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Restricted independence in displacement function for better estimation of cyclicity *

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Abstract

Since the independence of focal values is a sufficient condition to give a number of limit cycles arising from a center-focus equilibrium, in this paper we consider a restricted independence to a parametric curve, which gives a method not only to increase the lower bound for the cyclicity of the center-focus equilibrium but also to be available when those focal values are not independent. We apply the method to a nondegenerate cubic center-focus variety and prove that the cyclicity reaches its an upper bound.

Keywords: center-focus variety; cyclicity; focal value; independence; power sequence.

AMS 2010 Classification: 34C07; 34C23

1 Introduction

As it appears in the books [3, 5, 19] and articles [4, 6, 13, 14, 18, 20], the discussion on the center-focus equilibria is one of the most important problems in ordinary differential equations. A center-focus equilibrium is an equilibrium at which the linear part of the differential system has a pair of nonvanished pure imaginary eigenvalues. The main interest on the research of these equilibria is the determination of the kind

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of their stability and on their *cyclicity* at this equilibrium, i.e., the greatest number of limit cycles which may arise from a Hopf bifurcation at these equilibria.

We generally consider the family of analytic systems

$$\dot{x} = \alpha x - y + P(x, y, \boldsymbol{\lambda}), \quad \dot{y} = x + \alpha y + Q(x, y, \boldsymbol{\lambda}) \tag{1.1}$$

with a standardized linear part, where $\alpha \in \mathbb{R}$ and $\boldsymbol{\lambda} := (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m$ are parameters, $P(x, y, \boldsymbol{\lambda})$ and $Q(x, y, \boldsymbol{\lambda})$ are real analytic functions of x and y starting at least with terms of degree two and depending polynomially on $\boldsymbol{\lambda}$. This family, denoted by $LC(\alpha, \boldsymbol{\lambda})$ for short, is referred as a family of linear centers, which has either a center or a focus at the origin $O : (0, 0)$ clearly. Its *center-focus variety* is

$$\mathcal{CF} := \{(\alpha, \boldsymbol{\lambda}) \in \mathbb{R} \times \mathbb{R}^m : \alpha = 0\} \cong \mathbb{R}^m.$$

In this variety the so-called center-focus equilibrium O needs to be identified between focus (called a *weak focus*) and center, which is decided if using finitely many focal values (see [3] for more details). Focal values come from the coefficients of the displacement function $\Pi(\rho) := h(\rho) - \rho$, where h is the Poincaré return map

$$h(\rho) := e^{2\pi\alpha}\rho + \sum_{i=2}^{+\infty} \mathrm{g}_i(\alpha, \boldsymbol{\lambda})\rho^i \tag{1.2}$$

and g_i's are analytic functions of α and $\boldsymbol{\lambda}$ such that $\mathrm{g}_2(0, \boldsymbol{\lambda}) \equiv 0$. Let $g_i(\boldsymbol{\lambda}) := \mathrm{g}_i(0, \boldsymbol{\lambda})$ for all $i = 2, 3, ...$ Since P, Q are assumed to be polynomially dependent on $\boldsymbol{\lambda}$, $g_i \in \mathbb{R}[\boldsymbol{\lambda}]$ (the ring of real polynomials in the variable $\boldsymbol{\lambda}$) for all $i \geq 2$ and $g_{2k} \in \langle g_3, ..., g_{2k-1}\rangle$ (the ideal generated by $g_3, ..., g_{2k-1}$ over the ring $\mathbb{R}[\boldsymbol{\lambda}]$) for all $k \geq 2$, as indicated in [1, 3]. Those g_{2i+1}'s are called the *focal values*, which are algebraically equivalent to the Lyapunov quantities ([15, 16]). Note that $\mathcal{CF}$ contains the subset

$$\mathcal{C} := V(g_3, ..., g_{2i+1}, ...),$$

where $V(g_3, ..., g_{2i+1}, ...)$ denotes the algebraic variety of $\langle g_3, ..., g_{2i+1}, ...\rangle$ and, by [7, p. 3], is actually the set of all common zeros of all g_{2i+1}'s ($i \geq 1$). As in [3, p. 11], $\mathcal{C}$ is called the *center variety* of the family $LC(\alpha, \boldsymbol{\lambda})$ because the center-focus O of system $LC(0, \boldsymbol{\lambda}')$ is a *center* if and only if $\boldsymbol{\lambda}' \in \mathcal{C}$. For any $\boldsymbol{\lambda}' \in \mathbb{R}^m \backslash \mathcal{C}$ there exists an integer $k \geq 1$ such that $g_{2k+1}(\boldsymbol{\lambda}') \neq 0$ and $g_i(\boldsymbol{\lambda}') = 0$ for all $i = 2, ..., 2k$, for which we call O a *weak focus of multiplicity* k in system $LC(0, \boldsymbol{\lambda}')$.

As usual the *cyclicity* of a center-focus is the maximal number of limit cycles emerging from it in the phase portrait when we change slightly the parameters of the system (see [3, 8, 10, 17] and references therein). More precisely, the greatest number of limit cycles bifurcated from O is called the *cyclicity* of system $LC(0, \boldsymbol{\lambda}')$ at O (perturbed within the family $LC(\alpha, \boldsymbol{\lambda})$) and denoted by $\mathcal{N}(\boldsymbol{\lambda}')$. In particular, $\mathcal{N}(\boldsymbol{\lambda}')$ is denoted by $\mathcal{N}_c(\boldsymbol{\lambda}')$ (resp. $\mathcal{N}_f(\boldsymbol{\lambda}')$) and called *center cyclicity* (resp. *focus cyclicity*) *of system* $LC(0, \boldsymbol{\lambda}')$ if $\boldsymbol{\lambda}' \in \mathcal{C}$ (resp. $\boldsymbol{\lambda}' \in \mathbb{R}^m \backslash \mathcal{C}$).

The cyclicity $\mathcal{N}(\boldsymbol{\lambda}')$ is decided not only by the multiplicity of the center-focus equilibrium, but also by the greatest number of independent sign changes in the displacement function near $(0, \boldsymbol{\lambda}')$. For multi-parametric families there are more difficulties in finding the greatest number of nonvanished focal values, but one usually gives its a lower bound. For such an independence, a well-known method is to check the following conditions:

(ID$_k$-1) every neighborhood of $\boldsymbol{\lambda}'$ contains a $\boldsymbol{\mu}' \in V(g_3, ..., g_{2k-1})$ such that $g_{2k+1}(\boldsymbol{\mu}') \neq 0$, and

(ID$_k$-2) for each positive integer $\ell \leq k-1$ and each $\boldsymbol{\mu}' \in V(g_3, ..., g_{2\ell+1})$ satisfying that $g_{2\ell+3}(\boldsymbol{\mu}') \neq 0$, every neighborhood of $\boldsymbol{\mu}'$ contains a $\boldsymbol{\mu}'' \in V(g_3, ..., g_{2\ell-1})$ such that $g_{2\ell+1}(\boldsymbol{\mu}'')g_{2\ell+3}(\boldsymbol{\mu}') < 0$.

Note that there is only **(ID$_1$-1)** if $k = 1$. As indicated in [2], **(ID$_k$-1)** and **(ID$_k$-2)** are known as conditions for the first k focal values g_{2j+1}, $j = 1, ..., k$, to be independent, under which $\mathcal{N}(\boldsymbol{\lambda}') \geq k$ as shown in [12]. Another method ([3]) is to determine the rank $r(\boldsymbol{\lambda}')$ of the Jacobian matrix

$$\frac{\partial(g_3, ..., g_{2j+1})}{\partial \boldsymbol{\lambda}}\bigg|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}'},$$

where all $g_3, ..., g_{2j+1}$ vanish at $\boldsymbol{\lambda}'$, which asserts $\mathcal{N}(\boldsymbol{\lambda}') \geq r(\boldsymbol{\lambda}')$ because it implies the existence of an independent subsequence of $r(\boldsymbol{\lambda}')$ members in $\{g_3, ..., g_{2j+1}\}$ as indicated in ([2]). However, it is not easy to verify the independence of focal values or compute the rank of the Jacobian with many parameters. Besides, conditions **(ID$_k$-1)** and **(ID$_k$-2)** are strong sufficient conditions for the independence, which remind us to find weaker ones. The rank of the Jacobian gives a lower bound for $\mathcal{N}(\boldsymbol{\lambda}')$, but this bound may not be the best.

In this paper we give a method to increase the lower bound for the cyclicity $\mathcal{N}(\boldsymbol{\lambda}')$ of system (1.1). The method is to find an appropriate curve passing through $(0, \boldsymbol{\lambda}')$ in the space $\mathbb{R}^{m+1}$ of parameters $(\alpha, \boldsymbol{\lambda})$, on which those focal values depend on a single variable in such a way that we can determine easily the number of independent sign changes in the displacement function on the curve, called an *restricted independence* to the curve. Such a restriction may give a larger number of independent sign changes than the rank of the Jabobian, from which we can find more limit cycles bifurcating from the origin O. The result about this method is given in section 2. In section 3 we give some corollaries for easier applications, and practical application of our method with an example. This example has a nonvanished 4th order focal value but does not satisfy the independence condition of focal values, from which one cannot assert that the cyclicity of O is 4. However, using our method, we prove that the cyclicity is exactly 4. Finally, in section 4 we apply the method to a five-parametric family of cubic systems for finding its $\mathcal{N}(\boldsymbol{\lambda}')$.

