

PERIODS OF CONTINUOUS MAPS ON CLOSED SURFACES

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ABSTRACT. The objective of the present work is to present what information on the set of periodic points of a continuous self-map on a closed surface can be obtained using the action of this map on the homological groups of the closed surface.

1. INTRODUCTION

Along this work by a *closed surface* we denote a connected compact surface with or without boundary, orientable or not. More precisely, an *orientable connected compact surface without boundary of genus $g \geq 0$* , \mathbb{M}_g , is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of g copies of the torus if $g \geq 2$. An *orientable connected compact surface with boundary of genus $g \geq 0$* , $\mathbb{M}_{g,b}$, is homeomorphic to \mathbb{M}_g minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

A *non-orientable connected compact surface without boundary of genus $g \geq 1$* , \mathbb{N}_g , is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of g copies of the real projective plane if $g > 1$. A *non-orientable connected compact surface with boundary of genus $g \geq 1$* , $\mathbb{N}_{g,b}$, is homeomorphic to \mathbb{N}_g minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{N}_{g,0} = \mathbb{N}_g$.

Let $f : \mathbb{X} \rightarrow \mathbb{X}$ be a continuous map on a closed surface \mathbb{X} . A point $x \in \mathbb{X}$ is periodic of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \dots, n - 1$. We denote by $\text{Per}(f)$ the set of periods of all periodic points of f . The aim of the present paper is to provide some information on $\text{Per}(f)$.

Let A be an $n \times n$ complex matrix. A $k \times k$ *principal submatrix* of A is a submatrix lying in the same set of k rows and columns, and a $k \times k$ *principal minor* is the determinant of such a principal submatrix. There are $\binom{n}{k}$ different $k \times k$ principal minors of A , and the sum of these is denoted by $E_k(A)$. In particular, $E_1(A)$ is the trace of A , and $E_n(A)$ is the determinant of A , denoted by $\det(A)$.

It is well known that the characteristic polynomial of A is given by

$$\det(tI - A) = t^n - E_1(A)t^{n-1} + E_2(A)t^{n-2} - \dots + (-1)^n E_n(A).$$

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Our main result is state in the following theorem.

Theorem 1. *Let \mathbb{X} be a closed surface and let $f : \mathbb{X} \rightarrow \mathbb{X}$ be a continuous map and let A and (d) be the integral matrices of the endomorphisms $f_{*i} : H_i(\mathbb{X}, \mathbb{Q}) \rightarrow H_i(\mathbb{X}, \mathbb{Q})$ induced by f on the i -th homology group of \mathbb{X} , $i = 1, 2$.*

If \mathbb{X} is either $\mathbb{M}_{g,b}$ with $b > 0$, or $\mathbb{N}_{g,b}$ with $b \geq 0$, then the following statements hold.

- (a) *If $E_1(A) \neq 1$, then $1 \in \text{Per}(f)$.*
- (b) *If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $\text{Per}(f) \cap \{1, 2\} \neq \emptyset$.*

If $\mathbb{X} = \mathbb{M}_{g,b}$ with $b = 0$, then the following statement hold.

- (c) *If $E_1(A) \neq 1 + d$, then $1 \in \text{Per}(f)$.*
- (d) *If $E_1(A) = 1 + d$ and $E_2(A) \neq d^2 + 3d + 1$, then $\text{Per}(f) \cap \{1, 2\} \neq \emptyset$.*

If $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, then the following statement hold.

- (e) *If $2g + b - 1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, \dots, 2g + b - 1\}$ such that $E_k(A) \neq 0$, then $\text{Per}(f)$ has a periodic point of period a divisor of k .*

If $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, then the following statement hold.

- (f) *If $g + b - 1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, \dots, g + b - 1\}$ such that $E_k(A) \neq 0$, then $\text{Per}(f)$ has a periodic point of period a divisor of k .*

Theorem 1 is proven in section 2.

Similar results tote ones obtained in Theorem 1 but for homeomorphisms on closed surfaces where obtained by Franks and Llibre in [3], and by the authors in [4].

