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ON THE INTEGRABILITY OF THE 5-DIMENSIONAL LORENZ SYSTEM FOR THE GRAVITY-WAVE ACTIVITY

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ABSTRACT. We consider the 5-dimensional Lorenz system

$$\begin{aligned} U' &= -VW + bVZ, \\ V' &= UW - bUZ, \\ W' &= -UV, \\ X' &= -Z, \\ Z' &= bUV + X \end{aligned}$$

where $b \in \mathbb{R}\backslash\{0\}$ and the derivative is with respect to T. This system describes coupled Rosby waves and gravity waves. First we prove that the number of functionally independent global analytic first integrals of this differential system is two. This solves an open question in the paper *On the analytic integrability of the* 5*-dimensional Lorenz system for the gravity-wave activity*, Proc. Amer. Math. Soc. **142** (2014), 531–537, where it was proved that this number was two or three. Moreover, we characterize all the invariant algebraic surfaces of the system, and additionally we show that it has only two functionally independent Darboux first integrals.

1. Introduction and statement of the main results

E.N. Lorenz constructed in [9] a 5-dimensional 1-parameter differential system in $\mathbb{R}^5$ which describes coupled Rosby waves and gravity waves:

$$\begin{aligned} U' &= -VW + bVZ, \\ V' &= UW - bUZ, \\ W' &= -UV, \\ X' &= -Z, \\ Z' &= bUV + X. \end{aligned} \tag{1}$$

He studied its slow manifolds and in this paper we are interested in studying its global analytic integrability, its algebraic invariant surfaces and its Darboux first integrals. More precisely, we want to know *what is the maximal number of functionally independent either global analytic or Darboux first integrals that system* (1) *can exhibit?*. This question has been considered for many other differential equations and other classes of first integrals not necessarily analytic or Darboux; see for instance [6, 7, 10] and the references therein.

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Let Ω be an open subset of $\mathbb{R}^5$ invariant by the flow of the differential system (1). A *first integral* of the differential system (1) in Ω is a C^1-function H satisfying

$$\begin{aligned}(-VW+bVZ)\frac{\partial H}{\partial U}+(UW-bUZ)\frac{\partial H}{\partial V}\\ -UV\frac{\partial H}{\partial W}-Z\frac{\partial H}{\partial X}+(bUV+X)\frac{\partial H}{\partial Z}\equiv 0, \quad \text{in } \Omega.\end{aligned} \tag{2}$$

Let $H_1\colon \Omega_1 \to \mathbb{R}$ and $H_2\colon \Omega_2 \to \mathbb{R}$ be two first integrals of the 5-dimensional Lorenz system (1). They are *functionally independent* in $\Omega_1\cap\Omega_2$ if their gradients are linearly independent over a full Lebesgue measure subset of $\Omega_1\cap\Omega_2$.

From [9] we know that the 5-dimensional Lorenz system (1) has the polynomial first integrals

$$H_1 = U^2+V^2 \quad \text{and} \quad H_2 = V^2+W^2+X^2+Z^2. \tag{3}$$

To study the existence of Darboux first integrals we will use the well-known Darboux theory of integrability. The Darboux theory of integrability in dimension 5 is based on the existence of invariant algebraic hypersurfaces (or Darboux polynomials). For more details see [2, 3] and [5]. This theory is one of the best theories for studying the existence of first integrals for the polynomial differential systems.

A *Darboux polynomial* of system (1) is a polynomial $f\in\mathbb{C}[U,V,W,X,Z]\setminus\mathbb{C}$ such that

$$\begin{aligned}(-VW+bVZ)\frac{\partial f}{\partial U}+(UW-bUZ)\frac{\partial f}{\partial V}\\ -UV\frac{\partial f}{\partial W}-Z\frac{\partial f}{\partial X}+(bUV+X)\frac{\partial f}{\partial Z}=Kf,\end{aligned} \tag{4}$$

for some polynomial K called the *cofactor* of f and with degree at most one.

Note that $f=0$ is an *invariant algebraic hypersurface* for the flow of system (1). A *polynomial first integral* (a first integral which is a polynomial) is a Darboux polynomial with zero cofactor. We recall that if $f\notin\mathbb{R}[U,V,W,X,Z]$ is a Darboux polynomial then there exists another Darboux polynomial $\bar f$ (the conjugate of f) with cofactor $\bar K$ (the conjugate of K).

An *exponential factor* $F=F(U,V,W,X,Z)$ of system (1) is a function of the form $F=\exp(g_0/g_1)\notin\mathbb{C}$ with $g_0,g_1\in\mathbb{C}[U,V,W,X,Z]$ coprime satisfying that

$$\begin{aligned}(-VW+bVZ)\frac{\partial F}{\partial U}+(UW-bUZ)\frac{\partial F}{\partial V}\\ -UV\frac{\partial F}{\partial W}-Z\frac{\partial F}{\partial X}+(bUV+X)\frac{\partial F}{\partial Z}=LF,\end{aligned}$$

for some polynomial $L=L(U,V,W,X,Z)$ called the *cofactor* of F and with degree at most one. We recall that if $F\notin\mathbb{R}[U,V,W,X,Z2]$ is an exponential factor then there exists another exponential factor $\bar F$ (the conjugate of F) with cofactor $\bar L$ (the conjugate of L).

A *Darboux first integral* G of system (1) is a first integral of the form

$$G{=}f_1^{\lambda_1}\cdots f_p^{\lambda_p}F_1^{\mu_1}\cdots F_q^{\mu_q}, \tag{5}$$

where $f_1,\ldots,f_p$ are Darboux polynomials and $F_1,\ldots,F_q$ are exponential factors and $\lambda_j,\mu_k\in\mathbb{C}$ for $j=1,\ldots,p$ and $k=1,\ldots,q$. Note that a Darboux first integral is always a real function due to the fact that if there are complex Darboux polynomials or complex exponential factors always also appear their conjugates.

In [8] the authors studied system (1) from the view point of the analytic integrability. We recall that when H is an analytic function we say that H is an *analytic first integral* and when the domain of definition is $\mathbb{R}^5$ then H is called a *global analytic first integral*. Additionally, they show that when $b = 0$ the system is completely integrable with four functionally independent first integrals (not necessarily analytic), and that when $b \neq 0$ the number of functionally independent global analytic first integrals is either two or three.

