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CENTERS AND LIMIT CYCLES OF POLYNOMIAL DIFFERENTIAL SYSTEMS OF DEGREE 4

REBIHA BENTERKI[1] AND JAUME LLIBRE[2]

Abstract. In this paper we classify the phase portraits in the Poincaré disc of the centers of the generalized class of Kukles systems

$$\dot{x} = -y, \quad \dot{y} = x + ax^3y + bxy^3,$$

symmetric with respect to the y-axis, and we study, using the averaging theory up to sixth order, the limit cycles which bifurcate from the periodic solutions of these centers when we perturb them inside the class of all polynomial differential systems of degree 4.

1. Introduction and statement of the main results

Two of the classical and difficult problems in the qualitative theory of polynomial differential systems in $\mathbb{R}^2$ is the characterization of their centers, and the study of the limit cycles which can bifurcate from their periodic orbits when we perturb them inside some class of polynomial differential equations.

Our work is related with the class of polynomial differential systems of the form

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x, y), \tag{1}$$

having a center at the origin, where $Q_n(x, y)$ is a homogeneous polynomial of degree n, and in the study of the number of limit cycles which bifurcate from the periodic orbits of these centers when they are perturbed inside the class of all polynomial differential systems of degree n.

Differential polynomial systems (1) were called *Kukles homogeneous systems* in [7]. The centers of systems (1) started to be studied by Volokitin and Ivanov in [18].

For $n = 1$ the differential systems (1) are linear, they can have centers, but the perturbation of these centers inside the class of linear differential systems cannot produce limit cycles, because it is well known that linear differential systems cannot have isolated periodic solutions in the set of all periodic solutions.

For $n = 2$ the phase portraits of system (1) are a particular class of the phase portraits studied in [3], where it is proved that such systems have no centers.

In [2, 4, 14, 19, 20, 21] are characterized the centers and the phase portraits of linear systems with homogeneous nonlinearities of degree 3, so in particular the phase portraits of systems (1) with $n = 3$. The limit cycles that bifurcate from the periodic orbits of the centers of systems (1) with $n = 3$ when they are perturb

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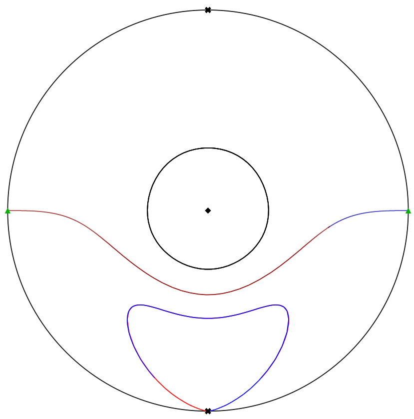

FIGURE 1. Case $a > 0$ and $b = 0$. The separatrices of this phase portrait are the circle of the infinity; and an orbit A which connects the two separatrices inside the Poincaré disc of the two saddles at infinity, localized at the origins of the local charts U_1 and V_1. Therefore this phase portrait has two canonical regions. The canonical region limited by the orbit A and the part of infinity containing the origin of U_2 is filled by the periodic orbits of the center; and the canonical region limited by the orbit A and the part of infinity containing the origin of V_2 is filled by an elliptic sector of the infinite singular point localized at the origin of V_2.

inside the class of all cubic polynomial differential systems were studied inside the more general articles [5, 10, 11].

Giné in [7] proved that for $n = 4$ system (1) has a center at the origin if and only if its vector field is symmetric about one of the coordinate axes.

The first objective of this paper is to study the phase portraits of the centers of systems (1) with $n = 4$ which are symmetric with respect to the y–axis, i.e. the phase portraits of the systems

$$\dot{x} = -y, \quad \dot{y} = x + ax^3y + bxy^3. \tag{2}$$

The second objective is to study the limit cycles that bifurcate from the periodic solutions of the centers of systems (2) when they are perturbed inside the class of all quartic polynomial differential systems.

For the definition of the global phase portrait of a polynomial differential system in the Poincaré disc see section 2, where we provide the notations, definitions and basic results which we need for reaching our two objectives.

Our first main result is the following.

Theorem 1. *A polynomial differential system* (2) *with* $a^2+b^2 \neq 0$ *has a phase portrait in the Poincaré disc topologically equivalent to one of the three phase portraits of Figures 1, 2 and 3.*

Theorem 1 is proved in section 4.

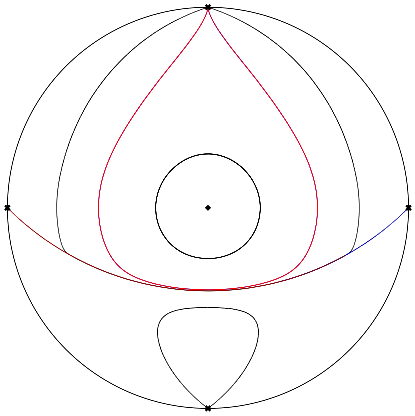

FIGURE 2. Case $a = 0$ and $b > 0$. The separatrices of this phase portrait are the circle of the infinity; an orbit A which connects the two separatrices of the hyperbolic sector of the infinite singular points localized at the origin of U_2; and an orbit B which connects the two separatrices which are inside the Poincaré disc of the two saddles at infinity, these saddles are the origins of the local charts U_1 and V_1. So this phase portrait has three canonical regions. The canonical region limited by the orbit A is filled by the periodic orbits surrounding the center; the canonical region limited by the orbits A, B and the infinity is filled by orbits which start and end at the origin of the local chart U_2; and the canonical region limited by the orbit B and the infinity is filled by an elliptic sector of the infinite singular point localized at the origin of V_2.

We write the perturbed quartic polynomial differential system of system (2) as

$$
\begin{aligned}
\dot{x} &= -y + \sum_{s=1}^{6} \varepsilon^s \sum_{0\le i+j\le 4} a_{ij}^{(s)} x^i y^j, \\
\dot{y} &= x + ax^3y + bxy^3 + \sum_{s=1}^{6} \varepsilon^s \sum_{0\le i+j\le 4} b_{ij}^{(s)} x^i y^j,
\end{aligned}
\tag{3}
$$

where i and j are non–negative integers. For the definition of the averaging theory of order $k = 1, \ldots, 6$ see section 5. In what follows we state our second main result.

