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AN INVERSE APPROACH TO THE CENTER-FOCUS PROBLEM FOR POLYNOMIAL DIFFERENTIAL SYSTEM WITH HOMOGENOUS NONLINEARITIES

JAUME LLIBRE[1], RAFAEL RAMÍREZ[2] AND VALENTÍN RAMÍREZ[3]

Abstract. We consider polynomial vector fields of the form

$$\mathcal{X} = (-y + X_m)\frac{\partial}{\partial x} + (x + Y_m)\frac{\partial}{\partial y},$$

where $X_m = X_m(x, y)$ and $Y_m = Y_m(x, y)$ are homogenous polynomials of degree m. It is well–known that $\mathcal{X}$ has a center at the origin if and only if $\mathcal{X}$ has an analytic first integral of the form

$$H = \frac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j,$$

where $H_j = H_j(x, y)$ is a homogenous polynomial of degree j.

The classical center-focus problem already studied by H. Poincaré consists in distinguishing when the origin of $\mathcal{X}$ is either a center or a focus. In this paper we study the inverse center-focus problem. In particular for a given analytic function H defined in a neighborhood of the origin we want to determine the homogenous polynomials X_m and Y_m in such a way that H is a first integral of $\mathcal{X}$ and consequently the origin of $\mathcal{X}$ will be a center. Moreover, we study the case when

$$H = \frac{1}{2}(x^2 + y^2)\left(1 + \sum_{j=1}^{\infty} \Upsilon_j\right),$$

where Υ_j is a convenient homogenous polynomial of degree j for $j \geq 1$.

The solution of the inverse center problem for polynomial differential systems with homogenous nonlinearities, provides a new mechanism to study the center problem, which is equivalent to Liapunov's Theorem and Reeb's criterion.

1. Introduction

Let

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial},$$

be the real planar polynomial vector field associated to the real planar polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

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where the dot denotes derivative with respect to an independent variable here called the time t, and P and Q are real coprime polynomials in $\mathbb{R}[x, y]$. We say that polynomial differential system (1) has *degree* $m = \max\{\text{degP}, \text{degQ}\}$.

In what follows we assume that origin $O := (0, 0)$ is a singular or equilibrium point, i.e. $P(0, 0) = Q(0, 0) = 0$.

The equilibrium point O is a *center* if there exists an open neighborhood U of O where all the orbits contained in $U \setminus \{O\}$ are periodic.

Assume that the origin of the polynomial differential system (1) is a center. It is well–known that, after a linear change of variables and a constant scaling of the time variable (if necessary), system (1) can be written in one of the next three forms:

$$\begin{aligned} \dot{x} &= -y + X(x, y), & \dot{y} &= x + Y(x, y), \\ \dot{x} &= y + X(x, y), & \dot{y} &= Y(x, y), \\ \dot{x} &= X(x, y), & \dot{y} &= Y(x, y), \end{aligned} \tag{2}$$

where $X(x, y)$ and $Y(x, y)$ are polynomials without constant and linear terms defined in a neighborhood of the origin. Then the origin O of the polynomial differential system 1 is called *linear type*, *nilpotent* or *degenerate* if after a linear change of variables and a scaling of the time it can be written as the first, second and third system of (2), respectively.

We shall study the differential system of the linear type

$$\dot{x} = -y + X_m(x, y), \quad \dot{y} = x + Y_m(x, y) \tag{3}$$

where $X_m = X_m(x, y)$ and $Y_m = Y_m(x, y)$ are homogenous polynomial of degree m. The classical Poincaré center-focus problem asks about conditions on the coefficients of X_m and Y_m under which all trajectories of (3) situated in a small open neighborhood of the origin are closed. This problem was solved for system (3) for $m = 2, 3$ (see for instance [2, 16, 20, 21, 22]).

The necessary and sufficient condition for O to be a center for analytic vector fields was obtained by Liapunov (see for instance [12, 9]) and Poincaré for the polynomial vector fields.

Theorem 1 (Poincaré-Liapunov Theorem)**.** *A planar analytic differential system*

$$\dot{x} = -y + \sum_{j=2}^{\infty} X_j(x, y), \quad \dot{y} = x + \sum_{j=2}^{\infty} Y_j(x, y), \tag{4}$$

has a center at the origin if and only if it has a first integral of the form

$$H = \sum_{j=2}^{\infty} H_j(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j(x, y), \tag{5}$$

where X_j, Y_j and H_j are homogenous polynomials of degree j.

Now we shall recall the *Reeb's criterion* for solving the center problem.

We need the following definitions and concept. The definition of *integrating factor* is standard and well-known, we recall here the definition of *inverse integrating factor* of system (1). A function $V = V(x, y)$ is an inverse integrating factor of system (1) in an open subset $U \subset \mathbb{R}^2$ if $V \in C^1(U), V \not\equiv 0$ in U and

$$\frac{\partial \left(\dfrac{P}{V}\right)}{\partial x} + \frac{\partial \left(\dfrac{Q}{V}\right)}{\partial y} = 0 \Longleftrightarrow P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = V\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right).$$

The first integral F associated to the inverse integrating factor V is given by the line integral

$$F(x,y) = \int_\gamma \left(-\frac{P}{V}dy + \frac{Q}{V}dx\right),$$

We note that $\{V = 0\}$ is formed by orbits of system (1). The function $1/V$ defines an integrating factor in $U\backslash\{V = 0\}$ of system (1) which allows to compute a first integral for system (1) in $U\backslash\{V = 0\}$.

We consider now the relation between the existence of a center and the existence of an integrating factor, for analytic vector fields. The main result is given by the following theorem which is equivalent to Theorem 1.

Theorem 2 (Reeb 's criterion [18])**.** *The analytic differential system* (4) *has a center at the origin if and only if there is a local nonzero analytic integrating factor of the form* $V = 1 + h.o.t.$ *in a neighborhood of the origin.*

Consequently, to show that a singular point is a center we have two basic mechanisms: the Poincaré–Liapunov's Theorem and the Reeb's criterion.

The main objective of the present paper is to propose a new mechanism to solve the center problem for systems (3). We shall analyze the center problem from the inverse point of view (see for instance [10, 19]). Indeed given an analytic function H of the form (5) we shall determine the homogenous polynomials X_m and Y_m in (3) in such a way that the function H is a first integral of the differential system (3).

2. Preliminary results

In the proofs of the main results that we proposed in this paper it plays an important role the following results .

The proof of the next result can be found in particular in [16].

Theorem 3. *For system* (4) *there exists a formal power series*

$$W = \sum_{n=2}^{\infty} W_n := \frac{1}{2}(x^2 + y^2) + \sum_{n=3}^{\infty} W_n(x,y),$$

where $W_j = W_j(x,y)$ *is a homogenous polynomial of degree* j *such that*

$$\begin{aligned} \frac{dW}{dt} &= \left(x + \frac{\partial W_3}{\partial x} + \frac{\partial W_4}{\partial x} + \dots\right)\left(-y + \sum_{j=2}^{\infty} X_j(x,y)\right) \\ &\quad + \left(y + \frac{\partial W_3}{\partial y} + \frac{\partial W_4}{\partial y} + \dots\right)\left(x + \sum_{j=2}^{\infty} Y_j(x,y)\right) \qquad (6) \\ &= \sum_{j=0}^{\infty} v_j (x^2 + y^2)^{j+1}, \end{aligned}$$

where v_j are the Poincaré-Liapunov constants. This derivative is calculated on the solutions of system (4)*.*

Clearly if the constants $v_j = 0$ for $j \in \mathbb{N}$ then there exists a first integral

$$H := \frac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j,$$

where H_j is a homogenous polynomial of degree j. Consequently the origin is a center. If there exist a first non–zero Liapunov constant v_j, then in view of the relation

$$\frac{dW}{dt} = v_j(x^2 + y^2)^{j+1} + \ldots,$$

the origin is a stable focus if $v_j < 0$ and unstable if $v_j > 0$.

As usual the *Poisson bracket* of the functions $f(x, y)$ and $g(x, y)$ is defined as

$$\{f, g\} := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

We will need the following result.

Proposition 4. *The following relation holds*

$$\int_0^{2\pi} \{H_2, \Psi\}|_{x=\cos t,\, y=\sin t}\, dt = 0,$$

for arbitrary C^1 function $\Psi = \Psi(x, y)$ defined in the interval $[0, 2\pi]$.

Proof. Indeed, if we change $x = \cos t$, $y = \sin t$ then it is easy to show that

$$\{H_2, \Psi\}|_{x=\cos t,\, y=\sin t} = x\frac{\partial \Psi}{\partial y} - y\frac{\partial \Psi}{\partial x}\bigg|_{x=\cos t,\, y=\sin t} = \frac{d\Psi(\cos t,\, \sin t)}{dt}.$$

Hence,

$$\int_0^{2\pi} \{H_2, \Psi\}|_{x=\cos t,\, y=\sin t}\, dt = \Psi(\cos t, \sin t)|_{t=0}^{t=2\pi} = 0.$$

□

The following result due to Liapunov (see Theorem 1, page 276 of [12]).

Theorem 5. *If all the roots $\lambda_1, \ldots, \lambda_n$ of the equation*

$$\begin{vmatrix} p_{11} - \lambda & p_{21} & \ldots & p_{n1} \\ p_{12} & p_{22} - \lambda & \ldots & p_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ p_{1n} & p_{2n} & \ldots & p_{nn} - \lambda \end{vmatrix} = 0$$

are such that the relation

$$\lambda = m_1\lambda_1 + \ldots + m_n\lambda_n,$$

is not vanishing for arbitrary non-negative integers $m_1, \ldots, m_n$ linked by the expression

$$m = m_1 + \ldots + m_n \neq 0.$$

Then for arbitrary given homogenous polynomial $U = U(x_1, \ldots, x_n)$ *of degree* m *there exists a unique homogenous polynomial* $V = V(x_1, \ldots, x_n)$ *of degree* m *which is a solution of the equation*

$$\sum_{j=1}^{n} (p_{j1}x_1 + \ldots + p_{jn}x_n) \frac{\partial V}{\partial x_j} = U.$$

In particular, for $n = 2$ *the partial differential equation*

$$x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x} := \{H_2, V\} = U, \tag{7}$$

has a unique solution V *if and only if*

$$\lambda_1 m_1 + \lambda_2 m_2 = i(m_1 - m_2) \neq 0, \quad \textit{with} \quad m = m_1 + m_2.$$

A simple consequence of Theorem 5 is the following result.

Corollary 6. *Let* $U = U(x, y)$ *be a homogenous polynomial of degree* m. *The linear partial differential equation* (7) *has a unique homogenous polynomial solution* V *of degree* m *if* m *is odd; and if* V *is a homogenous polynomial solution when* m *is even, then any other homogenous polynomial solution is of the form* $V + c(x^2 + y^2)^{m/2}$ *with* $c \in \mathbb{R}$. *Moreover, for* m *even these solutions exist if and only if* $\int_0^{2\pi} U(x, y)|_{x=\cos t,\, y=\sin t}\, dt = 0.$

3. Statement of the main results

The main results are divided in three subsections

3.1. Differential system (3) with local analytic first integral of the form $H = (x^2 + y^2)/2 + h.o.t.$.

We state and solve the following inverse problem of the center for differential system (3).

Problem 1 *Determine the polynomial planar vector fields of degree* m

$$\mathcal{X} = (-y + X_m)\frac{\partial}{\partial x} + (x + Y_m)\frac{\partial}{\partial y},$$

where $X_m = X_m(x, y)$ *and* $Y_m = Y_m(x, y)$ *are homogenous polynomial of degree* m, *for which the given analytic function* (5) *is a local analytic first integral.*

The inverse problem 1 has been solved in the following main theorem which provide the expressions of the polynomial differential systems (3) in function of its first integral (5).

Theorem 7. *A polynomial differential system* (3) *associated to polynomial vector field* $\mathcal{X}$ *has the function* (5) *as a local analytic first integral if and only if the system can be written as*

$$\dot{x} = -y + \{H_{m+1}, x\} + g_{m-1}\{H_2, x\}, \quad \dot{y} = x + \{H_{m+1}, y\} + g_{m-1}\{H_2, y\}, \tag{8}$$

where $g_{m-1} = g_{m-1}(x, y)$ *is an arbitrary homogenous polynomial of degree* $m - 1$ *and the infinite numbers of the following partial differential equations hold*

$$\mathcal{X}_m(H_k) + \{H_2, H_{m+k-1}\} = 0 \qquad \textit{for} \qquad k \geq 2, \tag{9}$$

where

$$\mathcal{X}_m = (\{H_{m+1}, x\} + g_{m-1}\{H_2, x\})\frac{\partial}{\partial x} + (\{H_{m+1}, y\} + g_{m-1}\{H_2, y\})\frac{\partial}{\partial y},$$

i.e. the polynomial vector field $\mathcal{X}$ has the first integral (5) *if and only if it can be written as* $\mathcal{X} = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y} + \mathcal{X}_m$, *where the vector field $\mathcal{X}_m$ satisfies* (9).

Theorem 8. *Differential system* (3) *has a center at the origin if and only if it can be written as*

$$\dot{x} = \frac{1+g_{m-1}}{1+\Lambda}\{H,x\}, \qquad \dot{y} = \frac{1+g_{m-1}}{1+\Lambda}\{H,y\}, \tag{10}$$

where H is an analytic functions such that

$$H = \int_\gamma \left(\frac{1+\Lambda}{1+g_{m-1}}dH_{m+1} + (1+\Lambda)dH_2\right), \tag{11}$$

and $\Lambda = \Lambda(x,y)$ is an analytic function in the neighborhood of the origin satisfying the partial differential equation

$$\left\{H_{m+1}, \frac{1+\Lambda}{1+g_{m-1}}\right\} + \{H_2, \Lambda\} = 0. \tag{12}$$

Thus $\displaystyle\int_0^{2\pi} \left\{H_{m+1}, \frac{1+\Lambda}{1+g_{m-1}}\right\}\Big|_{x=\cos t, y=\sin t} dt = 0$. *Moreover, this theorem is equivalent to Theorem 7, i.e. differential system* (8) *coincides with differential system* (10), (11) *and condition* (9) *coincides with condition* (11), (12).

Theorems 7 and 8 have the following corollary.