In this paper first we prove that the differential system (1) has only two global analytic first integrals. After we characterize the Darboux first integrals of system (1). We recall that the class of Darboux first integrals and the class of global analytic first integrals have intersection but they are different classes. We also characterize the invariant algebraic surfaces and the so-called exponential factors.

Our main result is the following one.

Theorem 1. *The following statements hold for the differential system* (1) *with* $b \neq 0$.

(a) *Any global analytic first integral must be a function in the variables* H_1 *and* H_2 *given in* (3).
(b) *It has two irreducible Darboux polynomials* $U+iV$ *and* $U-iV$ *with non–zero cofactors* $i(W-bZ)$ *and* $-i(W-bZ)$*, respectively.*
(c) *It has only two functionally independent Darboux first integrals.*

Theorem 1 (a) completes the characterization of the global analytic first integrals of system (1) which was unfinished in [8]. Theorem 1 (b) characterizes all its invariant algebraic hypersurfaces, and Theorem 1 (c) characterizes all the Darboux first integrals of system (1).

Theorem 1 (a) is proved in section 2. Theorem 1 (b) is proved in section 3 and Theorem 1 (c) is proved in section 4.

2. Proof of Theorem 1(a)

We prove the following proposition which is exactly the statement (a) of Theorem 1.

Proposition 2. *Any global analytic first integral must be a function in the variables* $H_1 = U^2 + V^2$ *and* $H_2 = V^2 + W^2 + X^2 + Y^2$.

Proof. Let H be an analytic first integral of system (1) with $b \neq 0$. Then by definition we have that H must satisfy (2). We expand H in Taylor series

$$H = \sum_{j=m}^{\infty} H_j(U, V, W, X, Z)$$

where $m \geq 1$ is a positive integer, and H_j for $j = m, m+1, \dots,$ are homogeneous polynomials of degree j. Comparing the homogeneous polynomials in (2) of the same degree, we get

$$-Z\frac{\partial H_m}{\partial X} + X\frac{\partial H_m}{\partial Z} = 0, \tag{6}$$

$$\begin{aligned} -Z\frac{\partial H_{j+1}}{\partial X} + X\frac{\partial H_{j+1}}{\partial Z} &= V(W-bZ)\frac{\partial H_j}{\partial U} - U(W-bZ)\frac{\partial H_j}{\partial V} \\ &\quad + UV\frac{\partial H_j}{\partial W} - bUV\frac{\partial H_j}{\partial Z}, \end{aligned} \tag{7}$$

for $j = m, m+1, \ldots$. The characteristic equation associated with the linear partial differential equation (6) has the first integral $X^2 + Z^2$, so by the method of characteristic curves for solving linear partial differential equations we get that the general solution of (6) is

$$H_m(U,V,W,X,Z) = \tilde{G}_m(A,U,V,W),$$

where $\tilde{G}_m$ must be a polynomial function in its variables because H_m is a homogeneous polynomial of degree m in the variables U, V, W, X, Z.

For $j \geq m$ since X^2+Z^2 is a first integral of the characteristic equation associated with $-Z\partial H_{j+1}/\partial X + X\partial H_{j+1}/\partial Z = 0$ we make the change of variables

$$A = X^2 + Z^2, \quad Z = Z \tag{8}$$

Then equation (7) becomes the ordinary differential equation

$$\begin{aligned} \sqrt{A - Z^2}\frac{d\tilde{H}_{j+1}}{dZ} &= V(W - bZ)\frac{\partial \tilde{H}_j}{\partial U} - U(W - bZ)\frac{\partial \tilde{H}_j}{\partial V} \\ &\quad + UV\frac{\partial \tilde{H}_j}{\partial W} - 2bUVZ\frac{\partial \tilde{H}_j}{\partial A}, \end{aligned} \tag{9}$$

where for $j \geq m$, $\tilde{H}_j$ is H_j written in the variables A, U, V, W, Z instead of the variables U, V, W, X, Z. Note that for $j = m$ we have $\tilde{H}_m = \tilde{G}_m(A,U,V,W)$ and

$$\begin{aligned} \frac{d\tilde{H}_{m+1}}{dZ} &= \bigg(V(W - bZ)\frac{\partial \tilde{G}_m}{\partial U} - U(W - bZ)\frac{\partial \tilde{G}_m}{\partial V} + UV\frac{\partial \tilde{G}_m}{\partial W} \\ &\quad - 2bUVZ\frac{\partial \tilde{G}_m}{\partial A}\bigg)\frac{1}{\sqrt{A - Z^2}} = \frac{s_1 + s_2 Z}{\sqrt{A - Z^2}}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} s_1 &= s_1(U,V,W,A) = VW\frac{\partial \tilde{G}_m}{\partial U} - UW\frac{\partial \tilde{G}_m}{\partial V} + UV\frac{\partial \tilde{G}_m}{\partial W}, \\ s_2 &= s_2(U,V,W,A) = -bV\frac{\partial \tilde{G}_m}{\partial U} + bU\frac{\partial \tilde{G}_m}{\partial V} - 2bUV\frac{\partial \tilde{G}_m}{\partial A}. \end{aligned}$$

Integrating this ordinary differential equation with respect to Z, we get

$$\tilde{H}_{m+1}(A,U,V,W,Z) = s_1 \arctan\frac{Z}{\sqrt{A - Z^2}} - s_2\sqrt{A - Z^2} + \tilde{G}_{m+1}(A,U,V,W),$$

where $\tilde{G}_{m+1}$ is an integrating constant with respect to Z. Since $H_{m+1} = H_{m+1}(U,V,W,X,Z)$ is a homogeneous polynomial of degree $m+1$, we must have $s_1 = 0$, that is

$$VW\frac{\partial \tilde{G}_m}{\partial U} - UW\frac{\partial \tilde{G}_m}{\partial V} + UV\frac{\partial \tilde{G}_m}{\partial W} = 0. \tag{11}$$