Theorem 2. *For $\varepsilon \neq 0$ sufficiently small the number of limit cycles of the differential system* (3) *obtained using the averaging theory of order*

(a) *one and two is* 0*,*
(b) *three and four is* 1*,*
(c) *five is* 2*,*
(d) *six is* 5*.*

Theorem 2 is proved in section 6.

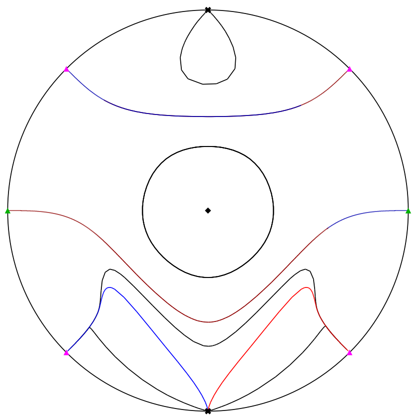

FIGURE 3. Case $a > 0$ and $b < 0$. The separatrices of this phase portrait are the circle of the infinity; an orbit A which connects the two separatrices inside the Poincaré disc of the two saddle-nodes at infinity, these two saddles-nodes are the ones which are closed to the origin of U_2; an orbit B which connects the two separatrices which are inside the Poincaré disc of the two saddles at infinity, these saddles are the origins of the local charts U_1 and V_1; and the two separatrices C and D of the hyperbolic sector of the infinite singular point which is located at the origin of V_2. This phase portrait has five canonical regions. The canonical region limited by the orbit A and the infinity is filled by an elliptic sector of the infinite singular point localized at the origin of U_2; the canonical region limited by the orbits A, B and the infinity is filled by the periodic orbits of the center; the canonical region limited by the orbits B, C, D and the infinity is filled with orbits which start at the saddle-node close to the left of the origin of V_2 and end at the saddle-node close to the right of the origin of V_2; the canonical region limited by the separatrix C and the infinity is filled with orbits which start in the saddle-node close to the left of the origin of V_2 and end at the origin of V_2; and the canonical region limited by the separatrix D and the infinity is filled with orbits which start at the origin of V_2 and end in the saddle-node close to the right of the origin of V_2.

2. Preliminaries

In this section we introduce the basic definitions and notations that we will need for the analysis of the local phase portraits of the finite and infinite singular points of the polynomial differential systems (2), and also for doing their phase portraits in the Poincaré disc.

We denote by $\mathcal{P}_n(\mathbb{R}^2)$ the set of polynomial vector fields on $\mathbb{R}^2$ of the form $\mathcal{X}(x, y) = (P(x, y), Q(x, y))$ where P and Q are real polynomials in the variables x and y such that the maximal degree of P and Q is n.

2.1. **Singular points.** A point $q \in \mathbb{R}^2$ is said to be a singular point of the vector field $\mathcal{X}$ if $P(q) = Q(q) = 0$.

If $\Delta = P_x(q)Q_y(q) - P_y(q)Q_x(q)$ and $T = P_x(q) + Q_y(q)$, then the singular point q is said to be *elementary* if either $\Delta \neq 0$, or $\Delta = 0$ and $T \neq 0$.

Let q be an elementary singular point with $\Delta \neq 0$. If the two eigenvalues of the matrix

$$\begin{pmatrix} P_x(q) & P_y(q) \\ Q_x(q) & Q_y(q) \end{pmatrix} \tag{4}$$

have real part non–zero then this singular point is called *hyperbolic*. In this case q is a saddle if $\Delta < 0$; a node if $T^2 \geq 4\Delta > 0$ (stable if $T < 0$, unstable if $T > 0$), a focus if $4\Delta > T^2 > 0$ (stable if $T < 0$, unstable if $T > 0$). If $\Delta \neq 0$ but q is not hyperbolic then $T = 0 < \Delta$ and q is either a weak focus or a center. For more details see, for instance, Theorem 2.15 of [6].

Let q be an elementary singular point with $\Delta = 0$ and $T \neq 0$. Then q is called a *semi–hyperbolic* singular point. The local phase portraits of a semi–hyperbolic singular point can be studied using Theorem 2.19 of [6].

When $\Delta = T = 0$ but the Jacobian matrix (4) at the singular point q is not the zero matrix, we say that q is *nilpotent*. The local phase portraits at a nilpotent singular point can be studied using Theorem 3.5 of [6].

Finally, if the Jacobian matrix at the singular point q is identically zero, and q is isolated inside the set of all singular points, then we say that q is *linearly zero*. The study of the local phase portraits of such singular points needs special changes of variables called blow-ups, see for more details Chapter 3 of [6], or [1].

2.2. **Poincaré compactification.** Let $\mathcal{X} \in P_n(\mathbb{R}^2)$ be any planar vector field of degree n. The *Poincaré compactified vector field* $p(\mathcal{X})$ corresponding to $\mathcal{X}$ is an analytic vector on $\mathbb{S}^2$ defined as follows (see, for instance [11] or Chapter 5 of [6]). Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at point y. We identify the plane $T_{(0,0,1)}\mathbb{S}^2$ with the $\mathbb{R}^2$ where we have our vector field $\mathcal{X}$. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \longrightarrow \mathbb{S}^2$, i.e. to each point q of the $T_{(0,0,1)}\mathbb{S}^2$ the map associates the two intersection points of the straight line, joining q with $(0,0,0)$, with the sphere $\mathbb{S}^2$. This map provides two copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern hemisphere. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the infinity of $\mathbb{R}^2$. In order to extend X' to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X} \in P_n(\mathbb{R}^2)$; $p(\mathcal{X})$ is the only analytic extension of $y_3^n\mathcal{X}'$ to $\mathbb{S}^2$. In short, on $\mathbb{S}^2 \setminus \mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$, and knowing the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behavior of $\mathcal{X}$ at infinity. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disc*, and it is denoted by $\mathbb{D}^2$.