Corollary 9. *Under the assumptions of Theorem 8 the next statements hold.*

(a) *If m is odd then $H(-x,-y) = H(x,y)$. Hence the phase portrait of system* (3) *having a center at the origin is symmetric with respect to the origin, and has an odd number of centers.*

(b) *If system* (3) *has a center at the origin then*

$$\int_0^{2\pi} \left(\frac{\partial \mathcal{X}_m(x)}{\partial x} + \frac{\partial \mathcal{X}_m(y)}{\partial y}\right)\Big|_{x=\cos t,\, y=\sin t} dt = 0. \tag{13}$$

We note that statement (b) of Corollary 9 follows from Theorem 2 of [8].

Remark 10. (a) *From Theorem 7 it follows that a polynomial differential system* (3) *has a center at the origin if and only if it can be written into the form* (8) *with the complementary conditions* (9). *Consequently we have a new mechanism to solve the center problem for polynomial differential systems of the form* (3) *which is equivalent to the Poincaré–Liapunov's Theorem and to the Reeb's criterion.*

(b) *From Theorem 8 it follows the equivalence between Reeb's criterion and Theorem 7, but the criterion which we propose provides a complementary information of the structure of the differential system* (3) *having a center and the structure of its integrating factor and the integral form of the first integral H.*

(c) *We note that the Poincaré–Liapunov first integral given in* (5) *for systems* (3) *is computed in* (11).

(d) *Given a system* (3) *we can determine the homogeneous polynomials* H_{m+1} *and* g_{m-1} *which appear in* (8)*. Now solving with respect to the function* Λ *the linear partial differential equation of first order* (12)*, if we get an analytic solution defined in a neighborhood of the origin, then we have solved the center problem for systems* (3)*.*

The proofs of the results of the subsection 3.1 are given in section 4.

3.2. **Differential system (3) with local analytic first integral of the form** $H = (x^2 + y^2)/2\,(1 + h.o.t.)$**.** We say that differential system (1) has a *weak center* at the origin if this system has a local analytic first integral of the form

$$H = \frac{1}{2}(x^2 + y^2)\left(1 + \sum_{j=1}^{\infty} \Upsilon_j(x, y)\right),$$

where Υ_j is a convenient homogenous polynomial of degree j.

The interest for this type of centers becomes in particular from the fact that any linear type center is locally a weak center (see Remark 22).

In the study of weak centers plays a fundamental role the following polynomial differential system of degree m

$$\dot{x} = -y(1 + \Phi) + x\varphi, \qquad \dot{y} = x(1 + \Phi) + y\varphi, \tag{14}$$

where $\Phi = \Phi(x, y)$ and $\varphi = \varphi(x, y)$ are convenient polynomials of degree at most $m - 1$. It is easy to observe that the singular points of this differential system are on the intersection of the curves

$$(x^2 + y^2)\varphi(x, y) = 0, \qquad (x^2 + y^2)(1 + \Phi(x, y)) = 0.$$

Thus, by Bezout Theorem, the maximum number of singular points of system (14) is $(m - 1)^2 + 1$. In particular if $\Phi = 0$ then the only critical point is the origin.

The aim of the following results is to study the existence of the weak centers for polynomial differential systems with homogenous nonlinearities.

Proposition 11. *Assume that the polynomial differential system* (3) *has a center at the origin of coordinates. Then this center is a weak center if and only if this system can be written as*

$$\begin{aligned} \dot{x} = &\ -y\left(1 + g_{m-1} + \frac{m+1}{2}\Upsilon_{m-1}\right) + \frac{x}{2}\{\Upsilon_{m-1}, H_2\}, \\ \dot{y} = &\ x\left(1 + g_{m-1} + \frac{m+1}{2}\Upsilon_{m-1}\right) + \frac{y}{2}\{\Upsilon_{m-1}, H_2\}, \end{aligned} \tag{15}$$

where $g_{m-1} = g_{m-1}(x, y)$ *and* $\Upsilon_{m-1} = \Upsilon_{m-1}(x, y)$ *are homogenous polynomials of degree* $m - 1$*, i.e. system* (3) *can be written as system* (14) *with* Φ *and* φ *such that*

$$\Phi = g_{m-1} + \frac{m+1}{2}\Upsilon_{m-1}, \qquad \varphi = \frac{1}{2}\{\Upsilon_{m-1}, H_2\}.$$

Now we introduce the following concepts that we need further.

Let $\mathbb{R}[x, y]$ be the ring of all real polynomials in the variables x and y, and let

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} \tag{16}$$

be a polynomial vector field of degree m. Let $g = g(x, y) \in \mathbb{R}[x, y]$. Then $g = 0$ is an *invariant algebraic curve* of $\mathcal{X}$ if

$$\mathcal{X}g = P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} = Kg,$$

where $K = K(x, y)$ is a polynomial of degree at most $m - 1$, which is called the *cofactor* of $g = 0$. If the polynomial g is irreducible in $\mathbb{R}[x, y]$, then we say that the invariant algebraic curve $g = 0$ is *irreducible*, and that its *degree* is the degree of the polynomial g. A first integral F of the polynomial vector field (16) is called *Darboux* if

$$F = e^{G(x,y)/h(x,y)} g_1^{\lambda_1}(x, y) \dots g_r^{\lambda_r}(x, y),$$

where G, h, $g_1, ..., g_r, g, h$ are polynomials and $\lambda_1, \dots, \lambda_r$ are complex constants. For more details on the so–called Darboux theory of integrability see for instance Chapter 8 of [6].

We introduce the following definition. We say that polynomial vector field $\mathcal{X}$ of degree m is *quasi–Darboux integrable* if there exist r polynomial partial integrals $g_1, \dots, g_r$ and s non-polynomial partial integrals $f_1, \dots, f_s$ analytic in $D \subseteq \mathbb{R}^2$ satisfying

$$\mathcal{X}(f_j) = P\frac{\partial f_j}{\partial x} + Q\frac{\partial f_j}{\partial y} = K_j f_j,$$

where $K_j = K_j(x, y)$ is a convenient polynomials of degree $m - 1$, for $j = 1, \dots, s$ such that the function

$$F = e^{G(x,y)/h(x,y)} g_1^{\lambda_1}(x, y) \dots g_r^{\lambda_r}(x, y) f_1^{\kappa_1} \dots f_s^{\kappa_s},$$

where $\lambda_1, \dots, \lambda_r, \kappa_1, \dots, \kappa_s$, are complex constants, is a first integral. We observe that a generalization of the Darboux theory was developed in the paper [7], which evidently contain the above definition, but for our aim we shall use the definition of quasi–Darboux integrable.

Conjecture 12. *Differential system* (15) *with a weak center at the origin is quasi–Darboux integrable.*

This conjecture is supported by several facts which we give below.

Example 13. *Polynomial differential system with weak center at the origin*

$$\begin{aligned} \dot{x} =& \quad -y\left(1 + x^4 + 4x^2y^2 - y^4\right) = -y(1 + 3x^2y^2 - y^4) - x^2y(x^2 + y^2), \\ \dot{y} =& \quad x((1 + 2y^2(x^2 - y^2) = x(1 + 3x^2y^2 - y^4) - xy^2(x^2 + y^2), \end{aligned}$$

is quasi-Darboux integrable.

Indeed, this system has the following polynomial partial integrals

$$g_1 = x^2y^2 + (1 + y^2)^2, \quad g_2 = x^2y^2 + (1 - y^2)^2, \quad g_3 := H_2 = (x^2 + y^2)/2,$$

such that

$$\dot{g}_1 = 2xy\left(2 + x^2 - 3y^2\right) g_1, \quad \dot{g}_2 = 2xy\left(-2 + x^2 - 3y^2\right) g_2, \quad \dot{g}_3 = -2xyg_3^2.$$

It is easy to show that the analytic in $\mathbb{R}^2$ function $f = 4 - 2g_3 \log \dfrac{g_1}{g_2}$ is a partial integral such that $\dot{f} = -4xyg_3 f$. Consequently we have the first integral $F = \dfrac{H_2}{\sqrt{f}}$ which has the Taylor expansion at the origin $F = H_2(1 + h.o.t.)$.

Theorem 14. *Polynomial differential system* (15) *under the assumption*

$$\Upsilon_{m-1} = -2\beta\, g_{m-1} + \frac{2}{m+1}\,\theta(H_2),$$

with

$$\theta(H_2) = \begin{cases} \nu\, H_2^{(m-1)/2} & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases}$$

i.e., the polynomial differential system

$$\text{(17)} \quad \begin{aligned} \dot{x} = & \; -y\,(1 + (1-(m+1)\beta)\,g_{m-1} + \theta(H_2)) + \beta\, x\,\{H_2, g_{m-1}\}, \\ \dot{y} = & \; x\,(1 + (1-(m+1)\beta)\,g_{m-1} + \theta(H_2)) + \beta\, y\{H_2, g_{m-1}\}, \end{aligned}$$

where $H_2 = (x^2+y^2)/2$, $g_{m-1} = g_{m-1}(x,y)$ *is a homogenous polynomial of degree* m *and* β *and* ν *are constants, is quasi-Darboux integrable. Moreover a first integral* F *is given in the following.*

(a) *If* $(1-(m+1)\beta)\beta(1-2\beta) \neq 0$ *and* m *is even, then*

$$F = \frac{H_2}{(1 + (1-2\beta)\,g_{m-1})^{2\beta/(1-2\beta)}}.$$

The algebraic curves $H_2 = 0$ *and* $1+(1-2\beta)\,g_{m-1} = 0$ *are invariant curves with cofactors* $2\beta\{H_2, g_{m-1}\}$ *and* $(1-2\beta)\{H_2, g_{m-1}\}$, *respectively.*

(b) *If* $(1-(m+1)\beta)\beta(1-2\beta) \neq 0$ *and* m *is odd, then*

$$F = \frac{H_2}{\left(1 + \dfrac{(1-2\beta)}{1-(m+1)\beta}\left((1-(m+1)\beta)\,g_{m-1} + \nu\, H_2^{(m-1)/2}\right)\right)^{2\beta/(1-2\beta)}}.$$

The algebraic curves

$$H_2 = 0 \quad \text{and} \quad 1 + \frac{(1-2\beta)}{1-(m+1)\beta}\left((1-(m+1)\beta)\,g_{m-1} + \nu\, H_2^{(m-1)/2}\right) = 0$$

are invariant curves with cofactors $2\beta\{H_2, g_{m-1}\}$ *and* $(1-2\beta)\{H_2, g_{m-1}\}$, *respectively*

(c) *If* $\beta = 1/2$ *and* m *is even, then* $F = H_2 e^{-g_{m-1}}$.

(d) *If* $\beta = 1/2$ *and* m *is odd, then* $F = H_2 e^{-g_{m-1} + 2\nu/(m-1)\, H_2^{(m-1)/2}}$.

In the cases (c) and (d) $H_2 = 0$ *is invariant curve with cofactor* $\{H_2, g_{m-1}\}/2$.

(e) *If* $\beta = 0$, *then* $F = H_2$.

(f) *If* $\beta = 1/(m+1)$ *and* m *is even, then*

$$F = \frac{H_2}{\left(1 + \dfrac{m-1}{m+1} g_{m-1}\right)^{2/(m-1)}}.$$

The algebraic curves $H_2 = 0$ *and* $1+(m-1)\,g_{m-1}/(m+1) = 0$ *are invariant curves with cofactors* $2\{H_2, g_{m-1}\}/(m+1)$ *and* $(1-m)/(m+1)\{H_2, g_{m-1}\}$, *respectively.*

(g) *If* $\beta = 1/(m+1)$ *and* m *is odd, then*

$$F = \frac{H_2}{\left(1 + \dfrac{m-1}{m+1} g_{m-1} - \nu \dfrac{m-1}{2} H_2^{(m-1)/2} \log\, H_2\right)^{2/(m-1)}}.$$

The algebraic curve $H_2 = 0$ and the non-polynomial curve defined in $\mathbb{R}^2\setminus\{0\}$ by

$$f(x,y) := 1 + \frac{m-1}{m+1} g_{m-1} - \nu \frac{m-1}{2} H_2^{(m-1)/2} \log H_2 = 0,$$

are invariant curves with cofactors $2\{H_2, g_{m-1}\}/(m+1)$ and $(1-m)/(m+1)\{H_2, g_{m-1}\}$, respectively.

Remark 15. *We observe that the function $\Omega : f(x,y) = 1 + \dfrac{m-1}{m+1} g_{m-1} - \nu \dfrac{m-1}{2} H_2^{(m-1)/2} \log H_2$ is an analytic in $\mathbb{R}^2\setminus\{0\}$, which in polar coordinates becomes $f(r\cos\theta, r\sin\theta) = 1 + \dfrac{m-1}{m+1} r^{m-1} g_{m-1}(\cos\theta, \sin\theta) - \nu(m-1) r_2^{m-1} \log r$, and satisfies*

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = 1, \quad \lim_{r\to 0} \frac{\partial^k}{\partial r^k} \left(f(r\cos\theta, r\sin\theta) \right) = 0,$$

for $k = 1, \ldots, m-2$.

We recall that the singular point of system (1) located at the origin is an *isochronous center* if all the periodic solutions in a neighborhood of the origin has the same period.

A center O of system (1) is a *uniform isochronous center* if the equality $x\dot{y} - y\dot{x} = \kappa(x^2 + y^2)$ holds for a nonzero constant κ; or equivalently in polar coordinates (r,θ) such that $x = r\cos\theta$, $y = r\sin\theta$, we have that $\dot{\theta} = \kappa$ (see for instance [11]).

Proposition 16. *A polynomial differential system (3) has a uniform isochronous center at the origin if and only if this system can be written as (17) with $\theta(H_2) = 0$ and $\beta = 1/(m+1)$, i.e.*

$$\begin{aligned} \dot{x} &= -y + \frac{x}{m+1}\{H_2, g_{m-1}\}, \\ \dot{y} &= x + \frac{y}{m+1}\{H_2, g_{m-1}\}, \end{aligned} \tag{18}$$

where $H_2 = (x^2+y^2)/2$ and $g_{m-1} = g_{m-1}(x,y)$ is a homogenous polynomial of degree $m-1$. Moreover, this system has the first integral

$$F = \frac{H_2}{\left(1 + \dfrac{m-1}{m+1} g_{m-1}\right)^{2/(m-1)}},$$

which has the following Taylor expansion $F := H = H_2(1 + h.o.t.)$ at the origin the coordinates Consequently the uniform center (18) is a weak center.

