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Study of the period function of a two-parameter family of centers

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Abstract. In this paper we study the period function of $\ddot{x} = (1+x)^p - (1+x)^q$, with $p, q \in \mathbb{R}$ and $p > q$. We prove three independent results. The first one establishes some regions in the parameter space where the corresponding center has a monotonous period function. This result extends the previous ones by Miyamoto and Yagasaki for the case $q = 1$. The second one deals with the bifurcation of critical periodic orbits from the center. The third one is addressed to the critical periodic orbits that bifurcate from the period annulus of each one of the three isochronous centers in the family when perturbed by means of a one-parameter deformation. These three results, together with the ones that we obtained previously on the issue, leads us to propose a conjectural bifurcation diagram for the global behaviour of the period function of the family.

1 Introduction and setting of the problem

In the present paper we study the bifurcation diagram of the period function associated to a family of potential centers. Recall that a singular point p of an analytic vector field $X = f(x,y)\partial_x + g(x,y)\partial_y$ is a *center* if it has a punctured neighbourhood that consist entirely of periodic orbits surrounding p. The largest neighbourhood with this property is called *period annulus* and henceforth it will be denoted by $\mathscr{P}$. From now on $\partial\mathscr{P}$ will denote the boundary of $\mathscr{P}$ after embedding it into $\mathbb{RP}^2$. Clearly the center p belongs to $\partial\mathscr{P}$, and in what follows we will call it the *inner boundary* of the period annulus. We also define the *outer boundary* of the period annulus to be $\Pi := \partial\mathscr{P} \setminus \{p\}$. Note that Π is a non-empty compact subset of $\mathbb{RP}^2$. The *period function* of the center assigns to each periodic orbit in $\mathscr{P}$ its period. If the period function is constant, then the center is said to be *isochronous*. Since the period function is defined on the set of periodic orbits in $\mathscr{P}$, in order to study its qualitative properties usually the first step is to parametrize this set. This can be done by taking an analytic transverse section to X on $\mathscr{P}$, for instance an orbit of the orthogonal vector field $X^{\perp}$. If $\{\gamma_s\}_{s\in(a,b)}$ is such a parametrization, then $s \longmapsto T(s) := \{\text{period of } \gamma_s\}$ is an analytic map that provides the qualitative properties of the period function that we are concerned about. In particular the existence of *critical periods*, which are isolated critical points of this function, i.e. $\hat{s} \in (a,b)$ such that $T'(s) = \alpha(s-\hat{s})^k + \mathrm{o}\big((s-\hat{s})^k\big)$ with $\alpha \neq 0$ and $k \geqslant 1$. In this case we shall say that $\gamma_{\hat{s}}$ is a *critical periodic orbit* of multiplicity k of the center. One can readily see that this definition does not depend on the particular parametrization of the set of periodic orbits used. We say that the period function of a center is *monotonous increasing* (respectively, *decreasing*) if there are no critical periodic orbits on $\mathscr{P}$ and, for any two periodic orbits γ_1 and γ_2 with $\gamma_1 \subset \mathrm{Int}(\gamma_2)$, the period of γ_2 is greater (respectively, smaller) than the one of γ_1.

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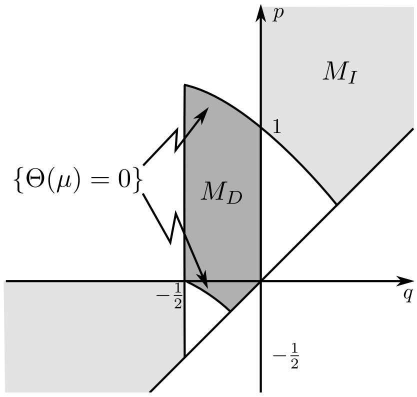

Figure 1: Monotonicity regions according to Theorem A.

The problem of bounding the number of critical periodic orbits is analogous to the problem of bounding the number of limit cycles, which is related to the well known Hilbert's 16th Problem (see [1, 7, 20, 25] and references there in) and its various weakened versions. Questions related to the behaviour of the period function have been extensively studied by a number of authors. Let us quote for instance the problems of isochronicity (see [6, 13, 17]), monotonicity (see [3, 4, 22]) or bifurcation of critical periodic orbits (see [5, 21, 23]).

In this paper we consider the two-parameter family of potential differential systems given by

$$X_\mu \quad \begin{cases} \dot{x} = -y, \\ \dot{y} = (1+x)^p - (1+x)^q, \end{cases} \tag{1}$$

where $\mu := (q,p)$ with $p, q \in \mathbb{R}$. This is a well defined analytic differential system on the half plane $\{x > -1\}$. The singular point at the origin is a non-degenerated center if $p > q$ and a hyperbolic saddle if $p < q$. Our goal in this paper is to provide a global study of the qualitative properties of the period function of the center, so we will consider X_μ with $\mu \in \Lambda := \{(q,p) \in \mathbb{R}^2 : p > q\}$. We became interested in this problem because of the previous results by Miyamoto and Yagasaki on the issue. Both authors proved, see [18], that the period function is monotonous when $q = 1$ and $p \in \mathbb{N}$. As it often occurs, they came across the period function when studying the solutions of an elliptic Neumann problem and needed this monotonicity property to prove a bifurcation result. Later Yagasaki improved the result showing in [24] the monotonicity of the period function for $q = 1$ and any $p \in \mathbb{R}$ with $p > 1$. We will prove three main results on the period function of the family $\{X_\mu\}_{\mu \in \Lambda}$. The first one, Theorem A, establishes some regions in the parameter space where the corresponding center has a monotonous period function. This result extends the previous ones by Miyamoto and Yagasaki [18, 24]. The second one, Theorem B, deals with the bifurcation of critical periodic orbits from the inner boundary of $\mathscr{P}$, i.e., the center. Finally Theorem C is addressed to the bifurcation of critical periodic orbits from the interior of the period annulus of an isochronous center. These three results, together with the ones that we obtained in [14] concerning the bifurcation of critical periodic orbits from the outer boundary of $\mathscr{P}$, leads us to propose a conjectural bifurcation diagram for the global behaviour of the period function of $\{X_\mu\}_{\mu \in \Lambda}$. We will explain it in detail at the end of this section once we state precisely our main results.

Let us begin with the statement of the monotonicity result. To this end we define

$$\Theta(\mu) := 2p^4 + p^3(3+4q) + p^2(9q^2+9q-1) + p(4q^3+9q^2+2q-3) + (1+q)^2(2q^2-q-1). \tag{2}$$

Then, denoting the light grey region in Figure 1 by M_I and the dark grey region by M_D, we will prove the following result:

Theorem A. *The period function of the center at the origin of the potential differential system* (1) *is monotonous increasing* (*respectively, decreasing*) *in case that* $\mu \in M_I$ (*respectively,* $\mu \in M_D$).

In order to state the results about the bifurcation of critical periodic orbits it is first necessary to recall the definition of criticality given in [8].

Definition 1.1. Let $\mathscr{U}$ be a subset of $\mathbb{R}^n$ and consider an analytic family $\{Y_\mu\}_{\mu\in\mathscr{U}}$ of planar vector fields with a center. Fix some $\hat{\mu}\in\mathscr{U}$ and consider an invariant set L of $Y_{\hat{\mu}}$. If L is compact, then criticality of the pair $(L, Y_{\hat{\mu}})$ with respect to the deformation Y_μ is the smallest integer $\mathcal{N}_L$ having the property that there exist a neighbourhood V of L and $\delta>0$ such that, for every $\mu\in\mathscr{U}$ with $\|\mu-\hat{\mu}\|<\delta$, the vector field Y_μ has no more than $\mathcal{N}_L$ critical periodic orbits inside V. For a general set L, not necessary compact, we define

$$\text{Crit}\left((L, Y_{\hat{\mu}}), Y_\mu\right) := \sup\{\mathcal{N}_K : K\subset L, K \text{ is a compact invariant set of } Y_{\hat{\mu}}\}$$

to be the *criticality* of the pair $(L, Y_{\hat{\mu}})$ with respect to the deformation Y_μ. □

Roughly speaking, the criticality $\text{Crit}\left((L, Y_{\hat{\mu}}), Y_\mu\right)$ is the maximal number of critical periodic orbits that tend to L as $\mu\to\hat{\mu}$. The ultimate aim in the study of the global behaviour of the period function of a given family $\{Y_\mu\}_{\mu\in\mathscr{U}}$ is to decompose the parameter space $\mathscr{U}=\cup U_i$ in such a way that if μ_1 and μ_2 belong to the same set U_i, then the corresponding period functions are qualitatively the same. The set $\cup\partial U_i$ consists of those parameters $\hat{\mu}\in\mathscr{U}$ for which some critical periodic orbit emerges or disappears as μ tends to $\hat{\mu}$. There are three different places from where a critical periodic orbit may bifurcate (see [16] for details):

(1) The inner boundary of the period annulus (i.e., the center itself),

(2) the "interior" of the period annulus, or

(3) the outer boundary of the period annulus.

In the present paper we shall study the first and second type of bifurcation inside the family $\{X_\mu\}_{\mu\in\Lambda}$ given in (1), but the later only in the particular case that the unperturbed system $X_{\hat{\mu}}$ has an isochronous center. In this regard we point out that the study when the unperturbed system is not isochronous is out of reach for the moment. Its counterpart in the context of Hilbert's 16th Problem is the so-called *blue-sky* bifurcation of a semi-stable limit cycle, which is usually undetectable (see [19] for instance). As we already mentioned, the third type of bifurcation inside $\{X_\mu\}_{\mu\in\Lambda}$ was the subject of a previous paper [14]. That being said, let $\{Y_\mu\}_{\mu\in\mathscr{U}}$ be an analytic family of vector fields such that, for each $\mu\in\mathscr{U}$, Y_μ has a center at $p_\mu\in\mathbb{R}^2$. Fixed $\hat{\mu}\in\mathscr{U}$, the following definition is addressed to the cases: (1) $L=p_{\hat{\mu}}$, (2) $L=\mathscr{P}_{\hat{\mu}}$ with $p_{\hat{\mu}}$ an isochronous center, and (3) $L=\Pi_{\hat{\mu}}=\partial\mathscr{P}_{\hat{\mu}}\setminus\{p_{\hat{\mu}}\}$.

Definition 1.2. Let L be an invariant set of $Y_{\hat{\mu}}$. We say that $\hat{\mu}$ is a *local regular value of the period function at* L if $\text{Crit}\left((L, Y_{\hat{\mu}}), Y_\mu\right)=0$, otherwise $\hat{\mu}$ is a *local bifurcation value of the period function at* L. □

Taking this definition into account, we will prove the following result concerning the bifurcation of critical periodic orbits from the center:

Theorem B. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be the family of vector fields in (1) and consider the period function of the center at the origin. Define $\Delta_1(\mu)=2p^2+2q^2+7pq-p-q-1$. Then the set $\{\mu\in\Lambda:\Delta_1(\mu)\neq 0\}$ consists of local regular values of the period function at the inner boundary of the period annulus. In addition,*

(a) If $\Delta_1(\hat{\mu})>0$, then the period function of $X_{\hat{\mu}}$ is increasing near the inner boundary.

