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BIFURCATION DIAGRAMS FOR HAMILTONIAN NILPOTENT CENTERS OF LINEAR PLUS CUBIC HOMOGENEOUS POLYNOMIAL VECTOR FIELDS

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Abstract. Following the work done in [8] we provide the bifurcation diagrams for the global phase portraits in the Poincaré disk of all Hamiltonian nilpotent centers of linear plus cubic homogeneous planar polynomial vector fields.

1. Introduction

To distinguish when a singular point of a real planar polynomial differential system is a focus or a center is one of the main problems in the qualitative theory of differential systems. The definition of a *center* mainly goes back to Poincaré, who in [17] defines a *center* for a vector field on the real plane as a singular point having a neighborhood filled with periodic orbits with the exception of the singular point.

There are three types of centers for analytic differential systems. An analytic system having a center can be written in one of the following forms after an affine change of variables and a rescaling of the time variable:

$\dot{x} = -y + P(x,y)$, $\dot{y} = x + Q(x,y)$, called a *linear type center*,
$\dot{x} = y + P(x,y)$, $\dot{y} = Q(x,y)$, called a *nilpotent center*,
$\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, called a *degenerate center*,

where $P(x,y)$ and $Q(x,y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. Poincaré [18] and Lyapunov [14] provide an algorithm for the characterization of linear type centers, see also Chazy [5] and Moussu [16]. There is also an algorithm for the characterization of nilpotent and some class of degenerate centers due to Chavarriga *et al.* [4], Cima and Llibre [6], Giacomini *et al.* [10], and Giné and Llibre [11].

The study of centers of polynomial differential systems started with the characterization of the centers of quadratic ones, and these studies are historically traced back to mainly Kapteyn [12, 13] and Bautin [1]. For more recent works see Schlomiuk [19] and Żołądek [24]. Even though the centers of polynomial differential systems with degrees higher than 2 are not classified completely, there are many partial results. For instance the linear type centers of cubic polynomial differential systems of the form linear with homogeneous nonlinearities of degree 3 were characterized by Malkin [15], and by Vulpe and Sibiirski [22]. On the other hand, for systems with higher

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degree homogeneous nonlinearities the linear type centers are not fully characterized, but see Chavarriga and Giné [2, 3] for some of the main results. Despite these advances the path to characterize and classify the centers of all polynomial differential systems of degree 3 and greater is long. We note that there are some interesting results in some subclasses of cubic systems due to the works of Rousseau and Schlomiuk [20], and Żołądek [25, 26].

In [21] Vulpe provides all the global phase portraits of quadratic polynomial differential systems having a center. Then the bifurcation diagrams for the global phase portraits of these systems is given in [19]. The global phase portraits of linear type and nilpotent centers of polynomial differential systems having linear plus cubic homogeneous terms are presented in [7] (see also [9]) and in [8] respectively. In this work we provide the bifurcation diagrams for the global phase portraits of the latter.

We say that two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one onto the other which sends orbits to orbits preserving or reversing the direction of the flow. In [8] the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields with only linear and cubic homogeneous terms having a nilpotent center at the origin are given by the following theorem:

Theorem 1. *A Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms has a nilpotent center at the origin if and only if, after a linear change of variables and a rescaling of its independent variable, it can be written as one of the following six classes:*

(I) $\dot{x} = ax + by$, $\dot{y} = -\dfrac{a^2}{b}x - ay + x^3$, *with* $b < 0$.

(II) $\dot{x} = ax + by - x^3$, $\dot{y} = -\dfrac{a^2}{b}x - ay + 3x^2y$, *with* $a > 0$ *and* $b \neq 0$.

(III) $\dot{x} = ax + by - 3x^2y + y^3$, $\dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + 3xy^2$, *with either* $a = b = 0$ *and* $c < 0$, *or* $c = 0$, $ab \neq 0$, *and* $a^2/b - 6b > 0$.

(IV) $\dot{x} = ax + by - 3x^2y - y^3$, $\dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + 3xy^2$, *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0$, $a \neq 0$, *and* $b < 0$.

(V) $\dot{x} = ax+by-3\mu x^2y+y^3$, $\dot{y} = \left(c-\dfrac{a^2}{b+c}\right)x-ay+x^3+3\mu xy^2$, *with either* $a = b = 0$ *and* $c < 0$, *or* $c = 0$, $b \neq 0$, *and* $(a^4 - b^4 - 6a^2b^2\mu)/b > 0$.

(VI) $\dot{x} = ax+by-3\mu x^2y-y^3$, $\dot{y} = \left(c-\dfrac{a^2}{b+c}\right)x-ay+x^3+3\mu xy^2$, *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0$, $b \neq 0$, *and* $(a^4 + b^4 + 6a^2b^2\mu)/b < 0$.

where $a, b, c, \mu \in \mathbb{R}$. *Moreover the global phase portraits of these six families of systems are topologically equivalent to the following of Figure 1:*

(a) 1.1 *for systems (I) and (IV);*
(b) 1.2 *for systems (II);*
(c) 1.3, 1.4 *or* 1.5 *for systems (II);*
(d) 1.2, 1.6, 1.7 *or* 1.8 *for systems (V);*
(e) 1.9, 1.10, 1.11 *or* 1.12 *for systems (VI).*