This proposition has the following corollary.

Corollary 17. *The solutions of (18) in polar coordinates (r,θ) are*

$$\theta = t, \qquad r = \left(\frac{C}{1 - \dfrac{C(m-1)}{m+1} g_{m-1}(\cos t, \sin t)} \right)^{1/(m-1)},$$

where C is an arbitrary constant.

Note that Proposition 16 characterizes the form of the polynomial uniform isochronous centers with homogeneous nonlinearities and improves previous results of Conti [4].

Remark 18. *Differential system* (17) *with* $\beta = 1/(m+1)$ *and* m *odd, i.e.*

$$\dot{x} = \quad -y(1+\nu\, H^{(m-1)/2}) + \frac{x}{m+1}\{H_2, g_{m-1}\},$$
$$\dot{y} = \quad x(1+\nu\, H^{(m-1)/2}) + \frac{y}{m+1}\{H_2, g_{m-1}\},$$

has one singular point if $\nu > 0$ *and at most* $(m-1)^2+1$ *singular points if* $\nu = -(2/a^2)^{(m-1)/2}$ *which are on the circle* $x^2+y^2=a^2$*. Moreover, its first integral*

$$F = \frac{H_2}{\left(1+\dfrac{m-1}{m+1}g_{m-1} - \nu\dfrac{m-1}{2}H_2^{(m-1)/2}\log\, H_2\right)^{2/(m-1)}},$$

is non-analytic at the origin.

Other particular cases of differential systems with isochronous centers are the systems which satisfy the Cauchy–Riemann conditions (see for instance [4]).

Proposition 19 (Cauchy-Riemann condition for a center)**.** *Let* O *be a center of* (1)*. Then* O *is isochronous center if* P *and* Q *satisfy the Cauchy-Riemann equations*

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \tag{19}$$

A center of system (1) for which (19) holds is called a *holomorphic center,* which is an isochronous center, see for more details [13, 14].

Proposition 20. *A differential system* (3) *with a center at the origin has a holomorphic isochronous center at the origin if and only if it can be written as* (17) *with* $\beta = 1/(2m)$ *i.e.,*

$$\begin{aligned} \dot{x} &= \quad -y\left(1+\frac{m-1}{2m}g_{m-1} + \Theta(H_2)\right) + \frac{x}{2m}\{H_2, g_{m-1}\},\\ \dot{y} &= \quad x\left(1+\frac{m-1}{2m}g_{m-1} + \Theta(H_2)\right) + \frac{y}{2m}\{H_2, g_{m-1}\}, \end{aligned} \tag{20}$$

and $g_{m-1} = g_{m-1}(x,y)$ *is a homogenous polynomial of degree* $m-1$ *such that*
(21)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)g_{m-1} = \begin{cases} -\dfrac{\nu(m-1)(m+1)m}{2}H_2^{(m-3)/2} & \text{if } m \text{ is odd,}\\ 0 & \text{if } m \text{ is even,}\end{cases}$$

where ν *is a constant. Moreover, differential system* (20) *has the first integral*

$$F = \begin{cases} \dfrac{H_2}{\left(1+\dfrac{m-1}{m}g_{m-1}\right)^{1/(m-1)}} & \text{if } m \text{ is even,}\\[2ex] \dfrac{H_2}{\left(1+\dfrac{m-1}{m}g_{m-1}+\nu\, H_2^{(m-1)/2}\right)^{1/(m-1)}} & \text{if } m \text{ is odd,}\end{cases} \tag{22}$$

and has the following expansion at the origin $F := H = H_2(1 + h.o.t.)$. *Thus the holomorphic isochronous center is a weak center.*

Proposition 20 characterizes the holomorphic isochronous centers for the polynomial differential systems with homogenous nonlinearities.

The following result goes back to Poincaré and Liapunov see [15, 12, 17].

Theorem 21 (Poincaré normal form of a nondegenerate center)**.** *For a polynomial differential system* (4)*, there exists a local analytic change of coordinates*

$$u = x + h.o.t., \quad v = y + h.o.t., \tag{23}$$

and an analytic function $\Psi = \Psi(u^2 + v^2)$ *such that the coordinate change* (23) *transforms system* (4) *into the form*

$$\dot{u} = -\frac{\partial H}{\partial v}, \qquad \dot{v} = \frac{\partial H}{\partial u}, \tag{24}$$

where $H = \dfrac{1}{2}\displaystyle\int \left(1 + \Psi(u^2 + v^2)\right) d(u^2 + v^2)$. *Without loss of generality we can assume that* $\Psi(0, 0) = 0$.

Remark 22. *From Propositions 16 and 20 it follows that all the uniform isochronous centers and all the holomorphic isochronous centers for polynomial differential systems with homogenous nonlinerities are always weak centers.*

It is important to observe that there is not a relation between isochronous centers and weak centers, i.e. there exist isochronous centers which are not weak centers and weak centers which are not isochronous centers.

Finally we remark that any linear type center after an analytic change of variables is locally a weak center Indeed from (24) *it follows that there exists a local first integral* $F = u^2 + v^2$.

Proposition 23. *A polynomial differential system* (14) *with a first integral of the form* $F = (x^2 + y^2)\Omega(x, y)$ *where* $\Omega(0, 0) = 1/2$*, after the local analytic change of coordinates*

$$u = x\sqrt{\Omega} \qquad v = y\sqrt{\Omega}, \tag{25}$$

in a neighborhood of the origin becomes

$$\dot{u} = (1 + \tilde{\Phi}(u, v))v, \qquad \dot{v} = -(1 + \tilde{\Phi}(u, v))u, \tag{26}$$

which has the local first integral u^2+v^2*, where* $\tilde{\Phi}(u, v) = \Phi(x, y)\mid_{x=R_1(u,v),\, y=R_2(u,v)}$*, and* $x = R_1(u, v), \quad y = R_2(u, v)$ *is the inverse transformation of* (25).

The next corollary follows from (26) and from the definition of isochronous center.

Corollary 24. *A polynomial differential system* (26) *has an isochronous center at the origin if and only if*

$$\int_0^T \left.\frac{d\vartheta}{1 + \tilde{\Phi}(u, v)}\right|_{u=R\cos\vartheta,\, y=R\sin\vartheta} = T.$$

Remark 25. *The useful of Corollary 24 is very limited. The problem is that, although u and v are given by explicitly simple functions of x and y, this is not necessarily the case for the inverse change of coordinates, i.e. it may be impossible to give a simple expression of x and y as function of u and v. Since the function $1+\tilde{\Phi}(u,v)$ in* (26) *is first obtained in (x,y)–coordinates, it may be difficult in the practice to apply Corollary 24.*

In view of the relation $\dfrac{u}{v}=\dfrac{y}{x}$ it follows that if we pass to the polar coordinates

$$u = R\cos\vartheta, \quad v = R\sin\vartheta, \qquad x = r\cos\theta, \quad y = r\sin\theta,$$

then $\tan\vartheta = \tan\theta$ and consequently $\dot{\vartheta} = \dot{\theta}$.

From Remark 25 and Proposition 11 we get the following corollary.

Corollary 26. *A polynomial differential system* (15) *has an isochronous center at the origin if and only if*

$$\int_0^T \frac{d\theta}{1+g_{m-1}+\dfrac{m+1}{2}\Upsilon_{m-1}}\Bigg|_{x=r\cos\theta,\, y=r\sin\theta} = T.$$

Another important subclass of differential systems (3) with a weak center is formed by the differential systems satisfying the so called *weak condition for a center* (see for instance [1]).

Proposition 27 (Weak condition of the center for polynomial differential systems (3))**.** *The origin is a center of a polynomial differential system* (3) *if there exists $\mu \in \mathbb{R}$ such that*

$$(x^2+y^2)\left(\frac{\partial X_m}{\partial x}+\frac{\partial Y_m}{\partial y}\right) = \mu\,(xX_m + yY_m)\,, \tag{27}$$

and either $m=2k$ is even; or $m=2k-1$ is odd and $\mu \neq 2k$; or $m=2k-1$ is odd, $\mu = 2k$ and (13) *holds.*

In [5] the author proved that if $\mu = 2m$ then system (3) has the rational first integral

$$\frac{x^2+y^2-2\,(xY_m - yX_m)}{(x^2+y^2)^m}.$$

Proposition 27 can be improved as follow.

Theorem 28. *If a polynomial differential system* (3) *satisfies* (27) *and* (13)*, then it has a weak center at the origin and can be written in the form* (17) *with $\beta = \lambda/2$, i.e.*

$$\begin{aligned}
\dot{x} &= \left(1+\frac{2-(m+1)\lambda}{2}g_{m-1}+\Theta(H_2)\right)\{H_2,\,x\}+\frac{\lambda\,x}{2}\{H_2,g_{m-1}\} \\
&= H_2^{1/\lambda}\{F,x\}, \\
\dot{y} &= \left(1+\frac{2-(m+1)\lambda}{2}g_{m-1}+\Theta(H_2)\right)\{H_2,\,y\}+\frac{\lambda\,x}{2}\{H_2,g_{m-1}\} \\
&= H_2^{1/\lambda}\{F,y\}
\end{aligned} \tag{28}$$

where $\lambda = 2/\mu$, being μ the constant with appears in (27)*. The first integral F is such that*

(a) *If* $\lambda(\lambda-1)(\lambda-2/(m+1))\neq 0$, *then*

$$F=\begin{cases}\dfrac{H_2}{(1+(1-(m+1)\lambda)\,g_{m-1})^{\lambda/(1-\lambda)}}, & \textit{if } m \textit{ is even.}\\ \dfrac{H_2}{\left(1+\dfrac{(1-\lambda)}{1-(m+1)\lambda/2}\left((1-(m+1)\lambda/2)\,g_{m-1}+\nu\,H^{(m-1)/2}\right)\right)^{\lambda/(1-\lambda)}} & \textit{if } m \textit{ is odd}\end{cases}$$

(b) *If* $\lambda=1$, *then*

$$F=\begin{cases}H_2e^{-g_{m-1}} & \textit{if } m \textit{ is even}\\ H_2e^{-g_{m-1}+2/(m-1)\nu\,H_2^{(m-1)/2}} & \textit{if } m \textit{ is odd.}\end{cases}$$

(c) *If* $\lambda=0$ *then* $F=H_2$.
(d) *If* $\lambda=2/(m+1)$ *then*

$$F=\begin{cases}\dfrac{H_2}{\left(1+\dfrac{m-1}{m+1}g_{m-1})\right)^{2/(m-1)}}, & \textit{if } m \textit{ is even}\\ \dfrac{H_2}{\left(1+\dfrac{m-1}{m+1}g_{m-1}+\dfrac{m-1}{2}\nu\,H^{(m-1)/2}\log H_2\right)^{2/(m-1)}} & \textit{if } m \textit{ is odd.}\end{cases}$$

Moreover the weak center at the origin of system (28) *is*

(i) *isochronous if* $\lambda=1/m$ *and*

$$\int_0^{2\pi}\frac{G_{m-1}(\cos\theta,\sin\theta)}{\sqrt{(CG_{m-1}(\cos\theta,\sin\theta))^2+C}}d\theta=0, \tag{29}$$

where $G_{m-1}=\dfrac{m-1}{m+1}g_{m-1}+\theta(H_2)$ *and* $C\neq 0$ *is a constant;*

(ii) *a uniform isochronous if* $\Theta(H_2)=0$ *and* $\lambda=2/(m+1)$ *and*

(iii) *a holomorphic isochronous if* $\lambda=1/m$ *and* g_{m-1} *satisfies* (21).

From Proposition 28 it follows the next result due to Devlin [5], note that

$$X_m=\left(\frac{2-(m+1)\lambda}{2}g_{m-1}+\theta(H_2)\right)\{H_2,\,x\}+\frac{\lambda\,x}{2}\{H_2,g_{m-1}\},$$
$$Y_m=\left(\frac{2-(m+1)\lambda}{2}g_{m-1}+\theta(H_2)\right)\{H_2,\,y\}+\frac{\lambda\,y}{2}\{H_2,g_{m-1}\},$$

thus for $\mu=1/m$ we get $xY_{m-1}-yX_{m-1}=(x^2+y^2)\left(\dfrac{m-1}{2m}g_{m-1}+\theta(H_2)\right)=$ $(x^2+y^2)G_{m-1}$, and doing the Taylor expansion of the function

$$\frac{G_{m-1}(\cos\theta,\sin\theta)}{\sqrt{(CG_{m-1}(\cos\theta,\sin\theta))^2+C}}$$

with respect to the variable C in a neighborhood of zero, we get condition (b) of the next corollary.

Corollary 29. *If* (27) *and* (13) *hold, then the polynomial differential system* (3) *admits at the origin an isochronous center if and only if*

(a) *either* $\Omega(\theta) := xY_{m-1} - yX_{m-1}|_{x=\cos\theta,\, y=\sin\theta} = 0$,

(b) *or* $\mu = 2m$ *and* $\displaystyle\int_0^{2\pi} \Omega^k(\theta)d\theta = 0$ *for all odd* $k \geq 1$.

The results of subsection 3.2 are proved in section 5.

3.3. **Polynomial vector fields with homogenous nonlinearities having a focus at the origin.** We state the following inverse problem.

Problem 30. *Determine the polynomial planar vector fields*

$$\mathcal{X} = (-y + X_m)\frac{\partial}{\partial x} + (x + Y_m)\frac{\partial}{\partial y},$$

where $X_m = X_m(x,y)$ *and* $Y_m = Y_m(x,y)$ *are homogenous polynomials of degree* m, *for which*

$$\mathcal{X}(W) = \sum_{j=2}^{\infty} v_j (x^2 + y^2)^{j+1}, \tag{30}$$

where $W = \displaystyle\sum_{j=2}^{\infty} W_j$, $W_2 = \frac{1}{2}(x^2+y^2)$, *and* $W_j = W_j(x,y)$ *is a homogenous polynomial of degree* j, v_j *is a Liapunov constant, and not all the* v_j *are zero.*

The solution of Problem 30 is given in the following propositions.

