

THE PSEUDO-HOPF BIFURCATION FOR PLANAR DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS

JUAN CASTILLO¹, JAUME LLIBRE² AND FERNANDO VERDUZCO¹

ABSTRACT. The creation or destruction of a crossing limit cycle when a sliding segment changes its stability, is known as pseudo-Hopf bifurcation. In this paper, under generic conditions, we find an unfolding for such bifurcation, and we prove the existence and uniqueness of a crossing limit cycle for this family.

1. INTRODUCTION

The study of limit cycles is one of the most important problem in the qualitative theory of ordinary differential equations, however, the proof of their existence are generally very complicated. A large list of papers about the arising of limit cycles in piecewise smooth systems in the plane can be found in the literature of recent years, and in these some techniques has been developed to find them. In smooth systems there is a well known mechanism to search for the occurrence of limit cycles, the Hopf bifurcation theorem, see [13, 19]. There are analogous results for piecewise smooth systems, for the case of continuous systems see for example [6, 7, 26, 27], and for the case of discontinuous systems see [1, 8, 11, 12, 14, 18]. In the discontinuous ones we can have more than one limit cycle, either all crossing cycles or including one sliding cycle, and in fact, the determination of the number of limit cycle has been the subject of several recent papers, see [2, 3, 4, 10, 15, 16, 17, 20, 22, 23, 24].

When the appearance of more than one limit cycle is considered, often the mechanism to obtain one of them is by the collision of *two invisible tangencies*. This is, the creation or destruction of one crossing limit cycle occurs when a sliding segment changes its stability, this phenomenon is presented without demonstration in [18] and called *pseudo-Hopf bifurcation*. The appearance of a crossing limit cycle may occur in cases where there is not sliding segment, see [9, 21, 25].

In this paper we find an unfolding for the *pseudo-Hopf bifurcation* for planar discontinuous piecewise linear (DPWL) systems with two zones separated by a straight line. We prove the existence and uniqueness of a crossing limit cycle for all possible dynamic scenarios. It is important to mention that the unfolding found has seven parameters, but at moment that the dynamics on each zone be established, it reduces to five. However, in our result it will not be necessary to establish a priori the dynamics in each zone.

2010 *Mathematics Subject Classification.* Primary 34A30, 34C05, 34C25, 34C07, 37G15.

Key words and phrases. Pseudo-Hopf bifurcation, discontinuous piecewise linear differential systems, sliding segment, unfolding.

The rest of the paper is organized as follows. In section 2 we define the mathematical concepts used. In section 3 we state the main results. In section 4 we find the unfolding. The existence, uniqueness and stability of the crossing limit cycle is given in section 5. Finally, in section 6 we give the conclusions of this work.

2. PRELIMINARIES

Consider the planar DPWL system with two zones separated by the straight line $\Sigma = \{x \in \mathbb{R}^2 : \sigma(x) = c^T x - c_0 = 0\}$,

$$(1) \quad \dot{x} = f(x) = \begin{cases} f^-(x) = A_1 x + b_1 & \text{if } \sigma(x) < 0, \\ f^+(x) = A_2 x + b_2 & \text{if } \sigma(x) > 0, \end{cases}$$

where A_i are 2×2 matrices and $b_i \in \mathbb{R}^2$ for $i = 1, 2$.

We distinguish three open regions in the straight line Σ :

The *sliding region*: $\Sigma_s = \{x \in \Sigma : c^T f^-(x) > 0 \text{ and } c^T f^+(x) < 0\}$,
the *escaping region*: $\Sigma_e = \{x \in \Sigma : c^T f^-(x) < 0 \text{ and } c^T f^+(x) > 0\}$, and
the *crossing region*: $\Sigma_c = \{x \in \Sigma : (c^T f^-(x)) (c^T f^+(x)) > 0\}$.

Any segment contained in $\Sigma_s \cup \Sigma_e$ is called a *sliding segment*. The solutions on $\Sigma_s \cup \Sigma_e$ can be constructed by the Filippov's convex method, see [5]. Filippov's method takes a simple convex combination $f_s(x)$ of the two vector fields $f^\mp(x)$ to each sliding point $x \in \Sigma_s \cup \Sigma_e$, i.e.

$$f_s(x) = \frac{\tilde{f}_s(x)}{\Delta(x)},$$

where $\tilde{f}_s(x) = (c^T f^-(x)) f^+(x) - (c^T f^+(x)) f^-(x)$ and $\Delta(x) = c^T (f^-(x) - f^+(x)) \neq 0$. f_s is called the *sliding vector field*, while \tilde{f}_s is called the *regularized sliding vector field*.

A point $x \in \mathbb{R}^2$ is an *equilibrium point* of f^- if $f^-(x) = 0$ and $\sigma(x) < 0$. The equilibrium point is *virtual* for f^- if $f^-(x) = 0$ and $\sigma(x) > 0$.

A point $\tilde{x} \in \Sigma$ is a *pseudo-equilibrium* of (1) if $f_s(\tilde{x}) = 0$. The pseudo-equilibrium is *admissible* if $\tilde{x} \in \Sigma_s \cup \Sigma_e$, or *virtual* if $\tilde{x} \in \Sigma_c$.

A point \hat{x} is a *boundary equilibrium* of (1) if

$$f^-(\hat{x}) f^+(\hat{x}) = 0, \text{ and } f_s(\hat{x}) = 0.$$

Since the three kinds of regions in Σ are relatively open, their boundaries are the called *tangency points*: $q \in \Sigma$ such that $c^T f^-(q) = 0$ or $c^T f^+(q) = 0$ (see [12, 18]). That is, points where one of the two vector fields is tangent to Σ . In particular, the boundary equilibria are tangency points, since they are located on the boundary of the sliding region where one of the vector fields vanishes. The simplest tangency is the fold singularity, which is defined as follows.