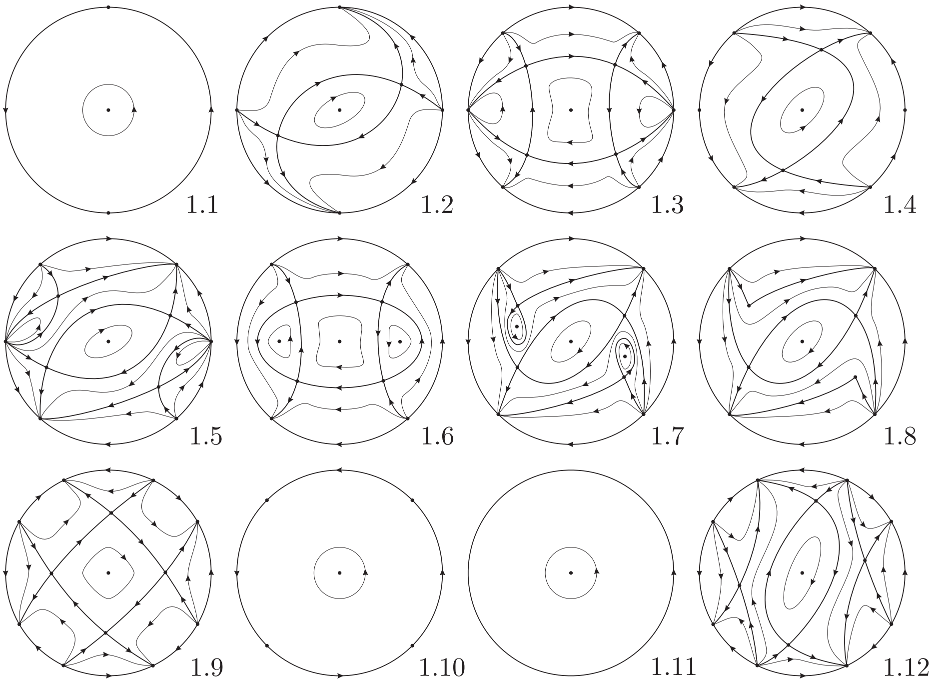

FIGURE 1. Global phase portraits of the vector fields in Theorem 1 which have a nilpotent center at the origin. The separatrices are in bold.

Note that for the above systems we have $a = 0$ whenever $b = 0$. Before stating our main result we make the following remark.

Remark 2. *A system in class (V) with $a = c = 0$ can be transformed to a system inside the same class with $a = b = 0$ and $c \neq 0$ doing the change $(x, y) \mapsto (y, x)$, $c \mapsto b$ and $\mu \mapsto -\mu$. Hence when $c = 0$ we can assume $a \neq 0$. Similarly we can assume $a \neq 0$ in systems (VI) whenever $c = 0$(in this case the change of variables is $(x, y) \mapsto (-y, x)$).*

On the other hand, via the rescaling of the variables $(x, y, t) \mapsto (x/\sqrt{|a|}, y/\sqrt{|a|}, |a|t)$ and the parameter $b \mapsto b/|a|$ we can assume $a = 1$ in the families of systems $(III) - (VI)$ when $c = 0$.

Finally a further change of variables $(x, y, t) \mapsto (-y, x, -t)$ together with $b \mapsto 1/b$ and $\mu \mapsto -\mu$ allows to assume $b > 0$ for systems (V) when $b \neq 0$.

Taking into account Remark 2, when $c = 0$ we will assume $a = 1$ for classes $(III) - (VI)$, and $b > 0$ for systems (V) throughout the rest of this paper. Our main result is the following:

Theorem 3. *The global phase portraits of Hamiltonian planar polynomial vector fields with linear plus cubic homogeneous terms having a nilpotent center at the origin are topologically equivalent to the following ones of Figure 1 using the notation of Theorem 1.*

(a) *For systems (I) and (IV) the phase portrait is* 1.1.
(b) *For systems (II) the phase portrait is* 1.2.

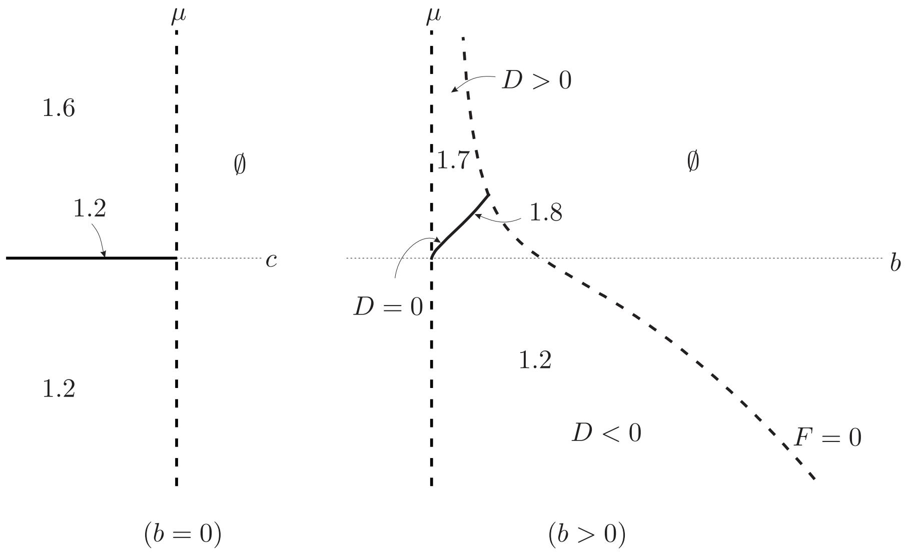

FIGURE 2. The bifurcation diagrams for systems (V) when $b = 0$ and when $b > 0$. Note that when $b > 0$ we have $c = 0$. In the figure $F = 1 - b^4 - 6b^2\mu$.

