LIMIT CYCLES FOR A VARIANT OF A GENERALIZED RICCATI EQUATION

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Abstract. In this paper we provide a lower bound for the maximum number of limit cycles surrounding the origin of systems $(\dot{x}, \dot{y} = \ddot{x})$ given by a variant of the generalized Riccati equation

$$\ddot{x} + \varepsilon x^{2n+1}\dot{x} + bx^{4n+3} = 0,$$

where $b > 0$, $b \in \mathbb{R}$, $n$ is a non-negative integer and $\varepsilon$ is a small parameter. The tool for proving this result uses Abelian integrals.

1. Introduction and statement of the main results

Some variants of the generalized Riccati equation

$$\ddot{x} + \alpha x^{2n+1}\dot{x} + x^{4n+3} = 0,$$  \hspace{1cm} (1)

have been studied for several authors, see for instance [8], [5], and the references quoted there. In the first paper the authors studied mainly the following variant of equation (1)

$$\ddot{x} + (2n + 3)x^{2n+1}\dot{x} + x^{4n+3} + \omega^2 x = 0,$$

showing numerically that such differential equation exhibits isochronous oscillations. In the second paper the authors study the variant of equation (1)

$$\ddot{x} + (2n + 3)x^{2n+1}\dot{x} + x^{4n+3} + \omega^2 x^{2n+1} = 0,$$

and they find the analytical expression of some particular solutions.

In the present paper we will study the following variant of the generalized Riccati equation (1)

$$\ddot{x} + \varepsilon x^{2n+1}\dot{x} + bx^{4n+3} + \varepsilon a(x + yq(x)) = 0,$$  \hspace{1cm} (2)

where $a, b \in \mathbb{R}$ with $b \neq 0$, $n$ is a non-negative integer, $\varepsilon$ is a small parameter and $q(x)$ is a polynomial of degree $m$.

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Equation (1) can be written as
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -bx^{4n+3} - \varepsilon a(x + yq(x)) - \varepsilon x^{2n+1}y,
\end{align*}
(3)
or equivalently in the form
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial y} + \varepsilon P(x, y), \\
\dot{y} &= \frac{\partial H}{\partial x} + \varepsilon Q(x, y),
\end{align*}
(4)

where
\begin{equation}
H(x, y) = \frac{y^2}{2} + \frac{bx^{4n+4}}{4n+4},
\end{equation}
(5)
and
\begin{align*}
P(x, y) &= 0, \\
Q(x, y) &= -a(x + yq(x)) - x^{2n+1}y.
\end{align*}

Observe that for \( b > 0 \) there is a family of ovals \( \gamma_h \subset H^{-1}(h) \) continuously depending on a parameter \( h > 0 \) and varying in the compact components of \( H^{-1}(h) \). Moreover all the ovals \( \gamma_h \) fill up the plane \( \mathbb{R}^2 \) when \( h \) varies on all positive real numbers. These ovals are periodic orbits of the Hamiltonian system (4) with \( \varepsilon = 0 \).

The objective of this paper is to find the maximum number of values of \( h \) (that we denote by \( h^{\ast} \)) for which it bifurcate from \( \gamma_h^{\ast} \) a limit cycle of the differential system (4) for \( |\varepsilon| > 0 \) sufficiently small.

**Theorem 1.** For \( |\varepsilon| > 0 \) sufficiently small there are systems (3) with \( b > 0 \) having \( m \) limit cycles \( \Gamma_{h_m} \) that when \( \varepsilon \to 0 \) tend to periodic orbits \( \gamma_{h_m}^{\ast} \) of the Hamiltonian system (3) with \( b > 0 \) and \( \varepsilon = 0 \). Moreover there are polynomials \( q(x) \) for which the differential system (3) with \( b > 0 \) and \( |\varepsilon| > 0 \) sufficiently small has exactly \( m \) limit cycles.

The proof of Theorem 1 is given in section 2.

Note that Theorem 1 is closely related to the weakened 16th Hilbert problem proposed by Arnold in [1, 2] which in its turn is closely related to determining an upper bound for the number of limit cycles of a perturbed Hamiltonian system of the form in (4). For other papers on limit cycles see for instance [6, 7] and the references quoted there.

**2. Proof of Theorem 1**

To prove Theorem 1 we will use use the following Theorem whose proof can be obtained, for instance, in [4].
Theorem 2. Assume that there is a family of ovals \( \gamma_h \subset H^{-1}(h) \), continuously depending on \( h \in (a, b) \). Define the Abelian integral as

\[
I(h) = \int_{\gamma_h} P(x, y) \, dy - Q(x, y) \, dx.
\]

If there exists an \( h^* \in (a, b) \) such that \( I(h^*) = 0 \) and \( I'(h^*) \neq 0 \) then for \( \varepsilon \) sufficiently small, the Hamiltonian system (4) has at most one limit cycle \( \Gamma_{h^*} \) which tends to \( \gamma_{h^*} \) as \( \varepsilon \to 0 \).