2. PROOF OF THEOREM 1

Let $f : \mathbb{X} \rightarrow \mathbb{X}$ be a continuous map and let \mathbb{X} be either $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$. Then the *Lefschetz number* of f is defined by

$$L(f) = \text{trace}(f_{*0}) - \text{trace}(f_{*1}) + \text{trace}(f_{*2}).$$

For continuous self-maps f defined on \mathbb{X} the Lefschetz fixed point theorem states (see for instance [1]).

Theorem 2. *If $L(f) \neq 0$ then f has a fixed point.*

With the objective of studying the periodic points of f we shall use the Lefschetz numbers of the iterates of f , i.e. $L(f^n)$. Note that if $L(f^n) \neq 0$ then f^n has a fixed point, and consequently f has a periodic point of period a divisor of n . In order to study the whole sequence $\{L(f^n)\}_{n \geq 1}$ it is defined the formal *Lefschetz zeta function* of f as

$$(1) \quad Z_f(t) = \exp \left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n \right).$$

The Lefschetz zeta function is in fact a generating function for the sequence of the Lefschetz numbers $L(f^n)$.

Let f be a continuous self-map defined on $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$, respectively. For a closed surface the homological groups with coefficients in \mathbb{Q} are linear vector spaces over \mathbb{Q} . We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in \mathbb{Q} , i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \cdot^{n_k} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*k} : H_k(\mathbb{M}_{g,b}, \mathbb{Q}) \rightarrow H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ by f on the homological group $H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ are $f_{*0} = (1)$, $f_{*2} = (d)$ where d is the *degree* of the map f if $b = 0$, $f_{*2} = (0)$ if $b > 0$, and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix (see for additional details [6, 7]).

We recall that the homological groups of $\mathbb{N}_{g,b}$ with coefficients in \mathbb{Q} , i.e.

$$H_k(\mathbb{N}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \cdot^{n_k} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = g + b - 1$ and $n_2 = 0$; and the induced linear maps are $f_{*0} = (1)$ and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix (see again for additional details [6, 7]).

From the work of Franks in [2] we have for a continuous self-map of a closed surface that its Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - tf_{*1})}{\det(I - tf_{*0})\det(I - tf_{*2})},$$

where in $I - tf_{*k}$ the I denotes the $n_k \times n_k$ identity matrix, and $\det(I - tf_{*2}) = 1$ if $f_{*2} = (0)$. Then for a continuous map $f : \mathbb{M}_{g,b} \rightarrow \mathbb{M}_{g,b}$ we have

$$(2) \quad Z_f(t) = \begin{cases} \frac{\det(I - tA)}{(1-t)(1-dt)} & \text{if } b = 0, \\ \frac{\det(I - tA)}{1-t} & \text{if } b > 0, \end{cases}$$

and for a continuous map $f : \mathbb{N}_{g,b} \rightarrow \mathbb{N}_{g,b}$ we have

$$(3) \quad Z_f(t) = \frac{\det(I - tA)}{1-t}.$$

Proof of Theorem 1. Combining the expressions (1) and (2) if $\mathbb{X} = \mathbb{M}_{g,b}$ and $b > 0$, and the expressions (1) and (3) if $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, we obtain the

following equalities

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log(Z_f(t)) \\
&= \log\left(\frac{\det(I - tA)}{1 - t}\right) \\
&= \log\left(\frac{1 - E_1(A)t + E_2(A)t^2 - \dots + (-1)^m E_m(A)t^m}{1 - t}\right) \\
&= \log(1 - E_1(A)t + E_2(A)t^2 - \dots) - \log(1 - t) \\
&= \left(-E_1(A)t + \left(E_2(A) - \frac{E_1(A)^2}{2}\right)t^2 - \dots\right) - \left(-t - \frac{t^2}{2} - \dots\right) \\
&= (1 - E_1(A))t + \left(\frac{1}{2} - \frac{E_1(A)^2}{2} + E_2(A)\right)t^2 + O(t^3).
\end{aligned}$$

Here $n_1 = 2g + b - 1$ if $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, or $n_1 = g + b - 1$ if $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$. Therefore we have

$$L(f) = 1 - E_1(A) \quad \text{and} \quad L(f^2) = 1 - E_1(A)^2 + 2E_2(A).$$

Hence, if $E_1(A) \neq 1$ then $L(f) \neq 0$, and by Theorem 2 statement (a) follows.