(c) *For systems (III) the phase portraits are* 1.3, 1.4 *and* 1.5 *when $b = 0$, $b < 0$ and $b > 0$ respectively.*
(d) *For systems (V) with $b = 0$ the phase portrait is* 1.2 *and* 1.6 *when $\mu \leq 0$ and $\mu > 0$ respectively, and with $b > 0$ the phase portraits are* 1.2, 1.7 *and* 1.8 *when $D < 0$, $D > 0$ and $D = 0$ respectively. Here $D = -b^2 - 6b^2\mu + 4(1 - b^4)\mu^3 + 3b^2\mu^4$, and the corresponding bifurcation diagrams are shown in Figure 2.*
(e) *For systems (VI) with $\mu > -1/3$ the phase portrait is* 1.11, *with $\mu = -1/3$ the phase portrait is* 1.10, *and with $\mu < -1/3$ the phase portraits are* 1.9 *if $b = 0, 1$ and* 1.12 *otherwise. The corresponding bifurcation diagrams are shown in Figure 3.*

We remark that all the equations controlling the bifurcations of the global phase portraits described in Theorem 3 are algebraic curves.

Observe that each of the classes (I), (II) and (IV) have a unique global phase portrait. Also the bifurcation diagram of the phase portraits of systems (III) is trivial and follows directly from the work done in [8]. Consequently it remains to prove the last two statements of Theorem 3, and we will prove them in the following sections. We note that the explicit expressions of the finite singular points of classes (V) and (VI) are complicated, making it difficult to study their types or even their existence on the real plane. Therefore we will follow different approaches in determining the bifurcation diagrams for these last two classes.

2. Bifurcation diagram for systems (V)

Recall that for systems (V) we have $b \geq 0$. According to [8] when $a = b = 0$ systems (V) have the global phase portraits (up to topological equivalence)

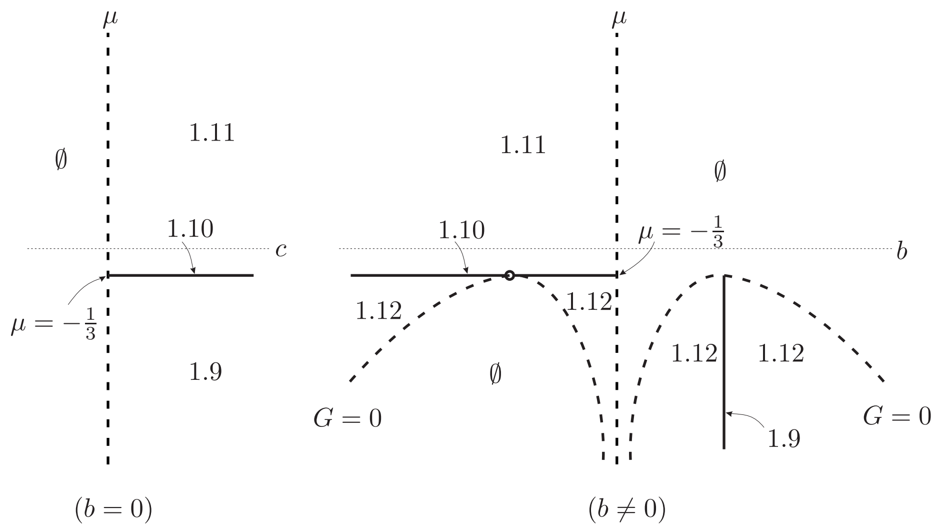

FIGURE 3. The bifurcation diagram for systems (VI) when $b = 0$ and when $b \neq 0$. Note that when $b \neq 0$ we have $c = 0$. In the figure $G = 1 + b^4 + 6b^2\mu$.

1.2 and 1.6 of Figure 1 when $\mu \leq 0$ and $\mu > 0$ respectively. On the other hand, when $b > 0$ there are three possible phase portraits: 1.2, 1.7 and 1.8 of Figure 1. The information in [8] is not enough to determine exactly when each phase portrait is achieved by these systems.

When $b > 0$, by Remark 2 systems (V) can be written as

$$\dot{x} = x + by - 3\mu x^2 y + y^3, \tag{1a}$$

$$\dot{y} = -x/b - y + x^3 + 3\mu x y^2, \tag{1b}$$

with

$$1 - b^4 - 6b^2\mu > 0. \tag{2}$$

We see that each of these three phase portraits has a different number of finite singular points, hence we will use this property to distinguish them. The explicit expressions of the finite singular points are complicated, and since we are only interested in the number of finite singular points we will make use of Yang's work [23] on the number of real roots of polynomials depending on their coefficients.

First we equate (1a) to zero, solve for x and get

$$x_{1,2} = \frac{1 \pm \sqrt{1 + 12b\mu y^2 + 12\mu y^4}}{6\mu y}. \tag{3}$$

We see that (3) is not defined when $\mu y = 0$, so we need to address this case separately.

When $y = 0$ we have (1a) equal to zero if and only if $x = 0$, so we can assume $y \neq 0$ because we are not interested in the origin. On the other hand, when $\mu = 0$ we can easily calculate the finite singular points of systems (1) and see that other than the origin there are only two, namely

$\pm\big(b^{1/3}\sqrt{(1-b^{4/3})/b}, -\sqrt{(1-b^{4/3})/b}\big)$. Note that these points are real because when $\mu=0$ inequality (2) yields $b<1$. Therefore when $\mu=0$ the global phase portrait of systems (1) is topologically equivalent to 1.2 of Figure 1.

Now we can assume $\mu y\neq 0$, substitute (3) into (1b) and obtain

$$\begin{aligned}\dot y_{1,2}=&\frac{1}{54b\mu^3y^3}\Big(b+9\mu(b^2-\mu)y^2+9b\mu(1-3\mu^2)y^4\\&\pm\sqrt{1+12b\mu y^2+12\mu y^4}\big(b+3\mu(b^2-3\mu)y^2+3b\mu(1+9\mu^2)y^4\big)\Big),\end{aligned}$$

where $\dot y_1$ and $\dot y_2$ denote $\dot y$ with x substituted by x_1 and x_2 respectively. Each root of $\dot y_1$ and $\dot y_2$ will be paired with at most one x by (3). Therefore the number of roots of $\dot y_1$ and $\dot y_2$ provides important information on the number of finite singular points of systems (1). So we compute the product $\dot y_1\dot y_2$ and obtain the sextic polynomial

$$\begin{aligned}-\frac{1}{27b^2\mu^3}\big(&b^2(1+9\mu^2)^2y^6+3b(1+9\mu^2)(b^2-2\mu+3b^2\mu^2)y^4\\&+3(b^4+3\mu^2+6b^4\mu^2-18b^2\mu^3)y^2-b(1-b^4-6b^2\mu)\big).\end{aligned}\tag{4}$$

Then we study the relation between the number of roots of (4) and the number of finite singular points of systems (1).