We first write the polynomial \( q(x) = \sum_{i=0}^{n} q_i x^i \). Note that the unperturbed system (2) (with \( \varepsilon = 0 \)) is Hamiltonian with the Hamiltonian \( H \) given in (5). The periodic orbits of the unperturbed system (2) with \( h > 0 \) are the ovals \( \gamma_h \). Now we will use Theorem 2 and so we shall compute the Abelian integral \( I(h) \) given in (6). We have

\[
I(h) = \int \int_{H(x,y) \leq h} \frac{\partial}{\partial y} (x^{2n+1} y + a(x + yq(x))) \, dx \, dy
= \int \int_{H(x,y) \leq h} (x^{2n+1} + aq(x)) \, dx \, dy
= \int \int_{H(x,y) \leq h} x^{2n+1} \, dx \, dy + \sum_{i=0}^{m} q_i \int \int_{H(x,y) \leq h} x^i \, dx \, dy
= 2 \int_{-\pi}^{\pi} x^{2n+1} \left( 2h - \frac{b}{2(n+1)} x^{4(n+1)} \right)^{1/2} \, dx
+ 2a \sum_{i=0}^{m} q_i \int_{-\pi}^{\pi} x^i \left( 2h - \frac{b}{2(n+1)} x^{4(n+1)} \right)^{1/2} \, dx,
\]

where

\[
\pi = \left( \frac{4h(n+1)}{b} \right)^{1/(4(n+1))}.
\]

Note that for any integer \( j \) we have

\[
\int_{-\pi}^{\pi} x^j \left( 2h - \frac{b}{2(n+1)} x^{4(n+1)} \right)^{1/2} \, dx = \frac{2^{j-n}}{2^{j-n+2}} (1 + (-1)^j) b^{-\frac{j+1}{4n+4}} (n + 1)^{\frac{j+1}{4n+4}} \sqrt{\pi} \Gamma \left( \frac{j+4n+5}{4n+4} \right) \bar{h}^{\frac{j+4n+5}{4n+4}} (j + 1) \Gamma \left( \frac{j+6n+7}{4n+4} \right),
\]

being \( \Gamma(\cdot) \) the Gamma function. If \( \bar{h} = h^{1/(4(n+1))} \) and

\[
B_{j,n} = \frac{2^{j-n}}{2^{j-n+2}} (1 + (-1)^j) b^{-\frac{j+1}{4n+4}} (n + 1)^{\frac{j+1}{4n+4}} \sqrt{\pi} \Gamma \left( \frac{j+4n+5}{4n+4} \right)
\]

\[
\frac{(j + 1) \Gamma \left( \frac{j+6n+7}{4n+4} \right)}{(j + 1) \Gamma \left( \frac{j+6n+7}{4n+4} \right)}
\]
Then
\[ J(h) = I(h) = 2h^{4n+5} \left( B_{2n+1,n}h^{2n+1} + a \sum_{i=0}^{m} q_i B_{i,n}h^i \right). \]

Note that \( J(h) \) has at most \( m \) simple positive zeros if \( m \geq 2n + 1 \), and by generalized Descartes theorem (see the Appendix) \( J(h) \) has at most \( m \) simple positive zeros if \( m < 2n + 1 \).

Since the coefficients \( q_i \) are arbitrary, we can choose a perturbation \( q(x) \) in such a way that \( J(h) \) has exactly \( m \) simple positive zeros, and consequently there are differential systems (3) with \( m \) limit cycles. This concludes the proof of the theorem.

3. Appendix

We recall the Descartes Theorem about the number of zeros of a real polynomial (for a proof see for instance [3]).

**Descartes Theorem** Consider the real polynomial \( p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r} \) with \( 0 \leq i_1 < i_2 < \cdots < i_r \) and \( a_{i_j} \neq 0 \) real constants for \( j \in \{1, 2, \cdots, r\} \). When \( a_{i_j}a_{i_{j+1}} < 0 \), we say that \( a_{i_j} \) and \( a_{i_{j+1}} \) have a variation of sign. If the number of variations of signs is \( r-1 \), then \( p(x) \) has at most \( m \) positive real roots. Moreover, it is always possible to choose the coefficients of \( p(x) \) in such a way that \( p(x) \) has exactly \( r-1 \) positive real roots.

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