Proposition 31. *Consider the polynomial vector field*

$$\mathcal{X} = (-y + X_{2k-2})\frac{\partial}{\partial x} + (x + Y_{2k-2})\frac{\partial}{\partial y}$$

where X_{2k-2} *and* Y_{2k-2} *are homogenous polynomials of degree* $m = 2k-2$ *for which* (30) *holds for a given function* W. *Then the polynomial differential system associated to* $\mathcal{X}$ *is*

$$\begin{aligned} \dot{x} =& \quad -y + \{W_{2k-1}, x\} + g_{2k-3}\{W_2, x\} := -y + \mathcal{X}_{2k-2}(x), \\ \dot{y} =& \quad x + \{W_{2k-1}, y\} + g_{2k-3}\{W_2, y\} := x + \mathcal{X}_{2k-2}y, \end{aligned} \tag{31}$$

where $W_2 = \frac{1}{2}(x^2+y^2)$ *and* $g_{2k-3} = g_{2k-3}(x,y)$ *is an arbitrary homogenous polynomial of degree* $2k-3$, *satisfying*

$$\begin{aligned} \mathcal{X}_{2k-2}(W_{2j-2k+5}) + \{W_2, W_{2j+2}\} =& \quad v_j\,(x^2+y^2)^{j+1}, \\ \mathcal{X}_{2k-2}(W_{2j+2-2k}) + \{W_2, W_{2j-1}\} =& \quad 0 \qquad \text{for} \quad j > k-1 \\ W_{2j-1} = 0, \qquad W_{2j} = \nu_j(x^2+y^2)^{j+1}, \qquad v_j =& \quad 0 \quad \text{for} \quad j = 1, \ldots, k-1. \end{aligned} \tag{32}$$

Remark 32. *A polynomial differential system with homogenous nonlinearities having a center at the origin can be written as system* (8), *consequently this system gives a necessary condition in order to have a linear type center, but this condition is not sufficient. Indeed from Proposition 31 it follows that the system with a focus at the origin can be written as* (31) *which is equivalent to* (8), *but not all systems* (31) *satisfy the conditions* (9) *for providing a center. In particular it is well known* *(see [2]) that any quadratic differential system with a center or a focus at the origin can be written as the systems*

$$\begin{aligned} \dot{x} = & \quad -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} = & \quad x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2, \end{aligned} \tag{33}$$

or equivalently (see (8) for $m = 2$) in to the form

$$\dot{x} = \{H_2 + H_3, x\} + g_1\{H_2, x\}, \quad \dot{y} = \{H_2 + H_3, y\} + g_1\{H_2, y\},$$

with

$$\begin{aligned} H_3 = & \quad \frac{1}{3}(\lambda_2 + \lambda_5)x^3 + \lambda_3 x^2 y - \frac{1}{3}(\lambda_4 + \lambda_6)y^3 - \lambda_2 xy^2, \\ g_1 = & \quad \lambda_4 y - \lambda_5 x. \end{aligned}$$

So this proves that the systems of the form (8) not always has a center at the origin.

Proposition 33. *Consider the polynomial vector field*

$$\mathcal{X} = (-y + X_{2k-1})\frac{\partial}{\partial x} + (x + Y_{2k-1})\frac{\partial}{\partial y},$$

where $X_{2k-1} = X_{2k-1}(x, y)$ and $Y_{2k-1} = Y_{2k-1}(x, y)$ for $k \geq 2$ are homogenous polynomials of degree $2k - 1$ for which (30) holds for a given function W. Then the polynomial differential system associated to $\mathcal{X}$ is

$$\begin{aligned} \dot{x} = & \quad -y + \{W_{2k}, x\} + g_{2k-2}\{W_2, x\} + v_{k-1}x\,(x^2 + y^2)^{k-1} \\ := & \quad -y + \mathcal{X}_{2k-1}(x, \\ \dot{y} = & \quad x + \{W_{2k}, y\} + g_{2k-2}\{W_2, y\} + v_{k-1}y\,(x^2 + y^2)^{k-1} \\ := & \quad x + \mathcal{X}_{2k-1}(y), \end{aligned} \tag{34}$$

where $g_{2k-2} = g_{2k-2}(x, y)$ is a homogenous polynomial of degree $2k - 2$, satisfying

$$\begin{aligned} \mathcal{X}_{2k-1}(W_{2j-2k}) + \{W_2, W_{2j-2}\} = & \quad v_{j-2}(x^2 + y^2)^{j-1} \qquad \text{for} \quad j > k, \\ W_{2j} = \nu_j (x^2 + y^2)^{j+1}, \qquad v_j = & \quad 0 \quad \text{for} \quad j = 1, \ldots, k-1, \\ W_{2j+1} = & \quad 0 \qquad \text{for} \quad j \geq 1, \end{aligned} \tag{35}$$

where ν_j is a constant for $j = 1, \ldots, k - 1,$.

Corollary 34. *The Liapunov constants for a polynomial vector field of degree m associated to differential system (31) or (34) can be computing as follows.*

$$v_j = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{X}_{2k-2}(W_{2j-2k+5})|_{x=\cos t,\, y=\sin t}\, dt, \tag{36}$$

where $j \geq k - 1$, if $m = 2k - 2$, and

$$v_j = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{X}_{2k-1}(W_{2j-2k})|_{x=\cos t,\, y=\sin t}\, dt, \tag{37}$$

where $j > k$ if $m = 2k - 1$.

The results of this subsection are proved in section 6.

4. The Proofs of Subsection 3.1

Proof of Theorem 7 . Consider a general polynomial vector field of degree m that we write as

$$\mathcal{X}=\left(\sum_{j=0}^{m}X_j(x,y)\right)\frac{\partial}{\partial x}+\left(\sum_{j=0}^{m}Y_j(x,y)\right)\frac{\partial}{\partial y},$$

where X_j and Y_j for $j=0,1,\ldots,m$ are homogenous polynomials of degree j. Since the analytic first integral H starts with $H_2=(x^2+y^2)/2$, without loss of generality this implies that $X_0(x,y)=Y_0(x,y)=0$, $X_1(x,y)=-y$ and $Y_1(x,y)=x$. Hence the following infinite number of equations follow

$$\begin{aligned}
0= \quad & \frac{dH}{dt}=\left(x+\frac{\partial H_3}{\partial x}+\ldots\right)(-y+X_2+X_3+\ldots)\\
&+\left(y+\frac{\partial H_3}{\partial y}+\ldots\right)(x+Y_2+Y_3+\ldots)\\
= \quad & xX_2+yY_2+\{H_2,H_3\}\\
&+xX_3+yY_3+\frac{\partial H_3}{\partial x}X_2+\frac{\partial H_3}{\partial y}Y_2+\{H_2,H_4\}\\
&+xX_4+yY_4+\frac{\partial H_3}{\partial x}X_3+\frac{\partial H_3}{\partial y}Y_3+\frac{\partial H_4}{\partial x}X_2+\frac{\partial H_4}{\partial y}Y_2+\{H_2,H_5\}+\ldots\\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots\\
&+xX_m+yY_m+\frac{\partial H_3}{\partial x}X_{m-1}+\frac{\partial H_3}{\partial y}Y_{m-1}+\ldots+\frac{\partial H_n}{\partial x}X_2+\frac{\partial H_m}{\partial y}Y_2+\{H_2,H_{m+1}\}\\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

Consequently
(38)

$$\begin{aligned}
xX_2+yY_2+\{H_2,H_3\}= &\quad 0,\\
xX_3+yY_3+\frac{\partial H_3}{\partial x}X_2+\frac{\partial H_3}{\partial y}Y_2+\{H_2,H_4\}= &\quad 0,\\
xX_4+yY_4+\frac{\partial H_3}{\partial x}X_3+\frac{\partial H_3}{\partial y}Y_3+\frac{\partial H_4}{\partial x}X_2+\frac{\partial H_4}{\partial y}Y_2+\{H_2,H_5\}= &\quad 0,\\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad & \quad \vdots\\
xX_m+yY_n+\frac{\partial H_3}{\partial y}Y_{m-1}+\ldots+\frac{\partial H_m}{\partial x}X_2+\frac{\partial H_m}{\partial y}Y_2+\{H_2,H_{m+1}\}= &\quad 0,\\
\frac{\partial H_3}{\partial x}X_m++\frac{\partial H_3}{\partial y}Y_m+\ldots+\frac{\partial H_{m+1}}{\partial x}X_2+\frac{\partial H_{m+1}}{\partial y}Y_2+\{H_2,H_{m+2}\}= &\quad 0,\\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad & \quad \vdots
\end{aligned}$$

The first equation of (38) can be rewritten as follows

$$x\left(X_2+\frac{\partial H_3}{\partial y}\right)+y\left(Y_2-\frac{\partial H_3}{\partial x}\right)=0.$$

Solving it with respect to X_2 and Y_2 we obtain

$$X_2 = -\frac{\partial H_3}{\partial y} - yg_1 = \{H_3, x\} + g_1\{H_2, x\} := \mathcal{X}_2(x),$$
$$Y_2 = \frac{\partial H_3}{\partial x} + xg_1 = \{H_3, y\} + g_1\{H_2, y\} := \mathcal{X}_2(y),$$

where $g_1 = g_1(x, y)$ is an arbitrary homogenous polynomial of degree one. By substituting these polynomials into the second equation of (38) we get

$$x\left(X_3 - \frac{\partial H_4}{\partial y} + g_1\frac{\partial H_3}{\partial y}\right) + y\left(Y_3 - \frac{\partial H_4}{\partial x} - g_1\frac{\partial H_3}{\partial x}\right) = 0.$$

By solving this equation with respect to X_3 and Y_3 we have

$$X_3 = -\frac{\partial H_4}{\partial y} - g_1\frac{\partial H_3}{\partial y} - yg_2 = \{H_4, x\} + g_1\{H_3, x\} + g_2\{H_2, x\} := \mathcal{X}_3(x),$$
$$Y_3 = \frac{\partial H_4}{\partial x} + g_1\frac{\partial H_3}{\partial x} + xg_2 = \{H_4, y\} + g_1\{H_3, y\} + g_2\{H_2, y\} := \mathcal{X}_3(y),$$

where $g_2 = g_2(x, y)$ is an arbitrary homogenous polynomial of degree two. By continuing this process we obtain $X_4, Y_4, \ldots, X_m, Y_m$

$$X_m = \{H_{m+1}, x\} + g_1\{H_m, x\} + \ldots + g_{m-1}\{H_2, x\} := \mathcal{X}_m(x),$$
$$Y_m = \{H_{m+1}, y\} + g_1\{H_m, y\} + \ldots + g_{m-1}\{H_2, y\} := \mathcal{X}_m(y),$$

where $g_j = g_j(x, y)$ is an arbitrary homogenous polynomial of degree j. In particular if $X_j = Y_j = 0 \quad \text{for} \quad 1 < j \leq m-1$ then for simplicity we assume that

$$H_k = 0 \quad \text{for} \quad 3 \leq k \leq m \qquad \text{and} \qquad g_j = 0 \qquad \text{for} \quad 1 < j < m-1. \tag{39}$$

Consequently X_m and Y_m becomes

$$X_m = \{H_{m+1}, x\} + g_{m-1}\{H_2, x\} := \mathcal{X}_m(x),$$
$$Y_m = \{H_{m+1}, y\} + g_{m-1}\{H_2, y\} := \mathcal{X}_m(y),$$

By inserting the previous relations in the remain equations we get the conditions (9). Thus the proof of the theorem follows. □

Proof of Theorem 8. Now we assume that conditions (9) hold, and we shall prove that they imply condition (11). Condition (9) in view of (39) becomes

$$\{H_2, H_{m+k-1}\} = 0,$$

for $3 \leq k \leq m$. Hence

$$H_{m+k-1} = \begin{cases} 0, & \text{if } m+k-1 \text{ is odd} \\ a_k H^{(m+k-1)/2}, & \text{if } m+k-1 \text{ is even.} \end{cases}$$

where a_k is a constant.

For simplicity we assume that

$$H_{m+2} = H_{m+3} = \ldots = H_{2m-1} = 0, \quad H_{2m} \neq 0.$$

Consequently for $m+1\leq k\leq 2m-1$ from (9) we get

$$
\begin{aligned}
0 = \;& \mathcal{X}_m\left(H_{m+1}\right)+\{H_2,H_{2m}\}\\
= \;& \{H_{m+1},H_{m+1}\}+g_{m-1}\{H_2,H_{m+1}\}+\{H_2,H_{2m}\},\\
0 = \;& \{H_2,H_{m+l-1}\} \qquad \text{for} \quad m+1<l<2m,\\
0 = \;& \mathcal{X}_m\left(H_{2m}\right)+\{H_2,H_{3m-1}\}\\
= \;& \{H_{m+1},H_{2m}\}+g_{m-1}\{H_2,H_{2m}\}+\{H_2,H_{3m-1}\}.
\end{aligned} \tag{40}
$$

Again, for simplicity we assume that

$$H_{2m+1}=H_{2m+2}=\ldots=H_{3m-2}=0, \qquad H_{3m-1}\neq 0$$

After some computations from the first equation of (40) we get that

$$
\begin{aligned}
\frac{\partial H_{2m}}{\partial x} &= -g_{m-1}\frac{\partial H_{m+1}}{\partial x}+x\lambda_{2m-2},\\
\frac{\partial H_{2m}}{\partial y} &= -g_{m-1}\frac{\partial H_{m+1}}{\partial y}+y\lambda_{2m-2}.
\end{aligned} \tag{41}
$$

where $\lambda_{2m-2}=\lambda_{2m-2}(x,y)$ is an arbitrary homogenous polynomial of degree $2m-2$, which we choose in such a way that $\dfrac{\partial^2 H_{2m}}{\partial x\partial y}=\dfrac{\partial^2 H_{2m}}{\partial y\partial x}$. Hence λ_{2m-2} must be a solution of the first order partial differential equation

$$\{H_{m+1},g_{m-1}\}+\{\lambda_{2m-2},H_2\}=0. \tag{42}$$

In view of Corollary 6 this equation has a solution if and only if

$$\int_0^{2\pi}\{H_{m+1},g_{m-1}\}\big|_{x=\cos t,\,y=\sin t}\,dt=0.$$

After the integration (41) we obtain

$$H_{2m}=\int_\gamma\left(-g_{m-1}dH_{m+1}+\lambda_{2m-2}dH_2\right),$$

where γ is an oriented curve. Analogously from the last equation of (40) we get

$$
\begin{aligned}
\frac{\partial H_{3m-1}}{\partial x} &= (g_{m-1}^2+\lambda_{2m-2})\frac{\partial H_{m+1}}{\partial x}+x\lambda_{3m-3},\\
\frac{\partial H_{3m-1}}{\partial y} &= (g_{m-1}^2+\lambda_{2m-2})\frac{\partial H_{m+1}}{\partial y}+y\lambda_{3m-3},
\end{aligned} \tag{43}
$$

where $\lambda_{3m-3}=\lambda_{3m-3}(x,y)$ is an arbitrary homogenous polynomial of degree $3m-3$ which we choose in such a way that

$$\{g_{m-1}^2+\lambda_{2m-2},H_{m+1}\}+\{\lambda_{3m-3},H_2\}=0. \tag{44}$$

In view of Corollary 6 this equation has a solution if and only if

$$\int_0^{2\pi}\{g_{m-1}^2+\lambda_{2m-2},H_{m+1}\}\big|_{x=\cos t,\,y=\sin t}\,dt=0.$$