First we claim that the number of finite singular points cannot be less than the number of roots of (4). Now we prove our claim. If we define

$$\begin{aligned}s_1=&b+3\mu(b^2-3\mu)y^2+3b\mu(1+9\mu^2)y^4,\\s_2=&b+9\mu(b^2-\mu)y^2+9b\mu(1-3\mu^2)y^4,\\s_3=&1+12b\mu y^2+12\mu y^4,\end{aligned}$$

then we have $\dot y_{1,2}=(s_2\pm\sqrt{s_3}s_1)/54b\mu^3y^3$, and polynomial (4) can be rewritten as

$$\frac{1}{2916b^2\mu^6y^6}(s_2^2-s_3s_1^2).\tag{5}$$

The number of finite singular points are less than the number of roots of (4) only when $s_3<0$ because then (3) become complex. If $s_3<0$ then (5) is zero if and only if $s_1=s_2=0$. But if we subtract s_2 from s_1 we obtain

$$6b\mu\big(-b+(9\mu^2-1)y^2\big)y^2.\tag{6}$$

Since we have $b>0$ and $\mu y\neq 0$, (6) is zero if and only if $y=\pm\sqrt{b/(9\mu^2-1)}$, where $9\mu^2-1$ must be positive. Then we substitute these y into s_1 and s_2, and see that they are roots of these two polynomials provided that we have

$$1-9\mu^2+54b^2\mu^3=0.\tag{7}$$

We note that equation (7) further requires $\mu>0$ because we have $9\mu^2-1>0$. But this means that $s_3>0$, and this is a contradiction to the assumption that $s_3<0$. Hence we conclude that $s_3\geq 0$, so $x_{1,2}$ are real whenever y is a root of (4), and this proves the claim.

Second we consider the case in which the number of finite singular points could be greater than the number of roots of (4). This is only possible when $\dot y_1$ and $\dot y_2$ have common roots which produce distinct x in (3), so we must have $s_3 > 0$ and $s_1 = s_2 = 0$ for a common root. We have seen that this occurs if and only if equation (7) is satisfied with $\mu > 1/3$. Moreover in this case the number of common roots of s_1 and s_2 is two due to the fact that they have the same constant terms whereas their second order terms are different since $\mu \neq 0$.

In short the number of finite singular points of systems (1) is equal to the number of real roots of polynomial (4) unless $1 - 9\mu^2 + 54b^2\mu^3 = 0$ and $\mu > 1/3$, in which case there are two more singular points.

As we mentioned earlier we will determine the number of roots of (4) following [23], where the author provides the detailed analysis of the number of real roots of sextic polynomials.

We first need to compute the "discriminant sequence" $\{D_1, \dots, D_6\}$ of (4) accordingly (see [23] for definitions and details). Then we will determine the "sign list" $[\mathrm{sign}(D_1), \dots, \mathrm{sign}(D_6)]$ of the discriminant sequence, where the sign function is

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

And finally we need to construct the associated "revised sign list" $[r_1, \dots, r_6]$ which will give all the information about the number of real and complex roots of our polynomial. Given any sign list $[s_1, \dots, s_n]$, the revised sign list $[r_1, \dots, r_n]$ is obtained as follows:

If $s_k \neq 0$ we write $r_k = s_k$.
If $[s_i, s_{i+1}, \dots, s_{i+j}]$ is a section of the given sign list such that $s_{i+1} = \dots = s_{i+j-1} = 0$ with $s_i s_{i+j} \neq 0$, then in place of $[r_{i+1}, \dots, r_{i+j-1}]$ we write the $(j-1)$-tuple

$$[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, -s_i, \dots].$$

Note that this way there are no zeros between nonzero elements of the revised sign list.

The elements of the discriminant sequence of a sextic polynomial of the form

$$x^6 + px^4 + qx^3 + rx^2 + sx^2 + t$$

given in [23] are:

$$D_1 = 1, \quad D_2 = -p, \quad D_3 = 24rp - 8p^3 - 27q^2,$$

$$\begin{aligned} D_4 = {} & 32p^4r - 12p^3q^2 + 96p^3t + 324prq^2 - 224r^2p^2 - 288ptr - 120qp^2s \\ & + 300ps^2 - 81q^4 + 324tq^2 - 720qsr + 384r^3, \end{aligned}$$

$$\begin{aligned} D_5 = {} & -4p^3q^2r^2 - 1344ptr^3 + 24p^4q^2t + 144pq^2r^3 + 1440ps^2r^2 + 162q^4tp \\ & - 5400rts^2 + 1512prtsq + 16p^4r^3 - 192p^4t^2 + 72p^5s^2 - 128r^4p^2 \\ & + 256r^5 + 1875s^4 - 64p^5rt + 592p^3tr^2 + 432rt^2p^2 - 616rs^2p^3 \\ & + 558q^2p^2s^2 + 1080s^2tp^2 - 2400ps^3q - 324pt^2q^2 - 1134tsq^3 \\ & + 648q^2tr^2 + 1620q^2s^2r - 1344qsr^3 + 3240qst^2 + 12p^3q^3s - 1296pt^3 \\ & - 27q^4r^2 + 81q^5s + 1728t^2r^2 - 56p^4rsq - 72p^3tsq + 432r^2p^2sq \end{aligned}$$