(b) If $\Delta_1(\hat{\mu})<0$, then the period function of $X_{\hat{\mu}}$ is decreasing near the inner boundary.

Finally if $\Delta_1(\hat{\mu})=0$, then the criticality of $((0,0), X_{\hat{\mu}})$ with respect to the deformation X_μ is one. In particular, $\hat{\mu}$ is a local bifurcation value of the period function at the inner boundary.

In our third main result we study the critical periodic orbits that bifurcate from the period annulus of an isochronous center $X_{\hat{\mu}}$ when perturbed by means of a one-parameter deformation inside $\{X_\mu\}_{\mu\in\Lambda}$. We will prove the following:

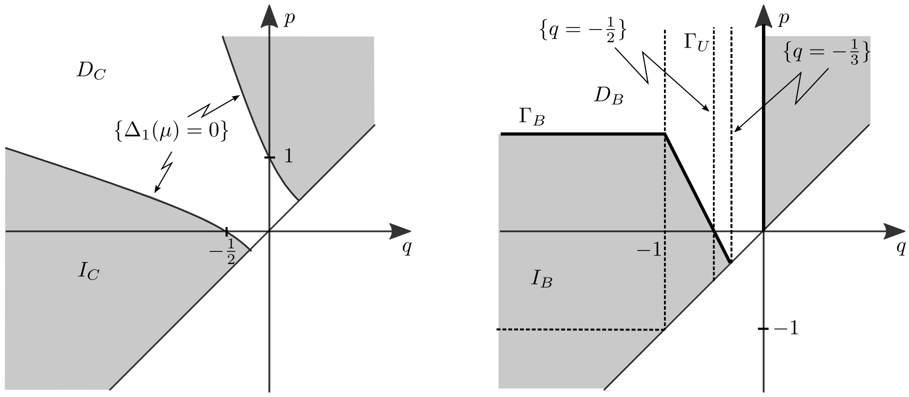

Figure 2: Bifurcation diagram of the period function at the inner boundary (left) and at the outer boundary (right), according to Theorem C and Theorem 1.3, respectively.

Theorem C. *The center at the origin of the differential system in (1) is isochronous if, and only if, $\mu \in \{(-3,1),(-1/2,0),(0,1)\}$. Moreover, if $\hat{\mu}$ is the parameter value of one of these isochronous centers and $\mu \longmapsto \mu(\varepsilon)$ is any germ of analytic curve in Λ with $\mu(0)=\hat{\mu}$, then Crit$\,((\mathscr{P}_{\hat{\mu}},X_{\hat{\mu}}),X_{\mu(\varepsilon)}) \leqslant 1$. Finally, for each isochronous center, there exists a germ of analytic curve for which this upper bound is achieved.*

We expect of course that Crit$\,((\mathscr{P}_{\hat{\mu}},X_{\hat{\mu}}),X_{\mu})=1$ for any $\mu \in \{(-3,1),(-1/2,0),(0,1)\}$. In relation to this, but in the context of Hilbert's 16th Problem, it is to be quoted the result of L. Gavrilov [9], which shows that the problem of finding the cyclicity of a period annulus with respect to a multi-parameter deformation can be always reduced to the "simpler" problem of finding the cyclicity with respect to a one-parameter deformation.

For completeness, we give at this point the precise statement of our result in [14], which constitutes the counterpart of Theorem B for the behaviour of the period function near the outer boundary of $\mathscr{P}$. To this end, let us denote

$$\Gamma_B := \{\mu\in\Lambda : q=0\}\cup\{\mu\in\Lambda : p=1, q\leqslant -1\}\cup\{\mu\in\Lambda : p+2q+1=0, q\geqslant -1\}$$

and

$$\Gamma_U := \{\mu\in\Lambda : (2q+1)(3q+1)(q+1)(p+1)=0\}.$$

(Here the subscripts B and U stand for bifurcation and unspecified, respectively.) The curve Γ_B splits the parameter space Λ into three connected components, see Figure 2. Denoting by D_B the uncoloured component and by I_B the union of the two other components in dark grey, by [14, Theorem E] we have:

Theorem 1.3. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be the family of vector fields in (1) and consider the period function of the center at the origin. Then the set $\Lambda\setminus(\Gamma_B\cup\Gamma_U)$ corresponds to local regular values of the period function at the outer boundary of the period annulus. In addition,*

(a) If $\hat{\mu}\in I_B\setminus\Gamma_U$ then the period function of $X_{\hat{\mu}}$ is increasing near the outer boundary.

(b) If $\hat{\mu}\in D_B\setminus\Gamma_U$ then the period function of $X_{\hat{\mu}}$ is decreasing near the outer boundary.

Moreover the parameters in Γ_B are local bifurcation values of the period function at the outer boundary of the period annulus. Finally, Crit$((\Pi_{\hat{\mu}},X_{\hat{\mu}}),X_\mu)=1$ *for all $\hat{\mu}=(\hat{q},1)$ with $\hat{q}<-3$ and $\hat{\mu}=(\hat{q},-2\hat{q}-1)$ with $\hat{q}\in(-\frac{3}{5},-\frac{1}{3})\setminus\{-\frac{1}{2}\}$.*

The combination of the three main results proved in the present paper, together with Theorem 1.3, leads us to propose the conjectural bifurcation diagram that we display in Figure 3 for the global behaviour of the period function of the differential system (1). This conjecture claims in particular, and it constitutes

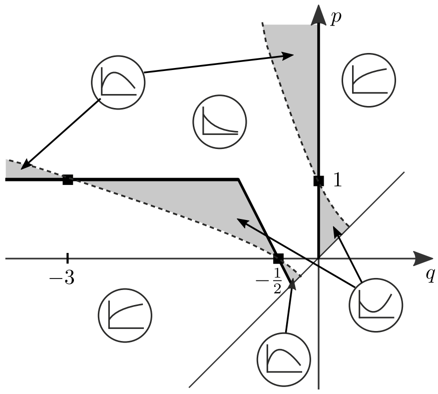

Figure 3: Conjectural bifurcation diagram for the period function of the differential system (1), where the solid and dotted curves consist of local bifurcation values at the outer and inner boundary, respectively. The parameters in the grey region correspond to systems with exactly one critical periodic orbit.

the key point, that there are no parameters for which two critical periodic orbits collide disappearing in the interior of the period annulus. Of course there are points in the conjecture already proved. For instance Theorem A covers in large part the uncoloured region in Figure 3, where the monotonicity of the period function is conjectured. On the other hand, a straightforward application of Bolzano's Theorem, comparing the sign of the derivative of the period function near the inner and outer boundary of $\mathscr{P}$, shows that if μ belongs to the grey region in Figure 3, then the period function of X_μ has at least one critical periodic orbit.

The paper has three additional sections, each one dedicated to prove one of the main results. For reader's convenience we advance that the three sections are essentially independent.

2 Proof of Theorem A

This section is devoted to prove Theorem A, but before we introduce the notation that we will use henceforth. Consider a potential differential system

$$X \quad \begin{cases} \dot{x} = -y, \\ \dot{y} = V'(x), \end{cases}$$

where V is an analytic function on some interval I that contains $x = 0$. In what follows we shall use sometimes the vector field notation $X = -y\partial_x + V'(x)\partial_y$ to refer the above differential system. We suppose $V'(0) = 0$ and $V''(0) > 0$, so that the origin is a non-degenerated center, and we shall denote the projection of its period annulus $\mathscr{P}$ on the x-axis by $\mathcal{I} = (x_\ell, x_r)$. Thus $x_\ell < 0 < x_r$. The corresponding Hamiltonian function is given by $H(x, y) = \frac{1}{2}y^2 + V(x)$, where we fix that $V(0) = 0$. Then $H(\mathscr{P}) = (0, h_0)$, with $h_0 \in (0, +\infty]$, and in this case we will say that h_0 is the energy level of the outer boundary of $\mathscr{P}$. We define in addition

$$g(x) := \operatorname{sgn}(x)\sqrt{V(x)} = x\sqrt{\frac{V(x)}{x^2}},$$

which is clearly an analytic diffeomorphism from $\mathcal{I}$ to $(-\sqrt{h_0}, \sqrt{h_0})$. It is well-known, see [5,15] for instance, that the period $T(h)$ of the periodic orbit γ_h inside the energy level $\{H(x, y) = h\}$ is given by

$$T(h) = \int_{\gamma_h} \frac{dx}{y} = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})'(\sqrt{h}\sin\theta)d\theta,$$

where the definite integral follows by using the polar coordinates that bring the oval $\gamma_h \subset \{\frac{1}{2}y^2 + V(x) = h\}$ to the circle of radius $\sqrt{h}$. The period function T is analytic on $(0, h_0)$ and it can be extended analytically at $h = 0$. In what follows we shall consider a potential differential system depending on a parameter $\mu \in \Lambda$ and we shall use the previous notations with a subscript μ.

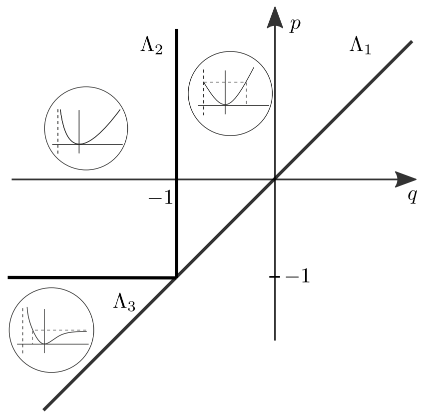

Figure 4: Bifurcation diagram of the graph of V_μ.

For the potential differential system (1) under consideration we have $\mu = (q, p)$ and

$$V_\mu(x) := \int_1^{x+1} (u^p - u^q)\, du.$$

Clearly the origin is a centre for all $\mu \in \Lambda$ because $V''(0) = p - q > 0$. Note that $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ with

$$\Lambda_1 := \{\mu \in \Lambda : -1 < q < p\},\ \Lambda_2 := \{\mu \in \Lambda : q \leqslant -1 \leqslant p\} \text{ and } \Lambda_3 := \{\mu \in \Lambda : q < p < -1\}.$$

The next result is a straightforward observation and we do not show it for the sake of shortness.