After the integration (41) we obtain

$$H_{3m-1}=\int_\gamma\left((g_{m-1}^2+\lambda_{2m-2})dH_{m+1}+\lambda_{3m-3}dH_2\right),$$

By continuing this process and for simplicity we assume that

$$\begin{aligned} H_{3m} = H_{3m+1} = \ldots = H_{4m-3} = & \quad 0, \qquad H_{4m-2} \neq 0, \\ H_{4m-1} = H_{4m} = \ldots = H_{5m-4} = & \quad 0, \qquad H_{5m-3} \neq 0, \\ \vdots \quad \vdots \quad \vdots \quad \vdots & \quad \vdots \qquad \vdots \end{aligned}$$

From the equation

$$\{H_{m+1}, H_{3m-1}\} + g_{m-1}\{H_2, H_{3m-1}\} + \{H_2, H_{4m-2}\} = 0,$$

and in view of (43), after some computations we deduce that

$$\begin{aligned} \frac{\partial H_{4m-2}}{\partial x} &= -(g_{m-1}^3 + g_{m-1}\lambda_{2m-2} + \lambda_{3m-3})\frac{\partial H_{m+1}}{\partial x} + x\lambda_{4m-4}, \\ \frac{\partial H_{4m-2}}{\partial y} &= -(g_{m-1}^2 + g_{m-1}\lambda_{2m-2} + \lambda_{3m-3})\frac{\partial H_{m+1}}{\partial y} + y\lambda_{4m-4}, \end{aligned}$$

where $\lambda_{4m-4} = \lambda_{4m-4}(x, y)$ is an arbitrary homogenous polynomial of degree $4m-4$ which we choose in such a way that

$$\{H_{m+1}, g_{m-1}^2 + g_{m-1}\lambda_{2m-2} + \lambda_{3m-3}\} + \{\lambda_{4m-42}, H_2\} = 0, \tag{45}$$

In view of Corollary 6 this equation has a solution if and only if

$$\int_0^{2\pi} \{H_{m+1}, g_{m-1}^2 + g_{m-1}\lambda_{2m-2} + \lambda_{3m-3}\}\big|_{x=\cos t,\, y=\sin t}\, dt = 0.$$

After the integration (41) we obtain

$$H_{4m-2} = \int_\gamma \left(-(g_{m-1}^3 + g_{m-1}\lambda_{2m-2} + \lambda_{3m-3}))dH_{m+1} + \lambda_{4m-4}dH_2\right),$$

By continuing this process we finally get that the function H is such that

$$\begin{aligned} H = \; & H_2 + H_{m+1} + H_{2m} + H_{3m-1} + H_{4m-2} + \ldots = H_2 + \sum_{k=0}^{\infty} H_{(1+k)m+1-k} \\ = \; & H_2 + H_{m+1} + \int_\gamma (-g_{m-1} + \lambda_{2m-2} + \lambda_{3m-3} + \lambda_{4m-4} + \ldots)\cdot \\ & \left(1 - g_{m-1} + g_{m-1}^2 - g_{m-1}^3 + \ldots\right) dH_{m+1} + \int_\gamma (\lambda_{2m-2} + \lambda_{3m-3} + \lambda_{4m-4})\, dH_2 \\ = \; & H_2 + H_{m+1} + \int_\gamma \frac{-g_{m-1} + \lambda_{2m-2} + \lambda_{3m-3} + \lambda_{4m-4} + \ldots}{1 + g_{m-1}}\, dH_{m+1} \\ & + \int_\gamma (\lambda_{2m-2} + \lambda_{3m-3} + \lambda_{4m-4})\, dH_2 \\ = \; & \int_\gamma \frac{1 + \lambda_{2m-2} + \lambda_{3m-3} + \lambda_{4m-4} + \ldots}{1 + g_{m-1}}\, dH_{m+1} \\ & + \int_\gamma (1 + \lambda_{2m-2} + \lambda_{3m-3} + \lambda_{4m-4})\, dH_2 \\ := \; & \int_\gamma \left(\frac{1 + \Lambda}{1 + g_{m-1}}\, dH_{m+1} + (1 + \Lambda)\, dH_2\right), \end{aligned}$$

where $\Lambda = \lambda_{2m-2} + \lambda_{3m-3} + \dots$. Here we use the expansion at the neighborhoods of the origin

$$1 - g_{m-1} + g_{m-1}^2 - g_{m-1}^3 + \dots = \frac{1}{1+g_{m-1}}.$$

In view of (42),(44),(45),..., we get

$$\{H_{m+1}, -g_{m-1} + g_{m-1}^2 + \lambda_{2m-2} - g_{m-1}^3 + \lambda_{3m-3} + \dots\} + \{\lambda_{2m-2} + \lambda_{3m-3} + \dots, H_2\}$$
$$= \{H_{m+1}, \frac{1+\Lambda}{1+g_{m-1}}\} + \{1+\Lambda, H_2\} = 0.$$

In short condition (12) holds.

Now we prove that from (11) it follows (9). Indeed, we suppose that the functions H, Λ and $\dfrac{1}{1+g_{m-1}}$ have the following expansion at the neighborhood of the origin

$$H = H_2 + \sum_{k=3}^{\infty} H_k, \qquad H_2 = 1/2(x^2+y^2),$$
$$\Lambda = \sum_{k=1}^{\infty} \Lambda_k, \quad \frac{1}{1+g_{m-1}} = 1 - g_{m-1} + g_{m-1}^2 - g_{m-1}^3 + \dots,$$

where H_j and Λ_j are homogenous polynomial of degree j. Consequently from (11) we obtain

$$\begin{aligned} dH = & \ dH_2 + \sum_{k=3}^{\infty} dH_k = \frac{1+\Lambda}{1+g_{m-1}} dH_{m+1} + (1+\Lambda) dH_2 \\ = & \ (1+\Lambda_1+\Lambda_2+\dots)\left(1 - g_{m-1} + g_{m-1}^2 + \dots\right) dH_{m+1} + (1+\Lambda_1+\Lambda_2+\dots) dH_2, \end{aligned}$$

Hence

$$\begin{aligned} dH_j = & \ \Lambda_{j-2} dH_2, \quad j = 3, \dots m, \\ dH_{m+1} = & \ dH_{m+1} + \Lambda_{m-1} dH_2, \\ dH_{m+j} = & \ \Lambda_{m+j-2} dH_2 + \Lambda_{j-1} dH_{m+1} \quad j = 2, \dots, m-1, \\ dH_{2m} = & \ \Lambda_{2m-2} dH_2 + \Lambda_{m-1} dH_{m+1} - g_{m-1} dH_{m+1}, \\ dH_{2m+j} = & \ \Lambda_{2m+j-2} dH_2 + \Lambda_{m+j-1} dH_{m+1} - g_{m-1}\Lambda_j dH_{m+1} \quad j = 1, \dots, m-2, \\ dH_{3m-1} = & \ \Lambda_{3m-3} dH_2 + \Lambda_{2m-2} dH_{m+1} - g_{m-1}\Lambda_{m-1} dH_{m+1} + g_{m-1}^2 dH_{m+1}, \\ \vdots \quad & \quad \vdots \quad \vdots \end{aligned}$$

where $dH_j = \dfrac{\partial H_j}{\partial x} dx + \dfrac{\partial H_j}{\partial y} dy$.

From the $m-3$ first equations it follows that

$$\begin{aligned} \frac{\partial H_j}{\partial x} = & \ x\Lambda_{j-2}, \\ \frac{\partial H_j}{\partial y} = & \ y\Lambda_{j-2}, \qquad \text{for} \quad j = 3, \dots, m. \end{aligned}$$

Thus $\{H_2, H_j\} = 0$. For simplicity we assume that $H_j = 0$, for $3 \le j \le m$. Consequently $\Lambda_j = 0, \quad 1 \le j \le m-2$. From the $m+1$ equation it follows that $\Lambda_{m-1} = 0$. By inserting in the next equations we get that $dH_{m+j} = \Lambda_{m+j-2} dH_2$ for

$j = 2, \ldots, m-1$. Again for simplicity we assume that $H_{m+j} = 0$ and consequently $\Lambda_{m+j-2} = 0$ for $j = 2, \ldots, m-1$. Thus the rest of equations becomes

$$\begin{aligned} dH_{2m} = &\ \Lambda_{2m-2} dH_2 - g_{m-1} dH_{m+1}, \\ dH_{2m+j} = &\ \Lambda_{2m+j-2} dH_2 \quad j = 1, \ldots, m-2, \\ dH_{3m-1} = &\ \Lambda_{3m-3} dH_2 + \Lambda_{2m-2} dH_{m+1} + g_{m-1}^2 dH_{m+1}, \\ \vdots \quad & \ \vdots \quad \vdots \end{aligned}$$

Again we assume that $H_{2m+j} = 0$ and $\Lambda_{2m+j-2} = 0$ for $j = 1, \ldots, m-2$.

Hence we get that

$$\begin{aligned} \frac{\partial H_{2m}}{\partial x} = &\ x\Lambda_{2m-2} - g_{m-1} \frac{\partial H_{m+1}}{\partial x}, \\ \frac{\partial H_{2m}}{\partial y} = &\ y\Lambda_{2m-2} - g_{m-1} \frac{\partial H_{m+1}}{\partial y}, \\ \frac{\partial H_{3m-1}}{\partial x} = &\ \left(g_{m-1}^2 + \Lambda_{2m-2}\right) \frac{\partial H_{m+1}}{\partial x} + x\Lambda_{3m-3}, \\ \frac{\partial H_{3m-1}}{\partial y} = &\ (g_{m-1}^2 + \Lambda_{2m-2}) \frac{\partial H_{m+1}}{\partial y} + y\Lambda_{3m-3}, \\ \vdots \quad & \ \vdots \quad \vdots \end{aligned}$$

From the equations

$$\frac{\partial H_{2m}}{\partial x} = x\Lambda_{2m-2} - g_{m-1} \frac{\partial H_{m+1}}{\partial x}, \quad \frac{\partial H_{2m}}{\partial y} = y\Lambda_{2m-2} - g_{m-1} \frac{\partial H_{m+1}}{\partial y},$$

it follows that

$$\begin{aligned} \{H_{2m}, H_2\} = &\ g_{m-1} \{H_2, H_{m+1}\}, \\ \{H_2, H_{m+1}\} \Lambda_{2m-2} = &\ \{H_{2m}, H_{m+1}\}. \end{aligned} \tag{46}$$

Thus we deduce the equation (9) for $k = m+1$, i.e. $\mathcal{X}_m(H_{m+1}) + \{H_2, H_{2m}\} = 0$.

On the other hand from the equations

$$\begin{aligned} \frac{\partial H_{3m-1}}{\partial x} = &\ \left(g_{m-1}^2 + \Lambda_{2m-2}\right) \frac{\partial H_{m+1}}{\partial x} + x\Lambda_{3m-3}, \\ \frac{\partial H_{3m-1}}{\partial y} = &\ (g_{m-1}^2 + \Lambda_{2m-2}) \frac{\partial H_{m+1}}{\partial y} + y\Lambda_{3m-3}, \end{aligned}$$

it follows that

$$\begin{aligned} \{H_{3m-1}, H_{m+1}\} = &\ \Lambda_{3m-3} \{H_2, H_{m+1}\}, \\ \{H_{3m-1}, H_2\} = &\ \Lambda_{2m-2} \{H_{m+1}, H_2\} + g_{m-1}^2 \{H_{m+1}, H_2\} \end{aligned}$$

and using (46) we get that

$$\begin{aligned} \{H_{3m-1}, H_2\} = &\ \Lambda_{2m-2} \{H_{m+1}, H_2\} + g_{m-1}^2 \{H_{m+1}, H_2\} \\ = &\ -\Lambda_{2m-2} \{H_2, H_{m+1}\} - g_{m-1} \left(g_{m-1} \{H_2, H_{m+1}\}\right) \\ = &\ g_{m-1} \{H_2, H_{2m}\} + \{H_{m+1}, H_{2m}\} = \mathcal{X}_m(H_{2m}), \end{aligned}$$

hence we get equation (9) for $k = 2m$, i.e. $\mathcal{X}_m(H_{2m}) + \{H_2, H_{3m-1}\} = 0$.

By continuing this process we finally obtain the relations (9). The equivalence between differential system (8) and the equations (10) and (11) is easy to show. In short the theorem is proved. □

Proof of Corollary 9. The proof of statement (a) follows from the proof of Theorem 8. Indeed, by considering that

$$H = H_2 + H_{m+1} + H_{2m} + H_{3m-1} + \ldots = H_2 + \sum_{k=0}^{\infty} H_{(1+k)m+1-k}.$$

Hence, if $m = 2n+1$ then

$$H = H_2 + \sum_{k=0}^{\infty} H_{(1+k)(2n+1)+1-k} = H_2 + \sum_{k=0}^{\infty} H_{2(n(1+k)+1)}.$$

Thus $H(-x,-y) = H(x,y)$, consequently the phase portrait of the differential system in this case is symmetric with respect to the origin. By considering that the origin is a center, then if the the differential system has more centers its number is odd.