The Poincaré compactifcation has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$. We say that two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(\mathcal{X})$ into orbits of the flow induced by $p(\mathcal{Y})$; preserving or reversing simultaneously the sense of all orbits.

As $\mathbb{S}^2$ is a differentiable manifold, for computing the expression of $p(\mathcal{X})$, we consider the six local charts $U_i = \{y_2 \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y_2 \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$; and the diffeomorphisms $F_i : U_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$ for

$i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$; $(-1, 0, 0)$; $(0, 1, 0)$; $(0, -1, 0)$; $(0, 0, 1)$ and $(0, 0, -1)$, respectively. If we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (so z represents different coordinates according to the local charts under consideration), then some easy computations give for $p(\mathcal{X})$ the following expressions:

$$z^n \Delta(z) \left(Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right)\right) \quad \text{in } U_1, \tag{5}$$

$$z^n \Delta(z) \left(P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right)\right) \quad \text{in } U_2, \tag{6}$$

$$z^n \Delta(z) \left(P\left(z_1, z_2\right), Q\left(z_1, z_2\right)\right) \quad \text{in } U_3,$$

where $\Delta(z) = \left(z_1^2 + z_2^2 + 1\right)^{-\frac{1}{2}(n-1)}$.

The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i = 1, 2$, $z_2 = 0$ always denotes the points of $\mathbb{S}^1$. In what follows we omit the factor $\Delta(z)$ doing a convenient scaling of the vector field $p(\mathcal{X})$. Thus we obtain a polynomial vector field in each local chart.

The singular points of $p(\mathcal{X})$ which are in the interior of the Poincaré disc are called the *finite* singular points, which correspond with the singular points of $\mathcal{X}$, and the singular points of $p(\mathcal{X})$ which are in $\mathbb{S}^1$ are called the *infinite* singular points of $\mathcal{X}$. We note that studying the infinite singular points of the local chart U_1, we obtain also the ones of the local chart V_1, and only remains to see if the origin of the local chart U_2, and consequently the origin of the local chart V_2, are infinite singular points.

2.3. **Local phase portraits on the Poincaré disc.** The first step in order to characterize all phase portraits of the polynomial differential systems (2) is to classify the local phase portraits at all finite and infinite singular points in the Poincaré disc. This is made by using the techniques described in subsection 2.1. In this way we shall provide all the local phase portraits at all the singular points of the Poincaré disc for all differential systems (2).

2.4. **Phase portraits on the Poincaré disc.** In this subsection we shall see how to characterize the global phase portraits in the Poincaré disc of the polynomial differential systems (2).

A *separatrix* of $p(\mathcal{X})$ is an orbit which is either a singular point, or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point. Neumann [15] proved that the set formed by all separatrices of $p(\mathcal{X})$; denoted by $S(p(\mathcal{X}))$ is closed.

The open connected components of $\mathbb{D}^2 \setminus S(p(\mathcal{X}))$ are called canonical regions of $p(\mathcal{X})$: We define a *separatrix configuration* as a union of $S(p(\mathcal{X}))$ plus one solution chosen from each canonical region. Two separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are said to be *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S(p(\mathcal{X}))$ into the trajectories of $S(p(\mathcal{Y}))$. The following result is due to Markus [13], Neumann [15] and Peixoto [16].

Theorem 3. *The phase portraits in the Poincaré disc of the two compactified polynomial differential systems $p(\mathcal{X})$ and $p(\mathcal{Y})$ are topologically equivalent if and only if their separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are topologically equivalent.*

3. Local phase portraits at the finite and infinite singular points

It is clear that the phase portrait of the linear polynomial differential system (2) with $a = b = 0$, is formed by all the invariant circles centered at the origin of coordinates.

In what follows we shall study the phase portraits of the quartic polynomial differential systems (2) with $(a, b) \neq (0, 0)$.

Remark 4. *System* (2) *is reversible because it does not change under the transformation $(x, y, t) \to (-x, y, -t)$. Hence we know that the phase portrait of system* (2) *is symmetric with respect to the y-axis.*

Remark 5. *By doing the following symmetries $(x, y, t, a, b) \to (-x, y, -t, -a, -b)$, we conclude that we only need to study the phase portrait of systems* (2) *when either $a > 0$, or $a = 0$ and $b > 0$.*

The way for studying the phase portraits of systems (2) is the following. First we shall characterize all the finite and infinite singular points of this system with their local phase portraits. After using the symmetry of the solutions with respect to the y–axis and the behavior of the vector field on the axes, we will determine their phase portraits in the Poincaré disc.

3.1. **Finite singular points.** For the planar quartic polynomial differential systems (2) the center at the origin is the unique finite singular point.

3.2. **Infinite singular points.** For studying the infinite singular points in the Poincaré disc, we use the definitions and notations given in subsection 2.2. We perform the analysis of the vector field at infinity.

Proposition 6. *System* (2) *in the local chart U_1*

(a) *has a unique semi-hyperbolic singular point, the origin q, which is a saddle if $a > 0$ and $b > 0$;*
(b) *has three semi-hyperbolic singular points: the origin q which is a saddle; and the two points $q_\pm = (\pm\sqrt{-a/b}, 0)$ which are saddle-nodes if $a > 0$ and $b < 0$, moreover q_+ (resp. q_-) has the two hyperbolic sectors in $z_2 > 0$ (resp. $z_2 < 0$), and the parabolic one in $z_2 < 0$ (resp. $z_2 > 0$);*
(c) *has a unique semi-hyperbolic singular point, the origin q, which is a saddle if $a > 0$ and $b = 0$;*
(d) *has a unique singular point, the origin q, which is a saddle if $a = 0$ and $b > 0$ system* (2).

