonic balance and iteration calculations of periodic solutions to $\ddot{y} + y^{-1} = 0$ *J. Sound Vibration* **306** 968–72

[12] Mickens R E 1996 *Oscillations in planar dynamic systems* Series on Advances in Mathematics for Applied Sciences **37** (River Edge, NJ: World Scientific Publishing Co. Inc.) 319 pp

[13] Mickens R E 2010 *Truly Nonlinear Oscillations. Harmonic balance, Parameter Expansions, Iteration, and Averaging Methods* (Hackensack, NJ: World Scientific Publishing Co. Pte. Ltd.) 238 pp

[14] Mickens R E and Wilkerson D 2009 Exact and approximate values of the period for a "truly nonlinear" oscillator: $\ddot{x} + x + x^{1/3} = 0$. *Adv. Appl. Math. Mech.* **1** 383–390

[15] Momeni M, Jamshidi N, Barari A and Ganji D D 2011 Application of He's energy balance method to Duffing-harmonic oscillators *Int. J. Comput. Math.* **88** 135–144.

[16] Müller P C 2008 Comments on the method of harmonic balance for nonlinear conservative single-degree-of-freedom systems *Conference Proceedings, 6th EUROMECH Nonlinear Dynamics Conference (ENOC 2008)* 4 pp.

[17] Rudin W 1987 *Real and complex analysis* Third ed. (New York: McGraw-Hill Book Co.) 416 pp

[18] Stokes A 1972 On the approximation of nonlinear oscillations *J. Differ Equations* **12** 535–58

[19] Urabe M 1964 Relations between periods and amplitudes of periodic solutions of $\ddot{x} + g(x) = 0$ *Funkcial. Ekvac.* **6** 63–88

[20] Urabe M 1965 Galerkin's procedure for nonlinear periodic systems *Arch. Rational Mech. Anal.* **20** 120–52