The characteristic equations associated with this last partial differential equation (11) is $g(B,C)$, with $B = U^2+V^2$, $C = V^2+W^2$ and g any continuous differentiable function. This forces that

$$H_m(U,V,W,X,Z) = \tilde{G}_m(A,U,V,W) = R_m(A,B,C),$$

with R_m a homogeneous polynomial in the variables A, B, C. So m must be even. Then s_2 becomes

$$s_2 = -2bUV\Big(\frac{\partial R_m}{\partial A} - \frac{\partial R_m}{\partial C}\Big) = -2bUV\tilde{S}_m, \quad \tilde{S}_m = \tilde{S}_m(A,B,C) = \frac{\partial R_m}{\partial A} - \frac{\partial R_m}{\partial C}.$$

Then

$$\tilde{H}_{m+1} = 2bUV\sqrt{A-Z^2}\tilde{S}_m + \tilde{G}_{m+1}(A,U,V,W). \tag{12}$$

Equation (9) with $j = m+1$ becomes

$$\begin{aligned}
\sqrt{A-Z^2}\frac{d\tilde{H}_{m+2}}{dZ} &= V(W-bZ)\frac{\partial \tilde{H}_{m+1}}{\partial U} - U(W-bZ)\frac{\partial \tilde{H}_{m+1}}{\partial V} + UV\frac{\partial \tilde{H}_{m+1}}{\partial W} \\
&\quad - 2bUVZ\frac{\partial \tilde{H}_{m+1}}{\partial A} \\
&= s_3 + s_4 Z + s_5\sqrt{A-Z^2} + s_6 Z\sqrt{A-Z^2} - \frac{Z}{\sqrt{A-Z^2}}s_7,
\end{aligned}$$

where

$$\begin{aligned}
s_3 &= s_3(U,V,W,A) = VW\frac{\partial \tilde{G}_{m+1}}{\partial U} - UW\frac{\partial \tilde{G}_{m+1}}{\partial V} + UV\frac{\partial \tilde{G}_{m+1}}{\partial W}, \\
s_4 &= s_4(U,V,W,A) = -bV\frac{\partial \tilde{G}_{m+1}}{\partial U} + bU\frac{\partial \tilde{G}_{m+1}}{\partial V} - 2bUV\frac{\partial \tilde{G}_{m+1}}{\partial A}, \\
s_5 &= s_5(U,V,W,A) = VW\Big(2bV\tilde{S}_m + 2bUV\frac{\partial \tilde{S}_m}{\partial U}\Big) - UW\Big(2bU\tilde{S}_m + 2bUV\frac{\partial \tilde{S}_m}{\partial V}\Big) \\
&\quad + 2bU^2V^2\frac{\partial \tilde{S}_m}{\partial W}, \\
s_6 &= s_6(U,V,W,A) = -bV\Big(2bV\tilde{S}_m + 2bUV\frac{\partial \tilde{S}_m}{\partial U}\Big) + bU\Big(2bU\tilde{S}_m + 2bUV\frac{\partial \tilde{S}_m}{\partial V}\Big) \\
&\quad - 4b^2U^2V^2\frac{\partial \tilde{S}_m}{\partial A}, \\
s_7 &= s_7(U,V,W,A) = -4b^2U^2V^2\tilde{S}_m.
\end{aligned}$$

Solving it we get

$$\begin{aligned}
\tilde{H}_{m+2} &= s_3 \arctan\frac{Z}{\sqrt{A-Z^2}} - s_4\sqrt{A-Z^2} + s_5 Z + \frac{s_6}{2}Z^2 \\
&\quad + \frac{s_7}{2}\log(Z^2 - A) + \tilde{G}_{m+2}(A,U,V,W),
\end{aligned} \tag{13}$$

where $\tilde{G}_{m+2}$ is an integrating constant with respect to Z. Since the polynomial $H_{m+2} = H_{m+2}(U,V,W,X,Z)$ is homogeneous of degree $m+2$, we must have $s_3 = s_7 = 0$. From $s_3 = 0$, as before for $s_1 = 0$, we get that $\tilde{G}_{m+1}$ must satisfy equation (11) with $\tilde{G}_{m+1}$ instead of $\tilde{G}_m$, and so $\tilde{G}_{m+1} = R_{m+1}(A,B,C)$, being R_{m+1} a homogeneous polynomial of degree $m+1$. From $s_7 = 0$ we must have $\tilde{S}_m = 0$. We write

$$R_m(A,B,C) = \sum_{l=0}^{m}\sum_{k=0}^{m-l} a_{k,l}B^l A^k C^{m-l-k}. \tag{14}$$

Hence

$$\begin{aligned}\tilde S_m &= \frac{\partial R_m}{\partial A} - \frac{\partial R_m}{\partial C}\\ &= \sum_{l=0}^{m} B^l \Big(\sum_{k=0}^{m-l} k a_{k,l} A^{k-1} C^{m-l-k} - \sum_{k=0}^{m-l} (m-l-k) a_{k,l} A^k C^{m-l-k-1} \Big)\\ &= \sum_{l=0}^{m} B^l \Big(\sum_{k=0}^{m-l-1} \big((k+1) a_{k+1,l} - (m-l-k) a_{k,l}\big) A^k C^{m-l-k-1} \Big) = 0,\end{aligned}$$

and so for each $l = 0, \dots, m$ we must have

$$\sum_{k=0}^{m-l-1} \big((k+1) a_{k+1,l} - (m-l-k) a_{k,l}\big) A^k C^{m-l-k-1} = 0.$$

That is

$$a_{k,l} = a_{k-1,l} \frac{m-l-k+1}{k} \quad \text{for} \quad k = 1, \dots, m-l$$

which yields

$$a_{k,l} = a_{0,l} \binom{m-l}{k}.$$

Hence, it follows from (15) that

$$\begin{aligned} R_m(A,B,C) &= \sum_{l=0}^{m} B^l \sum_{k=0}^{m-l} a_{0,l} \binom{m-l}{k} A^k C^{m-l-k}\\ &= \sum_{l=0}^{m} B^l a_{0,l} (A+C)^{m-l} = R_m(A+C, B)\\ &= R_m(V^2+W^2+X^2+Z^2, U^2+V^2).\end{aligned} \tag{15}$$