System (2) *in the local chart U_2*

(e) *has the origin as a singular point, which is linearly zero with one elliptic, one hyperbolic and two parabolic sectors, moreover when $a \geq 0$ and $b > 0$ (resp. $b < 0$) the hyperbolic sector together with two parabolic sector is in $z_2 > 0$ (resp. $z_2 < 0$), and the elliptic sector together with two parabolic sectors is in $z_2 < 0$ (resp. $z_2 > 0$), and if $a > 0$ and $b = 0$ then only there is a hyperbolic sector in $z_2 > 0$, and the elliptic sector together with two parabolic sectors is in $z_2 < 0$.*

Proof. From (5) the differential system (2) in the local chart U_1 is

$$\begin{aligned} \dot{u} &= au + bu^3 + u^2v^3 + v^3, \\ \dot{v} &= uv^4. \end{aligned} \tag{7}$$

If $a > 0$ and $b > 0$ the origin is the only infinite singular point of the differential system (7), which is a semi-hyperbolic singular point with eigenvalues a and 0. If $a > 0$ and $b < 0$ then there are two additional infinite singular points, namely $q_\pm = (\pm\sqrt{-a/b}, 0)$, which are semi-hyperbolic with eigenvalues $-2a$ and 0.

In order to obtain the local phase portraits at these semi-hyperbolic infinite singular points we use Theorem 2.19 of [6], and we obtain that the origin is a saddle. While for the singular points $q_\pm = (\pm\sqrt{-a/b}, 0)$ we obtain that they are saddle-nodes, located as it is described in the statement (b). Therefore the proofs of statements (a) and (b) are done.

If $a > 0$ and $b = 0$ system (2) becomes

$$\begin{aligned} \dot{u} &= au + u^2v^3 + v^3, \\ \dot{v} &= uv^4. \end{aligned} \tag{8}$$

The origin is the only infinite singular point of the differential system (8), which is a semi-hyperbolic singular point with eigenvalues a and 0. Applying Theorem 2.19 of [3] we conclude that the origin is a saddle. So statement (c) is proved.

If $a = 0$ and $b > 0$ system (2) becomes

$$\begin{aligned} \dot{u} &= bu^3 + u^2v^3 + v^3, \\ \dot{v} &= uv^4. \end{aligned} \tag{9}$$

The origin of this differential system is a linearly zero singular point. Using polar blowing up $(x, y) \to (\rho, \theta)$ where $x = \rho\cos\theta$ and $y = \rho\sin\theta$, system (9) writes

$$\begin{aligned} \dot{\rho} &= \rho^3\cos\theta(b\cos^3\theta + (1+\rho^2)\sin^3\theta), \\ \dot{\theta} &= -\rho^2\sin\theta(b\cos^3\theta + \sin^3\theta). \end{aligned} \tag{10}$$

We eliminated the common factor ρ^2 between $\dot{\rho}$ and $\dot{\theta}$ by doing a rescaling of the independent variable, we get the system

$$\begin{aligned} \dot{\rho} &= \rho\cos\theta(b\cos^3\theta + (1+\rho^2)\sin^3\theta), \\ \dot{\theta} &= -\sin\theta(b\cos^3\theta + \sin^3\theta). \end{aligned} \tag{11}$$

The zeros on $\rho = 0$ of the differential system (11) are located at $\theta_1 = 0$, $\theta_2 = \pi$, $\theta_3 = -\arctan\sqrt[3]{b}$, and $\theta_4 = -\arctan\sqrt[3]{b} + \pi$. The corresponding linear part for system (11) at $(0, \theta_j)$ for $j = 1, 2$ is

$$\begin{pmatrix} -b & 0 \\ 0 & b \end{pmatrix}.$$

Then we conclude that the point $(0, \theta_j)$ for $j = 1, 2$ is a saddle. For the singular point $(0, \theta_j)$ for $j = 3, 4$ the eigenvalues of its linear part are 0 and $3b/(1 + \sqrt[3]{b^2})$, so they are semi–hyperbolic singular points. In the differential system (11) we perform the translation $\theta = \alpha + \theta_j$ for $j = 3, 4$, i.e we put these singular points at the origin of coordinates.

By doing a Taylor expansion up to the third order in ρ and α for system (11), we get

$$\begin{aligned} \dot{\rho} &= \rho^3 \left(-\frac{b}{1+\sqrt[3]{b^2}} + O(\alpha) \right), \\ \dot{\alpha} &= \alpha \left(\frac{3b}{1+\sqrt[3]{b^2}} + O(\alpha) \right). \end{aligned} \tag{12}$$

Applying Theorem 2.19 of [6] to the origin of the differential system (12) we obtain that it is a saddle. Hence the singular points $(0, \theta_j)$ for $j = 3, 4$ are saddles. Now going back to system (8) through the changes of variables it follows the result of statement (d).

From (6) the differential system (2) in the local chart U_2 is

$$\begin{aligned} \dot{u} &= -bu^2 - v^3 - au^4 - u^2v^3, \\ \dot{v} &= -buv - au^3v - uv^4. \end{aligned} \tag{13}$$

In this chart we only need to study the singular point at the origin of system (13), and it is linearly zero singular point. We need to do blow-up's to describe the local behavior at this point. We perform the directional blow-up $(u, v) \to (u, w)$ with $w = v/u$ and have

$$\begin{aligned} \dot{u} &= -bu^2 - u^3w^3 - au^4 - u^5w^3, \\ \dot{w} &= u^2w^4. \end{aligned} \tag{14}$$

We eliminate the common factor u^2 between $\dot{u}$ and $\dot{w}$ by doing a rescaling of the independent variable, and we get the system

$$\begin{aligned} \dot{u} &= -b - uw^3 - au^2 - u^3w^3, \\ \dot{w} &= w^4. \end{aligned} \tag{15}$$

If $b > 0$ system (15) have no singular points. Going back through the change of variables to system (13), we see that the local phase portrait at the origin consists of one elliptic, one hyperbolic, and two parabolic sectors. The line at infinity separates the hyperbolic sector from the elliptic one, and this line is contained inside both parabolic sectors.