Lemma 2.1. *The projection on the x-axis of the period annulus $\mathscr{P}_\mu$ of the center at the origin of (1) is $\mathcal{I}_\mu = (-1, \rho(\mu))$ for $\mu \in \Lambda_1$, $\mathcal{I}_\mu = (-1, +\infty)$ for $\mu \in \Lambda_2$ and $\mathcal{I}_\mu = (\rho(\mu), +\infty)$ for $\mu \in \Lambda_3$, where*

$$\rho(\mu) := \left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}} - 1.$$

The proof of Theorem A will be an application of the following monotonicity criterion, see [22], and in its statement we use the previous notation.

Theorem 2.2 (Schaaf's criterion)**.** *Let $X = -y\partial_x + V'(x)\partial_y$ be an analytic potential differential system with a non-degenerated center at the origin and consider its period function $T(h)$. Then $T'(h) > 0$ for all $h \in (0, h_0)$ in case that*

$$(I_1) \qquad 5V'''(x)^2 - 3V''(x)V^{(4)}(x) > 0 \text{ for all } x \in \mathcal{I} \text{ with } V''(x) > 0,$$

and

$$(I_2) \qquad V'(x)V'''(x) < 0 \text{ for all } x \in \mathcal{I} \text{ with } V''(x) = 0.$$

On the other hand, $T'(h) < 0$ for all $h \in (0, h_0)$ in case that

$$(D) \qquad 5V'''(x)^2 - 3V''(x)V^{(4)}(x) < 0 \text{ for all } x \in \mathcal{I} \text{ with } V''(x) \geqslant 0.$$

The key point to apply Schaaf's criterion to the potential differential system X_μ given in (1) is that, as one can easily verify, we can write the "test functions" as

$$\begin{aligned}
&5V_\mu'''(x)^2 - 3V_\mu''(x)V_\mu^{(4)}(x) = (1+x)^{2q-4}P_\mu\big((1+x)^{p-q}\big),\\
&V_\mu'(x)V_\mu'''(x) = (1+x)^{2q-2}Q_\mu\big((1+x)^{p-q}\big),\\
&V_\mu''(x) = (1+x)^{q-1}R_\mu\big((1+x)^{p-q}\big),
\end{aligned}$$

with

$$P_\mu(z) := (q-1)q^2(1+2q) + pq(3p^2+3q^2-10pq+p+q+2)z + (p-1)p^2(1+2p)z^2,$$
$$Q_\mu(z) := q(q-1) + (p-p^2+q-q^2)z + p(p-1)z^2,$$

and $R_\mu(z) := -q + pz$. Accordingly we get the following result:

Lemma 2.3. *The conditions (I_1), (I_2) and (D) of Schaaf's monotonicity criterion applied to the potential differential system (1) are equivalent to*

$$(I_1') \qquad P_\mu(z) > 0 \text{ for any } z \in \varphi(\mathcal{I}_\mu) \text{ with } R_\mu(z) > 0,$$
$$(I_2') \qquad Q_\mu(z) < 0 \text{ for any } z \in \varphi(\mathcal{I}_\mu) \text{ with } R_\mu(z) = 0,$$
$$(D') \qquad P_\mu(z) < 0 \text{ for any } z \in \varphi(\mathcal{I}_\mu) \text{ with } R_\mu(z) \geqslant 0,$$

respectively, where $\varphi(x) := (1+x)^{p-q}$.

Moreover, taking Lemma 2.1 into account, we obtain:

Lemma 2.4. *Define $L_\mu = \{z \in \varphi(\mathcal{I}_\mu)\colon R_\mu(z) > 0\}$. Then*

(a) $L_\mu = \left(\frac{q}{p}, \frac{p+1}{q+1}\right)$ *if* $\mu \in \Lambda \cap \{q > 0\}$,

(b) $L_\mu = \left(0, \frac{p+1}{q+1}\right)$ $\mu \in \Lambda \cap \{p \geqslant 0, -1 < q \leqslant 0\}$ *or* $\mu \in \Lambda \cap \{p < 0, p+q > -1\}$,

(c) $L_\mu = (0, \infty)$ $\mu \in \Lambda \cap \{p \geqslant 0, q < -1\}$,

(d) $L_\mu = \left(0, \frac{q}{p}\right)$ *if* $\mu \in \Lambda \cap \{p < 0, q > -1, p+q \leqslant -1\}$ *or* $\mu \in \Lambda \cap \{q < -1, -1 < p < 0\}$,

(e) $L_\mu = \left(\frac{p+1}{q+1}, \frac{q}{p}\right)$ *if* $\mu \in \Lambda \cap \{p < -1\}$.

Lemma 2.5. *The potential differential system (1) verifies condition (I_1) of Schaaf's criterion if μ is inside the region 1, 8, 10, 11, 14 or 15 in Figure 5. On the other hand, it verifies condition (D) if μ is inside the region 5 or 7.*

Proof. By applying Lemmas 2.3 and 2.4, the first assertion is equivalent to require that the quadratic polynomial P_μ is positive on the interval L_μ. For each $\mu \in \Lambda$, let us define $\mathcal{Z}(\mu)$ to be the number of zeros of P_μ inside L_μ counted with multiplicities. The relevant information to study this number is given by the following expressions:

$$\operatorname{Disc}(P_\mu) = 3p^2(p-q)^2q^2(3p^2+3q^2-14pq+2p+2q+7),$$
$$P_\mu(0) = (q-1)q^2(1+2q),$$
$$P_\mu\left(\frac{q}{p}\right) = 5(p-q)^2q^2,$$
$$P_\mu(\infty) = (p-1)p^2(1+2p),$$
$$P_\mu\left(\frac{p+1}{q+1}\right) = (p-q)^2(1+q)^{-2}\Theta(\mu),$$

where Θ is defined in (2), $\operatorname{Disc}(P_\mu)$ denotes the discriminant of P_μ and $P_\mu(\infty)$ stands for the coefficient of maximum degree of P_μ. The zero level sets of these functions split the parameter space Λ into several connected components and it is clear that $\mathcal{Z}(\mu)$ is constant in each one. In this regard it is to be pointed out that if $\operatorname{Disc}(P_\mu) = 0$ then one can verify that the corresponding double root is outside L_μ. On account of this, in order to study $\mathcal{Z}(\mu)$ we can rule out the curve $\operatorname{Disc}(P_\mu) = 0$. We obtain in this way 15 connected components, that we display in Figure 5. It is clear that if $\mathscr{U}$ is one of these regions and P_μ is positive on

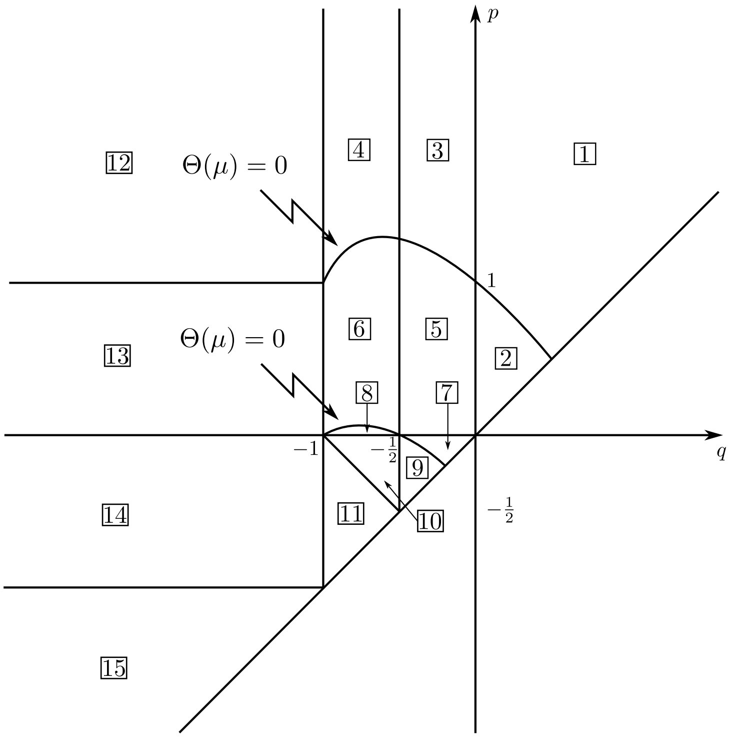

Figure 5: Sketch of the regions in the proof of Lemma 2.5.

L_μ for some $\mu = \hat{\mu} \in \mathscr{U}$, then the same is true for all the parameters $\mu \in \mathscr{U}$. Choosing one parameter inside each one of the 15 regions we prove that this is the case for the regions 1, 8, 10, 11, 14 and 15. This proves the first assertion.

Let us turn now to the second assertion. Thanks to Lemmas 2.3 and 2.4 again, condition (D) is equivalent to require that P_μ is negative on the interval $\hat{L}_\mu$, where $\hat{L}_\mu := L_\mu$ in case that q/p is not an endpoint of L_μ and $\hat{L}_\mu := L_\mu \cup \{q/p\}$ otherwise. (This follows from noting that $R_\mu(z) = 0$ if, and only if, $z = q/p$.) Arguing as we did with condition (I_1) one can show that this is the case for the parameters inside regions 5 and 7. This proves the result. ■

Proof of Theorem A. We claim that if $(q,p) \in \Lambda$ verifies $pq > 0$, then the potential differential system (1) satisfies condition (I_2) of Schaaf's criterion. Indeed, this follows by applying Lemma 2.3 and noting that $R_\mu(z) = 0$ if, and only if, $z = q/p$ and $Q_\mu(q/p) = -q(p-q)^2/p < 0$ when $pq > 0$.

On account of the claim and Lemma 2.5, we can assert that the potential differential system (1) verifies conditions (I_1) and (I_2) if $\mu = (q,p)$ is inside, see Figure 5, the union of the regions 1, 10, 11, 14 and 15, say R_I. Accordingly, by applying Theorem 2.2, we can assert that the derivative of the period function is strictly positive in case that $\mu \in R_I$. Note at this point, see also Figure 2, that $M_I \setminus R_I$ is the union of three segments, say ℓ_1, ℓ_2 and ℓ_3. Since one can easily verify that (I_1) and (I_2) are fulfilled in $\ell_1 \cup \ell_2 \cup \ell_3$ as well, the result concerning the set M_I follows.