2 Restricted independence

For $\boldsymbol{\lambda}' \in \mathbb{R}^m \setminus \mathcal{C}$ the equilibrium O is a weak focus of system $LC(0, \boldsymbol{\lambda}')$. Let $\zeta(\boldsymbol{\lambda}')$ be the multiplicity of the weak focus. Then $\zeta(\boldsymbol{\lambda}')$ gives an upper estimate for $\mathcal{N}_f(\boldsymbol{\lambda}')$ because

$$g_{2\zeta(\boldsymbol{\lambda}')+1}(\boldsymbol{\lambda}') \neq 0, \quad \text{and} \quad g_i(\boldsymbol{\lambda}') = 0 \quad \forall i < 2\zeta(\boldsymbol{\lambda}') + 1, \tag{2.1}$$

and the sequence of g_{2k+1}'s ($k = 1, ..., \zeta(\boldsymbol{\lambda}')$) may not be independent. In contrast, for $\boldsymbol{\lambda}' \in \mathcal{C}$, equilibrium O is a center of system $LC(0, \boldsymbol{\lambda}')$. Since $\mathbb{R}\{\boldsymbol{\lambda}\}_{\boldsymbol{\lambda}'}$, the ring of convergent power series at $\boldsymbol{\lambda}'$, is a Noetherian ring (see [9, p. 147]), every ideal in this ring is finitely generated, which implies the existence of a least integer $\iota(\boldsymbol{\lambda}') > 0$ satisfying that

$$\langle g_3, g_5, ..., g_{2\iota(\boldsymbol{\lambda}')+1} \rangle_{\boldsymbol{\lambda}'} = \langle g_3, ..., g_{2i+1}, ... \rangle_{\boldsymbol{\lambda}'} \quad \text{in } \mathbb{R}\{\boldsymbol{\lambda}\}_{\boldsymbol{\lambda}'}. \tag{2.2}$$

The integer $\iota(\boldsymbol{\lambda}')$, called the *multiplicity of the center* O, gives an upper estimate for $\mathcal{N}_c(\boldsymbol{\lambda}')$. In general, we see that the field $\mathbb{R}$ is a commutative Noetherian ring, which implies by the Hilbert Basis Theorem ([9, p. 144]) that $\mathbb{R}[\boldsymbol{\lambda}]$ is a Noetherian ring. Therefore, there exists the least integer $\iota_p \geq 1$ such that

$$\langle g_3, g_5, ..., g_{2\iota_p+1} \rangle = \langle g_3, ..., g_{2i+1}, ... \rangle \quad \text{in } \mathbb{R}[\boldsymbol{\lambda}]. \tag{2.3}$$

It follows that $V(g_3, ..., g_{2\iota_p+1}) = V(g_3, ..., g_{2i+1}, ...)$ and $\max\{\zeta(\boldsymbol{\lambda}'), \iota(\boldsymbol{\lambda}')\} \leq \iota_p$.

For $\boldsymbol{\lambda}' \in \mathbb{R}^m$ we need to discuss the sign changes among those focal values g_{2i+1} and the real part α of the eigenvalues near $(\alpha, \boldsymbol{\lambda}) = (0, \boldsymbol{\lambda}')$. For convenience, define

$$g_1(\alpha) := 2\pi\alpha$$

complementarily. Our strategy is to restrict those g_{2i+1}'s ($i = 0, 1, ..., \kappa(\boldsymbol{\lambda}')$), where $\kappa(\boldsymbol{\lambda}') = \zeta(\boldsymbol{\lambda}')$ (or $\iota(\boldsymbol{\lambda}')$) if $\boldsymbol{\lambda}' \in \mathbb{R}^m \setminus \mathcal{C}$ (or $\mathcal{C}$), to a curve in the space $\mathbb{R}^{m+1}$ of parameters $(\alpha, \boldsymbol{\lambda})$ to see independent sign changes in the displacement function. Consider a continuous curve Υ of the form

$$(\alpha(\eta), \boldsymbol{\lambda}(\eta)) := (d_0\eta^{\alpha_0}, \lambda_1' + d_1\eta^{\alpha_1}, ..., \lambda_m' + d_m\eta^{\alpha_m}) \tag{2.4}$$

in the parameter space $\mathbb{R}^{m+1}$, where $d_i \in \mathbb{R}$ and $\alpha_i > 0$ are indeterminate constants, $i = 0, ..., m$, and $(\lambda_1', ..., \lambda_m') = \boldsymbol{\lambda}'$. Clearly, $(\alpha(0), \boldsymbol{\lambda}(0)) = (0, \boldsymbol{\lambda}')$, i.e., the curve passes through the point $(0, \boldsymbol{\lambda}')$. The curve is of polynomial form if all α_i's are positive integers. Restricted to the curve Υ given in (2.4), the focal values are of the form

$$g_1(\alpha(\eta)) = c_0\eta^{w_0} + o(\eta^{w_0}), \quad g_{2i+1}(\boldsymbol{\lambda}(\eta)) = c_i\eta^{w_i} + o(\eta^{w_i}), \quad i = 1, ..., \kappa(\boldsymbol{\lambda}'), \tag{2.5}$$

where w_i's are positive constants depending on the α_i's and the c_i's are real constants depending on the d_i's such that, for each i, $c_i \neq 0$ if and only if $g_{2i+1}(\boldsymbol{\lambda}(\eta)) \not\equiv 0$. In particular,

$$c_{\kappa(\boldsymbol{\lambda}')} = g_{2\kappa(\boldsymbol{\lambda}')+1}(\boldsymbol{\lambda}') \neq 0 \quad \text{and} \quad w_{\kappa(\boldsymbol{\lambda}')} = 0$$

if $\boldsymbol{\lambda}' \in \mathbb{R}^m \setminus \mathcal{C}$.

Our method of restricted independence highly depends on the evolution in the power sequence $w := \{w_i\}$, where the w_i's are reals unless that all α_i's are chosen as integers, because then the w_i's are integers. It is worthy mentioning that the sequence $\{w_i\}$ may not be increasing although the corresponding g_{2i+1} is given by the coefficient of the term ρ^{2i+1} in the return map (1.2). Let

$$\Delta(i,j)w := \frac{w_i - w_j}{i-j},$$

called the *difference quotient of* w *between* i *and* j. For $i_0 < ... < i_k$ in $\{0, 1, ..., \kappa(\boldsymbol{\lambda}')\}$, the power sequence $\{w_i\}$ is said to be *ladder-likely degressive* on the scale $(i_0, ..., i_k)$ if there are constants $h_{i_0} > h_{i_1} > ... > h_{i_k} > 0$, called the *degressive rates*, such that

(LD) for each $\nu = 0, ..., k$,

$$\Delta(i_\nu, j)w \begin{cases} \leq -h_{i_\nu} & \forall j = 0, ..., i_\nu - 1, \\ \geq -h_{i_\nu} & \forall j = i_\nu + 1, ..., \kappa(\boldsymbol{\lambda}'). \end{cases}$$

Obviously, the sequence $\{7, 4, 2, 1\}$ is ladder-likely degressive on the scale $(0, 1, 2, 3)$, where we note that $\{3, 2, 1\}$, the sequence of differences between two consecutive terms, is strictly decreasing and we can choose the sequence $\{6, 5/2, 3/2, 1/2\}$ for degressive rates. Note that the concept of ladder-like degressiveness does not require the sequence $\{w_i\}$ to be decreasing but needs the existence of a decreasing subsequence of $\{w_i\}$ with weaker and weaker degressive rates correspondingly. For example, the sequence $\{7, 4, 2, 4, 1\}$ does not decrease but has a ladder-likely degressive scale $(0, 1, 4)$ with the sequence $(14, 5/2, 1/4)$ of degressive rates.

Considering the "=" in (**LD**), for each i_ν define

$$\Xi(i_\nu) := \{i_\nu\} \cup \{j \in \{0, ..., \kappa(\boldsymbol{\lambda}')\} : \ \Delta(i_\nu, j)w = -h_{i_\nu}\},$$

the set of all j's having the same slope $-h_{i_\nu}$ with respect to i_ν. Let

$$\begin{aligned} \hat{c}_{i_\nu} &:= \sum_{j \in \Xi(i_\nu)} c_j, \\ \mathcal{V} &:= \{i_\nu \in \{i_0, ..., i_k\} : \ \exists j \in \{\nu+1, ..., k\} \text{ such that } \hat{c}_{i_\nu}\hat{c}_{i_j} < 0 \\ &\qquad \text{and } \hat{c}_{i_l} = 0 \ \forall l = \nu+1, ..., j-1\}, \end{aligned} \tag{2.6}$$

where c_i is the leading coefficient of g_{2i+1}as given in (2.5). Clearly $\mathcal{V}$ is a set of indices for independent sign changes, i.e. an independence restricted to the parameterized curve Υ.

Theorem 2.1. *Suppose that the power sequence of* g_{2i+1}*'s, the focal values of family* $LC(\alpha, \boldsymbol{\lambda})$ *given in* (1.1) *near* $(0, \boldsymbol{\lambda}')$, *restricted to the parameterized curve* (2.4) *is ladder-likely degressive on* $(i_0, ..., i_k)$, *i.e., condition* (**LD**) *holds. Then* $\mathcal{N}(\boldsymbol{\lambda}') \geq \#\mathcal{V}$, *the cardinality of the set* $\mathcal{V}$.