Moreover, from (12) using that $\tilde S_m = 0$ and that $\tilde G_{m+1} = R_{m+1}(A,B,C)$ we get that

$$\tilde H_{m+1}(A,U,V,W,Z) = R_{m+1}(A,B,C).$$

Since H_{m+1} is a homogeneous polynomial of odd degree, we must have

$$\tilde G_{m+1} \equiv 0,$$

because $\tilde G_{m+1}$ cannot be of odd degree by its expression. Then from (13) using that $\tilde S_m = \tilde G_{m+1} = 0$, we obtain

$$\tilde H_{m+2} = \tilde G_{m+2}(A,U,V,W), \quad \text{and} \quad H_{m+2} = G_{m+2}(X^2+Z^2, U, V, W),$$

where H_{m+2} is a homogeneous polynomial of degree $m+2$.

Proceeding by induction we can show that

$$\begin{aligned} \tilde G_{m+2k}(A,U,V,W,Z) &= R_{m+2k}(A+C, B)\\ &= R_{m+2k}(V^2+W^2+X^2+Z^2, U^2+V^2),\\ \tilde G_{m+2k-1}(A,U,V,W,Z) &= 0,\end{aligned}$$

for $k = 1, \dots,$ where R_{m+2k} are homogeneous functions in $A+C, B$. This proves the theorem. □

3. Proof of Theorem 1(b)

To prove Theorem 1(b) we need to characterize the Darboux polynomials with non–zero cofactor.

We introduce the new variables $Y_1 = U + iV$ and $Y_2 = U - iV$ and rewrite system (1) in these new variables as

$$
\begin{aligned}
Y_1' &= iY_1(W - bZ),\\
Y_2' &= -iY_2(W - bZ),\\
W' &= \frac{i}{4}(Y_1^2 - Y_2^2),\\
X' &= -Z,\\
Z' &= -\frac{ib}{4}(Y_1^2 - Y_2^2) + X.
\end{aligned}
\tag{16}
$$

The proof of Theorem 1(b) can be reformulated as follows.

Theorem 3. *The unique irreducible Darboux polynomials of system* (16) *with non–zero cofactor are* Y_1 *and* Y_2 *with cofactors* $i(W - bZ)$ *and* $-i(W - bZ)$*, respectively.*

Let f be a Darboux polynomial of system (16) with non–zero cofactor. Then f satisfies

$$
\begin{aligned}
&iY_1(W - bZ)\frac{\partial f}{\partial Y_1} - iY_2(W - bZ)\frac{\partial f}{\partial Y_2} + \Big(\frac{i}{4}(Y_1^2 - Y_2^2)\Big)\frac{\partial f}{\partial W}\\
&\quad - Z\frac{\partial f}{\partial X} + \Big(-\frac{ib}{4}(Y_1^2 - Y_2^2) + X\Big)\frac{\partial f}{\partial Z} = Kf,
\end{aligned}
\tag{17}
$$

where $K = \alpha_0 + \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 W + \alpha_4 X + \alpha_5 Z$ with $\alpha_i \in \mathbb{C}$ for $i = 0, \ldots, 5$.

We write $K = \alpha_0 + \alpha_1 Y_1 + \alpha_2 Y_2 + K_1$ where $K_1 = \alpha_3 W + \alpha_4 X + \alpha_5 Z$.

We separate the proof into the next two propositions.

Proposition 4. *The unique irreducible Darboux polynomials of system* (16) *with cofactor* $K_1 \neq 0$ *are* Y_1 *and* Y_2 *with cofactors* $i(W - bZ)$ *and* $-i(W - bZ)$*, respectively.*

Proof. It is easy to check that the unique Darboux polynomials of system (16) of degree one with $K_1 \neq 0$ are Y_1 and Y_2 with cofactors $i(W - bZ)$ and $-i(W - bZ)$, respectively. We will see that they are the unique irreducible ones.

Let $f = f(Y_1, Y_2, W, X, Z)$ be an irreducible Darboux polynomial of system (16) with degree $n \geq 2$ and with cofactor $K_1 \neq 0$. We consider two different cases.

Case 1*:* $K_1 \neq i\alpha(W - bZ)$ *for any* $\alpha > 0$. We restrict system (16) to $Y_2 = 0$ that is we consider system

$$
\begin{aligned}
Y_1' &= iY_1(W - bZ),\\
W' &= \frac{i}{4}Y_1^2,\\
X' &= -Z,\\
Z' &= -\frac{ib}{4}Y_1^2 + X.
\end{aligned}
\tag{18}
$$

Let $g = g(Y_1, W, X, Z) = f(Y_1, 0, W, X, Z)$ that is, the polynomial f restricted to $Y_2 = 0$, so g satisfies

$$iY_1(W-bZ)\frac{\partial g}{\partial Y_1} + \frac{i}{4}Y_1^2\frac{\partial g}{\partial W} - Z\frac{\partial g}{\partial X} + \Big(-\frac{ib}{4}Y_1^2 + X\Big)\frac{\partial g}{\partial Z} = (\alpha_0 + \alpha_1 Y_1 + K_1)g.$$

We write g as sum of its homogeneous parts as $g = \sum_{i=0}^n g_i$ where $g_i = g_i(Y_1, W, X, Z)$ is a homogeneous polynomial of degree i. Clearly, g_n satisfies

$$iY_1(W-bZ)\frac{\partial g_n}{\partial Y_1} + \frac{i}{4}Y_1^2\frac{\partial g_n}{\partial W} - \frac{ib}{4}Y_1^2\frac{\partial g_n}{\partial Z} = (\alpha_1 Y_1 + K_1)g_n. \tag{19}$$

We consider two subcases.

Subcase 1.1*: g_n is not divisible by Y_1.* In this case if we restrict g_n to $Y_1 = 0$ and denote it by $\bar{g}_n$ then $\bar{g}_n \neq 0$. Moreover $\bar{g}_n$ satisfies (19) restricted to $Y_1 = 0$. So $\bar{g}_n$ satisfies $0 = K_1\bar{g}_n$, so $\bar{g}_n = 0$ which is not possible.

