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LIMIT CYCLES FOR A VARIANT OF A GENERALIZED RICCATI EQUATION

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Abstract. In this paper we provide a lower bound for the maximum number of limit cycles surrounding the origin of systems $(\dot{x}, \dot{y} = \ddot{x})$ given by a variant of the generalized Riccati equation

$$\ddot{x} + \varepsilon x^{2n+1}\dot{x} + bx^{4n+3} = 0,$$

where $b > 0$, $b \in \mathbb{R}$, n is a non–negative integer and ε is a small parameter. The tool for proving this result uses Abelian integrals.

1. Introduction and statement of the main results

Some variants of the generalized Riccati equation

$$\ddot{x} + \alpha x^{2n+1}\dot{x} + x^{4n+3} = 0, \tag{1}$$

have been studied for several authors, see for instance [8], [5], and the references quoted there. In the first paper the authors studied mainly the following variant of equation (1)

$$\ddot{x} + (2n+3)x^{2n+1}\dot{x} + x^{4n+3} + \omega^2 x = 0,$$

showing numerically that such differential equation exhibits isochronous oscillations. In the second paper the authors study the variant of equation (1)

$$\ddot{x} + (2n+3)x^{2n+1}\dot{x} + x^{4n+3} + \omega^2 x^{2n+1} = 0,$$

and they find the analytical expression of some particular solutions.

In the present paper we will study the following variant of the generalized Riccati equation (1)

$$\ddot{x} + \varepsilon x^{2n+1}\dot{x} + bx^{4n+3} + \varepsilon a(x + yq(x)) = 0, \tag{2}$$

where $a, b \in \mathbb{R}$ with $b \neq 0$, n is a non–negative integer, ε is a small parameter and $q(x)$ is a polynomial of degree m.

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Equation (1) can be written as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -bx^{4n+3} - \varepsilon a(x + yq(x)) - \varepsilon x^{2n+1} y, \end{aligned} \tag{3}$$

or equivalently in the form

$$\begin{aligned} \dot{x} &= -\frac{\partial H}{\partial y} + \varepsilon P(x,y), \\ \dot{y} &= \frac{\partial H}{\partial x} + \varepsilon Q(x,y), \end{aligned} \tag{4}$$

where

$$H(x,y) = \frac{y^2}{2} + \frac{bx^{4n+4}}{4n+4}, \tag{5}$$

and

$$P(x,y) = 0, \quad Q(x,y) = -a(x + yq(x)) - x^{2n+1} y.$$

Observe that for $b > 0$ there is a family of ovals $\gamma_h \subset H^{-1}(h)$ continuously depending on a parameter $h > 0$ and varying in the compact components of $H^{-1}(h)$. Moreover all the ovals γ_h fill up the plane $\mathbb{R}^2$ when h varies on all positive real numbers. These ovals are periodic orbits of the Hamiltonian system (4) with $\varepsilon = 0$.

The objective of this paper is to find the maximum number of values of h (that we denote by h^*) for which it bifurcate from γ_{h^*} a limit cycle of the differential system (4) for $|\varepsilon| > 0$ sufficiently small.

Theorem 1. *For $|\varepsilon| > 0$ sufficiently small there are systems* (3) *with $b > 0$ having m limit cycles $\Gamma_{h_m^*}$ that when $\varepsilon \to 0$ tend to periodic orbits $\gamma_{h_m^*}$ of the Hamiltonian system* (3) *with $b > 0$ and $\varepsilon = 0$. Moreover there are polynomials $q(x)$ for which the differential system* (3) *with $b > 0$ and $|\varepsilon| > 0$ sufficiently small has exactly m limit cycles.*

The proof of Theorem 1 is given in section 2.

Note that Theorem 1 is closely related to the weakened 16th Hilbert problem proposed by Arnold in [1, 2] which in its turn is closely related to determining an upper bound for the number of limit cycles of a perturbed Hamiltonian system of the form in (4). For other papers on limit cycles see for instance [6, 7] and the references quoted there.

2. Proof of Theorem 1

To prove Theorem 1 we will use use the following Theorem whose proof can be obtained, for instance, in [4].