A point $q \in \Sigma$ is a *fold singularity* of (1) if either

- (i) $c^T f^-(q) = 0$ and $c^T A_1 f^-(q) \neq 0$, or
- (ii) $c^T f^+(q) = 0$ and $c^T A_2 f^+(q) \neq 0$, or
- (iii) q is a hyperbolic focus of f^- or f^+ .

A fold singularity is a point with quadratic tangency with Σ , or is a boundary focus. A quadratic tangency point can be classified in visible or invisible as follows:

- (i) $q \in \Sigma$ is an *invisible* (*visible*) quadratic tangency point for f^- if

$$c^T f^-(q) = 0 \text{ and } r_1 = c^T A_1 f^-(q) > 0 (< 0).$$

- (ii) $q \in \Sigma$ is an *invisible* (*visible*) quadratic tangency point for f^+ if

$$c^T f^+(q) = 0 \text{ and } r_2 = c^T A_2 f^+(q) < 0 (> 0).$$

The case where system (1) has a quadratic tangency point for one vector field, and a boundary focus for the other one, at the same point on the switching line is called *fold-focus singularity*. When system (1) has a double boundary focus at the same point on the switching line, that is, when there is a boundary focus for both sides, this singularity is called the *focus-focus singularity*. Finally, a *fold-fold singularity* is when the DPWL system (1) has a double quadratic tangency at the same point on Σ .

For the case of the *invisible fold-fold singularity*, when the vectors $f^-(q_0)$ and $f^+(q_0)$ are antiparallel with $q_0 \in \partial\Sigma_c$, the singularity is called *fused-focus* in [18]. We are going to call *two-fold singularity* to the fold-fold, fold-focus or focus-focus singularities.

3. STATEMENTS OF THE MAIN RESULTS

The idea is to unfold the two-fold singularity q_0 in such way that two fold points, q_1 and q_2 , from f^- and f^+ , respectively, delimit a sliding segment, and when they change their relative position on Σ , after collapse in q_0 , the sliding segment change its stability. As will be proved in this article, for some configurations of the fold points, this change of stability in the sliding segment is accompanied by the birth or destruction of a crossing limit cycle. See Figure 1. With this idea, we assume that $f(x)$ satisfy the following generic hypothesis:

(H_0) The pairs of vectors $\{c, A_1^T c\}$ and $\{c, A_2^T c\}$ are linearly independent.

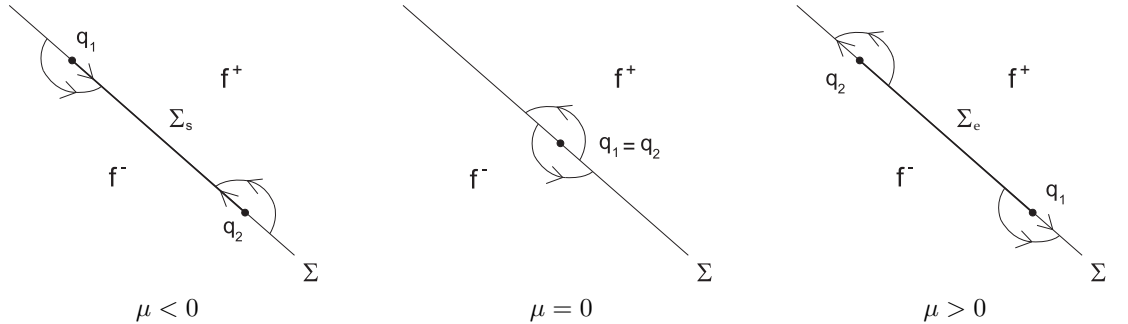


FIGURE 1. Change of stability of the sliding segment for the case of two invisible fold points.

Under Hypothesis (H_0) , the DPWL system (1) has two fold points q_1 and q_2 . This is clear because of, if we define the straight lines

$$L_i : c^T(A_i x + b_i) = 0, \text{ then } L_i \cap \Sigma = \{q_i\},$$

for $i = 1, 2$. Besides there exist γ_1 and γ_2 with $\gamma_2 \neq 0$ such that

$$A_2^T c = \gamma_1 c + \gamma_2 A_1^T c.$$

The first theorem in this section give us an unfolding for piecewise linear systems that satisfy the generic hypothesis (H_0) .

Theorem 1. *Under hypothesis (H_0) the change of coordinates*

$$(2) \quad y = h(x) = \begin{cases} \gamma_2 Q_1(x - q_1) & \text{if } \sigma(x) \leq 0, \\ Q_2(x - q_1) & \text{if } \sigma(x) \geq 0, \end{cases}$$

where $Q_1 = \begin{pmatrix} c^T \\ c^T A_1 \end{pmatrix}$ and $Q_2 = \begin{pmatrix} c^T \\ c^T A_2 \end{pmatrix}$, transforms the differential system (1) into the differential system

$$(3) \quad \dot{y} = F(y) = \begin{cases} F^-(y) = \begin{pmatrix} 0 & 1 \\ c_1 & c_2 \end{pmatrix} y + \begin{pmatrix} 0 \\ \gamma_2 r_1 \end{pmatrix} & \text{if } y_1 < 0, \\ F^+(y) = \begin{pmatrix} 0 & 1 \\ d_1 & d_2 \end{pmatrix} y + \begin{pmatrix} b \\ r_2 + d_2 b \end{pmatrix} & \text{if } y_1 > 0, \end{cases}$$

where

$$(4) \quad \begin{aligned} c_1 &= -\det(A_1), & c_2 &= \text{trace}(A_1), & d_1 &= -\det(A_2), \\ d_2 &= \text{trace}(A_2), & r_1 &= c^T A_1(A_1 q_1 + b_1), & r_2 &= c^T A_2(A_2 q_2 + b_2), \\ b &= c^T(A_2 q_1 + b_2). \end{aligned}$$

Theorem 1 is proved in section 4.