$$
\begin{aligned}
&- 648rq^2tp^2 - 486prq^3s,\\
D_6 =& - 32400ps^2t^3 - 3750pqs^5 + 16q^3p^3s^3 - 8640q^2p^3t^3 + 825q^2p^2s^4\\
&+ 108q^4p^3t^2 + 16r^3p^4s^2 - 64r^4p^4t - 4352r^3p^3t^2 + 512r^2p^5t^2\\
&+ 9216rp^4t^3 - 900rp^3s^4 - 17280t^3p^2r^2 - 192t^2p^4s^2 + 1500tp^2s^4\\
&- 128r^4p^2s^2 + 512r^5p^2t + 9216r^4pt^2 + 2000r^2s^4p + 108s^4p^5\\
&- 1024p^6t^3 - 4q^2p^3r^2s^2 - 13824t^4p^3 + 16q^2p^3r^3t + 8208q^2p^2r^2t^2\\
&- 72q^3p^3str + 5832q^3p^2st^2 + 24q^2p^4ts^2 - 576q^2p^4t^2r - 4536q^2p^2s^2tr\\
&- 72rp^4qs^3 + 320r^2p^4qst - 5760rp^3qst^2 - 576rp^5ts^2 + 4816r^2p^3s^2t\\
&- 120tp^3qs^3 + 46656t^3p^2qs - 6480t^2p^2s^2r + 560r^2qp^2s^3 - 2496r^3qp^2st\\
&- 3456r^2qpst^2 - 10560r^3s^2pt + 768sp^5t^2q + 19800s^3rqpt + 3125s^6\\
&- 46656t^5 - 13824r^3t^3 + 256r^5s^2 - 1024r^6t + 62208prt^4 + 108q^5s^3\\
&- 874q^4t^3 + 729q^6t^2 + 34992q^2t^4 - 630prq^3s^3 + 3888prq^2t^3\\
&+ 2250rq^2s^4 - 4860prq^4t^2 - 22500rts^4 + 144pr^3q^2s^2 - 576pr^4q^2t\\
&- 8640r^3q^2t^2 + 2808pr^2q^3st + 21384rq^3st^2 - 9720r^2q^2s^2t\\
&- 77760rt^3qs + 43200r^2t^2s^2 - 1600r^3qs^3 + 6912r^4qst - 27540pq^2t^2s^2\\
&- 27q^4r^2s^2 + 108q^4r^3t - 486q^5str + 162pq^4ts^2 - 1350q^3ts^3\\
&+ 27000s^3qt^2.
\end{aligned}
$$

We compute the discriminant sequence of the polynomial (4) and get

$$
\begin{aligned}
D_2 =&\frac{3A}{b(1+9\mu^2)}, \quad D_3 = \frac{216AB}{b^3(1+9\mu^2)^3}, \quad D_4 = \frac{2596BC}{b^4(1+9\mu^2)^6}\\
D_5 =&\frac{3888CDE^2}{b^6(1+9\mu^2)^{10}}, \quad D_6 = \frac{46656D^2E^4F}{b^9(1+9\mu^2)^{14}},
\end{aligned}
$$

where

$$
\begin{aligned}
A =& - b^2 + 2\mu - 3b^2\mu^2,\\
B =& (-4b^2 + \mu + 6b^2\mu^2 + 9b^4\mu^3)\mu,\\
C =& - b^2 + 2(1 - 4b^4)\mu - 27b^2\mu^2 - 18(1 - 2b^4)\mu^3 + 9b^2(7 + 2b^4)\mu^4\\
&+ 54(2 + 9b^4)\mu^5 - 81b^2(7 - 2b^4)\mu^6 - 486b^4\mu^7,\\
D =& - b^2 - 6b^2\mu^2 + 4(1 - b^4)\mu^3 + 3b^2\mu^4,\\
E =& 1 - 9\mu^2 + 54b^2\mu^3,\\
F =& 1 - b^4 - 6b^2\mu.
\end{aligned}
$$

Observe that we have $F > 0$ due to (2), and $b(1 + 9\mu^2) > 0$. Hence the sign list of this discriminant sequence is determined only by the signs of A, B, C, D and E. Note that $D_6 \geq 0$.

We have seen that systems (1) have six finite singular points other than the origin when