Theorem 2. *Assume that there is a family of ovals $\gamma_h \subset H^{-1}(h)$, continuously depending on $h \in (a,b)$. Define the Abelian integral as*

$$I(h) = \int_{\gamma_h} P(x,y)\,dy - Q(x,y)\,dx. \tag{6}$$

If there exists an $h^* \in (a,b)$ such that $I(h^*) = 0$ and $I'(h^*) \neq 0$ then for ε sufficiently small, the Hamiltonian system (4) has at most one limit cycle Γ_{h^*} which tends to γ_{h^*} as $\varepsilon \to 0$.

We first write the polynomial $q(x) = \sum_{i=0}^{n} q_j x^j$. Note that the unperturbed system (2) (with $\varepsilon = 0$) is Hamiltonian with the Hamiltonian H given in (5). The periodic orbits of the unperturbed system (2) with $h > 0$ are the ovals γ_h. Now we will use Theorem 2 and so we shall compute the Abelian integral $I(h)$ given in (6). We have

$$\begin{aligned}
I(h) &= \int\int_{H(x,y)\le h} \frac{\partial}{\partial y}(x^{2n+1}y + a(x+yq(x)))\,dx\,dy \\
&= \int\int_{H(x,y)\le h} (x^{2n+1} + aq(x))\,dx\,dy \\
&= \int\int_{H(x,y)\le h} x^{2n+1}\,dx\,dy + a\sum_{i=0}^{m} q_i \int\int_{H(x,y)\le h} x^i\,dx\,dy \\
&= 2\int_{-\overline{x}}^{\overline{x}} x^{2n+1}\left(2h - \frac{b}{2(n+1)}x^{4(n+1)}\right)^{1/2} dx \\
&\quad + 2a\sum_{i=0}^{m} q_i \int_{-\overline{x}}^{\overline{x}} x^i\left(2h - \frac{b}{2(n+1)}x^{4(n+1)}\right)^{1/2} dx,
\end{aligned}$$

where

$$\overline{x} = \left(\frac{4h(n+1)}{b}\right)^{1/(4(n+1))}.$$

Note that for any integer j we have

$$\int_{-\overline{x}}^{\overline{x}} x^j\left(2h - \frac{b}{2(n+1)}x^{4(n+1)}\right)^{1/2} dx =$$
$$\frac{2^{\frac{j-n}{2n+2}}\left(1+(-1)^j\right) b^{-\frac{j+1}{4n+4}}(n+1)^{\frac{j+1}{4n+4}}\sqrt{\pi}\,\Gamma\left(\frac{j+4n+5}{4n+4}\right)}{(j+1)\Gamma\left(\frac{j+6n+7}{4n+4}\right)} h^{\frac{j+4n+5}{4n+4}},$$

being $\Gamma(\cdot)$ the Gamma function. If $\overline{h} = h^{1/(4(n+1))}$ and

$$B_{j,n} = \frac{2^{\frac{j-n}{2n+2}}\left(1+(-1)^j\right) b^{-\frac{j+1}{4n+4}}(n+1)^{\frac{j+1}{4n+4}}\sqrt{\pi}\,\Gamma\left(\frac{j+4n+5}{4n+4}\right)}{(j+1)\Gamma\left(\frac{j+6n+7}{4n+4}\right)}$$

Then

$$J(\overline{h}) = I(h) = 2\overline{h}^{4n+5}\left(B_{2n+1,n}\overline{h}^{2n+1} + a\sum_{i=0}^{m} q_i B_{i,n}\overline{h}^{i}\right).$$

Note that $J(\overline{h})$ has at most m simple positive zeros if $m \geq 2n+1$, and by generalized Descartes theorem (see the Appendix) $J(\overline{h})$ has at most m simple positive zeros if $m < 2n+1$.

Since the coefficients q_i are arbitrary, we can choose a perturbation $q(x)$ in such a way that $J(\overline{h})$ has exactly m simple positive zeros, and consequently there are differential systems (3) with m limit cycles. This concludes the proof of the theorem.

3. Appendix

We recall the Descartes Theorem about the number of zeros of a real polynomial (for a proof see for instance [3]).

Descartes Theorem *Consider the real polynomial* $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ *with* $0 \leq i_1 < i_2 < \cdots < i_r$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in \{1, 2, \cdots, r\}$*. When* $a_{i_j}a_{i_{j+1}} < 0$*, we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* $r-1$*, then* $p(x)$ *has at most* m *positive real roots. Moreover, it is always possible to choose the coefficients of* $p(x)$ *in such a way that* $p(x)$ *has exactly* $r-1$ *positive real roots.*

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