Finally, Lemma 2.5 shows that the parameters inside the union of the regions 5 and 7, say R_D, satisfy condition (D) of Schaaf's criterion. Therefore, the derivative of the period function of the center at the origin of system (1) is negative for $\mu \in R_D$. Observe that $M_D \setminus R_D$ is the segment $(-1/2, 0) \times \{0\}$. Since one can easily verify that condition (D) also holds for parameters inside this segment, the result concerning M_D follows. This completes the proof of the result. ■

3 Proof of Theorem B

The linear part of the differential system (1) at the center depends on the parameters p and q. Since this is not very convenient in order to compute the period constants of a center, instead of X_μ we shall consider

$$\hat{X}_\mu \quad \begin{cases} \dot{u} = -v, \\ \dot{v} = \frac{1}{p-q}\left((1+u)^p - (1+u)^q\right). \end{cases} \tag{3}$$

One can verify that if $\mu \in \Lambda$, i.e., $p-q>0$, then coordinate transformation $\{u = x, v = \frac{1}{\sqrt{p-q}}y\}$ and the constant rescaling of time by $\frac{1}{\sqrt{p-q}}$ brings system (1) to (3). This of course guarantees that the properties of the period function that we are interested in do not change at all. Note that linear part of the differential system (3) at the center does not depend on the parameters because, following the obvious notation,

$$\hat{V}_\mu(u) := \frac{1}{p-q}\int_1^{u+1}(s^p - s^q)\,ds = \frac{1}{2}u^2 + \mathrm{o}(u^2).$$

The differential system (3) has the additional advantage that it is well-defined for all $(q,p) \in \mathbb{R}^2$, even for the straight line $p = q$, where it has the expression

$$\hat{X}_{q,q} \quad \begin{cases} \dot{u} = -v, \\ \dot{v} = (1+u)^q \log(1+u). \end{cases}$$

Observe in addition the symmetry $\hat{X}_{q,p} = \hat{X}_{p,q}$. Note finally that the projection of the period annulus of the center is the same interval for the differential systems (1) and (3). Thus, for the sake of simplicity, we shall keep denoting it by $\mathcal{I}_\mu$.

Proposition 3.1. *Let $\hat{T}_\mu(h)$ denote the period of the periodic orbit of system (3) inside the energy level $\frac{1}{2}v^2 + \hat{V}_\mu(u) = h$. Then, setting $\hat{T}_\mu(0) := 2\pi$, $\hat{T}_\mu(h)$ extends analytically at $h = 0$ and its Taylor development is given by $\hat{T}_\mu(h) = 2\pi + \sum_{i\geqslant 1}\Delta_i(\mu)h^i$ with $\Delta_i \in \mathbb{R}[\mu]$. Moreover,*

$$\begin{aligned}
\Delta_1(\mu) &= \pi\big(2p^2 + 2q^2 + 7pq - p - q - 1\big),\\
\Delta_2(\mu) &= \frac{5\pi}{24}\big(-23 + 4p^4 - 46q + 21q^2 + 44q^3 + 4q^4 + 4p^3(11+43q) +\\
&\quad + 3p^2(7 + 122q + 139q^2) + 2p(-23 + 42q + 183q^2 + 86q^3)\big),\\
\Delta_3(\mu) &= \frac{7\pi}{864}\big(-11237 - 1112p^6 - 33711q - 10641q^2 + 34903q^3 + 22434q^4\\
&\quad - 636q^5 - 1112q^6 + 12p^5(-53 + 803q) + 6p^4(3739 + 25888q + 27289q^2)\\
&\quad + p^3(34903 + 390273q + 734277q^2 + 336347q^3)\\
&\quad + 3p^2(-3547 + 88309q + 284637q^2 + 244759q^3 + 54578q^4)\\
&\quad + 3p(-11237 - 2951q + 88309q^2 + 130091q^3 + 51776q^4 + 3212q^5)\big).
\end{aligned}$$

Proof. We claim that $\hat{V}_\mu(u) = \sum_{k\geqslant 2}\hat{\alpha}_k(\mu)u^k$ with $\hat{\alpha}_k \in \mathbb{R}[\mu]$ and $\hat{\alpha}_2(\mu) = \frac{1}{2}$. To show this note first of all that $\hat{V}'_\mu(u) = \frac{1}{p-q}\sum_{k\geqslant 1}k!\alpha_k(\mu)u^k$ with

$$\alpha_k(q,p) := p(p-1)\cdots\big(p-(k-1)\big) - q(q-1)\cdots\big(q-(k-1)\big).$$

Since $\alpha_k(q,p) \in \mathbb{R}[q,p]$ and $\alpha_k(q,q) = 0$, we can assert that $\hat{\alpha}_{k+1}(p,q) = \frac{k!\alpha_k(q,p)}{(k+1)(p-q)} \in \mathbb{R}[q,p]$. This proves the validity of the claim because the fact that $\hat{\alpha}_2(p,q) = \frac{1}{2}$ is clear.

Let us define $\hat{g}_\mu(x) := \operatorname{sgn}(x)\sqrt{\hat{V}_\mu(x)}$ and suppose that the Taylor development of its inverse at $x=0$ is given by $\hat{g}_\mu^{-1}(x) = \sum_{k\geqslant 1}\beta_k(\mu)x^k$. Then, see for instance [5], it follows that

$$\hat{T}_\mu(h) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\hat{g}_\mu^{-1})'(\sqrt{h}\sin\theta)d\theta = \sum_{k\geqslant 0}\Delta_k(\mu)h^k \text{ with } \Delta_k(\mu) := 2\sqrt{2}(2k+1)\beta_{2k+1}(\mu)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2k}\theta d\theta.$$

Since one can easily verify that $\beta_1(\mu) = \frac{1}{\sqrt{2}}$, the above expression shows in particular that $\hat{T}_\mu(h)$ extends analytically to $h=0$ setting $\hat{T}_\mu(0) := 2\pi$. It shows moreover that $\Delta_k \in \mathbb{R}[\mu]$ if, and only if, $\beta_{2k+1}\in\mathbb{R}[\mu]$. Let us show that $\beta_k\in\mathbb{R}[\mu]$ for all $k\in\mathbb{N}$. To this end we note that, by definition, $V_\mu\left(\hat{g}_\mu^{-1}(x)\right) = x^2$ for all $x\in\mathcal{I}_\mu$. Hence

$$\sum_{k=2}^{\infty}\hat{\alpha}_k(\mu)\left(\sum_{i=1}^{\infty}\beta_i(\mu)x^i\right)^k = x^2 \text{ for all } x\approx 0.$$

Using this identity and taking the claim into account, one can prove by induction on k that $\beta_k\in\mathbb{R}[\mu]$ for all $k\in\mathbb{N}$. Therefore $\Delta_k\in\mathbb{R}[\mu]$ for all $k\in\mathbb{N}$. Finally the expression for Δ_1, Δ_2 and Δ_3 that we give in the statement can be easily computed following this approach by using a symbolic manipulator. This concludes the proof of the result. ∎

Next result of Cima, Mañosas and Villadelprat, see [6], provides a useful tool in order to study the isochronicity problem for a center of a potential differential system. It is given in terms of the existence of an *involution*, i.e., a function $\sigma\neq\mathrm{Id}$ such that $\sigma^2=\mathrm{Id}$.

Proposition 3.2. *Let V be an analytic function with $V(0)=0$ and suppose that $X=-y\partial_x+V'(x)\partial_y$ has a center at the origin. Let $\mathcal{I}$ be the projection of its period annulus on the x-axis. Then the origin is an isochronous center of period ω if, and only if, there exists an analytic involution σ on $\mathcal{I}$ such that $V(x)=\frac{\pi^2}{2\omega^2}\left(x-\sigma(x)\right)^2$.*

For a given ideal $\mathfrak{m}$ over $\mathbb{C}[x]$ we shall denote by $V(\mathfrak{m})$ the *complex variety* of $\mathfrak{m}$. The next result solves in particular the isochronicity problem in the family of centers under consideration. In its statement Δ_1, Δ_2 and Δ_3 are the period constants given in Proposition 3.1.

Theorem 3.3. *Define $\mu_1 := (-3,1)$, $\mu_2 := (-1/2,0)$, $\mu_3 := (0,1)$, $\mu_4 := (1,-3)$, $\mu_5 := (0,-1/2)$, $\mu_6 := (1,0)$ and $\mu_7 := (i/\sqrt{3},-i/\sqrt{3})$. Then the following holds:*

(a) The variety of the ideal $\mathfrak{m}_2 := (\Delta_1,\Delta_2)$ is $V(\mathfrak{m}_2)=\{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\bar{\mu}_7\}$.

(b) The variety of the ideal $\mathfrak{m}_3 := (\Delta_1,\Delta_2,\Delta_3)$ is $V(\mathfrak{m}_3)=\{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6\}$.

Moreover, the center at the origin of the differential system (3) is isochronous if, and only if, $\mu\in V(\mathfrak{m}_3)$.

Proof. The assertions in (a) and (b) can be proved with a symbolic manipulator, for instance using resultants. Let us prove the assertion concerning the isochronous centers of system (3). Clearly the necessity follows by definition, so we only need to show the sufficiency, i.e., if $\mu\in V(\mathfrak{m}_3)$ then the center is isochronous. To this end, taking advantage of the symmetry $\hat{X}_{(q,p)}=\hat{X}_{(p,q)}$, it suffices to show that μ_1, μ_2 and μ_3 correspond to isochronous centers. With this end in view, some easy computations show that $\sigma_1(x) := -\frac{x}{x+1}$, $\sigma_2(x) := 4+x-4\sqrt{x+1}$, and $\sigma_3(x) := -x$ are the involutions associated to $\hat{V}_{\mu_i}$ for $i=1,2,3$, respectively (i.e., such that $\hat{V}_{\mu_i}=\hat{V}_{\mu_i}\circ\sigma_i$). Finally the result follows by Proposition 3.2 after verifying that $\hat{V}_{\mu_i}(x)=\frac{1}{8}\left(x-\sigma_i(x)\right)^2$ for $i=1,2,3$. ∎

To study the criticality of the center at the origin, see Definition 1.1, we must parametrize the periodic orbits of the differential system near the center. To this end, since we deal with a Hamiltonian differential system, we use the energy level h of the periodic orbit. Then, to detect the critical periodic orbits that shrink or emerge from the center we analyse the Taylor development of the period function $\hat{T}_\mu(h)$ at $h=0$. In the statement of the next result Δ_1 is the first period constant, see Proposition 3.1, and μ_i, $i=1,2,\ldots,6$, are the parameters corresponding to isochronous centers of system (3), see Theorem 3.3.

Proposition 3.4. *The following holds:*

(*a*) *If $\Delta_1(\hat{\mu}) \neq 0$ with $\hat{\mu} \in \mathbb{R}^2$, then Crit$\big(((0,0),\hat{X}_{\hat{\mu}}),\hat{X}_\mu\big) = 0$. Moreover, if $\Delta_1(\hat{\mu})$ is positive (respectively, negative), then the period function of $\hat{X}_{\hat{\mu}}$ is increasing (respectively, decreasing) near the center.*

(*b*) *If $\Delta_1(\hat{\mu}) = 0$ with $\hat{\mu} \in \mathbb{R}^2 \setminus \{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6\}$, then Crit$\big(((0,0),\hat{X}_{\hat{\mu}}),\hat{X}_\mu\big) = 1$.*

Proof. Let $\hat{T}_\mu(h)$ denote the period of the periodic orbit of the differential system (3) inside the energy level $\frac{1}{2}v^2 + \hat{V}_\mu(u) = h$. Then, by Proposition 3.1, we have that

$$\hat{T}'_\mu(h) = \Delta_1(\mu) + 2\Delta_2(\mu)h + \mathrm{o}(h). \tag{4}$$

Clearly, if $\Delta_1(\hat{\mu}) \neq 0$, then there exist $\varepsilon > 0$ and an open neighbourhood $\mathscr{U}$ of $\hat{\mu}$ such that $\hat{T}'_\mu(h) \neq 0$ for all $h \in (0,\varepsilon)$ and $\mu \in \mathscr{U}$. This implies Crit$\big(((0,0),\hat{X}_{\hat{\mu}}),\hat{X}_\mu\big) = 0$ and proves (*a*) because the assertion concerning the monotonicity of the period function is trivial.