The proof of statement (b) follows from the proof of Theorem 8. Indeed, in this case differential system can be written as (8). Hence

$$\frac{\partial \mathcal{X}_m(x)}{\partial x} + \frac{\partial \mathcal{X}_m(y)}{\partial y} = \{H_2, H_{m+1}\}.$$

In view of Proposition 4 we get that (13) holds. Thus the corollary is proved. □

5. The Proofs of Subsection 3.2

Proof of Proposition 11 . Assume that polynomial differential equations (3) has a center at the origin, then they can be written as (8). On the other hand by considering that the center is a weak center, then we get that $H_{m+1} = H_2\Upsilon_{m-1}$. Hence

$$\begin{aligned}\dot{x} = &\ \{H_{m+1}, x\} + (1+g_{m-1})\{H_2, x\} = -H_2\frac{\partial \Upsilon_{m-1}}{\partial y} - y\,(1+\Upsilon_{m-1}+g_{m-1})\\ = &\ -\frac{x^2}{2}\frac{\partial \Upsilon_{m-1}}{\partial y} - \frac{y}{2}\left((m-1)\Upsilon_{m-1} - x\frac{\partial \Upsilon_{m-1}}{\partial x}\right) - y\,(1+\Upsilon_{m-1}+g_{m-1})\\ = &\ -y\left(1+\frac{m+1}{2}\Upsilon_{m-1}+g_{m-1}\right) + \frac{x}{2}\{\Upsilon_{m-1}, H_2\},\end{aligned}$$

here we apply the relation $x\dfrac{\partial \Upsilon_{m-1}}{\partial x} + y\dfrac{\partial \Upsilon_{m-1}}{\partial y} = (m-1)\Upsilon_{m-1}$.

Analogously we deduce the expression for $\dot{y}$. Hence, by comparing with (14) we obtain that

$$\Phi = g_{m-1} + \frac{m+1}{2}\Upsilon_{m-1}, \qquad \varphi = \frac{1}{2}\{\Upsilon_{m-1}, H_2\}.$$

This completes the proof of the proposition. □

Proof of Theorem 14. We shall study the following four cases:

(i) $\beta((m+1)\beta-1)(2\beta-1) \neq 0$,
(ii) $\beta = 1/2$,
(iii) $\beta = 0$,
(iv) $\beta = 1/(m+1)$.

For the case (i) by introducing the homogenous polynomial of degree $m-1$ such that

$$G_{m-1}=(1-\beta(m+1))\,g_{m-1}+\theta(H_2),$$

and by considering that $\{H_2,V\}=0$, we obtain that the differential system (17) becomes

$$\begin{aligned}\dot x=&\quad -y\,(1+G_{m-1})+\frac{\beta}{1-\beta\,(m+1)}\,x\,\{H_2,G_{m-1}\},\\ \dot y=&\quad x\,(1+G_{m-1})+\frac{\beta}{1-\beta\,(m+1)}\,y\{H_2,G_{m-1}\}.\end{aligned}$$

Hence we obtain that

$$\begin{aligned}\dot H_2=&\quad -\frac{2\beta}{(m+1)\beta-1}H_2\{H_2,G_{m-1}\},\\ \dot G_{m-1}=&\quad \left(1-\frac{1-2\beta}{(m+1)\beta-1}G_{m-1}\right)\{H_2,G_{m-1}\}.\end{aligned}$$

Thus $x^2+y^2=0$ and $1-\dfrac{1-2\beta}{(m+1)\beta-1}G_{m-1}=0$ are invariant algebraic curves of the polynomial vector field. Their cofactors are $\dfrac{2\beta\{H_2,g_{m-1}\}}{1-(m+1)\beta}$ and $\dfrac{1-2\beta}{1-(m+1)\beta}\{H_2,g_{m-1}\}$, respectively. Hence, the system has the Darboux first integral

$$F=\frac{H_2}{\left(1-\dfrac{1-2\beta}{(m+1)\beta-1}G_{m-1}\right)^{2\beta/(1-2\beta)}}.$$

Hence we deduce the first integrals F of statements (a) and (b). The Taylor expansion at the origin is $F:=H=H_2(1+h.o.t.)$. Consequently the origin is a weak center in this cases.

The case when $\beta=1/2$ by introducing the homogenous polynomial of degree $m-1$ such that

$$\begin{aligned}G_{m-1}=&\quad -(m-1)/2\;g_{m-1}+\Theta(H_2)\\ =&\quad \begin{cases}-(m-1)/2\;g_{m-1} & \text{if } m \text{ is even}\\ -(m-1)/2\;g_{m-1}+\nu\,H^{(m-1)/2} & \text{if } m \text{ is odd,}\end{cases}\end{aligned}$$

we get that differential system (17) becomes

$$\begin{aligned}\dot x=&\quad -y\,(1+G_{m-1})-\frac{x}{m-1}\{H_2,G_{m-1}\},\\ \dot y=&\quad x\,(1+G_{m-1})-\frac{y}{m-1}\{H_2,G_{m-1}\},\end{aligned}$$

thus we deduce that

$$\dot H_2=-\frac{2H_2}{m-1}\{H_2,G_{m-1}\},\quad \dot G_{m-1}=\{H_2,G_{m-1}\}.$$

Consequently $\dfrac{dH_2}{dG_{m-1}}=-\dfrac{2H_2}{m-1}$, thus $F=H_2e^{2/(m-1)\;G_{m-1}}$, is a first integral. This prove statements (c) and (d)

For the case when $\beta=0$ differential system (17) becomes

$$\dot x=-y\,(1+g_{m-1})+,\quad \dot y=x\,(1+g_{m-1})\,,$$

consequently $F = H_2$ is a first integral. This prove statement (e).

For the case when $\beta = 1/(m+1)$ we get that differential system (17) becomes

$$
\begin{aligned}
\dot{x} = &\quad -y\,(1+\Theta(H_2)) + \frac{x}{m+1}\,\{H_2, g_{m-1}\},\\
\dot{y} = &\quad x\,(1+\Theta(H_2)) + \frac{y}{m+1}\{H_2, g_{m-1}\},
\end{aligned}
\tag{47}
$$

hence

$$
\dot{H}_2 = \frac{2}{m+1}\,H_2\{H_2, g_{m-1}\}, \qquad \dot{g}_{m-1} = \{H_2, g_{m-1}\}\left(1 + \frac{m-1}{m+1}\,g_{m-1} + \Theta(H_2)\right).
$$

consequently

$$
\frac{dg_{m-1}}{dH_2} = \frac{m-1}{2H_2}\,g_{m-1} + (m+1)\frac{1+\Theta(H_2}{2H_2}.
$$

After the integration this linear system we get

$$
g_{m-1} = H_2^{(m+1)/2}\left(C + \frac{m+1}{2}\int \frac{(1+\theta(H_2)dH_2}{H_2^{(m+1)/2}}\right),
$$

where C is an arbitrary constant. Hence we get

$$
\frac{g_{m-1}}{H_2^{(m-1)/2}} - \frac{m+1}{2}\int \frac{(1+\theta(H_2)dH_2}{H_2^{(m+1)/2}} = C
$$

After some computation we deduce that the first integral is

$$
\begin{aligned}
F = &\quad \frac{H_2}{\left(1 + \dfrac{m-1}{m+1}g_{m-1} + \dfrac{m-1}{2}H^{(m-1)/2}\displaystyle\int \frac{\theta(H_2)}{H_2^{(m-1)/2}}dH_2\right)^{2/(m-1)}}\\
= &\quad \begin{cases}
\dfrac{H_2}{\left(1 + \dfrac{m-1}{m+1}\,g_{m-1}\right)^{2/(m-1)}} & \text{if } m \text{ is even},\\[2ex]
\dfrac{H_2}{\left(1 + \dfrac{m-1}{m+1}g_{m-1} + \nu\dfrac{m-1}{2}H_2^{(m-1)/2}\log H_2\right)^{2/(m-1)}} & \text{if } m \text{ is odd}.
\end{cases}
\end{aligned}
$$

It is easy to show that the curve $H_2 = 0$ and

$$
\begin{aligned}
1 + G = &\quad 1 + \frac{m-1}{m+1}g_{m-1} + \frac{m-1}{2}H^{(m-1)/2}\int \frac{1+\theta(H_2)}{H_2^{(m+1)/2}}dH_2 = 0\\
= &\quad \begin{cases}
1 + \dfrac{m-1}{m+1}\,g_{m-1} = 0 & \text{if } m \text{ is even},\\[2ex]
1 + \dfrac{m-1}{m+1}g_{m-1} + \nu\dfrac{m-1}{2}H_2^{(m-1)/2}\log H_2 = 0 & \text{if } m \text{ is odd}
\end{cases}
\end{aligned}
$$

are invariant curves of differential system (47). Indeed, the following relation holds

$$
\frac{d}{dt}(1+G) = (1+G)\{H_2, G\}, \qquad \frac{d}{dt}(H_2) = \frac{2H_2}{m-1}\{H_2, G\}.
$$

Clearly that $\{H_2, G\} = \dfrac{m-1}{m+1}\{H_2, g_{m-1}\}$, thus this cofactor is a polynomial of degree $m-1$. This prove statements (f) and (g). □

This completes the proof of the proposition.

Proof of Proposition 16. It is well known that if the origin is a center of (3), then this center is uniform if and only if

$$\dot{x} = -y + x\varphi, \qquad \dot{y} = x + y\varphi,$$

where $\varphi = \varphi(x, y)$ is a homogenous polynomial of degree $m-1$. Thus in view of Proposition 11 the uniform center is a particular case of a weak center (15) with

$$\Phi = g_{m-1} + \frac{m+1}{2}\Upsilon_{m-1} = 0, \qquad \varphi = \frac{1}{m+1}\{H_2, g_{m-1}\}.$$

Hence we easily get (18) and comparing with (17) it follows that

$$\Theta(H_2) = 0, \qquad \text{and} \quad \beta = \frac{1}{m+1}.$$

Hence the first integral F becomes (see case (f) of Theorem 14)

$$F = \frac{H_2}{\left(1 + \dfrac{m-1}{m+1}g_{m-1}\right)^{2/(m-1)}}.$$

The Taylor expansion at the origin is $F := H = H_2(1 + h.o.t.)$. This completes the proof of the proposition. □

Proof of Corollary 17. From the proof of Theorem 16 it follows that $\dot{\theta} = 1$ and from $F = C$ by solving with respect to r we get the proof of the corollary. □

Proof of Proposition 20. Polynomial differential system (3) with a center at the origin satisfies the Cauchy–Riemannn conditions if and only if the homogenous polynomial H_{m+1} and g_{m-1} are such that

$$\text{(48)} \qquad \begin{aligned} \Delta H_{m+1} + 2g_{m-1} &= \begin{cases} 0 & \text{if m is odd,} \\ \nu\, m(m+1)H_2^{(m-1)/2} & \text{if m is even.} \end{cases} \\ (x^2+y^2)g_{m-1} + 2mH_{m+1} &= \begin{cases} 0 & \text{if m is odd,} \\ 2m\nu\, H_2^{(m+1)/2} & \text{if m is even,} \end{cases} \end{aligned}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$. Indeed, from (19) with

$$P = -\frac{\partial H_{m+1}}{\partial y} - y(1 + g_{m-1}), \qquad Q = \frac{\partial H_{m+1}}{\partial x} + x(1 + g_{m-1}),$$

it follows that

$$\begin{aligned} y\frac{\partial g_{m-1}}{\partial x} + x\frac{\partial g_{m-1}}{\partial y} &= -2\frac{\partial^2 H_{m+1}}{\partial y \partial x}, \\ y\frac{\partial g_{m-1}}{\partial y} - x\frac{\partial g_{m-1}}{\partial x} &= \frac{\partial^2 H_{m+1}}{\partial x \partial x} - \frac{\partial^2 H_{m+1}}{\partial y \partial y}. \end{aligned}$$

Hence

$$(x^2+y^2)\frac{\partial g_{m-1}}{\partial x} = -2y\frac{\partial^2 H_{m+1}}{\partial x\partial y} - x\frac{\partial^2 H_{m+1}}{\partial x\partial x} + x\frac{\partial^2 H_{m+1}}{\partial y\partial y},$$
$$(x^2+y^2)\frac{\partial g_{m-1}}{\partial y} = -2x\frac{\partial^2 H_{m+1}}{\partial x\partial y} + y\frac{\partial^2 H_{m+1}}{\partial x\partial x} - y\frac{\partial^2 H_{m+1}}{\partial y\partial y}.$$

Consequently

$$(x^2+y^2)\frac{\partial g_{m-1}}{\partial x} = x\Delta H_{m+1} - 2m\frac{\partial H_{m+1}}{\partial x},$$
$$(x^2+y^2)\frac{\partial g_{m-1}}{\partial y} = y\Delta H_{m+1} - 2m\frac{\partial H_{m+1}}{\partial y},$$

here we apply the relation $x\dfrac{\partial H}{\partial x} + y\dfrac{\partial H_{m+1}}{\partial y} - H_{m+1} = mH_{m+1}$, or equivalently

$$\frac{\partial}{\partial x}\big((x^2+y^2)g_{m-1} + 2mH_{m+1}\big) = x\left(\Delta H_{m+1} + 2g_{m-1}\right),$$
$$\frac{\partial}{\partial y}\big((x^2+y^2)g_{m-1} + 2mH_{m+1}\big) = y\left(\Delta H_{m+1} + 2g_{m-1}\right).$$

Thus we have

$$x\frac{\partial}{\partial y}\left(\Delta H_{m+1} + 2g_{m-1}\right) - y\frac{\partial}{\partial x}\left(\Delta H_{m+1} + 2g_{m-1}\right) = 0.$$

Therefore (48) follows. Hence we get that

$$H_{m+1} = H_2\Upsilon_{m-1} := \begin{cases} -\dfrac{1}{m}H_2 g_{m-1} & \text{if m is odd,}\\ -\dfrac{1}{m}H_2 g_{m-1} + \nu\, H_2^{m+1/2} & \text{if m is even.}\end{cases}$$

Analogously to the proof of Proposition 11 we deduce differential system (20). By comparing with (17) we get that $\beta = \dfrac{1}{2m}$. Consequently the from Theorem 14 case (b) we get the first integral (22). In short the proposition is proved. □