Proof. Let i_s be the greatest member of $\mathcal{V}$ and $s' \in \{s+1, ..., k\}$ be the corresponding j given in the definition of $\mathcal{V}$. Clearly,

$$i_s < i_{s'} \quad \text{and} \quad \hat{c}_{i_s}\hat{c}_{i_{s'}} < 0.$$

From (1.2) we get

$$\begin{aligned} \Pi(\rho) &= g_1(\alpha)\rho(1+\Psi_0(\alpha)) + \sum_{i=2}^{+\infty} \mathrm{g}_i(\alpha, \boldsymbol{\lambda})\rho^i \\ &= \sum_{i=0}^{+\infty} g_{2i+1}\rho^{2i+1}\left(1+\Psi_{2i+1}(\alpha, \boldsymbol{\lambda}, \rho)\right), \end{aligned} \tag{2.7}$$

where $\Psi_0(\alpha)$ is an analytic function such that$\Psi_0(0) = 0$, and the functions Ψ_{2i+1}'s are analytic in $(\alpha, \boldsymbol{\lambda}, \rho)$ and vanish at $(0, \boldsymbol{\lambda}, 0)$. On the other hand, by (**LD**) we get $w_{i_\nu} - w_j \le -(i_\nu - j)h_{i_\nu}$, i.e. $w_{i_\nu} + i_\nu h_{i_\nu} \le w_j + jh_{i_\nu}$ for all $j = 0, ..., \kappa(\boldsymbol{\lambda}')$. This implies that for each $\nu \in \{0, ..., k\}$ we have

$$w_{i_\nu} + (i_\nu+1)h_{i_\nu} \le w_j + (j+1)h_{i_\nu} \qquad \forall j = 0, ..., \kappa(\boldsymbol{\lambda}'). \tag{2.8}$$

In what follows we use (2.7) and (2.8) to discuss in $\mathbb{R}^m \backslash \mathcal{C}$ and $\mathcal{C}$ separately.

For $\boldsymbol{\lambda}' \in \mathbb{R}^m \setminus \mathcal{C}$, by (2.3) and (2.7) we get

$$\Pi(\rho) = \sum_{i=0}^{\iota_p} g_{2i+1}\rho^{2i+1}\left(1+\Phi_{2i+1}(\alpha, \boldsymbol{\lambda}, \rho)\right), \tag{2.9}$$

where Φ_{2i+1}'s are analytic at $(\alpha, \boldsymbol{\lambda}, \rho)$ and vanish when $\alpha = \rho = 0$. Restricted to the curve $(\alpha, \boldsymbol{\lambda}) = (\alpha(\eta), \boldsymbol{\lambda}(\eta))$, from (2.9) we obtain

$$\begin{aligned} \Pi(\rho) \;&=\; \sum_{i=0}^{\zeta(\boldsymbol{\lambda}')} c_i\eta^{w_i}\rho^{2i+1}\left(1+H_{2i+1}(\eta, \rho)\right) \\ &\quad + \begin{cases} \displaystyle\sum_{i=\zeta(\boldsymbol{\lambda}')+1}^{\iota_p} g_{2i+1}(\alpha(\eta), \boldsymbol{\lambda}(\eta))\rho^{2i+1}\left(1+\Phi_{2i+1}(\alpha(\eta), \boldsymbol{\lambda}(\eta), \rho)\right) & \text{if } \zeta(\boldsymbol{\lambda}') < \iota_p, \\ 0 & \text{if } \zeta(\boldsymbol{\lambda}') = \iota_p, \end{cases} \end{aligned}$$

where $H_{2i+1}(\eta, \rho) \to 0$ as $(\eta, \rho) \to (0, 0)$. Then, for each $\nu = 0, 1, ..., k$, we have

$$\begin{aligned} \Pi(\eta^{h_{i_\nu}/2}) &= \sum_{i=0}^{\zeta(\boldsymbol{\lambda}')} c_i\eta^{w_i+(i+1/2)h_{i_\nu}}\left(1+\tilde{H}_{2i+1}(\eta)\right) \\ &\quad + \begin{cases} \displaystyle\sum_{i=\zeta(\boldsymbol{\lambda}')+1}^{\iota_p} g_{2i+1}(\alpha(\eta), \boldsymbol{\lambda}(\eta))\eta^{(i+1/2)h_{i_\nu}}\left(1+\tilde{H}_{2i+1}(\eta)\right) & \text{if } \zeta(\boldsymbol{\lambda}') < \iota_p, \\ 0 & \text{if } \zeta(\boldsymbol{\lambda}') = \iota_p, \end{cases} \end{aligned} \tag{2.10}$$

where

$$\tilde{H}_{2i+1}(\eta) := H_{2i+1}(\eta, \eta^{h_{i_\nu}/2}) \to 0 \qquad \text{as } \eta \to 0. \tag{2.11}$$

Applying (2.8) to (2.10) we get

$$\begin{aligned}\Pi(\eta^{h_{i_\nu}/2}) &= \eta^{w_{i_\nu}+(i_\nu+1/2)h_{i_\nu}}\sum_{i\in\Xi(i_\nu)} c_i\left(1+\tilde{H}_{2i+1}(\eta)\right)\\
&\quad+\sum_{i\in\{0,\dots,\zeta(\boldsymbol{\lambda}')\}\setminus\Xi(i_\nu)} c_i\eta^{w_i+(i+1/2)h_{i_\nu}}\left(1+\tilde{H}_{2i+1}(\eta)\right)\\
&\quad+\begin{cases}\displaystyle\sum_{i=\zeta(\boldsymbol{\lambda}')+1}^{\iota_p} g_{2i+1}(\alpha(\eta),\boldsymbol{\lambda}(\eta))\eta^{(i+1/2)h_{i_\nu}}\left(1+\tilde{H}_{2i+1}(\eta)\right) & \text{if } \zeta(\boldsymbol{\lambda}')<\iota_p,\\ 0 & \text{if } \zeta(\boldsymbol{\lambda}')=\iota_p,\end{cases}\\
&= \eta^{w_{i_\nu}+(i_\nu+1/2)h_{i_\nu}}\left\{\hat{c}_{i_\nu}+\sum_{i\in\Xi(i_\nu)} c_i\tilde{H}_{2i+1}(\eta)\right\}\\
&\quad+\sum_{i\in\{0,\dots,\zeta(\boldsymbol{\lambda}')\}\setminus\Xi(i_\nu)} c_i\eta^{w_i+(i+1/2)h_{i_\nu}}\left(1+\tilde{H}_{2i+1}(\eta)\right)\\
&\quad+\begin{cases}\displaystyle\sum_{i=\zeta(\boldsymbol{\lambda}')+1}^{\iota_p} g_{2i+1}(\alpha(\eta),\boldsymbol{\lambda}(\eta))\eta^{(i+1/2)h_{i_\nu}}\left(1+\tilde{H}_{2i+1}(\eta)\right) & \text{if } \zeta(\boldsymbol{\lambda}')<\iota_p,\\ 0 & \text{if } \zeta(\boldsymbol{\lambda}')=\iota_p,\end{cases}\\
&= \hat{c}_{i_\nu}\eta^{w_{i_\nu}+(i_\nu+1/2)h_{i_\nu}}\left(1+\Psi_f(\eta)\right) \qquad \forall\ i_\nu\in\mathcal{V}\cup\{i_{s'}\},\end{aligned} \tag{2.12}$$

where $\hat{c}_{i_\nu}$ is defined in (2.6) and

$$\begin{aligned}\Psi_f(\eta) &:= \sum_{i\in\Xi(i_\nu)}\frac{c_i}{\hat{c}_{i_\nu}}\tilde{H}_{2i+1}(\eta)+\sum_{i\in\{0,\dots,\zeta(\boldsymbol{\lambda}')\}\setminus\Xi(i_\nu)}\frac{c_i}{\hat{c}_{i_\nu}}\eta^{w_i-w_{i_\nu}+(i-i_\nu)h_{i_\nu}}\left(1+\tilde{H}_{2i+1}(\eta)\right)\\
&\quad+\begin{cases}\displaystyle\sum_{i=\zeta(\boldsymbol{\lambda}')+1}^{\iota_p}\frac{g_{2i+1}(\alpha(\eta),\boldsymbol{\lambda}(\eta))}{\hat{c}_{i_\nu}}\eta^{(i-i_\nu)h_{i_\nu}-w_{i_\nu}}\left(1+\tilde{H}_{2i+1}(\eta)\right) & \text{if } \zeta(\boldsymbol{\lambda}')<\iota_p,\\ 0 & \text{if } \zeta(\boldsymbol{\lambda}')=\iota_p.\end{cases}\end{aligned} \tag{2.13}$$

By (2.11) we have that $\Psi_f(\eta)\to 0$ as $\eta\to 0$.