In order to show (*b*) note that if $\Delta_1(\hat{\mu}) = 0$ with $\hat{\mu} \in \mathbb{R}^2 \setminus \{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6\}$ then, by Theorem 3.3, $\Delta_2(\hat{\mu}) \neq 0$. Hence, from (4) and by applying the Implicit Function Theorem, Crit$\big(((0,0),\hat{X}_{\hat{\mu}}),\hat{X}_\mu\big) \leqslant 1$. The fact that this upper bound is achieved follows the same way using that the gradient of Δ_1 does not vanish for parameter values with $\Delta_1(\mu) = 0$. So the result is proved. ■

The study of the criticality at the isochronous centers is the last ingredient for the proof of Theorem B. Our approach strongly relies in the following two general results of Chicone and Jacobs [5].

Theorem 3.5. *Let $\{Y_\mu\}_{\mu\in\Lambda}$ be an analytic family of analytic Hamiltonian differential systems with a non-degenerate center at the origin. Let H_μ be the Hamiltonian function with $H_\mu(0,0) = 0$. Let $T_\mu(h)$ denote the period of the periodic orbit of Y_μ inside the energy level $H_\mu = h$ and let $T_\mu(h) = \sum_{i=0}^{\infty}\Delta_i(\mu)h^i$ be its Taylor development at $h = 0$. If the center is isochronous for $\mu = \hat{\mu}$ and if, for all $i \in \mathbb{N}$, Δ_i is inside the ideal $(\Delta_1,\Delta_2,\ldots,\Delta_{k+1})$ over $\mathbb{R}\{\mu\}_{\hat{\mu}}$, the ring of convergent power series at $\hat{\mu}$, then Crit$\big(((0,0),Y_{\hat{\mu}}),Y_\mu\big) \leqslant k$. Moreover, if the gradients of $\Delta_1,\Delta_2,\ldots,\Delta_{k+1}$ are linearly independent at $\hat{\mu}$, then Crit$\big(((0,0),Y_{\hat{\mu}}),Y_\mu\big) = k$.*

The previous result is an adaptation of the Isochrone Bifurcation Theorem in [5] to Hamiltonian systems, for which it is more natural to parametrize the periodic orbits with the energy instead of the intersection point with the positive x-axis, and to the definition of criticality that we use in this paper. The proof is omitted because it follows verbatim the one Chicone and Jacobs. The next result is a particular case of [5, Theorem A.1].

Proposition 3.6. *Suppose that the ideal $\mathfrak{m} = (f_1,\ldots,f_r) \subset \mathbb{C}[x_1,x_2,\ldots,x_n]$ satisfies that $V(\mathfrak{m})$ is a finite set and that rank$(\nabla f_1(a),\nabla f_2(a),\ldots,\nabla f_r(a)) = n$ for all $a \in V(\mathfrak{m})$. Then $\mathfrak{m}$ is radical, i.e., $f \in \mathfrak{m}$ if, and only if, $f\big(V(\mathfrak{m})\big) = 0$.*

Proposition 3.7. *Let $\hat{X}_\mu$ be the differential system in (3) and let $\mu_i \in \Lambda$, $i = 1,2,\ldots,6$, be the parameters corresponding to the isochronous centers. Then Crit$\big(((0,0),\hat{X}_{\mu_i}),\hat{X}_\mu\big) = 1$ for $i = 1,2,\ldots,6$.*

Proof. We apply Proposition 3.1 and consider the ideal $\mathfrak{m} := (\Delta_1,\Delta_2)$ over $\mathbb{C}[q,p]$. Then, by Theorem 3.3 we have that $V(\mathfrak{m}) = \{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\bar{\mu}_7\}$. One can verify in addition that the gradients of Δ_1 and Δ_2 are linearly independent at any $\mu \in V(\mathfrak{m})$. Thus, by applying Proposition 3.6, $\mathfrak{m}$ is radical over $\mathbb{C}[q,p]$. We claim that if $f_k(\mu) := (3p^2+3q^2+2)\Delta_k(\mu)$, then $f_k\big(V(\mathfrak{m})\big) = 0$ for all $k \geqslant 3$. Indeed, that $f_k(\mu_i) = 0$ for $i = 1,2,\ldots,6$ follows due to the fact that, by Theorem 3.3, the center of $\hat{X}_{\mu_i}$ is isochronous. We have on the other hand that $f_k(\mu_7) = 0$ and $f_k(\bar{\mu}_7) = 0$ because these two parameters are the roots of $3p^2+3q^2+2 = 0$. This proves the claim. Consequently $f_k \in \mathfrak{m}$ for all $k \geqslant 3$. Thus, for each $k \geqslant 3$, there exist $A_k, B_k \in \mathbb{C}[q,p]$ such that $f_k = A_k\Delta_1 + B_k\Delta_2$, so that

$$\Delta_k(q,p) = \frac{A_k(q,p)}{3p^2+3q^2+2}\Delta_1(q,p) + \frac{B_k(q,p)}{3p^2+3q^2+2}\Delta_2(q,p) \text{ for all } k \geqslant 3.$$

Fix any μ_i, $i = 1,2,\ldots,6$. Then, since $3p^2+3q^2+2 \neq 0$ at $\mu = \mu_i$, the above equality shows that $\Delta_k \in \mathfrak{m}$ over the local ring $\mathbb{R}\{\mu\}_{\mu_i}$ for all $k \geqslant 3$. (Here we use that $\Delta_i \in \mathbb{R}[\mu]$.) Hence, by applying Theorem 3.5, Crit$\big(((0,0),\hat{X}_{\mu_i}),\hat{X}_\mu\big) = 1$. So the result is proved. ■

Proof of Theorem B. As we mentioned at the beginning of the present section, any differential system (1) with $p>q$ can be brought to (3) by means of a conjugation and a constant rescaling of time. On account of this, the result follows by applying Propositions 3.4 and 3.7. ∎

4 Proof of Theorem C

By applying Theorem 3.3 we know in particular that $\mu_1=(-3,1)$, $\mu_2=(-1/2,0)$ and $\mu_3=(0,1)$ are the parameters corresponding to the isochronous centers of the family $\{X_\mu,\mu\in\Lambda\}$ in (1). In this section we shall consider each one of these isochronous centers and we shall study the emergence or disappearance of critical periodic orbits from its period annulus when we perturb it inside $\{X_\mu,\mu\in\Lambda\}$ by means of a one-parameter deformation. In order to study this problem we recall the following definitions:

Definition 4.1. We say that two planar vector fields *commute* on $\mathscr{U}\subset\mathbb{R}^2$ if they are transversal and the Lie bracket $[X,Y]$ vanishes identically on $\mathscr{U}$. □

Definition 4.2. Let $\varphi\colon M\longrightarrow N$ be a diffeomorphism between manifolds M and N and let X and Y be vector fields on M and N, respectively. The *pull back* of Y by φ is the vector field φ^*Y on M defined by $(\varphi^*Y)(p):=(D\varphi^{-1})_{\varphi(p)}Y\big(\varphi(p)\big)$ for all $p\in M$. The *push forward* of X by φ is the vector field φ_*X on N defined by $(\varphi_*X)(p):=(D\varphi)_{\varphi^{-1}(p)}X\big(\varphi^{-1}(p)\big)$ for all $p\in N$. □

Suppose that X is an analytic vector field with a center at p. It is well known, see [2] and references therein, that p is an isochronous center if, and only if, there exists an analytic vector field Y on a neighbourhood $\mathscr{U}$ of p such that X and Y commute on $\mathscr{U}\setminus\{p\}$. In order to prove Theorem C it will be convenient to have a commutator of each X_{μ_i}, $i=1,2,3$, with $\mathscr{U}$ being the whole period annulus $\mathscr{P}_{\mu_i}$. With this end in view we prove the following general result for isochronous centers of potential systems:

Proposition 4.3. *Let $X=-y\partial_x+V'(x)\partial_y$ be a potential vector field with an isochronous center at the origin of period ω and let $\mathscr{P}$ be its period annulus. Define $h(x):=\frac{x-\sigma(x)}{2}$, where σ is the involution such that $V=V\circ\sigma$. Then $(r,\theta)=\varphi(x,y)$, defined by means of $\big\{h(x)=r\cos\theta,\frac{\omega}{2\pi}y=r\sin\theta\big\}$, is a coordinate transformation on $\mathscr{P}$ and the pull-back of*

$$U=r\partial_r-\frac{r\int_0^\theta(h^{-1})''(r\cos s)\cos s\,ds}{(h^{-1})'(r\cos\theta)}\partial_\theta$$

by φ is an analytic vector field on $\mathscr{P}$ that extends analytically to the origin, and commutes with X on $\mathscr{P}$.

Proof. Without loss of generality we assume that $\omega=2\pi$. Note that h is a diffeomorphism on the projection of the period annulus because $\sigma'(x)=\frac{V'(x)}{V'\big(\sigma(x)\big)}<0$. We claim that $(h^{-1})''$ is an even function and that

$$F(r,\theta):=\int_0^\theta(h^{-1})'(r\cos s)ds$$

is a circle diffeomorphism of degree one, i.e., $F(r,\theta+2\pi)=F(r,\theta)+2\pi$, for each fixed r. To show this observe first that $h\circ\sigma=-h$, so that $\sigma(h^{-1}(u))=h^{-1}(-u)$. Hence

$$u=h(h^{-1}(u))=\frac{h^{-1}(u)-\sigma(h^{-1}(u))}{2}=\frac{h^{-1}(u)-h^{-1}(-u)}{2}.$$

Accordingly the odd part of the function h^{-1} is the identity, so we can write $h^{-1}(u)=u+G(u)$, with G being an even function. In particular this shows that $(h^{-1})''$ is an even function. In addition,

$$F(r,\theta+2\pi)-F(r,\theta)=\int_\theta^{\theta+2\pi}\big(h^{-1}\big)'(r\cos s)ds=2\pi+\int_\theta^{\theta+2\pi}G'(r\cos s)ds=2\pi,$$

where the last equality follows by using that G' is odd. This proves the validity of the claim.