(i) polynomial (4) has six real distinct roots,

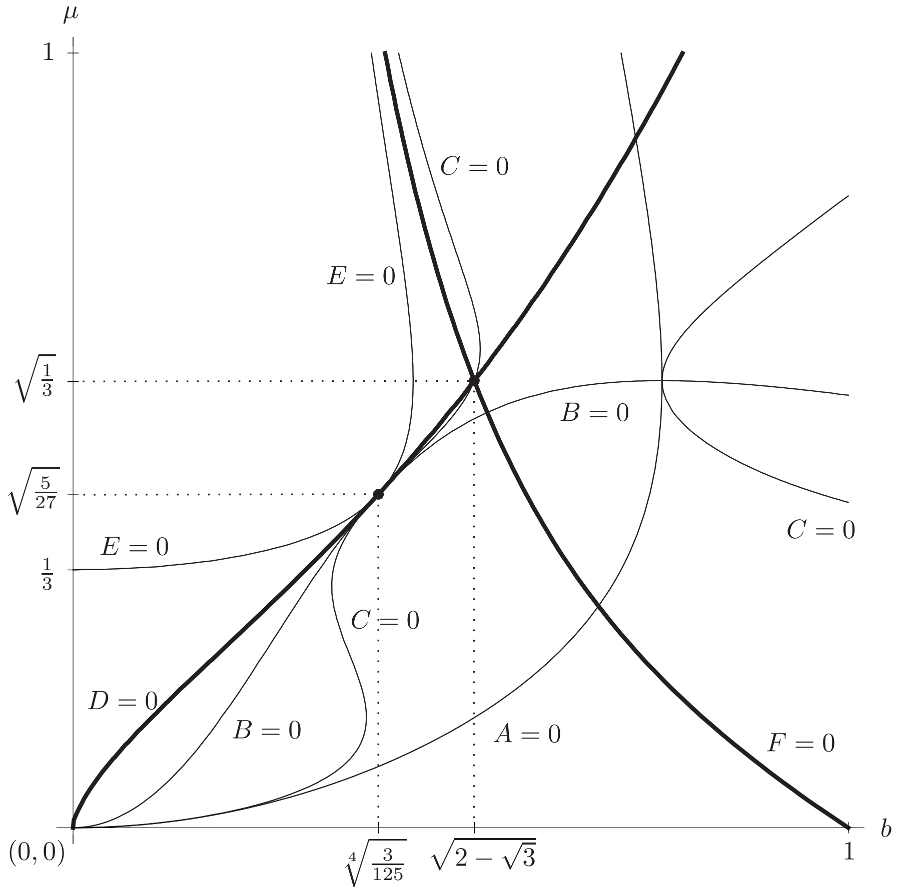

FIGURE 4. The graphs of $A=0$, $B=0$, $C=0$, $D=0$ and $E=0$ on the (b,μ)–plane.

(ii) it has four real distinct roots provided that $E=0$ and $\mu>1/3$, see (6).

According to [23] the only revised sign list in case (i) is $[1,1,1,1,1,1]$. Hence we need $D_i>0$ for all $i=1,\ldots,6$. Since A must be positive we get $\mu,B,C,D>0$. We also have $E\neq 0$. We plot the graphs of $A=0$, $B=0$, $C=0$, $D=0$, $E=0$ and $F=0$ in Figure 4 in the first quadrant of the (b,μ)–plane in order to study these inequalities. It is not difficult to prove that the curve $F=0$ does not intersect $E=0$, and that it intersects each of the remaining curves only once. Also note that $\mu>1/3$ when $E=0$.

Since we are only interested in the case $F>0$, which is to the left of the curve $F=0$ in Figure 4, we are not interested in the component of the curve $C=0$ which does not pass through the origin. We see that D is positive on the left and negative on the right of the curve $D=0$ in Figure 4. Moreover we have $A,B,C>0$ whenever $D,F>0$. Therefore case (i) characterized by the conditions $D>0$ and $E\neq 0$.

Due to [23] the unique revised sign list in case (ii) is $[1,1,1,1,0,0]$, so we need $A,B,C>0$. Figure 4 shows that these three inequalities and the equality $E = 0$ are satisfied only when $D > 0$. Hence case (ii) is characterized by the conditions $D > 0$ and $E = 0$.

We have shown that systems (1) have six finite singular points other than the origin independent of E, and that their global phase portraits are topologically equivalent to 1.7 of Figure 1 if and only if $D > 0$.

Now we study systems (1) having four finite singular points different from the origin. This can be achieved if and only if

(iii) either (4) has four real distinct roots provided that $E \neq 0$,
(iv) or (4) has two real distinct roots and $E = 0$.

Hence the possible revised sign lists that we need to study are $[1, 1, 1, 1, 0, 0]$ and $[1, 1, 0, 0, 0, 0]$ corresponding to cases (iii) and (iv) respectively.

In case (iii) we need $A, B, C > 0$ and $D = 0$. We again have $\mu > 0$ because $A > 0$. From Figure 4 we see that when $F > 0$ and $D = 0$ we have $A, B, C > 0$ unless $\mu = \sqrt{5/27}$. On the other hand case (iv) requires $B = E = 0$, which is possible only when $\mu = \sqrt{5/27}$, at which we have $D = 0$. Therefore whether $\mu = \sqrt{5/27}$ or not, systems (1) have four finite singular points additional to the origin if and only if $D = 0$. Consequently the global phase portrait is 1.8 of Figure 1 if and only if $D = 0$.

Finally it only remains to study systems (1) having only two additional finite singular points, in which case their global phase portraits are topologically equivalent to 1.2 of Figure 1. But as a trivial result of the above study this case can be realized if and only if $D < 0$.

In light of all the information that we obtained, for systems (V) we get the bifurcation diagram given in Figure 2.

3. Bifurcation diagram for systems (VI)

In [8] it is shown that for $\mu = -1/3$ and $\mu > -1/3$, the global phase portraits of systems (VI) are topologically equivalent to 1.10 and 1.11 of Figure 1, respectively. When $\mu < -1/3$, the unique global phase portrait is 1.9 of Figure 1 if $b = 0$, but there are two possibilities if $b \neq 0$: 1.9 and 1.12. We are going to distinguish these last two phase portraits using the facts that systems (VI) are Hamiltonian and that there are four finite singular points on the same energy level in the former but only two in the latter.