Remark 1. (a) If $q_2 \rightarrow q_1$ then $b \rightarrow 0$, i.e. at $b = 0$, the fold points collapse at q_0 .

(b) If $r_j = 0$ then the fold point q_j is a boundary equilibrium point, which must be a boundary focus, with eigenvalues $\alpha_j \pm i\beta_j$ for $j = 1, 2$.

(c) If $r_1 > 0$ then q_1 is an invisible fold point.

(d) If $r_2 < 0$ then q_2 is an invisible fold point.

The following corollary establishes the Σ -equivalence of the change of coordinates (2), see [12].

Corollary 1. *If $\gamma_2 > 0$ then $h(\Sigma_a) = \bar{\Sigma}_a$ for $a \in \{s, e, c\}$.*

Proof. For $x \in \Sigma$ we have $h(x) = \begin{pmatrix} 0 \\ \gamma_2 c^T A_1(x - q_1) \end{pmatrix} = \begin{pmatrix} 0 \\ c^T A_2(x - q_1) \end{pmatrix}$. Then

$$\begin{aligned} e_1^T F^-(h(x)) &= (1, 0) \begin{pmatrix} \gamma_2 c^T A_1(x - q_1) \\ (\cdot) \end{pmatrix} \\ &= \gamma_2 c^T A_1(x - q_1) + \gamma_2 c^T b_1 - \gamma_2 c^T b_1 \\ &= \gamma_2 c^T f^-(x), \end{aligned}$$

and

$$\begin{aligned}
 e_1^T F^+(h(x)) &= (1, 0) \begin{pmatrix} c^T A_2(x - q_1) + b \\ (\cdot) \end{pmatrix} \\
 &= c^T A_2(x - q_1) + c^T(A_2 q_1 + b_2) \\
 &= c^T f^+(x).
 \end{aligned}$$

□

Remark 2. The change of coordinates (2) classifies all the DPWL systems (1) that satisfy (H_0) into two classes: those systems that have a sliding segment ($\gamma_2 > 0$) and those that have a crossing segment ($\gamma_2 < 0$).

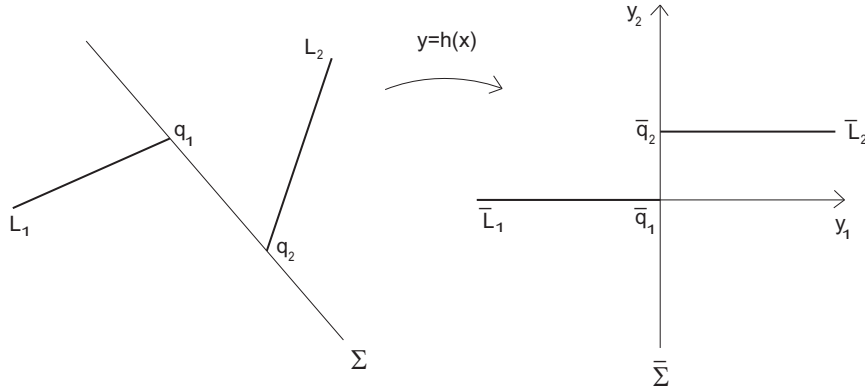


FIGURE 2. Change of coordinates (2).

From now on we will assume that $\gamma_2 > 0$. Figure 2 shows the effect of the orthogonalization of the change of coordinates (2). From the unfolding (3), for $b = 0$, we find nine different scenarios in which the two-fold singularity can be unfolded in a such way that is it possible to observe a change of stability in a sliding segment. See Figure 3.

The main theorem of the paper establishes that the unfolding (3) undergoes the pseudo-Hopf bifurcation only at four cases ($r_1 \geq 0$ and $r_2 \leq 0$).

Theorem 2 (Pseudo-Hopf bifurcation theorem). *Suppose that the DPWL system (1) satisfy (H_0) with $\gamma_2 > 0$. If $r_1 \geq 0$ and $r_2 \leq 0$, then for each b sufficiently small with $b\Lambda_0 < 0$, system (1) has a unique crossing limit cycle. If $\Lambda_0 < 0$ the limit cycle is stable, while if $\Lambda_0 > 0$ it is unstable, where*

$$\Lambda_0 = \begin{cases} \frac{c_2}{\gamma_2 r_1} - \frac{d_2}{r_2} & \text{if } r_1 > 0, r_2 < 0 \text{ (fused-focus),} \\ \alpha_2 & \text{if } r_1 > 0, r_2 = 0 \text{ (fold-focus),} \\ \alpha_1 & \text{if } r_1 = 0, r_2 < 0 \text{ (fold-focus),} \\ \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} & \text{if } r_1 = 0, r_2 = 0 \text{ (focus-focus).} \end{cases}$$