Since the origin is an isochronous center, by Proposition 3.2 we can write $V(x)=\frac{1}{2}h(x)^2$. Therefore, $X=-y\partial_x+h(x)h'(x)\partial_y$ and an easy computation shows that $(u,v)=\varphi_1(x,y):=\big(h(x),y\big)$ brings X to

$$\varphi_{1*}X=\frac{1}{(h^{-1})'(u)}\big(-v\partial_u+u\partial_v\big).$$

Hence, if $(r,\theta)=\varphi_2(u,v)$ denotes the usual polar coordinates given by $\{u=r\cos\theta, v=r\sin\theta\}$, we get

$$\big(\varphi_2\circ\varphi_1\big)_*X=\varphi_*X=\frac{1}{(h^{-1})'(r\cos\theta)}\partial_\theta.$$

Finally, if $(R,\phi)=\varphi_3(r,\theta):=\big(r,F(r,\theta)\big)$, then $\big(\varphi_3\circ\varphi\big)_*X=\partial_\phi$ because $\phi'=\frac{d}{dt}F(r,\theta)=F_\theta(r,\theta)\theta'=1$. (At this point we used that $\theta\longmapsto F(r,\theta)$ is a one-degree circle diffeomorphism.) Clearly, a commutator for ∂_ϕ is given by $\hat{U}:=R\partial_R$, i.e., $[\big(\varphi_3\circ\varphi\big)_*X,\hat{U}]=0$. Then,

$$0=\big(\varphi_3\circ\varphi\big)^*\big[\big(\varphi_3\circ\varphi\big)_*X,\hat{U}\big]=\big[X,\big(\varphi_3\circ\varphi\big)^*\hat{U}\big].$$

The pull-back of $\hat{U}$ by φ_3 is precisely the vector field U given in the statement because $r'=R'=R=r$ and $0=\phi'=F_r(r,\theta)r'+F_\theta(r,\theta)\theta'$. Thus $\varphi_3{}^*\hat{U}=U$ and so the above expression shows that $[X,\varphi^*U]=0$, as desired.

It remains to be shown that φ^*U is an analytic vector field on $\mathscr{P}\cup\{(0,0)\}$. To this end it suffices to prove that $(\varphi_2)^*U$ is an analytic vector field at the origin because $\varphi^*U=\big(\varphi_2\circ\varphi_1\big)^*U=(\varphi_1)^*(\varphi_2)^*U$ and φ_1 is a well-defined analytic diffeomorphism on $\mathscr{P}\cup\{(0,0)\}$. Note that

$$(\varphi_2)^*U=\left(x+y\frac{S(x,y)}{(h^{-1})'(x)}\right)\partial_x+\left(y-x\frac{S(x,y)}{(h^{-1})'(x)}\right)\partial_y,$$

where

$$S(x,y):=r\int_0^\theta(h^{-1})''(r\cos s)\cos s ds\Bigg|_{\left\{r=\sqrt{x^2+y^2},\theta=\arctan(y/x)\right\}}.$$

Since $h'(0)\neq0$, we must show that S is analytic at $(x,y)=(0,0)$. To this end we use that, on account of the claim, $u\longmapsto(h^{-1})''(u)$ is an even function, so we can write $(h^{-1})''(u)=\sum_{i=0}^\infty\beta_iu^{2i}$ for $u\approx0$. Thus, for $r\approx0$,

$$r\int_0^\theta(h^{-1})''(r\cos s)\cos s ds=\sum_{i=0}^\infty\beta_ir^{2i+1}\int_0^\theta\cos^{2i+1}s ds=\sum_{i=0}^\infty\beta_ir^{2i+1}\sin\theta P_i(\sin^2\theta),$$

where P_i is a polynomial of degree i. Thanks to the identity $\sin^2\theta+\cos^2\theta=1$, we can write $P_i(\sin^2\theta)=\hat{P}_i(\cos^2\theta,\sin^2\theta)$ with $\hat{P}_i$ being a homogenous polynomial of degree i. Therefore

$$r\int_0^\theta(h^{-1})''(r\cos s)\cos s ds=\sum_{i=0}^\infty\beta_ir\sin\theta\hat{P}_i(r^2\cos^2\theta,r^2\sin^2\theta)$$

and, consequently, $S(x,y)=y\sum_{i=0}^\infty\beta_i\hat{P}_i(x^2,y^2)$ for $(x,y)\approx(0,0)$. This shows the analyticity of S at the origin and completes the proof of the result. ∎

Now the desired commutators are given in the following result:

Lemma 4.4. *Consider the parameters $\mu_1=(-3,1)$, $\mu_2=(-1/2,0)$ and $\mu_3=(0,1)$ corresponding to the isochronous centers of the family $\{X_\mu,\mu\in\Lambda\}$ in (1). Setting $S(x,y)=(x+1)^2y^2+x(x+2)(x^2+2x+2)$, define*

$$U_1=\tfrac{S(x,y)}{4(x+1)}\partial_x+\tfrac{y(4+S(x,y))}{4(x+1)^2}\partial_y$$
$$U_2=(2+2x-2\sqrt{1+x}+y^2)\partial_x+\tfrac{y}{\sqrt{1+x}}\partial_y$$
$$U_3=x\partial_x+y\partial_y$$

Then, for $i=1,2,3$, U_i is an analytic vector field on $\mathscr{P}_{\mu_i}\cup\{(0,0)\}$ that commutes with X_{μ_i} on $\mathscr{P}_{\mu_i}$.

Proof. The commutators follow by applying Proposition 4.3 and to this end we need the involutions associated to each potential function. As we already mentioned,

$$\sigma_1(x) = -\frac{x}{x+1}, \ \sigma_2(x) = x + 4 - 4\sqrt{x+1} \text{ and } \sigma_3(x) = -x$$

are the involutions for μ_1, μ_2 and μ_3, respectively. By using these functions the result follows after some easy computations which are omitted for the sake of shortness. (Of course, alternatively, the reader may check that $[X_{\mu_i}, U_i] = 0$.) ■

Fix some $\hat{\mu} \in \{\mu_1, \mu_2, \mu_3\}$ and take a germ of analytic curve $\varepsilon \longmapsto \mu(\varepsilon)$ in the parameter space Λ such that $\mu(0) = \hat{\mu}$. Our first goal is to parametrize the set of periodic orbits of $X_{\mu(\varepsilon)}$ for $\varepsilon \approx 0$. To this end we consider the commutator U of $X_{\hat{\mu}}$ given by Lemma 4.4 and we proceed as follows. We choose an arbitrary point $\mathbf{x} \in \mathscr{P}_{\hat{\mu}}$ and we take the solution $\psi(s; \mathbf{x})$ of U such that $\psi(0; \mathbf{x}) = \mathbf{x}$. Then, for some open interval I, $\psi(\,\cdot\,; \mathbf{x}) \colon I \longrightarrow \mathbb{R}^2$ is an analytic transverse section to $X_{\hat{\mu}}$ on $\mathscr{P}_{\hat{\mu}}$. By continuity, this will be also the case for $X_{\mu(\varepsilon)}$ with $\varepsilon \approx 0$. Setting $\xi(s) = \psi(s; \mathbf{x})$ for the sake of shortness, we define $T(s; \varepsilon)$ to be the period of the periodic orbit of $X_{\mu(\varepsilon)}$ passing through the point $\xi(s)$. The function $T(s; \varepsilon)$ is analytic for $\varepsilon \approx 0$ and so we can consider its Taylor's series development at $\varepsilon = 0$,

$$T(s; \varepsilon) = \sum_{i=0}^{\infty} T_i(s)\varepsilon^i.$$

Notice that T_0 is constant because $X_{\mu(\varepsilon)}$ is isochronous for $\varepsilon = 0$. Then $T'(s; \varepsilon) = \sum_{i=1}^{\infty} T_i'(s)\varepsilon^i$. We can now give the fundamental result in order to prove Theorem C. In its statement we use the notation we have just introduced.

Theorem 4.5. *Take $\hat{\mu} \in \{\mu_1, \mu_2, \mu_3\}$ and set $\hat{\mu} = (\hat{q}, \hat{p})$. Let U be the commutator of $X_{\hat{\mu}}$ given by Lemma 4.4 and take a transverse section $\xi \colon I \longrightarrow \mathbb{R}^2$ to $X_{\hat{\mu}}$ on $\mathscr{P}_{\hat{\mu}}$ given by a solution of U. Then there exist analytic functions A_1 and A_2 on I such that:*

(a) *(A_1, A_2) is an ECT-system on I.*

(b) *If $\varepsilon \longmapsto \mu(\varepsilon)$ is a germ of analytic curve in Λ with $\mu(\varepsilon) = (\hat{q} + \kappa_1\varepsilon^\ell + \mathrm{o}(\varepsilon^\ell), \hat{p} + \kappa_2\varepsilon^\ell + \mathrm{o}(\varepsilon^\ell))$ such that $\kappa_1 \neq 0$ or $\kappa_2 \neq 0$, then the period function $T(s; \varepsilon)$ corresponding to the perturbation $X_{\mu(\varepsilon)}$ verifies $T_0' \equiv T_1' \equiv \cdots \equiv T_{\ell-1}' \equiv 0$ and $T_\ell'(s) = \kappa_1 A_1(s) + \kappa_2 A_2(s)$ for all $s \in I$.*

We remark that, for a given $\hat{\mu} \in \{\mu_1, \mu_2, \mu_3\}$, the functions A_1 and A_2 depend only on the commutator U. In particular, they do not depend on the germ of analytic curve chosen. In the previous statement we also use the following definition (we refer the reader to [12] for details):

Definition 4.6. Let $f_0, f_1, \ldots f_{n-1}$ be analytic functions on an open interval $I \subset \mathbb{R}$.

(a) $(f_0, f_1, \ldots f_{n-1})$ is a *complete Chebyshev system* (for short, a CT-system) on I if, for all $k = 1, 2, \ldots n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k - 1$ isolated zeros on I.

(b) $(f_0, f_1, \ldots f_{n-1})$ is a *extended complete Chebyshev system* (for short, a ECT-system) on I if, for all $k = 1, 2, \ldots n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k - 1$ isolated zeros on I counted with multiplicities.