When $b \neq 0$, due to Remark 2 systems (VI) are written as

$$\dot{x} = x + by - 3\mu x^2 y - y^3, \tag{8a}$$

$$\dot{y} = -x/b - y + x^3 + 3\mu x y^2, \tag{8b}$$

where

$$\frac{1 + b^4 + 6b^2\mu}{b} < 0, \tag{9}$$

with the Hamiltonian

$$H(x, y) = -\frac{x^4 + y^4}{4} - \frac{3\mu x^2 y^2}{2} + \frac{x^2}{2b} + \frac{by^2}{2} + xy.$$

Assume $\mu < -1/3$. We are going to look for the number $\mathcal{N}$ of distinct real solutions of the three equations $\dot{x} = 0$, $\dot{y} = 0$ and $H - h = 0$, where $h \in \mathbb{R} \setminus \{0\}$. Note that $h \neq 0$ because the only singular point of systems

(8) at which $H = 0$ is the origin. Indeed, evaluating H at a singular point (x_0, y_0) of systems (8) we get

$$H(x_0, y_0) = H(x_0, y_0) - \frac{y_0\dot{x} - x_0\dot{y}}{4} = \frac{(x_0 + by_0)^2}{4b} = h,$$

due to the fact that $\dot{x} = \dot{y} = 0$ at (x_0, y_0). Then we have $h = 0$ if and only if $x_0 + by_0 = 0$. But when $x_0 = -by_0$ we obtain

$$\dot{x} = -(1 + 3b^2\mu)y_0^3 = 0,$$
$$\dot{y} = -b(b^2 + 3\mu)y_0^3 = 0.$$

If $y_0 \neq 0$, then, since $b \neq 0$, we need to have $1 + 3b^2\mu = 0 = b^2 + 3\mu$. Hence we get $b^2 = -3\mu$ and $1 - 9\mu^2 = 0$, which is not possible because $\mu < -1/3$. So we have $y_0 = x_0 = 0$.

In order to simplify our calculations we multiply H by 4 and calculate the Gröbner basis of the three polynomials $\dot{x}$, $\dot{x}$ and $4H - h$. We see that it consists of 27 polynomials in the variables x and y. Due to the length of these polynomials we cannot present all of them here but we provide all the necessary information that we get from them. First of all there are 21 polynomials that do not contain x, and they are of degrees varying between two and six in y. In particular, 7 of these polynomials are of the form py^2+q, where p and q are constants in terms of the parameters b and μ. Second, there is another polynomial that is linear in x such that the coefficient of x is $809238528h$, which is different from zero. This means that whenever $p \neq 0$ in one of the 7 polynomials of the form $py^2 + q$ we have $\mathcal{N} \leq 2$, and therefore at most two singular points of systems (8) are on the same energy level.

We pick 4 of these 7 polynomials and call them P_1, P_2, P_3 and P_4. Due to the length of these polynomials, we will only provide the coefficients of their quadratic terms:

$$\begin{aligned}
p_1 = {} & h(b^4-1)^2\big(-27b^2(b^4-1)^2 + 16(b^4+1)^3h - 48b^2(b^4+1)^2h^2 \\
& + 48b^4(b^4+1)h^3 - 16b^6h^4\big),\\
p_2 = {} & h(b^4-1)\big(10368(b^4-1)^3 + 24(192\mu + 1399b^2 - 846b^6 - 233b^{10} \\
& - 256b^{14})h + 4(b^4-1)(520 + 5759b^4 + 4345b^8)h^2 - 2(155b^2 - 5046b^6 \\
& + 6139b^{10} + 864b^{14} + 6336\mu)h^3 + (b^4-1)(1592 + 13109b^4 + 3731b^8)h^4 \\
& + (5577b^2 - 466b^6 - 3399b^{10} + 880b^{14} + 7776\mu)h^5 - 2(b^4-1)(721 \\
& + 2240b^4 + 1151b^8)h^6 - 4(656b^2 + 203b^6 - 861b^{10} + 152b^{14} + 450\mu)h^7 \\
& + 4(b^4-1)(153 + 230b^4 + 445b^8)h^8 + 16(74b^2 + 43b^6 - 118b^{10} + 4b^{14} \\
& + 9\mu)h^9 - 32(b^4-1)(b^4-2)(6b^4-1)h^{10} + 64(b^4-1)b^2(3b^4+2)h^{11} \\
& - 64(b^4-1)b^4h^{12}\big),\\
p_3 = {} & h(b^4-1)\big(96(-149 + 328b^4 - 369b^8 + 126b^{12} - 192b^2\mu) + 2(24419b^2 \\
& - 15526b^6 - 2429b^{10} - 3584b^{14} + 8640\mu)h - (5136 - 14383b^4 + 26286b^8 \\
& - 29327b^{12} - 36864b^2\mu)h^2 + (6053b^2 + 10182b^6 - 20571b^{10} - 8144b^{14} \\
& - 37440\mu)h^3 + 2(b^4-1)(4923 + 21104b^4 + 11621b^8)h^4 + 4(4688b^2
\end{aligned}$$

$$+ 2065b^6 - 6775b^{10} + 520b^{14} + 1494\mu)h^5 - 4(b^4-1)(523 - 494b^4 + 1527b^8)h^6 - 16(254b^2 + 145b^6 - 402b^{10} + 12b^{14} + 27\mu)h^7 + 32(b^4-1)(6 - 47b^4 + 18b^8)h^8 - 192(b^4-1)b^2(2+3b^4)h^9 + 192(b^4-1)b^4h^{10}\big),$$

$$p_4 = h\big(254016(b^4-1)(119b^2 - 78b^6 + 23b^10 + 192\mu) - 96(13289 + 46216b^4 - 329710b^8 + 282592b^{12} - 37475b^16 - 479808b^2\mu + 329280b^6\mu - 225792\mu^2)h + 2(5152135b^2 - 3602421b^6 + 34453b^{10} + 2916057b^{14} - 2091776b^{18} + 13070016\mu - 5844672b^4\mu)h^2 + (7140592 + 9690755b^4 - 240715593b^8 - 1663991b^{12} + 107105735b^{16} - 3612672b^2\mu - 27095040\mu^2)h^3 - (123403996b^2 - 118144936b^6 - 7892771b^{10} + 91221456b^{14} - 551056b^{18} + 9119808\mu - 5507136b^4\mu)h^4 - 2(1165263 - 456814b^4 - 917952b^8 - 250482b^{12} + 761041b^{16} - 2709504\mu^2)h^5 - 4(b^4-1)(683024b^2 - 241779b^6 - 455259b^{10} + 45544b^{14} + 94590\mu)h^6 + 4(b^4-1)^2(45471 - 694b^4 + 137435b^8)h^7 + 16(b^4-1)(22918b^2 + 11021b^6 - 33866b^{10} - 292b^{14} - 657\mu)h^8 + 32(b^4-1)^2(146 + 6131b^4 + 438b^8)h^9 - 4672(b^4-1)^2b^2(2+3b^4)h^{10} + 4672(b^4-1)^2b^4h^{11}\big),$$

where p_i is the coefficient of the quadratic term of the polynomial P_i.