For $\boldsymbol{\lambda}'\in\mathcal{C}$, from (2.2) and (2.7) we get

$$\Pi(\rho)=\sum_{i=0}^{\iota(\boldsymbol{\lambda}')} g_{2i+1}\rho^{2i+1}\left(1+\Phi_{2i+1}(\alpha,\boldsymbol{\lambda},\rho)\right), \tag{2.14}$$

where Φ_{2i+1}'s are analytic at $(\alpha,\boldsymbol{\lambda}',\rho)$ and vanish when $\alpha=\rho=0$. Restricted to the curve $(\alpha,\boldsymbol{\lambda})=(\alpha(\eta),\boldsymbol{\lambda}(\eta))$, from (2.14) we obtain

$$\Pi(\rho)=\sum_{i=0}^{\iota(\boldsymbol{\lambda}')} c_i\eta^{w_i}\rho^{2i+1}\left(1+H_{2i+1}(\eta,\rho)\right),$$

where $H_{2i+1}(\eta,\rho)\to 0$ as $(\eta,\rho)\to(0,0)$. Then

$$\Pi(\eta^{h_{i_\nu}/2})=\sum_{i=0}^{\iota(\boldsymbol{\lambda}')} c_i\eta^{w_i+(i+1/2)h_{i_\nu}}\left(1+\tilde{H}_{2i+1}(\eta)\right) \quad \text{for } \nu=0,1,\dots,k, \tag{2.15}$$

where

$$\tilde{H}_{2i+1}(\eta) := H_{2i+1}(\eta, \eta^{h_{i_\nu}/2}) \to 0 \quad \text{as } \eta \to 0. \tag{2.16}$$

Applying (2.8) to (2.15) we get

$$\begin{aligned}
\Pi(\eta^{h_{i_\nu}/2}) &= \eta^{w_{i_\nu}+(i_\nu+1/2)h_{i_\nu}} \sum_{i\in\Xi(i_\nu)} c_i \left(1+\tilde{H}_{2i+1}(\eta)\right) \\
&\quad + \sum_{i\in\{0,\ldots,\iota(\boldsymbol{\lambda}')\}\setminus\Xi(i_\nu)} c_i \eta^{w_i+(i+1/2)h_{i_\nu}} \left(1+\tilde{H}_{2i+1}(\eta)\right) \\
&= \eta^{w_{i_\nu}+(i_\nu+1/2)h_{i_\nu}} \Big(\hat{c}_{i_\nu} + \sum_{i\in\Xi(i_\nu)} c_i \tilde{H}_{2i+1}(\eta)\Big) \\
&\quad + \sum_{i\in\{0,\ldots,\iota(\boldsymbol{\lambda}')\}\setminus\Xi(i_\nu)} c_i \eta^{w_i+(i+1/2)h_{i_\nu}} \left(1+\tilde{H}_{2i+1}(\eta)\right) \\
&= \hat{c}_{i_\nu} \eta^{w_{i_\nu}+(i_\nu+1/2)h_{i_\nu}} \left(1+\Psi_c(\eta)\right) \quad \forall\, i_\nu \in \mathcal{V}\cup\{i_{s'}\},
\end{aligned} \tag{2.17}$$

where $\hat{c}_{i_\nu}$ is defined in (2.6) and

$$\Psi_c(\eta) := \sum_{i\in\Xi(i_\nu)} \frac{c_i}{\hat{c}_{i_\nu}} \tilde{H}_{2i+1}(\eta) + \sum_{i\in\{0,\ldots,\iota(\boldsymbol{\lambda}')\}\setminus\Xi(i_\nu)} \frac{c_i}{\hat{c}_{i_\nu}} \eta^{w_i - w_{i_\nu} + (i-i_\nu)h_{i_\nu}} \left(1+\tilde{H}_{2i+1}(\eta)\right). \tag{2.18}$$

By (2.16) we have that $\Psi_c(\eta) \to 0$ as $\eta \to 0$.

Finally, for $\boldsymbol{\lambda}' \in \mathbb{R}^m$ we can choose $\varepsilon > 0$ small enough such that the point $(\alpha(\hat{\eta}), \boldsymbol{\lambda}(\hat{\eta}))$ on the parameterized curve, where $\hat{\eta} = \varepsilon^\beta$ and $\beta > 2/h_{i_{s'}}$ is a constant, lies in an arbitrarily small neighborhood of $(0, \boldsymbol{\lambda}')$. The monotonicity of $\{h_{i_\nu}\}$, given just before condition (**LD**), implies that

$$\hat{\eta}^{h_{i_\nu}/2} < \hat{\eta}^{h_{i_{s'}}/2} = \varepsilon^{\beta h_{i_{s'}}/2} < \varepsilon,$$

for all $i_\nu \in \mathcal{V}$. Let $\hat{\imath} := \#\mathcal{V}$, the cardinality of $\mathcal{V}$. It follows that the $\hat{\imath}+1$ points $\hat{\eta}^{h_{i_\nu}/2}$, $i_\nu \in \mathcal{V}\cup\{i_{s'}\}$, all lie in $(0,\varepsilon)$ and increase as ν increases. Thus, from (2.12) when $\boldsymbol{\lambda}' \in \mathbb{R}^m \setminus \mathcal{C}$, or from (2.17) when $\boldsymbol{\lambda}' \in \mathcal{C}$, we get

$$\Pi(\hat{\eta}^{h_{i_\nu}/2}) = \hat{c}_{i_\nu} \varepsilon^{\beta w_{i_\nu} + (i_\nu+1/2)\beta h_{i_\nu}} \left(1+\Phi(\varepsilon)\right) \quad \text{for each } i_\nu \in \mathcal{V}\cup\{i_{s'}\}, \tag{2.19}$$

where $\Phi(\varepsilon) := \Psi_f(\varepsilon^\beta)$ or $\Psi_c(\varepsilon^\beta)$, which tends to 0 as $\varepsilon \to 0$ as defined in (2.13) and (2.18). The formula (2.19) shows that, for sufficiently small $\varepsilon > 0$, $\Pi(\hat{\eta}^{h_{i_\nu}/2})$ has the same sign as $\hat{c}_{i_\nu}$ for each $i_\nu \in \mathcal{V}\cup\{i_{s'}\}$. Therefore, the sign of Π alters once at each of these $\hat{\imath}+1$ points, implying that the equation $\Pi(\rho) = 0$ has at least $\hat{\imath}$ roots in $(0,\varepsilon)$, i.e., $\mathcal{N}(\boldsymbol{\lambda}') \geq \hat{\imath}$. The proof is completed. □

In roughly speaking, $\#\mathcal{V}$ is a number of sign changes caused by those terms whose powers compose a ladder-likely degressive sequence. In order to apply Theorem 2.1, we need to find a curve $(\alpha, \boldsymbol{\lambda}) = (\alpha(\eta), \boldsymbol{\lambda}(\eta))$ passing through $(0, \boldsymbol{\lambda}')$ in the parameter space $\mathbb{R}^{m+1}$ such that the condition (**LD**) holds. Then, one can choose an appropriate $(\alpha, \boldsymbol{\lambda})$ near $(0, \boldsymbol{\lambda}')$ on the curve to obtain the number $\#\mathcal{V}$ of limit cycles of system $LC(\alpha, \boldsymbol{\lambda})$ near the center O. The curve Υ is found by solving the indeterminate α_i's (in the w_i's) and h_j's from the inequalities given in (**LD**), which will be illustrated in section 4.

3 Some corollaries

In this section we give two corollaries of Theorem 2.1 for easier applications in some cases. Those cases come from special cases of the condition (**LD**).

Suppose that the power sequence of g_{2i+1}'s at $(0, \boldsymbol{\lambda}')$ satisfies

$$\textbf{(D)} \qquad w_0 - w_1 \geq ... \geq w_{i-1} - w_i \geq ... \geq w_{\kappa(\boldsymbol{\lambda}')-1} - w_{\kappa(\boldsymbol{\lambda}')} > 0.$$

This means that the sequence $\{w_i\}$ is decreasing and the gaps between two consecutive terms become smaller and smaller.

For those "=" in condition **(D)**, we consider $w_{\varsigma-1} - w_\varsigma = w_\varsigma - w_{\varsigma+1}$ for some ς in $\{0, 1, ..., \kappa(\boldsymbol{\lambda}')\}$ and let integers $\varsigma_1, \varsigma_2 \in \{0, ..., \kappa(\boldsymbol{\lambda}')\}$ denote the indices such that $\varsigma_1 < \varsigma < \varsigma_2$ and

$$w_{\varsigma_1-1} - w_{\varsigma_1} > w_{\varsigma_1} - w_{\varsigma_1+1} = ... = w_{\varsigma_2-1} - w_{\varsigma_2} > w_{\varsigma_2} - w_{\varsigma_2+1}, \tag{3.1}$$

i.e. ς_1 and ς_2 are respectively the first left index and the first right index near ς which destroy the equality "=" in (D). Define $w_{-1} = w_{\kappa(\boldsymbol{\lambda}')+1} := +\infty$ complementarily. Then, for each $\varsigma \in \{0, 1, ..., \kappa(\boldsymbol{\lambda}')\}$, we define

$$\widehat{\Xi}(\varsigma) := \begin{cases} \{\varsigma_1, ..., \varsigma, ..., \varsigma_2\} & \text{if (3.1) holds,} \\ \{\varsigma\} & \text{if } w_{\varsigma-1} - w_\varsigma > w_\varsigma - w_{\varsigma+1}, \end{cases}$$

which can be used to find a scale $(i_0, i_1, ..., i_k)$, where $i_0 := 0$, $i_k := \kappa(\boldsymbol{\lambda}')$ and $i_1, ..., i_{k-1} \in \{1, ..., \kappa(\boldsymbol{\lambda}') - 1\}$ such that

$$\begin{array}{ll} \widehat{\Xi}(i_j) \neq \widehat{\Xi}(i_l) & \text{for } 0 \leq j \neq l \leq k, \\ \widehat{\Xi}(i_j) \neq \widehat{\Xi}(i_j - 1) & \text{for } 0 < j \leq k, \\ \cup_{\nu=0}^{k} \widehat{\Xi}(i_\nu) = \{0, 1, ..., \kappa(\boldsymbol{\lambda}')\}. & \end{array}$$

On the scale we define the set $\mathcal{V}$ as in (2.6), where

$$\hat{c}_{i_\nu} := \sum_{j \in \widehat{\Xi}(i_\nu)} c_j \qquad \text{for } \nu = 0, 1, ..., k,$$

and c_i is the leading coefficient of g_{2i+1}as given in (2.5).