Proof of Proposition 23. By Proposition 11 if a differential system (3) has the first integral $F = (x^2+y^2)\Omega$, then it can be written as system (14). Consequently $\dot{H}_2 = 2H_2\varphi$. On the other hand from the condition $\dot{F} = 0$ we get

$$\dot{\sqrt{\Omega}} = -\sqrt{\Omega}\varphi.$$

Consequently, if $u = x\sqrt{\Omega}$ and $v = y\sqrt{\Omega}$, then

$$\dot{u} = \dot{x}\sqrt{\Omega} + x\dot{\sqrt{\Omega}} = -(1+\Phi(x,y))y\sqrt{\Omega} = -(1+\tilde{\Phi})(u,v)\,v,$$

analogously we deduce that $\dot{v} = -(1+\tilde{\Phi})(u,v)\,u$. □

Proof of Corollary 26 . From (15) it follows that

$$x\dot{y} - y\dot{x} = 2H_2\left(1 + g_{m-1} + \frac{m+1}{2}\Upsilon_{m-1}\right),$$

which in polar coordinates (r,θ) becomes

$$\dot{\theta} = \left(1 + g_{m-1} + \frac{m+1}{2}H_{m+1}\right)\Big|_{x=r\cos\theta,\, y=r\sin\theta},$$

hence it is easy to obtain the proof of the corollary. □

Proof of Corollary 26. From (15) it follows that

$$x\dot{y} - y\dot{x} = (x^2 + y^2)\left(1 + g_{m-1} + \frac{m+1}{2}\Upsilon_{m-1}\right),$$

which in polar coordinates (r, θ) becomes $\dot{\theta} = 1 + g_{m-1} + \frac{m+1}{2}\Upsilon_{m-1}\Big|_{x=r\cos\theta,\, y=r\sin\theta}$.
Hence we easily obtain the proof of the corollary. □

Proof of Theorem 28. Since (27) and (13) hold, by Proposition 27 system has a center at the origin. Consequently, by Theorem 7 it can be written as

$$\begin{aligned} \dot{x} = & \quad -y + X_m = \{H_2 + H_{m+1}, x\} + g_{m-1}\{H_2, x\}, \\ \dot{y} = & \quad x + Y_m = \{H_2 + H_{m+1}, y\} + g_{m-1}\{H_2, y\}. \end{aligned} \tag{49}$$

It is easy to prove that relation (27) can be written as

$$\frac{\partial}{\partial x}\left(\frac{-y + X_m}{H_2^{\mu/2}}\right) + \frac{\partial}{\partial y}\left(\frac{x + Y_m}{H_2^{\mu/2}}\right) = 0.$$

Hence

$$\begin{aligned} \dot{x} = -y + X_m = & \quad \{H_2 + H_{m+1}, x\} + g_{m-1}\{H_2, x\} = H_2^{\mu/2}\{F, x\}, \\ \dot{y} = x + Y_m = & \quad \{H_2 + H_{m+1}, y\} + g_{m-1}\{H_2, y\} = H_2^{\mu/2}\{F, y\}, \end{aligned}$$

where F is a first integral which we determine below.

From (49) follows that

$$\frac{\partial X_m}{\partial x} + \frac{\partial Y_m}{\partial y} = \{H_2, g_{m-1}\}, \quad xX_m + yY_m = \{H_{m+1}, H_2\},$$

consequently

$$\lambda H_2\{H_2, g_{m-1}\} = \{H_{m+1}, H_2\} \Longrightarrow \{H_2, H_{m+1} + \lambda H_2 g_{m-1}\} = 0,$$

where $\lambda = 2/\mu$. Thus in view of Corollary 6 we get that

$$H_{m+1} := H_2\Upsilon_{m-1} = \begin{cases} -\lambda H_2 g_{m-1} & \text{if } m \text{ is even }, \\ -\lambda H_2 g_{m-1} + \nu H_2^{(m-1)/2} & \text{if } m \text{ is odd}, \end{cases}$$

where $\nu \in \mathbb{R}$.

Hence we get that

$$
\begin{aligned}
\dot x = &\ -y - \frac{\partial H_{m+1}}{\partial y} - y g_{m-1}\\
= &\ - H_2 \frac{\partial \Upsilon_{m-1}}{\partial y} - y\,(1+(1-\lambda)\,g_{m-1}+\Upsilon_{m-1}))\\
= &\ \frac{x}{2}\{\Upsilon_{m-1},H_2\} - y\left(1+(1-\lambda)\,g_{m-1}+\frac{m+1}{2}\Upsilon_{m-1})\right)\\
= &\ \frac{\lambda\, x}{2}\{H_2,g_{m-1}\} - y\left(1+(1-\frac{\lambda(m+1)}{2})\,g_{m-1}+\Theta(H_2))\right),\\
\dot y = &\ x + \frac{\partial H_{m+1}}{\partial x} + x g_{m-1}\\
= &\ H_2 \frac{\partial \Upsilon_{m-1}}{\partial x} + x\,(1+(1-\lambda)\,g_{m-1})+\Upsilon_{m-1})\\
= &\ \frac{y}{2}\{\Upsilon_{m-1},H_2\} + x\left(1+(1-\lambda)\,g_{m-1}+\frac{m+1}{2}\Upsilon_{m-1})\right)\\
= &\ \frac{\lambda\, y}{2}\{H_2,g_{m-1}\} + x\left(1+(1-\frac{\lambda(m+1)}{2})\,g_{m-1}+\Theta(H_2))\right),
\end{aligned}
\tag{50}
$$

where

$$
\Theta(H_2) = \begin{cases} 0 & \text{if } m \text{ is even,}\\ \nu\, H^{(m-1)/2} & \text{if } m \text{ is odd.}\end{cases}
$$

From this differential system we obtain the expression (28) for differential system (3) which satisfies the condition (13) and (27).

Now we show that differential system (28) is integrable. Indeed from equations (28) and (17) it follows that $\beta = \dfrac{\lambda}{2}$. Hence, from Theorem 14 after some computations we get the proof of statements (a), (b), (c) and (d) of Theorem 28.

The second condition (13) of the center given in Proposition 27 in this case takes the form

$$
\begin{aligned}
\int_0^{2\pi}\left(\frac{\partial \mathcal{X}_m(x)}{\partial x}+\frac{\partial \mathcal{X}_m(y)}{\partial y}\right)|_{x=\cos t,\,y=\sin t}dt = &\ \int_0^{2\pi}\{H_2,H_{m+1}\}\,|_{x=\cos t,\,y=\sin t}dt\\
= &\ \int_0^{2\pi}\frac{dH_{m+1}}{dt}\,|_{x=\cos t,\,y=\sin t}dt = 0.
\end{aligned}
$$

Now we shall prove statement (i). Indeed if in (28) we take $\lambda = 1/m$, then the differential system becomes

$$
\begin{aligned}
\dot x = &\ -y\,(1+G_{m-1}) + \frac{x}{m-1}\{H_2,G_{m-1}\},\\
\dot y = &\ x\,(1+G_{m-1}) + \frac{y}{m-1}\{H_2,G_{m-1}\},
\end{aligned}
$$

where $G_{m-1} = \dfrac{m-1}{2m}g_{m-1}+\theta(H_2)$. The first integral F takes the form $F^{m-1} = \dfrac{H_2^{m-1}}{(1+2G_{m-1})}$. Thus $\dfrac{H_2^{m-1}}{1+2G_{m-1}} = C$. Since G_{m-1} is a homogenous polynomial of degree $m-1$, then in polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ we have $\zeta^2 - 2\Psi(\theta)\zeta -$

$C=0$, with $\zeta=r^{m-1}$, where $G_{m-1}=\Psi(\varphi)r^{m-1}$. Hence $\zeta=C\Psi\pm\sqrt{C^2\Psi^2+C}$. Therefore, from the equation $\dot\varphi=1+G_{m-1}(r\cos\varphi,\,r\sin\varphi)=1+\zeta\Psi(\varphi)$, we get

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{1+\Psi(C\Psi\pm\sqrt{C^2\Psi^2+C})}&=\int_0^{2\pi}\frac{Cd\theta}{C+C^2\Psi^2\pm C\Psi\sqrt{C^2\Psi^2+C})}\\ =\int_0^{2\pi}\frac{Cd\theta}{\sqrt{C^2\Psi^2+C}(\sqrt{C^2\Psi^2+C}\pm\Psi C)}&=\int_0^{2\pi}\frac{(\sqrt{C^2\Psi^2+C}\mp\Psi C)d\theta}{\sqrt{C^2\Psi^2+C}}\\ =\quad 2\pi\mp C\int_0^{2\pi}\frac{\Psi d\theta}{\sqrt{C^2\Psi^2+C}}=2\pi,\end{aligned}$$

thus (29) holds. This completes the proof of statement (i). Now we prove statement (ii). If $\Theta(H_2)=0$, and $\lambda=2/(m+1)$ then differential system (50) becomes

$$\dot x=-y+\frac{x}{m+1}\{H_2,g_{m-1}\},\qquad \dot y=x+\frac{y}{m+1}\{H_2,g_{m-1}\}.$$

Thus $x\dot y-y\dot x=x^2+y^2$. So the center is a uniform isochronous center.

Finally we prove statement (iii). If $\lambda=1/m$ then differential system (28) can be written as (20). If g_{m-1} is a homogenous polynomial of degree $m-1$ which satisfies (21), then from Corollary 20 the proof follows. □

6. The Proofs of Subsection 3.3

Proof of Proposition 31. From (6) it follows that

$$\begin{aligned}0=&\quad\{W_2,W_3\}+xX_2+yY_2,\\ v_1(x^2+y^2)^2&=\{W_2,W_4\}+xX_3+yY_3\\ &+\frac{\partial W_3}{\partial x}X_2+\frac{\partial W_3}{\partial y}Y_2,\\ 0=&\quad\{W_2,W_5\}+xX_4+yY_4\\ &+\frac{\partial W_3}{\partial x}X_3+\frac{\partial H_3}{\partial y}Y_3\\ &+\frac{\partial W_4}{\partial x}X_2+\frac{\partial W_4}{\partial y}Y_2,\\ v_2(x^2+y^2)^3&=\{W_2,W_6\}+xX_5+yY_5\\ &+\frac{\partial W_3}{\partial x}X_4+\frac{\partial H_3}{\partial y}Y_4\\ &+\frac{\partial W_4}{\partial x}X_3+\frac{\partial W_4}{\partial y}Y_3+\frac{\partial W_5}{\partial x}X_2+\frac{\partial W_5}{\partial y}Y_2,\\ \vdots\qquad&\quad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\end{aligned}\tag{51}$$

$$\begin{aligned}
0 = \quad & \{W_2, W_{2k-3}\} + xX_{2k-2} + yY_{2k-2} \\
& + \frac{\partial W_3}{\partial x} X_{2k-3} + \frac{\partial W_3}{\partial y} Y_{2k-3} \\
& + \ldots + \frac{\partial W_{2k-4}}{\partial x} X_2 + \frac{\partial W_{2k-4}}{\partial y} Y_2, \\
v_{k-1}(x^2+y^2)^k \quad & = \{W_2, W_{2k}\} + xX_{2k-1} + yY_{2k-1} \\
& + \frac{\partial W_3}{\partial x} X_{2k-2} + \frac{\partial W_3}{\partial y} Y_{2k-2} \\
& + \ldots + \frac{\partial W_{2k-3}}{\partial x} X_2 + \frac{\partial W_{2k-3}}{\partial y} Y_2, \\
\vdots \qquad & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \ddots
\end{aligned}$$

First we solve Problem 30 for $m = 2$. Thus $X_j = Y_j = 0$ for $j > 2$. The first equation of (51) can be rewritten as follows

$$x\left(X_2 + \frac{\partial W_3}{\partial y}\right) + y\left(Y_2 - \frac{\partial W_3}{\partial x}\right) = 0.$$

Solving it with respect to X_2 and Y_2 we obtain

$$\begin{aligned}
X_2 = \quad & -\frac{\partial W_3}{\partial y} - yg_1 = \{W_3, x\} + g_1\{W_2, x\} := \mathcal{X}_2(x), \\
Y_2 = \quad & \frac{\partial W_3}{\partial x} + xg_1 = \{W_3, y\} + g_1\{W_2, y\} := \mathcal{X}_2(y),
\end{aligned}$$

where $g_1 = g_1(x, y)$ is an arbitrary homogenous polynomial of degree one. By substituting these polynomials into the rest of equations of (52) for $j = 1, 2, \ldots$ we get

$$\begin{aligned}
\mathcal{X}_2(W_{2j+1}) = \quad & v_j(x^2+y^2)^{j+1} + \{W_{2j+2}, W_2\}, \\
\mathcal{X}_2(W_{2j+2}) = \quad & \{W_{2j+3}, W_2\}.
\end{aligned}$$

Now we solve Problem 30 for $m = 3$. Thus $X_2 = Y_2 = X_j = Y_j = 0$ for $j > 3$. From the second equation of (51) we deduce that

$$x\left(X_3 - xv_1(x^2+y^2) - \frac{\partial W_4}{\partial y}\right) + y\left(Y_3 - yv_1(x^2+y^2) - \frac{\partial W_4}{\partial x}\right) = 0.$$

By solving this equation with respect to X_3 and Y_3 we have

$$\begin{aligned}
X_3 = \quad & -\frac{\partial W_4}{\partial y} - yg_2 + xv_1(x^2+y^2) \\
= \quad & \{W_4, x\} + xv_1(x^2+y^2) := \mathcal{X}_3(x), \\
Y_3 = \quad & \frac{\partial W_4}{\partial x} + xg_2 + yv_1(x^2+y^2) \\
= \quad & \{W_4, y\} + g_2\{W_2, y\} + yv_1(x^2+y^2) := \mathcal{X}_3(y),
\end{aligned}$$

where $g_2 = g_2(x, y)$ is an arbitrary homogenous polynomial of degree two. Inserting in the rest of the equations of (51) we get for $j = 1, 2, \ldots$ that

$$\begin{aligned}
\mathcal{X}_3(W_{2j}) = \quad & v_j(x^2+y^2)^{j+1} + \{W_{2j+2}, W_2\}, \\
\{W_{2j+1}, W_2\} = \quad & 0.
\end{aligned}$$