(Let us mention that, in these abbreviations, "T" stands for Tchebycheff, which in some sources is the transcription of the Russian name Chebyshev). □

To obtain an expression of T'_ℓ we shall apply a result of Grau and Villadelprat that appears in [11]. In order to state it, some additional notation must be introduced. Since $X_{\hat{\mu}}$ and U are transverse on $\mathscr{P}_{\hat{\mu}}$, there exist two analytic functions $\alpha = \alpha(x, y; \varepsilon)$ and $\beta = \beta(x, y; \varepsilon)$ such that

$$X_{\mu(\varepsilon)} = \alpha X_{\hat{\mu}} + \beta U.$$

Note that

$$\alpha = \frac{< X_{\mu(\varepsilon)}, U^\perp >}{< X_{\hat{\mu}}, U^\perp >} \text{ and } \beta = \frac{< X_{\hat{\mu}}^\perp, X_{\mu(\varepsilon)} >}{< X_{\hat{\mu}}^\perp, U >},$$

where $<\,,\,>$ stands for the scalar product and $X^\perp$ denotes the orthogonal vector field to X. Let us also denote the k-jet of $X_{\mu(\varepsilon)}$ at $\varepsilon = 0$ by $j^k\big(X_{\mu(\varepsilon)}\big)$. With this notation, by applying [11, Theorem 3.2] we get:

Lemma 4.7. *Let us assume that, for some $k \in \mathbb{N}$, $j^{k-1}(X_{\mu(\varepsilon)})$ has an isochronous center at the origin for all $\varepsilon \approx 0$. Then $T'_0 \equiv T'_1 \equiv \cdots \equiv T'_{k-1} \equiv 0$ and*

$$T'_k(s) = -\int_0^{T_0} U(\alpha_k)|_{(x,y)=\varphi(t;s)}\, dt \quad \text{for all } s \in I,$$

where $\varphi(t; s)$ is the solution of $X_{\hat{\mu}}$ with $\varphi(0; s) = \xi(s)$ and α_k is the kth term of $\alpha(x, y; \varepsilon) = \sum_{i\geqslant 0} \alpha_i(x, y)\varepsilon^i$, the Taylor development of α at $\varepsilon = 0$.

We point out that the assumption in [11, Theorem 3.2] is that there exists an analytic family of diffeomorphisms $\{\Phi_\varepsilon\}$, defined in a neighbourhood of $(0, 0)$, such that Φ_ε linearizes $j^{k-1}(X_{\mu(\varepsilon)})$ for each $\varepsilon \approx 0$. Here we replace it by the assumption that $j^{k-1}(X_{\mu(\varepsilon)})$ has an isochronous center at the origin for all $\varepsilon \approx 0$, which is more easy to verify. The next result shows that both conditions are equivalent:

Lemma 4.8. *Let $\Lambda \subset \mathbb{R}^n$ be an open set and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an analytic family of planar analytic vector fields with center at the origin. Then the center is isochronous for all $\lambda \in \Lambda$ if, and only if, for each $\lambda_0 \in \Lambda$ there exists a neighbourhood $\mathscr{U}$ of λ_0 and an analytic family of analytic diffeomorphisms $\{\Phi_\lambda\}_{\lambda\in\mathscr{U}}$, defined in a neighbourhood of $(0, 0)$, such that Φ_λ linearizes X_λ for each $\lambda \in \mathscr{U}$.*

Proof. First we show that the condition is necessary. Let $\varphi_\lambda(t; p)$ be the solution of X_λ with $\varphi_\lambda(0; p) = p$. In addition, for each $\lambda \in \Lambda$, let $\mathscr{P}_\lambda$ be the period annulus of the center at the origin of X_λ and let T_λ be the period of its periodic orbits. Finally, let $A_\lambda \in M_{2\times 2}$ be the Jacobian matrix of X_λ at the origin. Define

$$\Phi_\lambda(p) := \frac{1}{T_\lambda}\int_0^{T_\lambda} e^{-A_\lambda s}\varphi_\lambda(s; p)ds.$$

One can easily verify that $\Phi_\lambda\big(\varphi_\lambda(t; p)\big) = e^{A_\lambda t}\Phi_\lambda(p)$. Moreover, by applying the variational equations, the linear part of $p \longmapsto \varphi_\lambda(s; p)$ at $p = (0, 0)$ is $e^{A_\lambda s}p$. Consequently the Jacobian matrix of Φ_λ at $p = (0, 0)$ is the identity. Hence, for each $\lambda \in \Lambda$, the map Φ_λ linearizes X_λ in some neighbourhood U_λ of $(0, 0)$. Let us fix $\lambda_0 \in \Lambda$ and take a neighbourhood W of $(0, 0)$ inside $\mathscr{P}_{\lambda_0}$. Then there exists a neighbourhood V of λ_0 such that $W \subset \mathscr{P}_\lambda$ for all $\lambda \in V$. Now the result follows by applying the inverse function theorem to the map $\Phi : W \times V \longrightarrow \mathbb{R}^2 \times V$ given by $\Phi(p, \lambda) := (\Phi_\lambda(p), \lambda)$, which is analytic, thanks to the analytic dependence of solutions on initial conditions and parameters, and satisfies that its Jacobian matrix at $(p, \lambda) = (0, 0, \lambda_0)$ is the identity. This proves the result because the reverse implication is well-known (see [2] for instance). ∎

Lemma 4.7 constitutes the first ingredient in the proof of Theorem 4.5. The second one is a criterion of Grau, Mañosas and Villadelprat [10] that gives a sufficient condition for a collection of Abelian integrals to be an ECT-system. In order to state it precisely some previous definitions must be introduced. Suppose that $H(x, y) = A(x) + B(x)y^{2m}$ is an analytic function in some open subset of the plane that has a local minimum at the origin. Then there exists a punctured neighbourhood $\mathscr{P}$ of the origin foliated by ovals $\gamma_h \subset \{H(x, y) = h\}$. We set $H(0, 0) = 0$ and then the set of ovals γ_h inside $\mathscr{P}$ is parameterized by the energy levels $h \in (0, h_0)$ for some positive h_0. The projection of $\mathscr{P}$ on the x-axis is an interval (x_ℓ, x_r) with

$x_\ell < 0 < x_r$. Under these assumptions A has a zero of even multiplicity at $x = 0$, and it is easy to verify that there exist an analytic involution σ such that

$$A(x) = A\big(\sigma(x)\big) \text{ for all } x \in (x_\ell, x_r).$$

Given a function κ defined on $(x_\ell, x_r) \setminus \{0\}$, we define its σ-*balance* as

$$\mathscr{B}_\sigma\big(\kappa\big)(x) = \kappa(x) - \kappa\big(\sigma(x)\big).$$

Following this notation we can now state the criterion [10, Theorem B] as follows.

Theorem 4.9. *Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on (x_ℓ, x_r), and consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x) y^{2s-1} dx, \;\; i = 0, 1, \ldots, n-1.$$

Let σ be the involution associated to A and define $\ell_i := \mathscr{B}_\sigma\Big(\frac{f_i}{A' B^{\frac{2s-1}{2m}}}\Big)$. If $\big(\ell_0, \ell_1, \ldots, \ell_{n-1}\big)$ is a CT-system on $(0, x_r)$ and $s > m(n-2)$, then $(I_0, I_1, \ldots, I_{n-1})$ is an ECT-system on $(0, h_0)$.

The next result can also be found in [10]. It is very useful in order to apply the previous criterion to a collection of Abelian integrals not verifying the condition $s > m(n-2)$.

Lemma 4.10. *Let γ_h be an oval inside the level curve $\{A(x) + B(x)y^2 = h\}$, and we consider a function F such that F/A' is analytic at $x = 0$. Then, for any $k \in \mathbb{N}$,*

$$\int_{\gamma_h} F(x) y^{k-2} dx = \int_{\gamma_h} G(x) y^k dx,$$

where $G(x) = \frac{2}{k}\left(\frac{BF}{A'}\right)'(x) - \left(\frac{B'F}{A'}\right)(x)$.

Proof of Theorem 4.5. Fix some $\hat{\mu} \in \{\mu_1, \mu_2, \mu_3\}$ and, setting $\hat{\mu} = (\hat{q}, \hat{p})$, take a germ of analytic curve

$$\mu(\varepsilon) = \big(\hat{q} + \kappa_1 \varepsilon^\ell + \mathrm{o}(\varepsilon^\ell), \hat{p} + \kappa_2 \varepsilon^\ell + \mathrm{o}(\varepsilon^\ell)\big) \text{ with } \kappa_1 \neq 0 \text{ or } \kappa_2 \neq 0.$$

We consider the one-parameter perturbation $X_{\mu(\varepsilon)}$ and an easy computation shows that

$$X_{\mu(\varepsilon)} = X_{\hat{\mu}} + Z\varepsilon^\ell + \mathrm{o}(\varepsilon^\ell) \text{ with } Z := \big(\kappa_2 (x+1)^{\hat{p}} - \kappa_1 (x+1)^{\hat{q}}\big) \log(x+1) \partial_y.$$

Hence $j^{\ell-1}\big(X_{\mu(\varepsilon)}\big)$ is isochronous for all $\varepsilon \approx 0$ and, by applying Lemma 4.7, $T_0' \equiv T_1' \equiv \cdots \equiv T_{\ell-1}' \equiv 0$ and

$$T_\ell'(s) = -\int_0^{T_0} U(\alpha_\ell)|_{(x,y)=\varphi(t;s)}\, dt \;\; \text{for all } s \in I,$$

where $\varphi(t; s)$ is the solution of $X_{\hat{\mu}}$ with $\varphi(0; s) = \xi(s)$. Note also that, due to $\alpha = \frac{<X_{\mu(\varepsilon)}, U^\perp>}{<X_{\hat{\mu}}, U^\perp>} = \sum_{i \geqslant 0} \alpha_i \varepsilon^i$, we have $\alpha_\ell = \frac{<Z, U^\perp>}{<X_{\hat{\mu}}, U^\perp>}$.

Recall that $H(x, y) = \frac{1}{2}y^2 + V_{\hat{\mu}}(x)$ is the Hamiltonian associated to $X_{\hat{\mu}}$. Hence, by construction, the solution $\varphi(t; s)$ is inside the energy level $H(x, y) = H(\xi(s)) =: \eta(s)$. Since it is clear that $\eta\colon I \longrightarrow (0, h_0)$ is a diffeomorphism, to prove the result we can consider $J(h) := T_\ell'(\eta^{-1}(h))$ for $h \in (0, h_0)$. More precisely, the result will be proved once we show that there exist analytic functions J_1 and J_2 with (J_1, J_2) being an ECT-system on $(0, h_0)$ and such that $J(h) = \kappa_1 J_1(h) + \kappa_2 J_2(h)$. With this aim in view notice first that

$$J(h) = -\int_{\gamma_h} \frac{U(\alpha_\ell)(x, y)}{y} dx,$$

where as usual γ_h denotes the oval inside the energy level $H(x, y) = h$ for $h \in (0, h_0)$.