Corollary 3.1. $\mathcal{N}(\boldsymbol{\lambda}') \geq \#\mathcal{V}$ *if the power sequence of* g_{2i+1}*'s at* $(0, \boldsymbol{\lambda}')$ *satisfies condition* **(D)**.

Proof. Under condition **(D)**, define

$$\begin{array}{l} h_i := (w_{i-1} - w_{i+1})/2, \qquad \forall i = 1, ..., \kappa(\boldsymbol{\lambda}') - 1, \\ h_0 := 2(w_0 - w_1), \quad h_{\kappa(\boldsymbol{\lambda}')} := (w_{\kappa(\boldsymbol{\lambda}')-1} - w_{\kappa(\boldsymbol{\lambda}')})/2, \end{array} \tag{3.2}$$

i.e. h_i is the average of the differences $w_{i-1}-w_i$ and w_i-w_{i+1}. From **(D)** and (3.2), it is easy to see that $h_0 > h_i \geq h_j > h_{\kappa(\boldsymbol{\lambda}')} > 0$ for $i < j$ in $\{1, ..., \kappa(\boldsymbol{\lambda}')-1\}$. Moreover, $h_i = h_j$ if and only if $w_{i-1} - w_i = w_i - w_{i+1} = ... = w_j - w_{j+1}$. By the choice of the scale $(i_0, i_1, ..., i_k)$ given before this corollary, we have that $h_{i_0} > h_{i_1} > ... > h_{i_k} > 0$. We claim that this sequence $\{w_i\}$ is ladder-likely degressive on the scale $(i_0, i_1, ..., i_k)$. In fact, from **(D)** we see that for $j < i_\nu$

$$\begin{aligned}(i_\nu - j)h_{i_\nu} &= (i_\nu - j)\left(\frac{w_{i_\nu-1} - w_{i_\nu}}{2} + \frac{w_{i_\nu} - w_{i_\nu+1}}{2}\right) \leq (i_\nu - j)(w_{i_\nu-1} - w_{i_\nu}) \\ &\leq \sum_{i=j+1}^{i=i_\nu} (w_{i-1} - w_i) = w_j - w_{i_\nu},\end{aligned}$$

where "=" holds if and only if $w_j - w_{j+1} = ... = w_{i_\nu-1} - w_{i_\nu} = w_{i_\nu} - w_{i_\nu+1}$ because of **(D)**. Similarly, for $j > i_\nu$ we obtain

$$\begin{aligned}(i_\nu - j)h_{i_\nu} &= (i_\nu - j)\left(\frac{w_{i_\nu-1} - w_{i_\nu}}{2} + \frac{w_{i_\nu} - w_{i_\nu+1}}{2}\right) \leq (i_\nu - j)(w_{i_\nu} - w_{i_\nu+1}) \\ &= (j - i_\nu)(w_{i_\nu+1} - w_{i_\nu}) \leq \sum_{i=i_\nu+1}^{i=j} (w_i - w_{i-1}) = w_j - w_{i_\nu},\end{aligned}$$

where "=" holds if and only if $w_{i_\nu-1}-w_{i_\nu} = w_{i_\nu}-w_{i_\nu+1} = ... = w_{j-1}-w_j$. Thus, (2.8) holds in our case, which implies that the h_{i_ν}'s ($\nu = 0, 1, ..., k$) satisfy condition (**LD**) and therefore the claim is proved. Therefore, $\mathcal{N}(\boldsymbol{\lambda}') \geq \#\mathcal{V}$ by Theorem 2.1. □

Since condition **(D)** requires that the sequence of power-differences $\{w_0-w_1, ..., w_{\kappa(\boldsymbol{\lambda}')-1}-w_{\kappa(\boldsymbol{\lambda}')}\}$ is non-increasing, we have the following three cases:

(D0) there is no "=" in **(D)**;

(D1) "=" appears in **(D)** in discontinuous manner;

(D2) "=" appears continuously in **(D)**, i.e. there exists i such that $... \geq w_{i-1}-w_i = w_i - w_{i+1} = w_{i+1} - w_{i+2} \geq ....$

Corollary 3.2. *$\mathcal{N}(\boldsymbol{\lambda}') = \kappa(\boldsymbol{\lambda}')$ if either $c_ic_{i+1} < 0$ for all $i = 0, ..., \kappa(\boldsymbol{\lambda}') - 1$ in the case* **(D0)***, or $\hat{c}_{i_\nu}\hat{c}_{i_{\nu+1}} < 0$ for all $\nu = 0, ..., \kappa(\boldsymbol{\lambda}') - 1$ in the case* **(D1)***.*

Proof. In case **(D0)** choosing the scale $(i_0, i_1, ..., i_k)$ as $(0, 1, ..., \kappa(\boldsymbol{\lambda}'))$ and defining the h_{i_ν}'s as in (3.2), we compute $\hat{c}_{i_\nu} = c_{i_\nu}$ for all $\nu = 0, 1, ..., \kappa(\boldsymbol{\lambda}')$. In case **(D1)** choosing the scale $(i_0, i_1, ..., i_k)$ and defining the h_{i_ν}'s as above, we see that $k = \kappa(\boldsymbol{\lambda}')$ and $h_{i_0} > h_{i_1} > ... > h_{i_k} > 0$. Then the result follows from Corollary 3.1. □

Although the cyclicity $\mathcal{N}(\boldsymbol{\lambda}')$ may reach the upper estimate $\kappa(\boldsymbol{\lambda}')$ in cases **(D0)** and **(D1)**, for which Corollary 3.2 gives sufficient conditions, we do not have such a result yet in case **(D2)** because the equality $\widehat{\Xi}(i) = \widehat{\Xi}(i+1)$ known by the definition of $\widehat{\Xi}$ implies that $\#\mathcal{V} \le k < \kappa(\boldsymbol{\lambda}')$.

Remark that Theorem 6.6 of [11] can also be employed to case **(D0)** but does not work for case **(D1)**. On the other hand, $\hat{c}_{i_\nu} = c_{i_\nu}$ for all $\nu = 0, 1, ..., \kappa(\boldsymbol{\lambda}')$ in case **(D0)** but, there are some i such that $\hat{c}_{i_\nu} = c_{i_{\nu-1}} + c_{i_\nu} + c_{i_{\nu+1}}$ in case **(D1)**. For example, $\hat{c}_0 = c_0$, $\hat{c}_1 = c_1$, $\hat{c}_2 = c_1 + c_2 + c_3$ and $\hat{c}_3 = c_3$ when $w_0 - w_1 > w_1 - w_2 = w_2 - w_3 > w_3 - w_4 ....$

We consider the following family of polynomial differential systems

$$\dot{x} = \alpha x - y + \sum_{i=1}^{4} a_{2i+1} x(x^2 + y^2)^i, \quad \dot{y} = x + \alpha y + \sum_{i=1}^{4} a_{2i+1} y(x^2 + y^2)^i, \tag{3.3}$$

parameterized by $(\alpha, \boldsymbol{\lambda}) := (\alpha, \lambda_1, \lambda_2) \in \mathbb{R}^3$, where $a_3 := \lambda_1, a_5 := -\lambda_2^2, a_7 := 3\lambda_2$ and $a_9 := -1$. One can compute its focal values

$$g_1 = 2\pi\alpha, \quad g_3 = 2\pi\lambda_1, \quad g_5 = -2\pi\lambda_2^2, \quad g_7 = 6\pi\lambda_2, \quad g_9 = -2\pi, \tag{3.4}$$

where each g_{2i+1} is the remainder of the original g_{2i+1} divided by the Gröbner basis of ideal $\langle g_3, ..., g_{2i-1}\rangle$ in the order $\lambda_1 \prec \lambda_2$. Thus, $\kappa(\boldsymbol{\lambda}') = \zeta(\boldsymbol{\lambda}') = 4$, where $\boldsymbol{\lambda}' = (0, 0)$. Note that the independence condition of focal values, i.e. **(ID$_k$-1)** and **(ID$_k$-2)**, do not hold for $g_1, g_3, ..., g_7$ because $g_7 = 0$ if $g_5 = 0$, which implies that we cannot obtain 4 limit cycles by verifying the classical independence of focal values. Using our above mentioned method, we choose the curve

$$\Upsilon : \alpha = -\eta^9, \quad \lambda_1 = \eta^5, \quad \lambda_2 = \eta,$$

in the $(\alpha, \lambda_1, \lambda_2)$-space. Restricted to Υ those focal values given in (3.4) can be written in the form (2.5) taking

$$\begin{array}{lllll} w_0 = 9, & w_1 = 5, & w_2 = 2, & w_3 = 1, & w_4 = 0, \\ c_0 = -2\pi, & c_1 = 2\pi, & c_2 = -2\pi, & c_3 = 6\pi, & c_4 = -2\pi. \end{array}$$

One can check that

$$w_0 - w_1 > w_1 - w_2 > w_2 - w_3 = w_3 > 0,$$

and compute that

$$\hat{c}_0 = c_0 = -2\pi,\ \hat{c}_1 = c_1 = 2\pi,\ \hat{c}_2 = c_2 = -2\pi,\ \hat{c}_3 = c_2 + c_3 + c_4 = 2\pi,\ \hat{c}_4 = c_4 = -2\pi,$$

which implies by Corollary 3.2 in case **(D1)** that $\mathcal{N}(\boldsymbol{\lambda}') = 4$.