If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $L(f^2) = 2E_2(A) \neq 0$, and again by Theorem 2 we get that $\text{Per}(f) \cap \{1, 2\} \neq \emptyset$. So statement (b) is proved.

Let $\mathbb{X} = \mathbb{M}_{g,b}$ with $b = 0$. By (1) and (2) with $b = 0$ we obtain the following equalities

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log(Z_f(t)) \\
&= \log\left(\frac{\det(I - tA)}{(1 - t)(1 - dt)}\right) \\
&= \log\left(\frac{1 - E_1(A)t + E_2(A)t^2 - \dots + (-1)^m E_m(A)t^m}{(1 - t)(1 - dt)}\right) \\
&= \log(1 - E_1(A)t + E_2(A)t^2 - \dots) - \log((1 - t)(1 - dt)) \\
&= \left(-E_1(A)t + \left(E_2(A) - \frac{E_1(A)^2}{2}\right)t^2 - \dots\right) \\
&\quad - \left(- (1 + d)t - \left(\frac{d^2 + 1}{2}\right)t^2 - \dots\right) \\
&= (1 + d - E_1(A))t + \left(E_2(A) - \frac{E_1(A)^2}{2} - \frac{d^2 + 1}{2}\right)t^2 + O(t^3).
\end{aligned}$$

Here $n_1 = 2g$. Therefore we have

$$L(f) = 1 + d - E_1(A), \quad \text{and} \quad L(f^2) = 2E_2(A) - E_1(A)^2 - (d^2 + 1).$$

Hence, if $E_1(A) \neq 1 + d$ then $L(f) \neq 0$, and by Theorem 2 statement (c) follows.

If $E_1(A) = 1+d$ and $E_2(A) \neq d^2+d+1$, then $L(f^2) = 2E_2(A) - 2(d^2+d+1) \neq 0$, and again by Theorem 2 we get that $\text{Per}(f) \cap \{1, 2\} \neq \emptyset$. So statement (d) is proved.

Assume now that $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, $2g+b-1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, \dots, 2g+b-1\}$ such that $E_k(A) \neq 0$. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log \left(\frac{1-t + (-1)^k E_k(A) t^k + \dots + (-1)^{b-1} E_{2g+b-1}(A) t^{2g+b-1}}{1-t} \right) \\ &= \log \left(1 + \frac{(-1)^k E_k(A) t^k + \dots + (-1)^{b-1} E_{2g+b-1}(A) t^{2g+b-1}}{1-t} \right) \\ &= (-1)^k E_k(A) t^k + O(t^{k+1}). \end{aligned}$$

Hence, $L(f) = \dots = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k k E_k(A) \neq 0$. So, from Theorem 2, it follows the statement (e).

Suppose that $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, $g+b-1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, \dots, g+b-1\}$ such that $E_k(A) \neq 0$. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log \left(\frac{1-t + (-1)^k E_k(A) t^k + \dots + (-1)^{g+b-1} E_{g+b-1}(A) t^{g+b-1}}{1-t} \right) \\ &= \log \left(1 + \frac{(-1)^k E_k(A) t^k + \dots + (-1)^{g+b-1} E_{g+b-1}(A) t^{g+b-1}}{1-t} \right) \\ &= (-1)^k E_k(A) t^k + O(t^{k+1}). \end{aligned}$$

Again $L(f) = \dots = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k k E_k(A) \neq 0$. Therefore, from Theorem 2, it follows the statement (f). \square

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