If $b = 0$, the unique singular point of system (15) is the origin, whose linear part is again zero. Hence we do another blow-up $(u, w) \to (u, z)$ with $z = w/u$. Eliminating from the system $(\dot{u}, \dot{z})$ the common factor u by doing a rescaling of the independent variable, we get the system

$$\begin{aligned} \dot{u} &= -au - u^3z^3 - u^5z^3, \\ \dot{z} &= az + 2u^2z^4 + u^4z^4. \end{aligned} \tag{16}$$

The unique singular point of system (16) is the origin. The eigenvalues of the linear part of system (16) at the origin are $\pm a$, hence it is a saddle. Going back through the changes of variables up to system (13), we see that the local phase portrait at the origin of U_2 is the one described in statement (e) for $b = 0$. So statement (e) is proved . □

4. Phase portraits in the Poincaré disc

In this section we prove Theorem 1.

Let p be a center. The maximum region filled only with periodic orbits surrounding the center p is called the *period annulus* of p.

We denote by $\alpha(\gamma)$ the α-limit of the orbit γ, and by $\omega(\gamma)$ the ω-limit of the orbit γ.

A *graphic* is formed by a finite number of orbits γ_1,...,γ_n which are not singular points, and a finite number of singular points $p_1, ..., p_n$ such that $\alpha(\gamma_i) = p_i$, $\omega(\gamma_i) = p_i + 1$ for $i = 1, ..., n-1$, $\alpha(\gamma_n) = p_n$ and $\omega(\gamma_n) = p_1$. Possibly, some of the singular points p_i are identified.

Assume that $a > 0$ **and** $b < 0$. The singular point q_- is a saddle-node, q is a saddle, and q_+ is a saddle-node, and from statement (b) of Proposition 6 we obtain that the behavior of q_- in $U_1 \cap \mathbb{D}^2$ is a stable node (recall that $\mathbb{D}^2$ denotes the Poincaré disc), the behavior of q in $U_1 \cap \mathbb{D}^2$ is given by two hyperbolic sectors (the common separatrix of these two sectors is stable), and the behavior of q_+ in $U_1 \cap \mathbb{D}^2$ is again given by two hyperbolic sectors (the common separatrix of these two sectors is unstable). By the symmetry of the phase portrait of system (2) with respect to the y-axis, we obtain the local phase portraits at the singular points in the chart V_1. From statement (e) of Proposition 6 in $U_2 \cap \mathbb{D}^2$ the origin is a linearly zero singular point with one elliptic and two parabolic sectors, each one of these parabolic sector have one boundary at infinity and the other in the elliptic sector. In the origin of $V_2 \cap \mathbb{D}^2$ there are one hyperbolic and two parabolic sectors, each one of the parabolic sectors have one boundary at infinity and the other in the hyperbolic sector. This completes the study of the local phase portraits at all the singular points at infinity.

Now taking into account that we know the behavior of the vector field associated to system (2) on the axes (because $\dot{x}_{|x=0} = -y$ and $\dot{y}_{|y=0} = x$), and the symmetry with respect to the y-axis, the unstable separatrix at q_+ connects with the stable separatrix of the symmetric point of q_+. Using these argument the stable separatrix of q connects with the unstable one of its symmetric point. So we get the phase portrait described in the Figure 3.

Assume now that $a \geq 0$ **and** $b > 0$. From statements (a) and (d) of Proposition 6 the singular point q is a saddle, and the behavior of q in $U_1 \cap \mathbb{D}^2$ is given by two hyperbolic sectors (the common separatrix of these two sectors is stable). In $U_2 \cap \mathbb{D}^2$ the origin is a linearly zero singular point with one hyperbolic and two parabolic sectors, each one of these parabolic sector have one boundary at infinity and the other in the hyperbolic sector. In the origin of $V_2 \cap \mathbb{D}^2$ there are one elliptic and two parabolic sectors, each one of the parabolic sectors have one boundary at infinity and the other in the elliptic sector. This completes the study of the local phase portraits at all the singular points at infinity.

Now taking into account that we know the behavior of the vector field associated to system (2) on the axes, and by the symmetry with respect to the y-axis, the stable separatrix of q connects with the unstable one of its symmetric point, and by the same argument the two separatrices of the hyperbolic sector of the origin of the local chart U_2 connect, and they form the exterior boundary of the period annulus of the center. So we get the phase portrait described in the Figure 2.

Assume now that $a > 0$ **and** $b = 0$. The local phase portrait at the singular point q is the same as in the case $a \geq 0$ and $b > 0$. In $U_2 \cap \mathbb{D}^2$ the origin is a linearly zero singular point with one hyperbolic having its two separatrices on the infinity line. In the origin of $V_2 \cap \mathbb{D}^2$ there are one elliptic and two parabolic sectors, each one of the parabolic sectors have one boundary at infinity and the other in the elliptic sector. This completes the study of the local phase portraits at all the singular points at infinity. Now the same arguments than in the case $a \geq 0$ and $b > 0$ completes the phase portrait described in the Figure 1. Hence Theorem 1 is proved.

5. The averaging theory up to order 6

In this section we recall some results on the averaging theory that we shall use for studying the limit cycles which bifurcate from the periodic orbits of the centers of systems (2) when they are perturbed inside the class of all polynomial differential systems of degree 4.

We consider a nonlinear differential system of the form

$$\dot{x}(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{17}$$

where $F_i : \mathbb{R} \times D \to \mathbb{R}$ for $i = 0, 1, \cdots, k$, and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, are continuous functions, and T-periodic in the first variable, being D an open interval of $\mathbb{R}$, and ε a small parameter. From [12] we define the following functions $y_i(t,z)$ for $k = 1, 2, 3, 4, 5, 6$ associated to system (17):