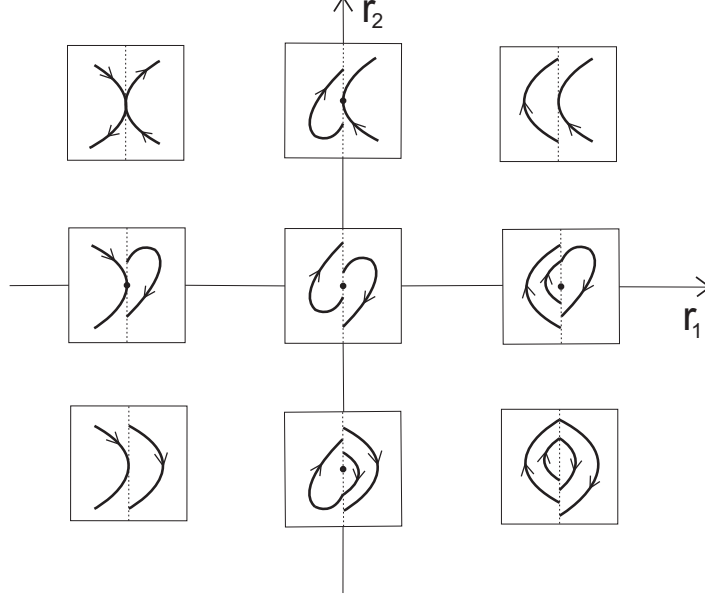


FIGURE 3. The two-fold singularity for the unfolding (3).

Theorem 2 is proved in section 5.

Remark 3. *It is not necessary to calculate the change of coordinates (2), nor the unfolding (3) to use Theorem 2, it is enough to calculate the expressions given in (4) from the original DPWL system (1).*

4. PROOF OF THEOREM 1

For $x \in \Sigma$ we have

$$h(x) = \begin{cases} \gamma_2 Q_1(x - q_1) = \begin{pmatrix} \gamma_2 c^T(x - q_1) \\ \gamma_2 c^T A_1(x - q_1) \end{pmatrix} = \begin{pmatrix} 0 \\ c^T A_2(x - q_1) \end{pmatrix} & \text{if } \sigma(x) \leq 0, \\ Q_2(x - q_1) = \begin{pmatrix} c^T(x - q_1) \\ c^T A_2(x - q_1) \end{pmatrix} = \begin{pmatrix} 0 \\ c^T A_2(x - q_1) \end{pmatrix} & \text{if } \sigma(x) \geq 0. \end{cases}$$

That is, h sends Σ on $y_1 = 0$. For $\sigma(x) \leq 0$ we have $\dot{y} = F^-(y) = \bar{A}_1 y + \bar{b}_1$, where

$$\bar{A}_1 = Q_1 A_1 Q_1^{-1} = \begin{pmatrix} c^T A_1 Q_1^{-1} \\ c^T A_1^2 Q_1^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c_1 & c_2 \end{pmatrix},$$

because of

$$Q_i Q_i^{-1} = \begin{pmatrix} c^T Q_i^{-1} \\ c^T A_i Q_i^{-1} \end{pmatrix} = I, \text{ for } i = 1, 2,$$

and

$$\bar{b}_1 = \gamma_2 Q_1(A_1 q_1 + b_1) = \begin{pmatrix} 0 \\ \gamma_2 r_1 \end{pmatrix}.$$

For $\sigma(x) \geq 0$ we have $\dot{y} = F^+(y) = \bar{A}_2 y + \bar{b}_2$, where

$$\bar{A}_2 = Q_2 A_2 Q_2^{-1} = \begin{pmatrix} c^T A_2 Q_2^{-1} \\ c^T A_2^2 Q_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ d_1 & d_2 \end{pmatrix},$$

and

$$\bar{b}_2 = Q_2(A_2 q_1 + b_2) = \begin{pmatrix} c^T(A_2 q_1 + b_2) \\ c^T A_2(A_2 q_1 + b_2) \end{pmatrix} = \begin{pmatrix} b \\ \tilde{b} \end{pmatrix}.$$

If $Q_2^{-1} = (v_1 \ v_2)$ then $Q_2 Q_2^{-1} = \begin{pmatrix} c^T \\ c^T A_2 \end{pmatrix} (v_1 \ v_2) = I$. Observe that it is possible to normalize c such that $\|v_2\| = 1$. We define $v = q_2 - q_1$, then

$$\tilde{b} = c^T A_2(A_2 q_1 + b_2) = c^T A_2(A_2(q_2 - v) + b_2) = c^T A_2(A_2 q_2 + b_2) - c^T A_2^2 v = r_2 - c^T A_2^2 v.$$

Besides there exist $s_1, s_2 \in \mathbb{R}$ such that $v = s_1 v_1 + s_2 v_2$, but $0 = c^T v = s_1 c^T v_1 + s_2 c^T v_2 = s_1$, then

$$(5) \quad v = s_2 v_2.$$

Therefore $c^T A_2^2 v = s_2 c^T A_2^2 v_2 = s_2 d_2$, and $c^T A_2 v = s_2 c^T A_2 v_2 = s_2$, that is

$$(6) \quad s_2 = c^T A_2 v = c^T A_2(q_2 - q_1) + c^T b_2 - c^T b_2 = -c^T(A_2 q_1 + b_2) = -b.$$

This completes the proof of theorem 1.

Remark 4. From (5) and (6) it follows that $\|q_2 - q_1\| = |b|$.

5. PROOF OF THEOREM 2

Consider the unfolding (3), i.e.

$$\dot{y} = \begin{cases} \begin{pmatrix} 0 & 1 \\ c_1 & c_2 \end{pmatrix} y + \begin{pmatrix} 0 \\ \gamma_2 r_1 \end{pmatrix} & \text{if } y_1 < 0, \\ \begin{pmatrix} 0 & 1 \\ d_1 & d_2 \end{pmatrix} y + \begin{pmatrix} b \\ r_2 + d_2 b \end{pmatrix} & \text{if } y_1 > 0. \end{cases}$$

We call ϕ_t and ψ_t the flow for $y_1 < 0$ and $y_1 > 0$, respectively. To prove the existence of a crossing limit cycle we are going to find $\hat{q}_1 = \begin{pmatrix} 0 \\ u \end{pmatrix}$ and $\hat{q}_2 = \begin{pmatrix} 0 \\ v \end{pmatrix}$, with $u > 0$ and $v < 0$, and times t_1, t_2 , such that the system

$$(7) \quad S_1 = \phi_{t_1}(\hat{q}_2) - \hat{q}_1 = 0,$$

$$(8) \quad S_2 = \psi_{t_2}(\hat{q}_1) - \hat{q}_2 = 0,$$

has a unique solution. See Figure 4.