4 Application to cubic systems

In this section we apply our method to a family of cubic polynomial differential systems with 5 parameters. Consider

$$\begin{aligned} \dot{x} &= \alpha x - y + (\lambda_1 - \lambda_3)x^2 + \lambda_2 xy + \lambda_3 y^2 - (9 + \lambda_2^2 + \lambda_3\lambda_4)x^2 y + 2y^3, \\ \dot{y} &= x + \alpha y - x^3 - (12 + \lambda_2^2 + \lambda_3^2 + \lambda_3\lambda_4)xy^2, \end{aligned} \tag{4.1}$$

parameterized by $(\alpha, \boldsymbol{\lambda}) := (\alpha, \lambda_1, ..., \lambda_4) \in \mathbb{R}^5$. It is easy to compute the first five nonzero focal values

$$g_1 = 2\pi\alpha, \quad g_3 = \frac{\pi}{4}\lambda_1\lambda_2, \quad g_5 = -\frac{\pi}{24}\lambda_2\lambda_3^3, \quad g_7 = \frac{\pi}{96}\lambda_2\lambda_3^2\lambda_4, \quad g_9 = -\frac{\Omega(\lambda_2)\pi}{960}\lambda_2\lambda_3, \quad (4.2)$$

where $\Omega(\lambda) := 6720 + 1265\lambda^2 + 61\lambda^4$ and, for a short statement, each g_{2i+1} is the remainder of the original g_{2i+1} divided by the Gröbner basis of ideal $\langle g_3, ..., g_{2i-1}\rangle$ in the order $\lambda_1 \prec \lambda_2 \prec \lambda_3 \prec \lambda_4$. Family (4.1) has the center variety $\mathcal{C} = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 := \{\boldsymbol{\lambda} \in \mathbb{R}^4 : \ \lambda_1 = \lambda_3 = 0\} \quad \text{and} \quad \Gamma_2 := \{\boldsymbol{\lambda} \in \mathbb{R}^4 : \ \lambda_2 = 0\}.$$

In fact, family $(4.1)|_{\alpha=0}$ is time-reversible for $\boldsymbol{\lambda} \in \Gamma_1 \cup \Gamma_2$, but on the contrary

$$V(g_3) \cap \mathbb{R}^4 \supsetneq V(g_3, g_5) \cap \mathbb{R}^4 = \Gamma_1 \cup \Gamma_2 = V(g_3, ..., g_{2i+1}, ...) \cap \mathbb{R}^4, \qquad (4.3)$$

by the expressions of g_3 and g_5.

Proposition 4.1. *For $\boldsymbol{\lambda} \in \mathbb{R}^4 \backslash \mathcal{C}$ the cyclicity $\mathcal{N}_f(\boldsymbol{\lambda})$ and the multiplicity $\zeta(\boldsymbol{\lambda})$ of O in the family* (4.1) *satisfy that either $\mathcal{N}_f(\boldsymbol{\lambda}) = \zeta(\boldsymbol{\lambda}) = 1$ or $\mathcal{N}_f(\boldsymbol{\lambda}) = \zeta(\boldsymbol{\lambda}) = 2$, which holds if either $\lambda_1\lambda_2 \neq 0$ or $\lambda_1 = 0 \neq \lambda_2\lambda_3$ correspondingly.*

Proof. The results can be proved by checking the independence condition of focal values, i.e. **(ID$_k$-1)** and **(ID$_k$-2)** for $k = 2$. Actually, by (4.3), the origin is a weak focus of multiplicity at most 2 when $\boldsymbol{\lambda} \in \mathbb{R}^4 \backslash \mathcal{C}$. For such a $\boldsymbol{\lambda}$, by the definitions of Γ_1 and Γ_2, there are only two cases: either $\lambda_1\lambda_2 \neq 0$ or $\lambda_1 = 0 \neq \lambda_2\lambda_3$. It is easy to check that g_1, g_3 and g_5 are independent at $(0, \boldsymbol{\lambda})$ when $\boldsymbol{\lambda}$ satisfies that $\lambda_1 = 0 \neq \lambda_2\lambda_3$, and that g_1 and g_3 are independent at $(0, \boldsymbol{\lambda})$ when $\boldsymbol{\lambda}$ satisfies that $\lambda_1\lambda_2 \neq 0$.

Meanwhile, the results of this proposition can also be proved by using our main theorem or corollaries. In fact, in the case that $\lambda_1 = 0 \neq \lambda_2\lambda_3$, consider the parametric curve

$$\alpha(\eta) := -\mathrm{sgn}(\lambda_2\lambda_3)\eta^{10}, \quad \lambda_1(\eta) := \mathrm{sgn}(\lambda_3)\eta^3, \quad \lambda_i(\eta) := \lambda_i + \eta, \qquad i = 2, 3, 4.$$

With the restriction to the curve, we can compute $w_0 = 10, w_1 = 3, w_2 = 0$ and $c_0 = -2\pi\mathrm{sgn}(\lambda_2\lambda_3), c_1 = \pi\lambda_2\mathrm{sgn}(\lambda_3)/4, c_2 = -\pi\lambda_2\lambda_3^3/24$. By Corollary 3.2, $\mathcal{N}_f(\boldsymbol{\lambda}) = 2$.

In the case that $\lambda_1\lambda_2 \neq 0$, we can prove $\mathcal{N}_f(\boldsymbol{\lambda}) = 1$ similarly. □

This proposition shows that, for $\boldsymbol{\lambda} \notin \Gamma_1 \cup \Gamma_2$, the origin O is a weak focus of multiplicity at most 2, and there are small perturbations such that exactly j limit cycles bifurcate from the weak focus of multiplicity j for $j = 1, 2$.

On the other hand, the origin O is a center of (4.1) if and only if $\alpha = 0$ and $\boldsymbol{\lambda} \in \Gamma_1 \cup \Gamma_2$. Clearly, every point in Γ_1 and Γ_2 can be written as $\boldsymbol{\lambda}^{(1)} := (0, \lambda_2^{(1)}, 0, \lambda_4^{(1)})$ and $\boldsymbol{\lambda}^{(2)} := (\lambda_1^{(2)}, 0, \lambda_3^{(2)}, \lambda_4^{(2)})$ respectively. In order to avoid a double discussion at the intersection of $\Gamma_1 \cap \Gamma_2$, we assume either $\lambda_1^{(2)} \neq 0$ or $\lambda_3^{(2)} \neq 0$.

Proposition 4.2. *For $\boldsymbol{\lambda}$ equal to $\boldsymbol{\lambda}^{(1)}$ or $\boldsymbol{\lambda}^{(2)}$ in $\mathcal{C}$, the cyclicity $\mathcal{N}_c(\boldsymbol{\lambda})$ and the multiplicity $\iota(\boldsymbol{\lambda})$ of the origin O in the family* (4.1) *have the results given in* Table 1*:*

For	if	then
$\boldsymbol{\lambda}^{(1)} := (0, \lambda_2^{(1)}, 0, \lambda_4^{(1)})$	$\lambda_4^{(1)} \neq 0,\ \vartheta > 0$	$\iota(\boldsymbol{\lambda}^{(1)}) = 4,\ \mathcal{N}_c(\boldsymbol{\lambda}^{(1)}) = 4$
	$\lambda_4^{(1)} \neq 0,\ \vartheta \leq 0$	$\iota(\boldsymbol{\lambda}^{(1)}) = 4,\ \mathcal{N}_c(\boldsymbol{\lambda}^{(1)}) \geq 2$
	$\lambda_4^{(1)} = 0$	$\iota(\boldsymbol{\lambda}^{(1)}) = 4,\ \mathcal{N}_c(\boldsymbol{\lambda}^{(1)}) \geq 2$
$\boldsymbol{\lambda}^{(2)} := (\lambda_1^{(2)}, 0, \lambda_3^{(2)}, \lambda_4^{(2)})$ where either $\lambda_1^{(2)} \neq 0$ or $\lambda_3^{(2)} \neq 0$	$\lambda_1^{(2)} = 0,\ \lambda_3^{(2)} \neq 0$	$\iota(\boldsymbol{\lambda}^{(2)}) = 2,\ \mathcal{N}_c(\boldsymbol{\lambda}^{(2)}) = 2$
	$\lambda_1^{(2)} \neq 0$	$\iota(\boldsymbol{\lambda}^{(2)}) = 1,\ \mathcal{N}_c(\boldsymbol{\lambda}^{(2)}) = 1$
Remark: $\vartheta := 5(\lambda_4^{(1)})^2 - 8\Omega(\lambda_2^{(1)})$ and Ω is given in (4.2).		

Table 1: The number of limit cycles bifurcating from the center O.