Subcase 1.2*: g_n is divisible by Y_1.* In this case we can write $g_n = Y_1^\ell h_n$ with $1 \le \ell \le n$ and h_n is a homogeneous polynomial of degree $n - \ell$. If we restrict h_n to $Y_1 = 0$ and denote it by $\bar{h}_n$ then $\bar{h}_n \neq 0$ and satisfies, after simplifying by Y_1^ℓ, we get

$$iY_1(W-bZ)\frac{\partial h_n}{\partial Y_1} + \frac{i}{4}Y_1^2\frac{\partial h_n}{\partial W} - \frac{ib}{4}Y_1^2\frac{\partial h_n}{\partial Z} = (\alpha_1 Y_1 + K_1 - i\ell(W-bZ))h_n. \tag{20}$$

Now if we restrict h_n to $Y_1 = 0$ and denote it by $\bar{h}_n$ then $\bar{h}_n \neq 0$. Moreover $\bar{h}_n$ satisfies (20) restricted to $Y_1 = 0$. So $\bar{h}_n$ satisfies

$$0 = (K_1 - i\ell(W-bZ))\bar{h}_n,$$

and by assumptions $K_1 - i\ell(W-bZ) \neq 0$ which yields $\bar{g}_n = 0$ which is not possible.

Case 2*: $K_1 = i\alpha(W-bZ)$ for some $\alpha > 0$.* We restrict system (16) to $Y_1 = 0$ that is we consider system

$$\begin{aligned} Y_2' &= -iY_2(W-bZ), \\ W' &= -\frac{i}{4}Y_2^2, \\ X' &= -Z, \\ Z' &= \frac{ib}{4}Y_2^2 + X. \end{aligned} \tag{21}$$

Let $g = g(Y_2, W, X, Z) = f(0, Y_2, W, X, Z)$ that is, the polynomial f restricted to $Y_1 = 0$, so g satisfies

$$-iY_2(W-bZ)\frac{\partial g}{\partial Y_2} - \frac{i}{4}Y_2^2\frac{\partial g}{\partial W} - Z\frac{\partial g}{\partial X} + \Big(\frac{ib}{4}Y_2^2 + X\Big)\frac{\partial g}{\partial Z} = (\alpha_0 + \alpha_2 Y_2 + K_1)g.$$

We write g as sum of its homogeneous parts as $g = \sum_{i=0}^n g_i$ where $g_i = g_i(Y_1, W, X, Z)$ is a homogeneous polynomial of degree i. Clearly, g_n satisfies

$$-iY_2(W-bZ)\frac{\partial g_n}{\partial Y_2} - \frac{i}{4}Y_2^2\frac{\partial g_n}{\partial W} + \frac{ib}{4}Y_2^2\frac{\partial g_n}{\partial Z} = (\alpha_2 Y_2 + K_1)g_n. \tag{22}$$

We consider two subcases.

Subcase 2.1*: g_n is not divisible by Y_2.* In this case if we restrict g_n to $Y_2 = 0$ and denote it by $\bar{g}_n$ then $\bar{g}_n \neq 0$. Moreover $\bar{g}_n$ satisfies (22) restricted to $Y_2 = 0$. So $\bar{g}_n$ satisfies $0 = K_1\bar{g}_n$ and so $\bar{g}_n = 0$ which is not possible.

Subcase 2.2: *g_n is divisible by Y_2.* In this case we can write $g_n = Y_2^{\ell} h_n$ with $1 \le \ell \le n$ and h_n is a homogeneous polynomial of degree $n-\ell$. If we restrict h_n to $Y_2 = 0$ and denote it by $\bar{h}_n$ then $\bar{h}_n \neq 0$ and satisfies, after simplifying by Y_2^{ℓ}, we get

$$
\begin{aligned}
&- iY_2(W-bZ)\frac{\partial h_n}{\partial Y_1} - \frac{i}{4}Y_2^2\frac{\partial h_n}{\partial W} + \frac{ib}{4}Y_2^2\frac{\partial h_n}{\partial Z} \\
&= (\alpha_2 Y_2 + K_1 + i\ell(W-bZ))h_n = i(\ell+\alpha)(W-bZ)h_n.
\end{aligned} \tag{23}
$$

Now if we restrict h_n to $Y_2 = 0$ and denote it by $\bar{h}_n$ then $\bar{h}_n \neq 0$. Moreover $\bar{h}_n$ satisfies (23) restricted to $Y_2 = 0$. So $\bar{h}_n$ satisfies

$$0 = i(\ell+\alpha)(W-bZ)\bar{h}_n,$$

and by assumptions $\ell + \alpha > 0$ which yields $\bar{h}_n = 0$ which is not possible. □

Now assume that $K_1 = 0$, so $K = \alpha_0 + \alpha_1 Y_1 + \alpha_2 Y_2$. We will prove the following proposition which together with Proposition 4 implies the proof of Theorem 1(b).

Proposition 5. *System* (16) *has no irreducible Darboux polynomials with cofactor* $K = \alpha_0 + \alpha_1 Y_1 + \alpha_2 Y_2 \neq 0$.

Proof. Let f be an irreducible Darboux polynomial with non-zero cofactor $K = \alpha_0 + \alpha_1 Y_1 + \alpha_2 Y_2 \neq 0$. Let $\tau\colon \mathbb{C}[Y_1, Y_2, W, X, Z] \to \mathbb{C}[Y_1, Y_2, W, X, Z]$ be the automorphism

$$\tau(Y_1, Y_2, W, X, Z) = (-Y_1, -Y_2, W, X, Z). \tag{24}$$

and consider the polynomial $g = f \cdot \tau f$ that is invariant by τ with a cofactor of the form $K_\tau = 2\alpha_0$. We consider two different cases.