We rename $A_1 = \begin{pmatrix} 0 & 1 \\ c_1 & c_2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 1 \\ d_1 & d_2 \end{pmatrix}$, and consider $\sigma(A_1) = \{\lambda_1, \lambda_2\}$ and $\sigma(A_2) = \{\delta_1, \delta_2\}$.

5.1. Fused-focus singularity: $r_1 > 0$ and $r_2 < 0$.

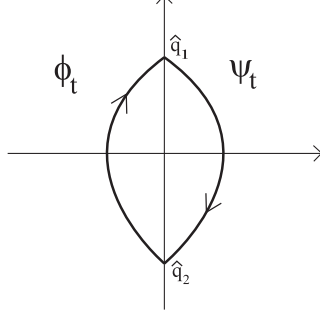


FIGURE 4. A crossing limit cycle.

5.1.1. *Case $\lambda_1 \lambda_2 \neq 0$, $\lambda_1 \neq \lambda_2$, $\delta_1 \delta_2 \neq 0$, $\delta_1 \neq \delta_2$.* In this case

$$S_1 = 0 \Leftrightarrow \begin{cases} e^{\lambda_1 t_1} = \frac{\gamma_2 r_1 + \lambda_1 u}{\gamma_2 r_1 + \lambda_1 v} = \frac{1 + \tilde{\lambda}_1 u}{1 + \tilde{\lambda}_1 v}, \\ e^{\lambda_2 t_1} = \frac{\gamma_2 r_1 + \lambda_2 u}{\gamma_2 r_1 + \lambda_2 v} = \frac{1 + \tilde{\lambda}_2 u}{1 + \tilde{\lambda}_2 v}, \end{cases}$$

where $\tilde{\lambda}_i = \frac{\lambda_i}{\gamma_2 r_1}$. Observe that for $u > 0$, $v < 0$, and $\tilde{\lambda}_i \in \mathbb{R}$ we have

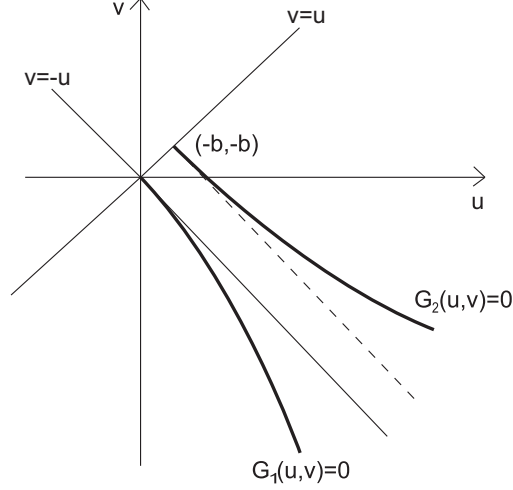
$$\frac{1 + \tilde{\lambda}_i u}{1 + \tilde{\lambda}_i v} > 0 \Leftrightarrow 1 + \tilde{\lambda}_i u > 0 \text{ and } 1 + \tilde{\lambda}_i v, \text{ for } i = 1, 2.$$

Therefore $e^{\lambda_i t_1} = \frac{1 + \tilde{\lambda}_i u}{1 + \tilde{\lambda}_i v} \Leftrightarrow e^{t_1} = \left(\frac{1 + \tilde{\lambda}_i u}{1 + \tilde{\lambda}_i v} \right)^{\frac{1}{\tilde{\lambda}_i}} = \frac{(1 + \tilde{\lambda}_i u)^{\frac{1}{\tilde{\lambda}_i}}}{(1 + \tilde{\lambda}_i v)^{\frac{1}{\tilde{\lambda}_i}}}$, then

$$S_1 = 0 \Leftrightarrow G_1(u, v) = \frac{(1 + \tilde{\lambda}_1 u)^{\frac{1}{\tilde{\lambda}_1}}}{(1 + \tilde{\lambda}_2 u)^{\frac{1}{\tilde{\lambda}_2}}} - \frac{(1 + \tilde{\lambda}_1 v)^{\frac{1}{\tilde{\lambda}_1}}}{(1 + \tilde{\lambda}_2 v)^{\frac{1}{\tilde{\lambda}_2}}} = 0.$$

Similarly

$$\begin{aligned} S_2 = 0 &\Leftrightarrow \begin{cases} e^{\delta_1 t_2} = \frac{r_2 + \delta_1(v+b)}{r_2 + \delta_1(u+b)} = \frac{1 + \tilde{\delta}_1(-v-b)}{1 + \tilde{\delta}_1(-u-b)}, \\ e^{\delta_2 t_2} = \frac{r_2 + \delta_2(v+b)}{r_2 + \delta_2(u+b)} = \frac{1 + \tilde{\delta}_2(-v-b)}{1 + \tilde{\delta}_2(-u-b)}, \end{cases} \\ &\Leftrightarrow G_2(u, v) = \frac{(1 + \tilde{\delta}_1(-v-b))^{\frac{1}{\tilde{\delta}_1}}}{(1 + \tilde{\delta}_2(-v-b))^{\frac{1}{\tilde{\delta}_2}}} - \frac{(1 + \tilde{\delta}_1(-u-b))^{\frac{1}{\tilde{\delta}_1}}}{(1 + \tilde{\delta}_2(-u-b))^{\frac{1}{\tilde{\delta}_2}}} = 0, \end{aligned}$$