We compute the resultant of p_1 and p_2 with respect to h, remove the nonzero constant and the repeating factors, and obtain

$$r_1 = b(b^4-1)(1+2b^2-b^4)(1-2b^2-b^4)(32 - 155b^2 + 138b^4 - 155b^6 + 32b^8)(32 + 155b^2 + 138b^4 + 155b^6 + 32b^8)(128 + 87b^4 + 128b^8)\big(b^2 - 6b^2\mu^2 + 4(1+b^4)\mu^3 - 3b^2\mu^4\big),$$

When $r_1 \neq 0$, due to the properties of the resultant we know that the coefficients p_1 and p_2 cannot be zero simultaneously, and as a result $\mathcal{N} \leq 2$. Therefore we are going to study the number $\mathcal{N}$ when $r_1 = 0$.

Since $b \neq 0$, we begin with $b^4 - 1 = 0$. If $b = 1$, systems (8) become

$$\dot{x} = x + y - 3\mu x^2 y - y^3, \quad \dot{y} = -x - y + x^3 + 3\mu x y^2. \tag{10}$$

Then we can explicitly calculate their finite singular points and we get

$$(0,0),\ \pm(\sqrt{M_1}M_2(1-3\mu), \sqrt{M_1}),\ \text{and}\ \pm(\sqrt{M_2}M_1(1-3\mu), \sqrt{M_2}),$$

where

$$M_{1,2} = 1 \pm \frac{\sqrt{3(3\mu^2 - 2\mu - 1)}}{1 - 3\mu}.$$

Observe that

$$3\mu^2 - 2\mu - 1 > 3\left(-\frac{1}{3}\right)^2 - 2\left(-\frac{1}{3}\right) - 1 = 0$$

and

$$3(3\mu^2 - 2\mu - 1) = 9\mu^2 - 6\mu - 3 < 9\mu^2 - 6\mu + 1 = (1 - 3\mu)^2$$

whenever $\mu < -1/3$, hence $M_{1,2}$ are positive. In addition, it is easy to check that we have $H = (3\mu+1)/\big(4(3\mu-1)\big)$ at the finite singular points other than the origin. This means that systems (10) have four finite singular points which are on the same energy level, and therefore their global phase portraits are topologically equivalent to 1.9 of Figure 1.

If $b = -1$ then we have

$$\frac{1+b^4+6b^2\mu}{b} = -2-6\mu > 0$$

whenever $\mu < -1/3$, which means that systems (8) cannot have a center at the origin (see (9)). So we have $b \neq -1$.

Now we study the case $1+2b^2-b^4 = 0$. Solving for b yields $b = \pm\sqrt{1+\sqrt{2}}$. When we substitute these values into p_1 (note that p_1 is an even polynomial in the variable b), equate it to zero and solve for h, we obtain $h = 1/\sqrt{2}$. Then we substitute both of these b and h into p_4 and get

$$4741632(\mu^2 - 2\sqrt{2}\mu - 1),$$

which is greater than zero for $\mu < -1/3$. This means that when $1+2b^2-b^4 = 0$ we have $p_4 \neq 0$, hence $\mathcal{N} \leq 2$.

If $1-2b^2-b^4 = 0$, we can show by repeating the same calculations that we did in the case with $1+2b^2-b^4 = 0$ that $\mathcal{N} \leq 2$.

Next is the case $32-155b^2+138b^4-155b^6+32b^8 = 0$. Following the same steps as in the last two cases is a little cumbersome here due to the higher degree of this polynomial in b. Instead we calculate the resultant of p_1 and p_3 with respect to h, and see that the only factor that does not appear in r_1 is

$$\begin{aligned} r_2 = {} & 4218421248 - 204309374976b^4 + 32568253073355b^8 - 5943760217597b^{12} \\ & - 18532611127177b^{16} - 3733079567170416b^{20} + 17150450881592176b^{24} \\ & - 21792980148802476b^{28} + 3576027216215016b^{32} - 818068919666836b^{36} \\ & + 3902515292160b^{40} - 354375696384b^{44} + 7247757312b^{48}. \end{aligned}$$

Then we calculate the resultant of $32-155b^2+138b^4-155b^6+32b^8$ and r_2 with respect to b and see that it is not zero. Therefore in this case even if $p_1 = p_2 = 0$, we have $p_3 \neq 0$, and consequently we have $\mathcal{N} \leq 2$.

The next two factors in r_1 cannot be zero for real b, so it only remains to study the case $w = b^2 - 6b^2\mu^2 + 4(1+b^4)\mu^3 - 3b^2\mu^4 = 0$. However this case is not possible because for $\mu < -1/3$ we have

$$w < b^2 - 6b^2\left(-\frac{1}{3}\right)^2 + 4(1+b^4)\left(-\frac{1}{3}\right)^3 - 3b^2\left(-\frac{1}{3}\right)^4 = -\frac{4}{27}(b^4-1)^4 \leq 0.$$

As a result of the above analysis we conclude that when $\mu < -1/3$ systems (8) have the global phase portrait 1.9 of Figure 1 if and only if $b = 1$. Therefore we obtain the bifurcation diagram for systems (VI) as shown in Figure 3.

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