Now we solve Problem 30 for $m > 3$. Thus $X_j = Y_j = 0$ for $2 \le j < m$ and $j > m$. First we study the case when $m = 2k - 2 \ge 4$. From (51) it follows that

$$
\begin{aligned}
\{W_3, W_2\} &= 0,\\
v_1(x^2+y^2)^2 + \{W_4, W_2\} &= 0,\\
\{W_5, W_2\} &= 0,\\
v_2(x^2+y^2)^3 + \{W_6, W_2\} &= 0,\\
\vdots \qquad\qquad & \vdots \qquad \vdots \qquad \vdots\\
v_{k-1}(x^2+y^2)^{k-1} + \{W_{2k-2}, W_2\} &= 0,\\
\{W_{2k-1}, W_2\} &= xX_{2k-2} + yY_{2k-2},\\
v_{k-1}(x^2+y^2)^{k} + \{W_{2k}, W_2\} &= \frac{\partial W_3}{\partial x}X_{2k-2} + \frac{\partial W_3}{\partial y}Y_{2k-2},\\
\{W_{2k+1}, W_2\} &= \frac{\partial W_4}{\partial x}X_{2k-2} + \frac{\partial W_4}{\partial y}Y_{2k-2},\\
\vdots \qquad\qquad & \vdots \qquad \vdots \qquad \vdots
\end{aligned} \tag{52}
$$

Consequently in view of Proposition 4, from (52) we get that

$$
W_{2j-1} = 0, \qquad W_{2j} = \nu_j (x^2+y^2)^{j+1}, \qquad v_j = 0 \quad \text{for} \quad j = 1, \dots, k-1.
$$

X_m and Y_m becomes

$$
\begin{aligned}
X_{2k-2} = &\quad \{H_{2k-1}, x\} + g_{2k-3}\{H_2, x\} := \mathcal{X}_{2k-2}(x),\\
Y_{2k-2} = &\quad \{H_{2k-1}, y\} + g_{2k-3}\{H_2, y\} := \mathcal{X}_{2k-2}(y).
\end{aligned}
$$

Substituting in the remaining equations of (52) we get the differential equations (32). Thus the proof of the proposition follows. $\square$

Proof of Proposition 33. From (6) it follows that

$$
\begin{aligned}
\{W_3, W_2\} &= 0,\\
v_1(x^2+y^2)^2 + \{W_4, W_2\} &= 0,\\
\{W_5, W_2\} &= 0,\\
v_2(x^2+y^2)^3 + \{W_6, W_2\} &= 0,\\
\vdots \qquad\qquad & \vdots \qquad \vdots\\
v_{k-1}(x^2+y^2)^{k-1} + \{W_{2k-2}, W_2\} &= 0,\\
\{W_{2k-1}, W_2\} &= 0,\\
v_{k-1}(x^2+y^2)^{k} + \{W_{2k}, W_2\} &= xX_{2k-1} + yY_{2k-1},\\
\{W_{2k+1}, W_2\} &= \frac{\partial W_3}{\partial x}X_{2k-1} + \frac{\partial W_3}{\partial y}Y_{2k-1},\\
v_{k}(x^2+y^2)^{k+1} + \{W_{2k+2}, W_2\} &= \frac{\partial W_4}{\partial x}X_{2k-1} + \frac{\partial W_4}{\partial y}Y_{2k-1},\\
\vdots \qquad\qquad & \vdots \qquad \vdots \qquad .
\end{aligned}
\tag{53}
$$

Consequently in view of Proposition 4, from (53) we get that $W_{2j+1} = 0$ for $j \geq 1$, $W_{2j} = \nu_j(x^2+y^2)^{j+1}$, $v_j = 0$ for $j = 1, \ldots, k-1$, and X_{2k-1} and Y_{2k-1} becomes

$$
\begin{aligned}
X_{2k-1} = &\ \{H_{2k}, x\} + g_{2k-2}\{H_2, x\} + v_k x(x^2+y^2)^{k-1} := \mathcal{X}_{2k-1}(x),\\
Y_{2k-1} = &\ \{H_{2k}, y\} + g_{2k-2}\{H_2, y\} + v_k y(x^2+y^2)^{k-1} := \mathcal{X}_{2k-1}(y).
\end{aligned}
$$

Substituting in the rest of equations of (53) we get the differential equations (35). Thus the proof of the proposition follows. □

Proof of Corollary 34 . It is easy to obtain from (32) and (35), by considering Proposition 4. Indeed, from (32) it follows that

$$
\int_0^{2\pi} \left(v_j(x^2+y^2)^{j+1} - \mathcal{X}_{2k-2}(W_{2j-2k+5}) - \{W_2, W_{2j+2}\}\right)\big|_{x=\cos t,\, y=\sin t}\, dt,
$$

consequently, by considering Corollary 6 we get

$$
v_j = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{X}_{2k-2}(W_{2j-2k+5})\big|_{x=\cos t,\, y=\sin t}\, dt,
$$

hence (36) follows.

Analogously, from (35) we obtain (37). Thus the corollary is proved. □

6.1. **Classification of quadratic and cubic planar differential system with a weak center.** For nondegenerate quadratic center (see for instance [2]) and cubic center (see for instance [16, 22]) the center problem has been solved in terms of algebraic equalities satisfied by the coefficients.

Proposition 35. *For the quadratic polynomial differential system* (33) *the origin is a center if and only if one of the following four conditions holds*

$$
(54)\qquad
\begin{array}{rrl}
(i) & \lambda_4=\lambda_5= & 0,\\
(ii) & \lambda_2=\lambda_5= & 0,\\
(iii) & \lambda_3-\lambda_6= & 0,\\
(iv) & \lambda_5= & 0,\quad \lambda_4+5(\lambda_3-\lambda_6)=0,\quad \lambda_3\lambda_6-2\lambda_6^2-\lambda_2^2=0.
\end{array}
$$

Theorem 36. *A quadratic polynomial differential system* (33) *has a weak center at the origin if and only if a linear change of coordinates x, y and a scaling of time t it can be written as one of the following systems*

$$
(55)\qquad
\begin{aligned}
\dot{x} =&\ -y\,(1+3\lambda_3 y)-\lambda_3 x^2,\\
\dot{y} =&\ x\,(1+3\lambda_3 y)-\lambda_3 xy;
\end{aligned}
$$

$$
(56)\qquad
\begin{aligned}
\dot{x} =&\ -y\,((1-\lambda_6 y)-\lambda_3 x^2,\\
\dot{y} =&\ x\,((1-\lambda_6 y)-\lambda_3 xy;
\end{aligned}
$$

$$
(57)\qquad
\begin{aligned}
\dot{x} =&\ -y\,(1+\lambda_2 x-\lambda_6 y)-x(\lambda_2 x+\lambda_6 y),\\
\dot{y} =&\ x\,(1+\lambda_2 x-\lambda_6 y)-y(\lambda_2 x+\lambda_6 y);
\end{aligned}
$$

$$
(58)\qquad
\begin{aligned}
\dot{x} =&\ -y-\lambda_3 x^2,\\
\dot{y} =&\ x-\lambda_3 xy;
\end{aligned}
$$

$$
(59)\qquad
\begin{aligned}
\dot{x} =&\ -y\,(1-\lambda_6 y)-2\lambda_6 xy,\\
\dot{y} =&\ x\,(1-\lambda_6 y)-2\lambda_6 y^2.
\end{aligned}
$$

Proof. Indeed, differential system (33) can be rewritten as (14) if and only if

$$
(60)\qquad \lambda_4+\lambda_6+3\lambda_3=0,\qquad \lambda_5+4\lambda_2=0.
$$

Consequently system (33) becomes

$$
\begin{aligned}
\dot{x} =&\ -y(1+\lambda_2 x-\lambda_6 y)-x(\lambda_3 x+\lambda_2 y),\\
\dot{y} =&\ x(1+\lambda_2 x-\lambda_6 y)-y(\lambda_3 x+\lambda_2 y),
\end{aligned}
$$

In view of (54) and taking into account the condition (60) we get $\lambda_4=\lambda_2=\lambda_5=0$ and $\lambda_6=-3\lambda_3$, then we obtain the differential system (55); $\lambda_5=\lambda_2=0$, and $\lambda_4+\lambda_6+3\lambda_3=0$, then we obtain the differential system (56); $\lambda_3=\lambda_6,\ \lambda_5=-4\lambda_2$, and $\lambda_4=-4\lambda_3$, then we obtain the differential system (57); $\lambda_5=\lambda_2=0$, $\lambda_4+5(\lambda_3-\lambda_6)=0$ and $\lambda_6(\lambda_3-2\lambda_6)=0$, then we have either $\lambda_5=\lambda_2=\lambda_6=0$, $\lambda_4+5\lambda_3=0$, or $\lambda_5=\lambda_2=0, \lambda_3=2\lambda_6$ and $\lambda_4+5\lambda_6=0$. Therefore we get the differential system (58) or (59). In short, the theorem is proved. □

Remark 37. *In the paper* [3] *the classification of isochronous quadratic centers is given. From this classification there are only two isochronous centers which are weak centers, that can be obtained from equation* (56) *for $\lambda_3=\lambda_6=-1$, and $\lambda_3=-1$, $\lambda_6=0$. These systems are*

$$
\dot{x}=-y+x^2-y^2,\quad \dot{y}=x+2xy,
$$

and

$$\dot{x} = -y + x^2, \quad \dot{y} = x + xy,$$

respectively.

Proposition 38. *For the cubic polynomial differential system*

$$\begin{aligned} \dot{x} = & \ -y + Ax^3 + Bx^2y + Cxy^2 + Dy^3, \\ \dot{y} = & \ x + Kx^3 + Lx^2y + M\,x\,y^2 + Ny^3, \end{aligned} \tag{61}$$

the origin is a center if and only if one of the following sets of conditions hold

$$\begin{aligned} (i) \qquad 3A + L + C + 3N = & \ 0, \\ (3A+L)(B+D+K+M) - 2(A-N)(B+M) = & \ 0, \\ 2(A+N)\left((3A+L)^2 - (B+M)^2\right) & \\ +(3A+L)(B+M)(B+K-D-M) = & \ 0, \\ (ii) \qquad 3A + L + C + 3N = & \ 0, \\ 2A + C - L - 2N = & \ 0, \\ B + 3D - 3K - M = & \ 0, \\ B + 5D + 5K + M = & \ 0, \\ (A+3N)(3A+N) - 16DK = & \ 0. \end{aligned} \tag{62}$$

Theorem 39. *A cubic polynomial differential system* (61) *has a weak center at the origin if and only if after a linear change of coordinates x, y and a scaling of time t it can be written as one of the following systems If $N \neq 0$,*

$$\begin{aligned} \dot{x} = & \ -y\left(1 + Kx^2 + (N+L)xy + \left(\frac{K-B}{2} - \frac{LB+LK}{2N}\right)y^2\right) \\ & +x\left(N(y^2 - x^2) + (K+B)xy\right), \\ \dot{y} = & \ x\left(1 + Kx^2 + (N+L)xy + \left(\frac{K-B}{2} - \frac{LB+LK}{2N}\right)y^2\right) \\ & +y\left(N(y^2 - x^2) + (K+B)xy\right); \end{aligned} \tag{63}$$

if $N = 0$, then we have either the system

$$\begin{aligned} \dot{x} = & \ -y\left(1 + Kx^2 - Dy^2\right) + (K+B)x^2y, \\ \dot{y} = & \ x\left(1 + Kx^2 - Dy^2\right) + (K+B)xy^2; \end{aligned} \tag{64}$$

or the system

$$\begin{aligned} \dot{x} = & \ -y\left(1 + Lxy - Bx^2 - Dy^2\right), \\ \dot{y} = & \ x\left(1 + Lxy - Bx^2 - Dy^2\right). \end{aligned} \tag{65}$$

Proof. Indeed, differential system (61) can be rewritten as (14) if and only if

$$L + C = A + N, \qquad M + D = B + K. \tag{66}$$

Consequently (61) becomes

$$\begin{aligned} \dot{x} = & \ -y\left(1 + (N-C)xy + Kx^2 - Dy^2\right) + x\left((L-N+C)x^2 + (B+K)xy + Ny^2\right), \\ \dot{x} = & \ x\left(1 + (N-C)xy + Kx^2 - Dy^2\right) + y\left((L-N+C)x^2 + (B+K)xy + Ny^2\right), \end{aligned}$$

By solving (66) together with (62) (i) we deduce that this system have no solution. Thus center of cubic system (61) under the conditions (62) (i) is not a weak center.

By solving (66) together with (62) (ii) we deduce that this system have three solution.

(i) If $N \neq 0$ then $A + N = 0$ and $L + C = 0,\ 2ND = N(B - K) + L(B + K)$. This solution provides the differential system (63).

If $N = 0$ then there exist two solutions.

(ii) If $N = 0$ then $A = 0$ and $L = 0$. This solution provides the differential system (64).

- (iii)] If $N = 0$ then $A = 0$ and $K + B = 0$. This solution provides the differential system (65).

  Thus the centers of cubic systems (61) under the conditions (62) (ii) generate three weak centers.

In short, the theorem is proved. □

Remark 40. *In the paper* [3] *the classification of the all isochronous cubic centers is given. From this classification there are two isochronous centers which are weak centers, that can be obtained from* (63) *for* $N = -1$, $B = K = 0$, *and* $N = -L = -1,\ K = B = 0$, *These cubic systems are*

$$\begin{aligned} \dot{x} = &\quad -y + x^3 - 3xy^2 = -y(1 + 2xy) + x(x^2 - y^2), \\ \dot{y} = &\quad x + 3x^2y - y^3 = x(1 + 2xy) + y(x^2 - y^2), \end{aligned}$$

and

$$\begin{aligned} \dot{x} = &\quad -y + x^3 - xy^2 = -y + x(x^2 - y^2), \\ \dot{y} = &\quad x + x + x^2y - y^3 = x + y(x^2 - y^2), \end{aligned}$$

respectively.

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain.
E-mail address: jllibre@mat.uab.cat

[2] Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Avinguda dels Països Catalans 26, 43007 Tarragona, Catalonia, Spain.
E-mail address: rafaelorlando.ramirez@urv.cat

[3] Universitat de Barcelona, Gran Vía de las Cortes Catalanas, 585 08007 Barcelona, Spain.
E-mail address: vramirsa8@alumnes.ub.edu