FIGURE 5. Curve solutions for system $S_1 = S_2 = 0$.

where $\tilde{\delta}_i = -\frac{\delta_i}{r_2}$. Observe that $G_1(0, 0) = 0$ and, for $b = 0$, $G_2(0, 0) = 0$. Moreover if $\lambda_i = \delta_i$ and $\frac{1}{\gamma_2 r_1} = -\frac{1}{r_2}$, then $\tilde{\lambda}_i = \tilde{\delta}_i$ for $i = 1, 2$, and

$$(9) \quad G_2(u, v) = G_1(-(v + b), -(u + b)).$$

That is, the curve $G_2(u, v) = 0$ can be obtained by the reflection of the curve $G_1(u, v) = 0$, with respect to the straight line $v = -u$ followed by the translation $(-b, -b)^T$. See Figure 5. Then it is sufficient to solve $G_1(u, v) = 0$.

Lemma 1. *There exists a smooth function $h : (-\varepsilon_1, 0] \rightarrow [0, \varepsilon_2)$ such that $G_1(h(v), v) = 0$ for each $v \in (-\varepsilon_1, 0]$. Furthermore $h(0) = 0$, $h'(0) = -1$ and $h''(0) = \frac{4c_2}{3\gamma_2 r_1}$.*

Proof. First we observe that

$$G_1(u, v) = 0 \Leftrightarrow H(u) = H(v),$$

where $H(z) = \frac{(1 + \tilde{\lambda}_1 z)^{\frac{1}{\tilde{\lambda}_1}}}{(1 + \tilde{\lambda}_2 z)^{\frac{1}{\tilde{\lambda}_2}}}$. We are going to distinguish two cases.

Real case: $\lambda_1, \lambda_2 \in \mathbb{R}$. In this case $H : \mathbb{R} \rightarrow \mathbb{R}$, and after some calculations $H(0) = 1$, $H'(0) = 0$, and $H''(0) = \frac{\lambda_2 - \lambda_1}{(\gamma_2 r_1)^2} > 0$, then H has a local minimum at $z = 0$. Then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that for each $v \in (-\varepsilon_1, 0)$ there exists a unique $u \in (0, \varepsilon_2)$ with $H(u) = H(v)$. That is, there exists a function $h : (-\varepsilon_1, 0] \rightarrow [0, \varepsilon_2)$ such that $h(0) = 0$ and

$$H(v) = H(h(v)) \Leftrightarrow G_1(h(v), v) = 0,$$

for each $v \in (-\varepsilon_1, 0]$. See Figure 6.

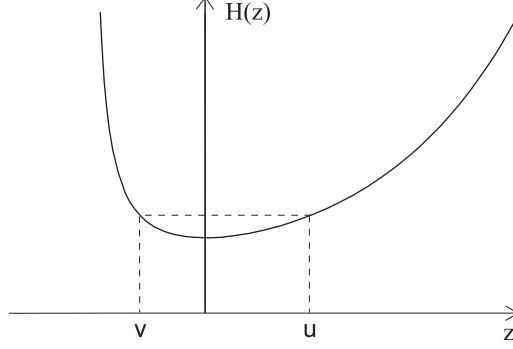


FIGURE 6. $u = h(v)$

From $H'(v) = H'(h(v))h'(v)$ we get

$$h'(v) = \frac{H'(v)}{H'(h(v))} < 0,$$

for each $v \in (-\varepsilon_1, 0]$, because of $H'(v) < 0$ and $H'(h(v)) > 0$. If we assume that $h'(0)$ exists, then

$$h'(0) = \lim_{v \rightarrow 0^-} h'(v) = \lim_{v \rightarrow 0^-} \frac{H'(v)}{H'(h(v))} = \lim_{v \rightarrow 0^-} \frac{H''(v)}{H''(h(v))h'(v)} = \frac{1}{h'(0)} \Leftrightarrow h'(0) = -1.$$

Therefore from $H''(v) = H''(h(v))(h'(v))^2 + H'(h(v))h''(v)$ we obtain

$$h''(v) = \frac{H''(v) - H''(h(v))(h'(v))^2}{H'(h(v))},$$

and again, if we assume that $h''(0)$ exists, then

$$\begin{aligned} h''(0) &= \lim_{v \rightarrow 0^-} h''(v) = \lim_{v \rightarrow 0^-} \frac{H''(v) - H''(h(v))(h'(v))^2}{H'(h(v))} \\ &= \lim_{v \rightarrow 0^-} \frac{H'''(v) - H'''(h(v))(h'(v))^3 - 2H''(h(v))h'(v)h''(v)}{H''(h(v))h'(v)} \\ &= -\frac{2H'''(0)}{H''(0)} - 2h''(0), \end{aligned}$$

but $H'''(0) = \frac{2(\lambda_1^2 - \lambda_2^2)}{(\gamma_2 r_1)^3}$, then $h''(0) = \frac{4}{3\gamma_2 r_1} (\lambda_1 + \lambda_2)$.