Proof. By the definition of Γ_1 and Γ_2 every focal value of family (4.1) is of the form

$$\lambda_2(f_1(\boldsymbol{\lambda})\lambda_1 + f_2(\boldsymbol{\lambda})\lambda_3), \tag{4.4}$$

where $f_1, f_2 \in \mathbb{R}[\boldsymbol{\lambda}]$. From (4.4) we see for $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(1)}$ that the least integer j such that

$$\langle g_3, ..., g_{2j+1}\rangle_{\boldsymbol{\lambda}^{(1)}} = \langle g_3, ..., g_{2i+1}, ...\rangle_{\boldsymbol{\lambda}^{(1)}}$$

in $\mathbb{R}\{\boldsymbol{\lambda}\}_{\boldsymbol{\lambda}^{(1)}}$ is 4, because every focal value lies in $\langle \lambda_1\lambda_2, \lambda_2\lambda_3\rangle_{\boldsymbol{\lambda}^{(1)}}$, $\lambda_1\lambda_2, \lambda_2\lambda_3 \in \langle g_3, g_5, g_7, g_9\rangle_{\boldsymbol{\lambda}^{(1)}}$ and $\lambda_2\lambda_3 \notin \langle g_3, g_5, g_7\rangle_{\boldsymbol{\lambda}^{(1)}}$. This implies $\iota(\boldsymbol{\lambda}^{(1)}) = 4$. For $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(2)}$ we consider the case that $\lambda_1^{(2)} = 0$ but $\lambda_3^{(2)} \neq 0$, and the case that $\lambda_1^{(2)} \neq 0$ separately. In the first case $\langle g_3\rangle_{\boldsymbol{\lambda}^{(2)}} \neq \langle g_3, ..., g_{2i+1}, ...\rangle_{\boldsymbol{\lambda}^{(2)}}$ and $\langle g_3, g_5\rangle_{\boldsymbol{\lambda}^{(2)}} = \langle g_3, ..., g_{2i+1}, ...\rangle_{\boldsymbol{\lambda}^{(2)}}$ in $\mathbb{R}\{\boldsymbol{\lambda}\}_{\boldsymbol{\lambda}^{(2)}}$, which implies that $\iota(\boldsymbol{\lambda}^{(2)}) = 2$. In the second case $\lambda_2^{(2)} \in \langle g_3\rangle_{\boldsymbol{\lambda}^{(2)}}$ in $\mathbb{R}\{\boldsymbol{\lambda}\}_{\boldsymbol{\lambda}^{(2)}}$ by the expression of g_3. Thus, $\langle g_3\rangle_{\boldsymbol{\lambda}^{(2)}} = \langle g_3, ..., g_{2i+1}, ...\rangle_{\boldsymbol{\lambda}^{(2)}}$ in $\mathbb{R}\{\boldsymbol{\lambda}\}_{\boldsymbol{\lambda}^{(2)}}$, which implies $\iota(\boldsymbol{\lambda}^{(2)}) = 1$.

First, consider $\boldsymbol{\lambda}^{(1)}$ in the case that $\lambda_4^{(1)} \neq 0$. Let

$$\begin{aligned} \alpha(\eta) := 0 + d_0\eta^{\alpha_0}, \quad \lambda_1(\eta) := 0 + d_1\eta^{\alpha_1}, \quad \lambda_2(\eta) &:= \lambda_2^{(1)} + d_2\eta^{\alpha_2}, \\ \lambda_3(\eta) := 0 + d_3\eta^{\alpha_3}, \quad \lambda_4(\eta) &:= \lambda_4^{(1)} + d_4\eta^{\alpha_4}, \end{aligned} \tag{4.5}$$

where the α_j's and the d_j's are undetermined. Then, we obtain

$$\begin{aligned} g_1(\alpha(\eta)) &= 2\pi d_0\eta^{\alpha_0}, \\ g_3(\boldsymbol{\lambda}(\eta)) &= \left(\pi d_1\lambda_2^{(1)} + \pi d_1 d_2\eta^{\alpha_2}\right)\eta^{\alpha_1}/4, \\ g_5(\boldsymbol{\lambda}(\eta)) &= -\left(\pi\lambda_2^{(1)}d_3^3 + \pi d_2 d_3^3\eta^{\alpha_2}\right)\eta^{3\alpha_3}/24, \\ g_7(\boldsymbol{\lambda}(\eta)) &= \left(\pi\lambda_2^{(1)}\lambda_4^{(1)}d_3^2 + \pi\lambda_2^{(1)}d_3^2 d_4\eta^{\alpha_4} + \pi d_2 d_3^2\lambda_4^{(1)}\eta^{\alpha_2} + \pi d_2 d_3^2 d_4\eta^{\alpha_2+\alpha_4}\right)\eta^{2\alpha_3}/96, \\ g_9(\boldsymbol{\lambda}(\eta)) &= -\left(\Omega(\lambda_2^{(1)} + d_2\eta^{\alpha_2})\pi\lambda_2^{(1)}d_3 + \Omega(\lambda_2^{(1)} + d_2\eta^{\alpha_2})\pi d_2 d_3\eta^{\alpha_2})\right)\eta^{\alpha_3}/960, \end{aligned}$$

which give the power sequence $\{w_i\}$ and the leading coefficients c_i's as follows:

$$\begin{aligned}
&w_0=\alpha_0, \quad w_1=\alpha_1, \quad w_2=3\alpha_3, \quad w_3=2\alpha_3, \quad w_4=\alpha_3,\\
&c_0=2\pi d_0, \quad c_1=\frac{\pi}{4}d_1\lambda_2^{(1)}, \quad c_2=-\frac{\pi}{24}d_3^3\lambda_2^{(1)}, \quad c_3=\frac{\pi}{96}d_3^2\lambda_2^{(1)}\lambda_4^{(1)}, \quad c_4=-\Omega(\lambda_2^{(1)})\frac{\pi}{960}d_3\lambda_2^{(1)},
\end{aligned}$$

when $\lambda_2^{(1)}\neq 0$, or

$$\begin{aligned}
&w_0=\alpha_0, \quad w_1=\alpha_1+\alpha_2, \quad w_2=3\alpha_3+\alpha_2, \quad w_3=2\alpha_3+\alpha_2, \quad w_4=\alpha_3+\alpha_2,\\
&c_0=2\pi d_0, \quad c_1=\frac{\pi}{4}d_1d_2, \quad c_2=-\frac{\pi}{24}d_2d_3^3, \quad c_3=\frac{\pi}{96}d_2d_3^2\lambda_4^{(1)}, \quad c_4=-\Omega(0)\frac{\pi}{960}d_2d_3,
\end{aligned}$$

when $\lambda_2^{(1)}=0$. We claim that there exist positive numbers α_i's in (4.5) such that

$$w_0-w_1>w_1-w_2>w_2-w_3=w_3-w_4>0. \tag{4.6}$$

In fact, (4.6) is equivalent to either $\alpha_0-\alpha_1>\alpha_1-3\alpha_3>\alpha_3$ when $\lambda_2^{(1)}\neq 0$ or $\alpha_0-\alpha_1-\alpha_2>\alpha_1-3\alpha_3>\alpha_3$ when $\lambda_2^{(1)}=0$, from which we can choose

$$\alpha_0=100, \quad \alpha_1=50, \quad \alpha_2=10, \quad \alpha_3=10,$$

for both cases. Thus, by (4.6) our system falls into the case **(D1)** . From the definition (2.6) we compute

$$\begin{aligned}
&\hat c_0=c_0=2\pi d_0, \quad \hat c_1=c_1=\pi d_1\lambda_2^{(1)}/4, \quad \hat c_2=c_2=-\pi d_3^3\lambda_2^{(1)}/24,\\
&\hat c_3=c_2+c_3+c_4=-\frac{\pi}{960}d_3\lambda_2^{(1)}\left(40d_3^2-10\lambda_4^{(1)}d_3+\Omega(\lambda_2^{(1)})\right), \quad \hat c_4=c_4=-\Omega(\lambda_2^{(1)})\frac{\pi}{960}d_3\lambda_2^{(1)},
\end{aligned}$$

when $\lambda_2^{(1)}\neq 0$, or

$$\begin{aligned}
&\hat c_0=c_0=2\pi d_0, \quad \hat c_1=c_1=\pi d_1d_2/4, \quad \hat c_2=c_2=-\pi d_2d_3^3/24,\\
&\hat c_3=c_2+c_3+c_4=-\pi d_2d_3\left(40d_3^2-10\lambda_4^{(1)}d_3+\Omega(0)\right)/960, \quad \hat c_4=c_4=-\Omega(0)\pi d_2d_3/960,
\end{aligned}$$

when $\lambda_2^{(1)}=0$. Then, if $\vartheta>0$ we can check that

$$\hat c_0\hat c_1<0, \quad \hat c_1\hat c_2<0, \quad \hat c_2\hat c_3<0, \quad \hat c_3\hat c_4<0,$$

where we choose in (4.5) either

$$d_0=-\mathrm{sign}(\lambda_2^{(1)}\lambda_4^{(1)}), \quad d_1=\mathrm{sign}(\lambda_4^{(1)}), \quad d_2=0, \quad d_3=\lambda_4^{(1)}/8, \quad d_4=0$$

when $\lambda_2^{(1)}\neq 0$, or

$$d_0=-\mathrm{sign}(\lambda_4^{(1)}), \quad d_1=\mathrm{sign}(\lambda_4^{(1)}), \quad d_2=1, d_3=\lambda_4^{(1)}/8, \quad d_4=0$$

when $\lambda_2^{(1)}=0$. Therefore $\mathcal{N}_c(\boldsymbol{\lambda}^{(1)})=4$ by Corollary 3.2. Similarly, if $\vartheta\leq 0$ we can check that

$$\hat c_0\hat c_1<0, \quad \hat c_1\hat c_2<0,$$

where we choose in (4.5) either

$$d_0=-\mathrm{sign}(\lambda_2^{(1)}), \quad d_1=1, \quad d_2=0, \quad d_3=1, \quad d_4=0$$

when $\lambda_2^{(1)} \neq 0$, or

$$d_0 = -1, \quad d_1 = 1, \quad d_2 = 1, \quad d_3 = 1, \quad d_4 = 0$$

when $\lambda_2^{(1)} = 0$. Therefore $\mathcal{N}_c(\boldsymbol{\lambda}^{(1)}) \geq 2$ by Corollary 3.1.