We must at this point particularize the proof for each one of the three isochronous centers. We will show in detail the computations for $\hat{\mu} = (-3, 1)$. In this case $\alpha_\ell(x, y) = (-\kappa_1 + \kappa_2(x+1)^4)f(x, y)$ with

$$f(x,y) = \frac{((x+1)^2y^2 + x(x+2)(x^2+2\,x+2))\log(x+1)}{((x+1)^2y^2 + (x^2+2\,x+2)^2(x+1)^2)(y^2+x^2(x+2)^2)}.$$

We then compute $U(\alpha_\ell) = \nabla\alpha_\ell \cdot U$, which yields to

$$\begin{aligned} J(h) = &\frac{\kappa_1}{16h(h+2)} \int_{\gamma_h} \frac{g_3(x)}{(x+1)^2}y^3 + \frac{g_4(x)}{(1+x)^4}y + \frac{g_5(x)}{(x+1)^6 y}dx \\ &- \frac{\kappa_2}{16h(h+2)} \int_{\gamma_h} (1+x)^2y^3 + g_1(x)y + \frac{g_2(x)}{(x+1)^2y}dx, \end{aligned}$$

where

$$\begin{aligned} g_1(x) &= 2(4x+6x^2+4x^3+x^4) + 4\log(x+1), \\ g_2(x) &= (4x+6x^2+4x^3+x^4)(4x+6x^2+4x^3+x^4-4\log(x+1)), \\ g_3(x) &= 4\log(x+1) - 1, \\ g_4(x) &= 4(2x^4+8x^3+12x^2+8x-1)\log(x+1) - (4x+6x^2+4x^3+x^4), \\ g_5(x) &= 4(x+1)^4(4x+6x^2+4x^3+x^4)\log(x+1) - (4x+6x^2+4x^3+x^4)^2. \end{aligned}$$

Here we used that, due to $V_{\mu_1}(x) = \frac{x^2(x+2)^2}{2(x+1)^2}$, we have $y^2 + \frac{x^2(x+2)^2}{(x+1)^2} = 2h$ for all $(x, y) \in \gamma_h$. Next we apply twice Lemma 4.10 to get

$$J(h) = \frac{-1}{12h(h+2)} \left(\kappa_1 \int_{\gamma_h} \frac{7y^3dx}{3(x+1)^2} + \kappa_2 \int_{\gamma_h} 8(x+1)^2y^3dx \right).$$

The projection of the period annulus $\mathscr{P}_{\mu_1}$ on the x-axis is $(-1, +\infty)$ and, according to Lemma 4.4, the involution associated to V_{μ_1} is $\sigma(x) = -\frac{x}{x+1}$. Next, setting $f_0(x) = \frac{1}{(x+1)^2}$ and $f_1(x) = (x+1)^2$, we will apply Theorem 4.9 with $A = V_{\mu_1}$, $B = \frac{1}{2}$, $m = 1$ and $s = n = 2$. Following its notation, we obtain

$$\ell_0(x) = \frac{8(x+1)}{x(x+2)} \text{ and } \ell_1(x) = \frac{8((x+1)^4 - 2x - x^2)}{x(x+1)(x+2)}.$$

Note that $\ell_0(x) \neq 0$ for all $x \in (0, +\infty)$. One can also verify that the wronskian of ℓ_0 and ℓ_1 does not vanish on $(0, +\infty)$ neither. Then by using a well-known result, see [10, Lemma 2.3] for instance, we can assert that (ℓ_0, ℓ_1) is an ECT-system on $(0, +\infty)$. Therefore, setting

$$J_1(h) = \frac{-7}{36h(h+2)} \int_{\gamma_h} \frac{y^3dx}{(x+1)^2} \text{ and } J_2(h) = \frac{-2}{3h(h+2)} \int_{\gamma_h} (x+1)^2y^3dx,$$

by applying Theorem 4.9 we can conclude that (J_1, J_2) is an ECT-system on $(0, h_0)$. Finally, on account of $T'_\ell(\eta^{-1}(h)) = J(h) = \kappa_1 J_1(h) + \kappa_2 J_2(h)$, the result follows for μ_1 taking $A_i(s) = J_i(\eta(s))$ for $i = 1, 2$.

Since the proof for $\mu_2 = (-1/2, 0)$ and $\mu_3 = (0, 1)$ follows exactly the same way, we omit it here for the sake of brevity. ■

Proof of Theorem C. The fact that the center of the differential system in (1) is isochronous if, and only if, $\mu \in \{\mu_1, \mu_2, \mu_3\}$ follows from Theorem 3.3. Fix some $\hat{\mu} \in \{\mu_1, \mu_2, \mu_3\}$ and take a germ of analytic curve $\varepsilon \longmapsto \mu(\varepsilon)$ in Λ with $\mu(0) = \hat{\mu}$. Let us set $\hat{\mu} = (\hat{q}, \hat{p})$ and note that there exists $\ell \in \mathbb{N}$ such that

$$\mu(\varepsilon) = \big(\hat{q} + \kappa_1\varepsilon^\ell + \mathrm{o}(\varepsilon^\ell), \hat{p} + \kappa_2\varepsilon^\ell + \mathrm{o}(\varepsilon^\ell)\big) \text{ with } \kappa_1 \neq 0 \text{ or } \kappa_2 \neq 0.$$

Then, by applying Theorem 4.5, the period function $T(s; \varepsilon)$ corresponding to the perturbation $X_{\mu(\varepsilon)}$ verifies $T'_0 \equiv T'_1 \equiv \cdots \equiv T'_{\ell-1} \equiv 0$ and $T'_\ell(s) = \kappa_1 A_1(s) + \kappa_2 A_2(s)$ for all $s \in I$. Moreover (A_1, A_2) is an ECT-system on I. Accordingly, by applying the Implicit Function Theorem, we can assert that, for each $\varepsilon \approx 0$, $T'(s;\varepsilon)$ has at most one zero on I counted with multiplicity. This proves that $\mathrm{Crit}\big((\mathscr{P}_{\hat{\mu}}, X_{\hat{\mu}}), X_{\mu(\varepsilon)}\big) \leqslant 1$.

Finally, in order to show that there exists a perturbation of $X_{\hat{\mu}}$ for which this upper bound is achieved, it suffices to consider

$$\mu(\varepsilon) = \big(\hat{q} + \kappa_1\varepsilon + \mathrm{o}(\varepsilon), \hat{p} + \kappa_2\varepsilon + \mathrm{o}(\varepsilon)\big)$$

taking κ_1 and κ_2 such that $\kappa_1 A_1(\hat{s}) + \kappa_2 A_2(\hat{s}) = 0$ for some $\hat{s} \in I$, i.e., $-\frac{\kappa_1}{\kappa_2} \in \big(\frac{A_2}{A_1}\big)(I)$. Here we use of course that A_1 and A_2 do not depend on the particular curve $\varepsilon \longmapsto \mu(\varepsilon)$ chosen. This proves the result. ■

References

[1] T.R. Blows and N.G. Lloyd, *The number of limit cycles of certain polynomial differential equations,* Proc. Roy. Soc. Edinburgh Sect. A **98** (1984), 215–239.

[2] J. Chavarriga and M. Sabatini, *A Survey of Isochronous Centers,* Qualitative Theory of Dynamical Systems **1** (1999), 1–70.

[3] C. Chicone, *The monotonicity of the period function for planar Hamiltonian vector fields,* J. Differential Equations **69** (1987), 310–321.

[4] C. Chicone, *Geometric methods of two-point nonlinear boundary value problem,* J. Differential Equations **72** (1988), 360–407.

[5] C. Chicone and M. Jacobs, *Bifurcation of critical periods for plane vector fields,* Trans. Amer. Math. Soc. **312** (1989), 433–486.

[6] A. Cima, F. Mañosas and J. Villadelprat, *Isochronicity for several classes of Hamiltonian systems,* J. Differential Equations **157** (1999), 373–413.

[7] J.P. Françoise and C.C. Pugh, *Keeping track of limit cycles,* J. Differential Equations **65** (1986), 139–157.

[8] A. Garijo and J. Villadelprat, *Algebraic and analytical tools for the study of the period function,* J. Differential Equations **254** (2014), 2464–2484.

[9] L. Gavrilov, *Cyclicity of period annuli and principalization of Bautin ideals,* Ergod. Th. & Dynam. Sys., **28**, (2008) 1497–1507.

[10] M. Grau, F. Mañosas and J. Villadelprat, *A Chebyshev Criterion for Abelian Integrals,* Transactions of the American Mathematical Society, **363** (2011) 109–129.

[11] M. Grau and J. Villadelprat, *Bifurcation of critical periods from Pleshkan's isochrones,* J. London Math. Soc. (2), **81** (2010) 142–160.

[12] S. Karlin and W. Studden, "Tchebycheff systems: with applications in analysis and statistics", Interscience Publishers, 1966.

[13] W.S. Loud, *Behaviour of the period of solutions of certain plane autonomous systems near centers,* Contrib. Differential Equations **3** (1964), 21–36.

[14] F. Mañosas, D. Rojas and J. Villadelprat, *The criticality of centers of potential systems at the outer boundary,* to appear in J. Differential Equations, (2015).

[15] F. Mañosas and J. Villadelprat, *Criteria to bound the number of critical periods*, J. Differential Equations, **246** (2009) 2415–2433.

[16] P. Mardešić, D. Marín, and J. Villadelprat, *The period function of reversible quadratic centers*, J. Differential Equations, **224** (2006) 120–171.

[17] P. Mardešić, L. Moser-Jauslin and C. Rousseau, *Darboux linearization and isochronous centers with a rational first integral,* J. Differential Equations **134** (1997), 216–268.

[18] Y. Miyamoto and K. Yagasaki, *Monotonicity of the first eigenvalue and the global bifurcation diagram for the branch of interior peak solutions,* J. Differential Equations, **254** (2013) 342–367.

[19] L. M. Perko, *Rotated vector fields.* J. Differential Equations **103**, no. 1 (1993), 127–145.

[20] R. Roussarie, *Bifurcation of Planar Vector Fields and Hilbert's Sixteenth Problem,* Progr. in Math., vol. 164, Birkhüser Verlag, Basel, 1998.

[21] C. Rousseau and B. Toni, *Local bifurcations of critical periods in the reduced Kukles system,* Can. J. Math. **49** (1997), 338–358.

[22] R. Schaaf, *A class of Hamiltonian systems with increasing periods,* J. Reine Angew. Math. **363** (1985), 96–109.

[23] J. Smoller and A. Wasserman, *Global bifurcation of steady-state solutions,* J. Differential Equations **39** (1981), 269–290.

[24] K. Yagasaki, *Monotonicity of the period function for* $u''-u+u^p = 0$ *with* $p \in \mathbb{R}$ *and* $p > 1$*,* J. Differential Equations, **255** (2013) 1988–2001.

[25] Yan-Qian Ye et al., "Theory of limit cycles," Transl. Math. Monographs, vol. 66, Amer. Math. Soc., Providence, R.I., 1984.