$$y_1(t,z) = \int_0^t F_1(s,z)ds,$$

$$\begin{aligned}
y_2(t,z) = & \int_0^t \Big(2F_2(s,z) + 2\partial F_1(s,z) y_1(s,z)\Big)ds,\\
y_3(t,z) = & \int_0^t \Big(6F_3(s,z) + 6\partial F_2(s,z) y_1(t,z)\\
& +3\partial^2 F_1(s,z) y_1(s,z)^2 + 3\partial F_1(s,z)\, y_2(s,z)\Big)ds,\\
y_4(t,z) = & \int_0^t \Big(24F_4(s,z) + 24\partial F_3(s,z) y_1(s,z)\\
& +12\partial^2 F_2(s,z) y_1(s,z)^2 + 12\partial F_2(s,z) y_2(s,z)\\
& +12\partial^2 F_1(s,z) y_1(s,z) y_2(s,z)\\
& +4\partial^3 F_1(s,z) y_1(s,z)^3 + 4\partial F_1(s,z) y_3(s,z)\Big)ds,\\
y_5(t,z) = & \int_0^t \Big(120F_5(s,z) + 120\partial F_4(s,z) y_1(s,z)\\
& +60\partial^2 F_3(s,z) y_1(s,z)^2 + 60\partial F_3(s,z) y_2(s,z)\\
& +60\partial^2 F_2(s,z) y_1(s,z) y_2(s,z) + 20\partial^3 F_2(s,z) y_1(s,z)^3\\
& +20\partial F_2(s,z) y_3(s,z) + 20\partial^2 F_1(s,z) y_1(s,z) y_3(s,z)\\
& +15\partial^2 F_1(s,z) y_2(s,z)^2 + 30\partial^3 F_1(s,z) y_1(s,z)^2 y_2(s,z)\\
& +5\partial^4 F_1(s,z) y_1(s,z)^4 + 5\partial F_1(s,z) y_4(s,z)\Big)ds,
\end{aligned}$$

$$
\begin{aligned}
y_6(t,z) = \quad & \int_0^t \Big(720F_6(s,z) + 720\partial F_5(s,z)y_1(s,z) \\
& +360\partial F_4(s,z)y_2(s,z) + 360\partial^2 F_4(s,z)y_1(s,z)^2 \\
& +120\partial F_3(s,z)y_3(s,z) + 360\partial^2 F_3(s,z)y_1(s,z)y_2(s,z) \\
& +120\partial^3 F_3(s,z)y_1(s,z)^3 + 30\partial F_2(s,z)y_4(s,z) \\
& +120\partial^2 F_2(s,z)y_1(s,z)y_3(s,z) + 30\partial^4 F_2(s,z)y_1(s,z)^4 \\
& +90\partial^2 F_2(s,z)y_2(s,z)^2 + 180\partial^3 F_2(s,z)y_1(s,z)^2 y_2(s,z) \\
& +6\partial F_1(s,z)y_5(s,z) + 30\partial^2 F_1(s,z)y_1(s,z)y_4(s,z) \\
& +60\partial^2 F_1(s,z)y_2(s,z)y_3(s,z) + 60\partial^3 F_1(s,z)y_1(s,z)^2 y_3(s,z) \\
& +60\partial^4 F_1(s,z)y_1(s,z)^3 y_2(s,z) + 90\partial^3 F_1(s,z)y_1(s,z)y_2(s,z)^2 \\
& +6\partial^5 F_1(s,z)y_1(s,z)^5\Big)ds.
\end{aligned}
$$

Here $\partial^k F_\ell(s,z)$ means the k-th partial derivative of the function $F_\ell(s,z)$ with respect to the variable z. Also from [12] we have the functions

$$
\begin{aligned}
f_1(z) = \quad & \int_0^T F_1(t,z)dt, \\
f_2(z) = \quad & \int_0^T \Big(F_2(t,z) + \partial F_1(t,z)y_1(t,z)\Big)dt, \\
f_3(z) = \quad & \int_0^T \Big(F_3(t,z) + \partial F_2(t,z)y_1(t,z) \\
& +\frac{1}{2}\partial^2 F_1(t,z)y_1(t,z)^2 + \frac{1}{2}\partial F_1(t,z)y_2(t,z)\Big)dt, \\
f_4(z) = \quad & \int_0^T \Big(F_4(t,z) + \partial F_3(t,z)y_1(t,z) \\
& +\frac{1}{2}\partial^2 F_2(t,z)y_1(t,z)^2 + \frac{1}{2}\partial F_2(t,z)y_2(t,z) \\
& +\frac{1}{2}\partial^2 F_1(t,z)y_1(t,z)y_2(t,z)dt + \frac{1}{6}\partial^3 F_1(t,z)y_1(t,z)^3 \\
& +\frac{1}{6}\partial F_1(t,z)y_3(t,z)\Big)dt, \\
f_5(z) = \quad & \int_0^T \Big(F_5(t,z) + \partial F_4(t,z)y_1(t,z) + \frac{1}{2}\partial^2 F_3(t,z)y_1(t,z)^2 \\
& +\frac{1}{2}\partial F_3(t,z)y_2(t,z) + \frac{1}{2}\partial^2 F_2(t,z)y_1(t,z)y_2(t,z) \\
& +\frac{1}{6}\partial^3 F_2(t,z)y_1(t,z)^3 + \frac{1}{6}\partial F_2(t,z)y_3(t,z) \\
& +\frac{1}{6}\partial^2 F_1(t,z)y_1(t,z)y_3(t,z) + \frac{1}{8}\partial^2 F_1(t,z)y_2(t,z)^2 \\
& +\frac{1}{4}\partial^3 F_1(t,z)y_1(t,z)^2 y_2(t,z) + \frac{1}{24}\partial^4 F_1(t,z)y_1(t,z)^4 \\
& +\frac{1}{24}\partial F_1(t,z)y_4(t,z)\Big)dt,
\end{aligned}
$$