Case 1: $\alpha_0 = 0$. In this case since $K \neq 0$ we must have $\alpha_1^2 + \alpha_2^2 \neq 0$. Note that g is a Darboux polynomial with zero cofactor, that is, it is a polynomial first integral. In view of Theorem 1(a) we must have $g = g(G_1, G_2)$, where

$$
\begin{aligned}
G_1 &= H_1(U,V) = H_1\Big(\frac{Y_1+Y_2}{2}, \frac{Y_1-Y_2}{2i}\Big) = Y_1 Y_2, \\
G_2 &= H_2(V,Y,X,Z) = H_2\Big(\frac{Y_1-Y_2}{2i}, Y, X, Z\Big) \\
&= X^2 + Z^2 + W^2 - \frac{1}{4}(Y_1^2 + Y_2^2 - 2Y_1Y_2).
\end{aligned} \tag{25}
$$

Since $\tau(Y_1) = -Y_1$, $\tau(Y_2) = -Y_2$ and $\tau(G_i) = G_i$ for $i = 1, 2$, we must have $f = f(G_1, G_2)$. But then f is a Darboux polynomial with zero cofactor, that is, $\alpha_1 = \alpha_2 = 0$ which is not possible. So, this case is not possible.

Case 2: $\alpha_0 \neq 0$. In this case, proceeding as in the proof of Proposition 4, let $\tilde{g} = \tilde{g}(Y_2, W, X, Z) = g(0, Y_2, W, X, Z)$ that is, the polynomial f restricted to $Y_1 = 0$, so $\tilde{g}$ satisfies

$$-iY_2(W-bZ)\frac{\partial \tilde{g}}{\partial Y_1} - \frac{i}{4}Y_2^2\frac{\partial \tilde{g}}{\partial W} - Z\frac{\partial \tilde{g}}{\partial X} + \Big(\frac{ib}{4}Y_2^2 + X\Big)\frac{\partial \tilde{g}}{\partial Z} = 2\alpha_0 \tilde{g}.$$

We write $\tilde{g}$ as sum of its homogeneous parts as $\tilde{g} = \sum_{i=0}^{n} \tilde{g}_i$ where $\tilde{g}_i = \tilde{g}_i(Y_2, W, X, Z)$ is a homogeneous polynomial of degree i. Clearly, $\tilde{g}_n$ satisfies

$$-iY_2(W-bZ)\frac{\partial \tilde{g}_n}{\partial Y_2} - \frac{i}{4}Y_2^2\frac{\partial \tilde{g}_n}{\partial W} + \frac{ib}{4}Y_2^2\frac{\partial \tilde{g}_n}{\partial Z} = 0.$$

Solving this linear partial differential equation we get that $\tilde{g}_n = \tilde{g}_n(X, bW + Z, -Y_2^2 + 4W((1-b^2)W - 2bZ))$, and since must be a polynomial of degree n we obtain

$$\tilde{g}_n = aX^{n-2k-\ell}(bW+Z)^\ell\big(-Y_2^2 + 4W((1-b^2)W - 2bZ)\big)^k.$$

Moreover $\tilde{g}_{n-1}$ satisfy

$$\begin{aligned}
&-iY_2(W-bZ)\frac{\partial \tilde{g}_{n-1}}{\partial Y_2} - \frac{i}{4}Y_2^2\frac{\partial \tilde{g}_{n-1}}{\partial W} + \frac{ib}{4}Y_2^2\frac{\partial \tilde{g}_{n-1}}{\partial Z}\\
&= aX^{-2k-\ell+n-1}(bW+Z)^{\ell-1}(-4b^2W^2+4W^2-8bZW-Y_2^2)^{k-1}\Big(\big(\ell X^2-\\
&2bW\alpha_0X - 2Z\alpha_0X + 2kZ^2 + \ell Z^2 - nZ^2 + 2bkWZ + b\ell WZ - bnWZ\big)Y_2^2\\
&+4W\Big(2kW^2Zb^3 + \ell W^2Zb^3 - nW^2Zb^3 - 2W^2X\alpha_0b^3 + 2kWX^2b^2+\\
&\ell WX^2b^2 + 6kWZ^2b^2 + 3\ell WZ^2b^2 - 3nWZ^2b^2 - 6WXZ\alpha_0b^2 + 4kZ^3b+\\
&2\ell Z^3b - 2nZ^3b - 2kW^2Zb - \ell W^2Zb + nW^2Zb + 2kX^2Zb + 2\ell X^2Zb-\\
&4XZ^2\alpha_0b + 2W^2X\alpha_0b - \ell WX^2 - 2kWZ^2 - \ell WZ^2 + nWZ^2 + 2WXZ\alpha_0\Big)\Big).
\end{aligned}$$

Restricting it to $Y_2 = 0$ and using that $\alpha_0 \neq 0$ we obtain that $\tilde{g}_n = 0$ and so g is divisible by Y_1. Proceeding in the same way we get that g must be divisible by Y_2 and so $g = Y_1Y_2h$. Therefore, h is a homogeneous polynomial of degree $n-2$. Then h also satisfies the same as g because Y_1Y_2 is a first integral. Proceeding as we did, we get that h is divisible by Y_1Y_2. Proceeding inductively, we get that n is even and $g = (Y_1Y_2)^{n/2}a_0$. So g is a first integral with zero cofactor, a contradiction. This concludes the proof. □

4. Proof of Theorem 1(c)

Note that it follows from Theorem 1(a) that the unique Darboux polynomials of system (1) with zero cofactor (that is, the polynomial first integrals) of system (1) are polynomials in the variables H_1 and H_2.

We introduce several auxiliary results. The first one was proved in [4].

Lemma 6. *Let f be a polynomial and $f = \prod_{j=1}^{s} f_j^{\alpha_j}$ its decomposition into irreducible factors in $\mathbb{C}[U,V,W,X,Z]$. Then f is a Darboux polynomial if and only if all the f_j are Darboux polynomials. Moreover, if K and K_j are the cofactors of f and f_j, then $K = \sum_{j=1}^{s}\alpha_jK_j$.*

In view of Theorem 1(a) if f is a Darboux polynomial (or $f = 0$ is an invariant algebraic surface) then $f = Y_1^{n_1}Y_2^{n_2}G_2^{n_3}$ where $n_1, n_2, n_3 \in \mathbb{N}\cup\{0\}$ (see (25)). The second result that we will need is proved in [1].