Complex case: $\lambda_1 = \lambda = \alpha + \beta i \in \mathbb{C}$ with $\beta > 0$. In this case $H : \mathbb{R} \rightarrow \mathbb{C}$. It is known that for $z_1, z_2 \in \mathbb{C}$ we have $\overline{z_1^{z_2}} = \bar{z}_1^{\bar{z}_2}$. Therefore if $w(z) = \left(1 + \tilde{\lambda}z\right)^{\frac{1}{\tilde{\lambda}}} = r(z)e^{i\theta(z)}$, then

$$H(z) = \frac{w(z)}{\bar{w}(z)} = e^{2i\theta(z)}.$$

That is

$$G_1(u, v) = 0 \Leftrightarrow H(u) = H(v) \Leftrightarrow \theta(u) = \theta(v).$$

We are going to find the real function $\theta(z)$. If $\tilde{\lambda} = \frac{1}{\gamma_2 r_1}(\lambda) = \tilde{\alpha} + \tilde{\beta}i$, then $1 + \tilde{\lambda}z = (1 + \tilde{\alpha}z) + i(\tilde{\beta}z) = r_0(z)e^{i\theta_0(z)}$. Therefore

$$\begin{aligned} (1 + \tilde{\lambda}z)^{\frac{1}{\tilde{\lambda}}} &= r(z)e^{i\theta(z)} \Leftrightarrow Ln(r_0 e^{i\theta_0}) = \lambda Ln(re^{i\theta}) \\ &\Leftrightarrow (\alpha Ln(r) - Ln(r_0) - \beta\theta) + i(\alpha\theta + \beta Ln(r) - \theta_0) = 0. \end{aligned}$$

Then

$$\theta(z) = \frac{1}{|\tilde{\lambda}|^2} \left(\alpha \arctan \left(\frac{\tilde{\beta}z}{1 + \tilde{\alpha}z} \right) - \frac{\beta}{2} Ln \left((1 + \tilde{\alpha}z)^2 + (\tilde{\beta}z)^2 \right) \right).$$

After some calculations, we find that $\theta(0) = \theta'(0) = 0$, and $\theta''(0) = -\frac{\beta}{\gamma_2 r_1} < 0$, then θ has a local maximum at $z = 0$. Following the same arguments than in the real case we prove that there exists a real function $u = h(v)$ such that $G_1(h(v), v) = 0$, for each $v \in (-\varepsilon_1, 0]$, where $h(0) = 0$, $h'(0) = -1$, and $h''(0) = \frac{4}{3\gamma_2 r_1}(\lambda + \bar{\lambda})$. This completes the proof of the lemma. \square

Existence and stability. From Lemma 1 and (9) it follows that the solutions of system (7-8), $h_1(v)$ and $h_2(u)$ are given by

$$\begin{aligned} u &= h_1(v) = -v + \lambda_0 v^2 - \lambda_0^2 v^3 + \dots, \\ v &= h_2(u) = -2b - u - \delta_0(b + u)^2 - \delta_0^2(b + u)^3 + \dots, \end{aligned}$$

where $\lambda_0 = \frac{2c_2}{3\gamma_2 r_1}$ and $\delta_0 = -\frac{2d_2}{3r_2}$. For $\varepsilon > 0$ we define the Poincaré map $P : (-\varepsilon, 0) \rightarrow (-\varepsilon, 0)$ given by

$$\begin{aligned} P(v, b) &= h_2(h_1(v)) = v - (\lambda_0 + \delta_0)v^2 + \mathcal{O}(|v|^3) + g_0(b) + \sum_{k=1} g_k(b)v^k \\ &= v - \frac{2}{3}\Lambda_0 v^2 + g_0(b) + \sum_{k=1} g_k(b)v^k + \mathcal{O}(|v|^3), \end{aligned}$$

where $g_0(b) = -2b + \mathcal{O}(|b|^2)$ and $g_k(b) = \mathcal{O}(|b|)$. Observe that the function

$$G(v, b) = P(v, b) - v = -\frac{2}{3}\Lambda_0 v^2 + g_0(b) + \sum_{k=1} g_k(b)v^k + \mathcal{O}(|v|^3),$$

satisfy $G(0, 0) = 0$ and $\frac{\partial G}{\partial b}(0, 0) = -2$. Then from the Implicit Function Theorem there exists a function

$$b = g(v) = -\frac{1}{3}\Lambda_0 v^2 + \mathcal{O}(|v|^3),$$

such that $G(v, g(v)) = 0$ for each $v \in (-\varepsilon, 0)$. In other words, for each b sufficiently small with $b\Lambda_0 < 0$, there exists $v \in (-\varepsilon, 0)$ such that $P(v, g(v)) = v$. That is, the unfolding (3) has a crossing limit cycle. Finally, to determine the stability of the limit cycle observe that

$$\frac{\partial}{\partial v} P(v, b) = 1 - \frac{4}{3}\Lambda_0 v + \sum_{k=1} k g_k(b)v^{k-1} + \mathcal{O}(|v|^2),$$

and for each $b = g(v)$ with $v \in (-\varepsilon, 0)$

$$\frac{\partial}{\partial v} P(v, g(v)) = 1 - \frac{4}{3} \Lambda_0 v + \mathcal{O}(|v|^2) = \begin{cases} < 1 & \text{if } \Lambda_0 < 0 \\ > 1 & \text{if } \Lambda_0 > 0. \end{cases}$$

5.1.2. *Improper nodes.* Assume $\lambda_1 = \lambda_2$ and $\delta_1 = \delta_2$. In this case

$$S_1 = 0 \Leftrightarrow \begin{cases} e^{\lambda_1 t_1} &= \frac{u}{v + t_1(\gamma_2 r_1 + \lambda_1 v)}, \\ e^{\lambda_1 t_1} &= \frac{\gamma_2 r_1}{\gamma_2 r_1 - \lambda_1 t_1(\gamma_2 r_1 + \lambda_1 v)}, \end{cases}$$

then $t_1 = \frac{\gamma_2 r_1(u - v)}{(\gamma_2 r_1 + \lambda_1 u)(\gamma_2 r_1 + \lambda_1 v)}$. That is