$$
\begin{aligned}
f_6(z) = \quad & \int_0^T \Big(F_6(t,z) + \partial F_5(t,z)y_1(t,z) + \frac{1}{2}\partial F_4(t,z)y_2(t,z) \\
& +\frac{1}{2}\partial^2 F_4(t,z)y_1(t,z)^2 + \frac{1}{6}\partial F_3(t,z)y_3(t,z) \\
& +\frac{1}{2}\partial^2 F_3(t,z)y_1(t,z)y_2(t,z) + \frac{1}{6}\partial^3 F_3(t,z)y_1(t,z)^3 \\
& +\frac{1}{24}\partial F_2(t,z)y_4(t,z) + \frac{1}{6}\partial^2 F_2(t,z)y_1(t,z)y_3(t,z) \\
& +\frac{1}{4}\partial^3 F_2(t,z)y_1(t,z)^2 y_2(t,z) + \frac{1}{8}\partial^2 F_2(t,z)y_2(t,z)^2 \\
& +\frac{1}{24}\partial^4 F_2(t,z)y_1(t,z)^4 + \frac{1}{120}\partial F_1(t,z)y_5(t,z) \\
& +\frac{1}{24}\partial^2 F_1(t,z)y_1(t,z)y_4(t,z) + \frac{1}{12}\partial^2 F_1(t,z)y_2(t,z)y_3(t,z) \\
& +\frac{1}{12}\partial^3 F_1(t,z)y_1(t,z)^2 y_3(t,z) + \frac{1}{12}\partial^4 F_2(t,z)y_1(t,z)^3 y_2(t,z) \\
& +\frac{1}{8}\partial^3 F_1(t,z)y_1(t,z)y_2(t,z)^2 + \frac{1}{120}\partial^5 F_1(t,z)y_1(t,z)^5 \Big) dt.
\end{aligned}
$$

The averaging theory for a differential system (17) works as follows, see [12] for more details. If the averaged function $f_1(z)$ is not the zero function, every simple zero of $f_1(z)$ provides a limit cycle of the differential system (17). If $f_1(z) \equiv 0$ but $f_2(z) \not\equiv 0$, then every simple zero of $f_2(z)$ provides a limit cycle of the differential system (17). If $f_1(z) \equiv 0$, $f_2(z) \equiv 0$ but $f_3(z) \not\equiv 0$, then every simple zero of $f_3(z)$ provides a limit cycle of the differential system (17), and so on.

6. Proof of Theorem 2

Consider system (2), we shall study which periodic solutions of its center become limit cycles when we perturb the center inside the class of polynomial differential systems of degree 4. This study will be done by applying the averaging theory described in section 5, we introduce a small parameter ε doing the scaling $x = \varepsilon X$, $y = \varepsilon Y$. Thus we get a differential system $(\dot{X}, \dot{Y})$. After that we perform the polar change of coordinates $X = r\cos\theta$, $Y = r\sin\theta$, and we pass system $(\dot{X}, \dot{Y})$ to a system $(\dot{r}, \dot{\theta})$. Now we take as independent variable the angle θ, and the system $(\dot{r}, \dot{\theta})$, becomes the differential equation $dr/d\theta$, and by doing a Taylor expansion truncated at 6-th order in ε we obtain an expression for $dr/d\theta$ similar to the one of the differential system (17). In short we have written our differential system (3) in the normal form (17) for applying the averaging theory.

We give only the expression of functions $F_1(r,\theta)$ and $F_2(r,\theta)$.

The explicit expressions of $F_i(r,\theta)$ for $i = 3,..,6$ are quite large so we omit them.

The functions $F_i(\theta, r)$ $i = 1, ..., 6$ and $R(t, x, \varepsilon)$ of system (17) are analytic, and since the independent variable θ appears through sinus and cosinus of θ, they are 2π–periodic. Hence the assumptions for applying the averaging theory described in section 5 are satisfied.

The expressions of $F_1(r,\theta)$ and $F_2(r,\theta)$ are

$$
\begin{aligned}
F_1(r,\theta) = \;& a_{00}^{(2)}\cos\theta + b_{00}^{(2)}\sin\theta + r(a_{10}^{(1)}\cos^2\theta \\
& +(a_{01}^{(1)}+b_{10}^{(1)})\cos\theta\sin\theta + b_{10}^{(1)}\sin^2\theta),\\
F_2(r,\theta) = \;& -(b_{00}^{(2)}a_{10}^{(1)} + b_{10}^{(1)}a_{00}^{(2)})\cos^3\theta \\
& -(2b_{10}^{(1)}b_{00}^{(2)} + b_{00}^{(2)}a_{01}^{(1)} + b_{01}^{(1)}a_{00}^{(2)} - 2a_{10}^{(1)}a_{00}^{(2)})\cos^2\theta\sin\theta \\
& +\cos\theta(a_{00}^{(3)} + (-2b_{01}^{(1)}b_{00}^{(2)} + b_{00}^{(2)}a_{10}^{(1)} + b_{10}^{(1)}a_{00}^{(2)})\sin^2\theta) \\
& +r^2((b_{20}^{(1)} + a_{11}^{(1)})\cos^2\theta\sin\theta + (b_{11}^{(1)} + a_{02}^{(1)})\cos\theta\sin^2\theta \\
& +a_{20}^{(1)}\cos^3\theta + b_{02}^{(1)}\sin^3\theta) + r(-b_{10}^{(1)}a_{10}^{(1)}\cos^4\theta - ((b_{10}^{(1)})^2 \\
& +(b_{01}^{(1)} - a_{10}^{(1)})a_{10}^{(1)} + b_{10}^{(1)}a_{01}^{(1)})\cos^3\theta\sin\theta + \sin^2\theta(b_{10}^{(2)} \\
& +b_{01}^{(1)}a_{01}^{(1)}\sin^2\theta) + \cos^2\theta(a_{10}^{(2)} + (-2b_{10}^{(1)}b_{01}^{(1)} + b_{10}^{(1)}a_{10}^{(1)} \\
& -b_{01}^{(1)}a_{01}^{(1)} + 2a_{10}^{(1)}a_{01}^{(1)})\sin^2\theta) + \cos\theta\sin\theta(b_{10}^{(2)} + a_{01}^{(2)} \\
& +\cos\theta\sin\theta(b_{10}^{(2)} + a_{01}^{(2)} + (-(b_{01}^{(1)})^2 + b_{01}^{(1)}a_{10}^{(1)} + a_{01}^{(1)}(b_{10}^{(1)} \\
& +a_{01}^{(1)})\sin^2\theta) + \sin\theta(b_{00}^{(3)} + (b_{00}^{(2)}a_{01}^{(1)} + b_{01}^{(1)}a_{00}^{(2)})\sin^2\theta \\
& +a_{01}^{(1)}a_{00}^{(2)}\sin\theta) + \frac{-2b_{00}^{(2)}a_{00}^{(2)}\cos\theta + ((a_{00}^{(2)})^2 - b_{00}^{(2)})\sin\theta}{2r}.
\end{aligned}
$$

Using the formulas given in section 2 we obtain the averaged function of first order

$$f_1(r) = (a_{10}^{(1)} + b_{01}^{(1)})r.$$

Clearly equation $f_1(r) = 0$ has no positive zeros. Thus the first averaged function does not provide any information about the limit cycles that bifurcate from the periodic solutions of the center when we perturb it.