Proposition 7. *The following statements hold.*

(a) *If $E = \exp(g_0/g)$ is an exponential factor for the polynomial system* (16) *and g is not a constant polynomial, then $g = 0$ is an invariant algebraic hypersurface.*

(b) *Eventually* e^{g_0} *can be an exponential factor, coming from the multiplicity of the infinite invariant hyperplane.*

In view of the above explanation and Proposition 7 if $F = \exp(h/g)$ is an exponential factor of system (16) then it must be of the form $F = \exp(h/(Y_1^{n_1}Y_2^{n_2}G_2^{n_3}))$ with $h \in \mathbb{C}[Y_1, Y_2, W, X, Z]$ and $n_1, n_2, n_3 \in \mathbb{N} \cup \{0\}$, with h coprime with Y_1 if $n_1 > 0$; h coprime with Y_2 if $n_2 > 0$ and h coprime with G_2 if $n_3 > 0$. Moreover, since the cofactor must have at most degree one we can write it as $L = \beta_0 + \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 W + \beta_4 X + \beta_5 Z$. So, h must satisfy,

$$\begin{aligned} &iY_1(W-bZ)\frac{\partial h}{\partial Y_1} - iY_2(W-bZ)\frac{\partial h}{\partial Y_2} + \frac{i}{4}(Y_1^2 - Y_2^2)\frac{\partial h}{\partial W} - Z\frac{\partial h}{\partial X} \\ &\quad + \Big(-\frac{ib}{4}(Y_1^2 - Y_2^2) + X\Big)\frac{\partial h}{\partial Z} - i(n_1 - n_2)(W - bZ)h \\ &= Y_1^{n_1}Y_2^{n_2}G_2^{n_3}(\beta_0 + \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 W + \beta_4 X + \beta_5 Z). \end{aligned} \tag{26}$$

We will prove that $n_1 = n_2$. Indeed, we consider two cases.

Case 1*:* $n_1 > n_2$. In this case, evaluating (26) on $Y_1 = 0$, and denoting by $\bar{h}$ the restriction of h to $Y_1 = 0$, that is $\bar{h} = \bar{h}(Y_2, W, X, Z) = h(0, Y_2, W, X, Z)$ we have

$$\begin{aligned} &-iY_2(W-bZ)\frac{\partial \bar{h}}{\partial Y_2} - \frac{i}{4}Y_2^2\frac{\partial \bar{h}}{\partial W} - Z\frac{\partial \bar{h}}{\partial X} + \Big(\frac{ib}{4}Y_2^2 + X\Big)\frac{\partial \bar{h}}{\partial Z} \\ &= i(n_1 - n_2)(W - bZ)\bar{h}. \end{aligned}$$

So $\bar{h}$ is either zero or a Darboux polynomial of system (21) with cofactor $i(n_1 - n_2)(W - bZ)$. The first case is not possible because h is coprime with Y_1. The other case is also not possible because the cofactor of $\bar{h}$ is of the form $i\alpha(W - bZ)$ for some $\alpha > 0$, and so it follows from the proof of Case 2 in Proposition 4 that $\bar{h} = 0$ which is not possible.

Case 2*:* $n_1 < n_2$. In this case, evaluating (26) on $Y_2 = 0$, and denoting by $\bar{h}$ the restriction of h to $Y_2 = 0$, we obtain

$$\begin{aligned} &iY_1(W-bZ)\frac{\partial \bar{h}}{\partial Y_2} + \frac{i}{4}Y_1^2\frac{\partial \bar{h}}{\partial W} - Z\frac{\partial \bar{h}}{\partial X} + \Big(-\frac{ib}{4}Y_1^2 + X\Big)\frac{\partial \bar{h}}{\partial Z} \\ &= i(n_1 - n_2)(W - bZ)\bar{h}. \end{aligned}$$

So $\bar{h}$ is either zero or a Darboux polynomial of system (18) with cofactor $i(n_1 - n_2)(W - bZ)$. The first case is not possible because h is coprime with Y_2. The other case is also not possible because the cofactor of $\bar{h}$ is of the form $-i\alpha(W - bZ)$ for some $\alpha > 0$, and so it follows from the proof of Case 1 in Proposition 4 that $\bar{h} = 0$ which is not possible.

Hence $n_1 = n_2$ and $F = \exp(h/(G_1^{n_1}G_2^{n_3}))$ and satisfies

$$\begin{aligned} &iY_1(W-bZ)\frac{\partial h}{\partial Y_1} - iY_2(W-bZ)\frac{\partial h}{\partial Y_2} + \frac{i}{4}(Y_1^2 - Y_2^2)\frac{\partial h}{\partial W} - Z\frac{\partial h}{\partial X} \\ &\quad + \Big(-\frac{ib}{4}(Y_1^2 - Y_2^2) + X\Big)\frac{\partial h}{\partial Z} \\ &= G_1^{n_1}G_2^{n_3}(\beta_0 + \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 W + \beta_4 X + \beta_5 Z). \end{aligned} \tag{27}$$

Now we need the following result whose proof can be found in [4].

Theorem 8. *Suppose that system* (16) *admits p Darboux polynomials f_i with cofactors k_i and q exponential factors F_j with cofactors ℓ_j. Then there exists $\lambda_j, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{q} \lambda_k k_i + \sum_{i=1}^{q} \mu_i \ell_i = 0$$

if and only if the function G given in (5) *(called of Darboux type) is a first integral of system* (16).

In view of Theorem 8 if G is a Darboux first integral it must be of the form

$$G = Y_1^{\lambda_1} Y_2^{\lambda_2} G_2^{\lambda_3} \exp(\mu_1 h/(G_1^{n_1} G_2^{n_3}))$$

whose cofactor K_G is

$$K_G = i(\lambda_1 - \lambda_2)(W - bZ) + \mu_1(\beta_0 + \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 W + \beta_4 X + \beta_5 Z).$$

Since G is a first integral we must have $K_G = 0$. This yields that either $\mu_1 = 0$ and $\lambda_1 = \lambda_2$ or $\mu_1 \neq 0$ and $\beta_0 = \beta_1 = \beta_2 = \beta_4 = 0$, $\beta_5 = -b\beta_3$ and $\lambda_1 - \lambda_2 = -i\beta_3\mu_1$. In the first case $G = G_1^{\lambda_1} G_2^{\lambda_2}$ and so G is a Darboux first integral in the variables G_1 and G_2.