Next, consider $\boldsymbol{\lambda}^{(1)}$ in the case that $\lambda_4^{(1)} = 0$. Let

$$\begin{aligned} \alpha(\eta) := 0 + d_0\eta^{\alpha_0}, \quad \lambda_1(\eta) &:= 0 + d_1\eta^{\alpha_1}, \quad \lambda_2(\eta) := \lambda_2^{(1)} + d_2\eta^{\alpha_2}, \\ \lambda_3(\eta) &:= 0 + d_3\eta^{\alpha_3}, \quad \lambda_4(\eta) := 0 + d_4\eta^{\alpha_4}. \end{aligned} \tag{4.7}$$

Then

$$\begin{aligned} g_1(\alpha(\eta)) &= 2\pi d_0\eta^{\alpha_0}, \\ g_3(\boldsymbol{\lambda}(\eta)) &= \left(\pi d_1\lambda_2^{(1)}\eta^{\alpha_1} + \pi d_1 d_2\eta^{\alpha_1+\alpha_2}\right)/4, \\ g_5(\boldsymbol{\lambda}(\eta)) &= -\left(\pi\lambda_2^{(1)}d_3^3\eta^{3\alpha_3} + \pi d_2 d_3^3\eta^{3\alpha_3+\alpha_2}\right)/24, \\ g_7(\boldsymbol{\lambda}(\eta)) &= \left(\pi\lambda_2^{(1)}d_3^2 d_4\eta^{2\alpha_3+\alpha_4} + \pi d_2 d_3^2 d_4\eta^{2\alpha_3+\alpha_2+\alpha_4}\right)/96, \\ g_9(\boldsymbol{\lambda}(\eta)) &= -\left(\Omega(\lambda_2^{(1)} + d_2\eta^{\alpha_2})\pi\lambda_2^{(1)}d_3\eta^{\alpha_3} + \Omega(\lambda_2^{(1)} + d_2\eta^{\alpha_2})\pi d_2 d_3\eta^{\alpha_2+\alpha_3})\right)/960, \end{aligned}$$

which gives the power sequence $\{w_i\}$ and the leading coefficients c_i's as follows:

$$\begin{aligned} &w_0 = \alpha_0, && w_1 = \alpha_1, && w_2 = 3\alpha_3, && w_3 = 2\alpha_3 + \alpha_4, && w_4 = \alpha_3, \\ &c_0 = 2\pi d_0, && c_1 = \frac{\pi}{4}d_1\lambda_2^{(1)}, && c_2 = -\frac{\pi}{24}d_3^3\lambda_2^{(1)}, && c_3 = \frac{\pi}{96}d_3^2 d_4\lambda_2^{(1)}, && c_4 = -\Omega(\lambda_2^{(1)})\frac{\pi}{960}d_3\lambda_2^{(1)}, \end{aligned}$$

when $\lambda_2^{(1)} \neq 0$ and

$$\begin{aligned} &w_0 = \alpha_0, && w_1 = \alpha_1 + \alpha_2, && w_2 = 3\alpha_3 + \alpha_2, && w_3 = 2\alpha_3 + \alpha_2 + \alpha_4, && w_4 = \alpha_3 + \alpha_2, \\ &c_0 = 2\pi d_0, && c_1 = \frac{\pi}{4}d_1 d_2, && c_2 = -\frac{\pi}{24}d_2 d_3^3, && c_3 = \frac{\pi}{96}d_2 d_3^2 d_4, && c_4 = -\Omega(0)\frac{\pi}{960}d_2 d_3, \end{aligned}$$

when $\lambda_2^{(1)} = 0$. We claim that there exist positive numbers α_i's in (4.7) such that

$$w_0 - w_1 > w_1 - w_2 > w_2 - w_3 < w_3 - w_4. \tag{4.8}$$

In fact, (4.8) is equivalent to either $\alpha_0 - \alpha_1 > \alpha_1 - 3\alpha_3 > \alpha_3 - \alpha_4 < \alpha_3 + \alpha_4$ when $\lambda_2^{(1)} \neq 0$, or $\alpha_0 - \alpha_1 - \alpha_2 > \alpha_1 - 3\alpha_3 > \alpha_3 - \alpha_4 < \alpha_3 + \alpha_4$ when $\lambda_2^{(1)} = 0$, from which we can choose

$$\alpha_0 = 13, \quad \alpha_1 = 8, \quad \alpha_2 = 3, \quad \alpha_3 = 2, \quad \alpha_4 = 1,$$

when $\lambda_2^{(1)} \neq 0$, and

$$\alpha_0 = 18, \quad \alpha_1 = 10, \quad \alpha_2 = 3, \quad \alpha_3 = 2, \quad \alpha_4 = 1,$$

when $\lambda_2^{(1)} = 0$. Thus (**LD**) holds on the scale $(i_0, i_1, i_2) := (0, 1, 2)$, where

$$h_{i_0} := 10, \quad h_{i_1} := 3.5, \quad h_{i_2} := 1.5,$$

when $\lambda_2^{(1)} \neq 0$, and

$$h_{i_0} := 10, \quad h_{i_1} := 4.5, \quad h_{i_2} := 2.5,$$

when $\lambda_2^{(1)} = 0$. Moreover "=" in (**LD**) holds only for $j = i_\nu$ ($\nu = 0, 1, 2$). From the definition of $\hat{c}_{i_\nu}$ we compute

$$\hat{c}_{i_0} = c_0 = 2\pi d_0, \quad \hat{c}_{i_1} = c_1 = \pi d_1 \lambda_2^{(1)}/4, \quad \hat{c}_{i_2} = c_2 = -\pi d_3^3 \lambda_2^{(1)}/24,$$

when $\lambda_2^{(1)} \neq 0$, or

$$\hat{c}_{i_0} = c_0 = 2\pi d_0, \quad \hat{c}_{i_1} = c_1 = \pi d_1 d_2/4, \quad \hat{c}_{i_2} = c_2 = -\pi d_2 d_3^3/24,$$

when $\lambda_2^{(1)} = 0$. We can check that

$$\hat{c}_{i_0}\hat{c}_{i_1} < 0, \quad \hat{c}_{i_1}\hat{c}_{i_2} < 0,$$

where we choose in (4.7) either

$$d_0 = -\mathrm{sign}(\lambda_2^{(1)}), \quad d_1 = d_2 = d_3 = d_4 = 1,$$

when $\lambda_2^{(1)} \neq 0$, or

$$d_0 = -1, \quad d_1 = d_2 = d_3 = d_4 = 1,$$

when $\lambda_2^{(1)} = 0$. Therefore $\mathcal{N}_c(\boldsymbol{\lambda}^{(1)}) \geq 2$ by Theorem 2.1.

We similarly consider $\boldsymbol{\lambda}^{(2)}$ and obtain $\mathcal{N}_c(\boldsymbol{\lambda}^{(2)}) = 2$ in the case that $\lambda_1^{(2)} = 0, \lambda_3^{(2)} \neq 0$ and $\mathcal{N}_c(\boldsymbol{\lambda}^{(2)}) = 1$ in the case that $\lambda_1^{(2)} \neq 0$. □

Proposition 4.2 implies that for some $\boldsymbol{\lambda}' \in \mathcal{C}$ there are 4 limit cycles bifurcating from the center O of system $(4.1)|_{(\alpha,\boldsymbol{\lambda})=(0,\boldsymbol{\lambda}')}$. However, it is impossible to obtain 4 limit cycles bifurcating from the center by either the well-known method of independent focal values, or the method given in [11, Theorem 6.6]. In fact, the focal values g_3, g_5, g_7, g_9 are not independent because $g_7 = 0$ when $g_5 = 0$. We can compute

$$\frac{\partial(g_3, g_5, g_7, g_9)}{\partial(\lambda_1, ..., \lambda_4)} = \begin{pmatrix} \frac{\pi}{4}\lambda_2 & \frac{\pi}{4}\lambda_1 & 0 & 0 \\ 0 & -\frac{\pi}{24}\lambda_3^3 & -\frac{\pi}{8}\lambda_2\lambda_3^2 & 0 \\ 0 & \frac{\pi}{96}\lambda_3^2\lambda_4 & \frac{\pi}{48}\lambda_2\lambda_3\lambda_4 & \frac{\pi}{96}\lambda_2\lambda_3^2 \\ 0 & -\frac{F(\lambda_2)\pi}{960}\lambda_3 & -\frac{G(\lambda_2)\pi}{960}\lambda_2 & 0 \end{pmatrix},$$

where $F(\lambda) := 6720 + 3795\lambda^2 + 305\lambda^4$ and $G(\lambda) := 6720 + 1265\lambda^2 + 61\lambda^4$, and obtain

$$\mathrm{rank}\left(\left.\frac{\partial(g_3, g_5, g_7, g_9)}{\partial(\lambda_1, ..., \lambda_4)}\right|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(1)}}\right) \leq 2, \quad \mathrm{rank}\left(\left.\frac{\partial(g_3, g_5, g_7, g_9)}{\partial(\lambda_1, ..., \lambda_4)}\right|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(2)}}\right) \leq 1,$$

which implies by [3, Theorem 1.3] that 2 limit cycles can be bifurcated from the center O for the case that $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(1)}$, and 1 limit cycle for the case that $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(2)}$. Moreover, from the expressions of the g_{2i+1}'s given in (4.2) we find that "=" always appears in (**D**), which implies that the result of [11, Theorem 6.6] cannot be used to find 4 limit cycles bifurcating from the center O.

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