Setting $a_{10}^{(1)} = -b_{01}^{(1)}$ we obtain $f_1(r) = 0$. So we can apply the averaging theory of second order, obtaining the averaged function of second order.

$$f_2(r) = (a_{10}^{(2)} + b_{01}^{(2)})r.$$

As for the first averaged function, the second one also does not provide information on the bifurcating limit cycles. Therefore the proof of statement (a) is done.

Doing $a_{10}^{(2)} = -b_{01}^{(2)}$ we get $f_2(r) = 0$, and then we can apply the averaging theory of third order, and its corresponding averaged function is

$$
\begin{aligned}
f_3(r) = \;& \frac{1}{2}(-b_{11}^{(1)}b_{00}^{(2)} + b_{01}^{(3)} - 2b_{00}^{(1)}a_{20}^{(1)} + 2b_{02}^{(1)}a_{00}^{(2)} + a_{11}^{(1)}a_{00}^{(2)} + a_{10}^{(3)})r \\
& +\frac{1}{8}(b_{21}^{(1)} + 3b_{03}^{(1)} + 3a_{30}^{(1)} + a_{12}^{(1)})r^3.
\end{aligned}
$$

Therefore $f_3(r)$ can have at most one positive real root. Hence statement (b) of the theorem is proved for $k = 3$.

In order to apply the averaging theory of fourth order, we need to have $f_3(r) = 0$ so we set $a_{10}^{(3)} = b_{11}^{(1)}b_{00}^{(2)} - b_{01}^{(3)} + 2b_{00}^{(1)}a_{20}^{(1)} - 2b_{02}^{(1)}a_{00}^{(2)} - a_{11}^{(1)}a_{00}^{(2)}$ and $a_{12}^{(1)} = -b_{21}^{(1)} - 3b_{03}^{(1)} - 3a_{30}^{(1)}$.

The resulting averaged function of fourth order is

$$f_4(r) = r(A_1 + A_2 r^2),$$

where

$$\begin{aligned} A_1 = \; & \frac{1}{2}\Big(b_{10}^{(1)}b_{11}^{(1)}b_{00}^{(2)} + b_{01}^{(4)} - b_{11}^{(2)}b_{00}^{(3)} - 2b_{02}^{(1)}b_{00}^{(2)}a_{10}^{(1)} + 2b_{10}^{(1)}a_{20}^{(1)}b_{00}^{(2)} \\ & -2b_{00}^{(3)}a_{20}^{(1)} - b_{00}^{(2)}a_{10}^{(1)}a_{11}^{(1)} + a_{10}^{(4)} + b_{11}^{(1)}a_{10}^{(1)}a_{00}^{(2)} - b_{00}^{(2)}b_{11}^{(2)} \\ & +a_{11}^{(1)}a_{00}^{(3)} + 2b_{02}^{(2)}a_{00}^{(2)} - 2b_{00}^{(2)}a_{20}^{(2)} + a_{00}^{(2)}a_{11}^{(2)} + 2b_{02}^{(1)}a_{00}^{(3)} \\ & +a_{01}^{(1)}a_{11}^{(1)}a_{00}^{(2)} + 2a_{10}^{(1)}a_{20}^{(1)}a_{00}^{(2)} + 2b_{02}^{(1)}a_{01}^{(1)}a_{00}^{(2)}\Big), \end{aligned}$$

$$\begin{aligned} A_2 = \; & \frac{1}{8}\Big(b_{20}^{(1)}b_{11}^{(1)} - b_{02}^{(1)}b_{11}^{(1)} + 3b_{10}^{(1)}b_{03}^{(1)} - 3ab_{00}^{(2)} - 3bb_{00}^{(2)} + b_{21}^{(2)} + 3b_{12}^{(2)} \\ & +2b_{12}^{(2)}a_{10}^{(1)} + 3b_{03}^{(1)}a_{01}^{(1)} - 2b_{20}^{(1)}a_{20}^{(1)} + a_{20}^{(1)}a_{11}^{(1)} + 2a_{30}^{(2)} + a_{12}^{(2)} \\ & +2b_{02}^{(1)}a_{02}^{(1)} + a_{11}^{(1)}a_{02}^{(1)} + 2a_{10}^{(1)}a_{21}^{(1)} + b_{10}^{(1)}a_{12}^{(1)} + a_{01}^{(1)}a_{12}^{(1)}\Big). \end{aligned}$$

In view of the expression of the polynomial $f_4(r)$ it follows immediately that $f_4(r)$ can have at most one positive real root. So statement (c) of the theorem is proved.

Solving $A_1 = 0$ and $A_2 = 0$ we obtain $f_4(r) = 0$, so we can apply the averaging theory of order 5, and its corresponding averaged function is of the form

$$f_5(r) = r(B_1 + B_2 r^2 + B_3 r^4),$$

We do not give the big expressions of the independent coefficients B_i for $i = 1, 2, 3$. It follows immediately that $f_5(r)$ can have at most two positive real roots. Thus, statement (c) of the theorem holds.

Solving $B_1 = 0$, $B_2 = 0$ and $B_3 = 0$ we obtain $f_5(r) = 0$, and we assume that a denominator $7a + 9b \neq 0$ which appears is not zero. Then applying the averaging theory of order six we obtain an averaged function of the form

$$f_6(r) = \left(C_1 + C_2 r + C_3 r^2 + C_4 r^3 + C_5 r^4 + C_6 r^5\right).$$

Where the coefficients C_i's for $i = 1, ...6$ are independent polynomials in $a_{ij}^{(s)}$ and $b_{ij}^{(s)}$. Hence $f_6(r)$ has at most has 5 simple positive zero. Hence statement (d) of the theorem is proved.

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