In the second case we have $F = \exp(h/(G_1^{n_1} G_2^{n_2}))$ with cofactor $L = \beta_3(W - bZ)$. Imposing this in (27) we get that h must satisfy

$$\begin{aligned} &iY_1(W - bZ)\frac{\partial h}{\partial Y_1} - iY_2(W - bZ)\frac{\partial h}{\partial Y_2} + \frac{i}{4}(Y_1^2 - Y_2^2)\frac{\partial h}{\partial W} - Z\frac{\partial h}{\partial X} \\ &+ \Big(-\frac{ib}{4}(Y_1^2 - Y_2^2) + X\Big)\frac{\partial h}{\partial Z} = \beta_3(W - bZ)G_1^{n_1} G_2^{n_2}. \end{aligned} \tag{28}$$

We will show that $\beta_3 = 0$. We consider different cases. If $\deg h \leq 2n_1 + 2n_3$ it follows from (28) that $\beta_3 = 0$. So, $\deg h \geq 2n_1 + 2n_3 + 1$. Moreover, if $\deg h = 2n_1 + 2n_3 + 1$ then it follows from (28) that

$$-Z\frac{\partial h}{\partial X} + X\frac{\partial h}{\partial Z} = \beta_3 G_1^{n_1} G_3^{n_3}(W - bZ).$$

With the change of variables in (8) and using the notation of Proposition 7 we have that the above equation is equivalent to

$$\sqrt{A - Z^2}\frac{d\tilde{h}}{dZ} = \beta_3(U^2 + V^2)^{n_1}(V^2 + W^2 + A^2)^{n_3}(W - bZ).$$

Proceeding again as in (10) with

$$\begin{aligned} s_1 &= \beta_3(U^2 + V^2)^{n_1}(V^2 + W^2 + A^2)^{n_3} W, \\ s_2 &= -b\beta_3(U^2 + V^2)^{n_1}(V^2 + W^2 + A^2)^{n_3}, \end{aligned}$$

we conclude that $s_1 = 0$ and so $\beta_3 = 0$. Hence, we can assume that $\deg h > 2n_1 + 2n_3 + 1$. We write $\deg h = 2n_1 + 2n_3 + 1 + k$ for some $k > 0$. Proceeding as in the proof of Proposition 7 we get that if we expand h is its homogeneous parts as $h = \sum_{j=m}^{2n_1+2n_3+1+k} h_j$, then m is even and $h_{2j} = h_{2j}(A + C, B)$, $h_{2j+1} = 0$ for $j = m/2, \ldots, n_1 + n_3 - 1$. Then $h_{2n_1+2n_3}$ satisfies (6) and proceeding as in proof of

Proposition 7 we get that $h_{2n_1+2n_3} = \tilde{G}_{2n_1+2n_3}(A,U,V,W)$ and $h_{2n_1+2n_3+1}$ satisfies

$$\frac{d\tilde{h}_{2n_1+2n_3+1}}{dZ} = \bigg(V(W-bZ)\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial U} - U(W-bZ)\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial V} + UV\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial W} - 2bUVZ\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial A} + \beta_3 G_1^{n_1}G_3^{n_3}(W-bZ)\bigg)\frac{1}{\sqrt{A-Z^2}} = \frac{s_1+s_2Z}{\sqrt{A-Z^2}},$$

which is (10) with

$$\begin{aligned} s_1 = s_1(U,V,W,A) &= VW\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial U} - UW\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial V} + UV\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial W} \\ &+ \beta_3(U^2+V^2)^{n_1}(V^2+W^2+A^2)^{n_3}W, \\ s_2 = s_2(U,V,W,A) &= -bV\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial U} + bU\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial V} - 2bUV\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial A} \\ &- b\beta_3(U^2+V^2)^{n_1}(V^2+W^2+A^2)^{n_3}. \end{aligned}$$

Proceeding again as in the proof of Proposition 7 we get that $s_1 = 0$ that is

$$VW\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial U} - UW\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial V} + UV\frac{\partial \tilde{G}_{2n_1+2n_3}}{\partial W} + \beta_3(U^2+V^2)^{n_1}(V^2+W^2+A^2)^{n_3}W = 0.$$

Introducing the change of variables $B = U^2+V^2$, $C = V^2+W^2$ with inverse change $U = \sqrt{B-V^2}, W = \sqrt{C-V^2}$ we rewrite $s_1 = 0$ as

$$\sqrt{B-V^2}\sqrt{C-V^2}\frac{d\hat{G}_{2n_1+2n_3}}{dV} = \beta_3\sqrt{C-V^2}B^{n_1}(C^2+A^2)^{n_3},$$

where $\hat{G}_{2n_1+2n_3} = \hat{G}_{2n_1+2n_3}(A,B,C) = \tilde{G}_{2n_1+2n_3}(A,U,V,W)$. Solving this differential equation we obtain

$$\hat{G}_{2n_1+2n_3} = \beta_3 B^{n_1}(C^2+A^2)^{n_3}\arctan\frac{V}{\sqrt{B-V^2}}.$$

Since $G_{2n_1+2n_3}$ must be a polynomial we must have $\beta_3 = 0$.

We have proved that $\beta_3 = 0$, and so $\beta_i = 0$ for $i = 0,\ldots,5$. This implies that $L = 0$ and h must be a polynomial first integral. In view of Theorem 1(a) we have $h = h(G_1,G_2)$. Moreover $\lambda_1 = \lambda_2$ because $\beta_3 = 0$ and so $G = G_1^{\lambda_1}G_2^{\lambda_3}\exp(h(G_1,G_2)/(G_1^{n_1}G_2^{n_3})$ which clearly implies that G is a Darboux first integral in the variables G_1 and G_2. This concludes the proof of Theorem 1(c).

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