$$S_1 = 0 \Leftrightarrow G_1(u, v) = (1 + \tilde{\lambda}_1 u) e^{\frac{1}{1 + \tilde{\lambda}_1 u}} - (1 + \tilde{\lambda}_1 v) e^{\frac{1}{1 + \tilde{\lambda}_1 v}} = 0,$$

where $\tilde{\lambda}_1 = \frac{\lambda_1}{\gamma_2 r_1}$. Similarly

$$S_2 = 0 \Leftrightarrow \begin{cases} e^{\delta_1 t_2} &= \frac{v + b}{u + b + t_2(r_2 + \delta_1(u + b))}, \\ e^{\delta_1 t_2} &= \frac{r_2}{r_2 - \delta_1 t_2(r_2 + \delta_1(u + b))}, \end{cases}$$

then $t_2 = \frac{-r_2(u - v)}{(r_2 + \delta_1(u + b))(r_2 + \delta_1(v + b))}$. That is

$$S_2 = 0 \Leftrightarrow G_2(u, v) = (1 + \tilde{\delta}_1(-v - b)) e^{\frac{1}{1 + \tilde{\delta}_1(-v - b)}} - (1 + \tilde{\delta}_1(-u - b)) e^{\frac{1}{1 + \tilde{\delta}_1(-u - b)}} = 0,$$

where $\tilde{\delta}_1 = -\frac{\delta_1}{r_2}$. As in the previous section

$$G_2(u, v) = G_1(-(v + b), -(u + b)),$$

and besides Lemma 1 is satisfied by $H(z) = (1 + \tilde{\lambda}_1 z) e^{\frac{1}{1 + \tilde{\lambda}_1 z}}$.

5.1.3. *Assume $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\delta_1 \neq 0$, $\delta_2 = 0$.* Then

$$S_1 = 0 \Leftrightarrow \begin{cases} e^{\lambda_1 t_1} &= \frac{1 + \tilde{\lambda}_1 u}{1 + \tilde{\lambda}_1 v}, \\ e^{\lambda_1 t_1} &= \frac{1 + \tilde{\lambda}_1 v + \lambda_1 t_1}{1 + \tilde{\lambda}_1 v}, \end{cases}$$

then $t_1 = \frac{u - v}{\gamma_2 r_1}$. That is

$$S_1 = 0 \Leftrightarrow G_1(u, v) = (1 + \tilde{\lambda}_1 u) e^{-\tilde{\lambda}_1 u} - (1 + \tilde{\lambda}_1 v) e^{-\tilde{\lambda}_1 v} = 0,$$

where $\tilde{\lambda}_1 = \frac{\lambda_1}{\gamma_2 r_1}$. Similarly to the previous cases $S_2 = 0 \Leftrightarrow G_2(u, v) = 0$, where

G_2 satisfy (9), and Lemma 1 is satisfied for $H(z) = (1 + \tilde{\lambda}_1 z) e^{-\tilde{\lambda}_1 z}$.

5.1.4. Assume $\lambda_1 = \lambda_2 = 0$, $\delta_1 = \delta_2 = 0$. Then $S_1 = 0 \Leftrightarrow u = h_1(v) = -v$, with $t_1 = -\frac{2v}{\gamma_2 r_1}$, and $S_2 = 0 \Leftrightarrow v = h_2(u) = -2b - u$, with $t_2 = -\frac{2(u+b)}{r_2}$.

5.2. **Focus-focus singularity:** $r_1 = 0$ and $r_2 = 0$. In this case $\lambda_{1,2} = \alpha_1 \pm i\beta_1$ and $\delta_{1,2} = \alpha_2 \pm i\beta_2$. Then

$$S_1 = 0 \Leftrightarrow \begin{cases} e^{\lambda_1 t_1} &= e^{\lambda_2 t_1}, \\ u &= e^{\lambda_1 t_1} v, \end{cases}$$

that is, $t_1 = \frac{\pi}{\beta_1}$ and $u = h_1(v) = -e^{\frac{\alpha_1 \pi}{\beta_1}} v$. Therefore

$$S_2 = 0 \Leftrightarrow \begin{cases} e^{\delta_1 t_2} &= e^{\delta_2 t_2}, \\ v &= e^{\delta_1 t_2} u + b(e^{\delta_1 t_2} - 1). \end{cases}$$

That is $t_2 = \frac{\pi}{\beta_2}$ and $v = h_2(u) = -e^{\frac{\alpha_2 \pi}{\beta_2}} u - b(e^{\frac{\alpha_2 \pi}{\beta_2}} + 1)$. For each $\varepsilon > 0$ we define the Poincaré map $P : (-\varepsilon, 0) \rightarrow (-\varepsilon, 0)$ given by

$$P(v) = h_2(h_1(v)) = e^{\Lambda_0 \pi} v - b(e^{\frac{\alpha_2 \pi}{\beta_2}} + 1),$$

which for each b such that $b\Lambda_0 < 0$ has the fixed point $\tilde{v} = \frac{b(e^{\frac{\alpha_2 \pi}{\beta_2}} + 1)}{e^{\Lambda_0 \pi} - 1} < 0$, which is stable if $\Lambda_0 < 0$ and unstable if $\Lambda_0 > 0$.

5.3. Invisible fold-focus singularity.

5.3.1. Assume $r_1 > 0$ and $r_2 = 0$. From the previous cases we know that, $S_1 = 0 \Leftrightarrow h_1(v) = -v + \lambda_0 v^2 - \lambda_0^2 v^3 + \dots$, and $S_2 = 0 \Leftrightarrow h_2(u) = -e^{\frac{\alpha_2 \pi}{\beta_2}} u - b(e^{\frac{\alpha_2 \pi}{\beta_2}} + 1)$. Then the Poincaré map is given by

$$P(v, b) = h_2(h_1(v)) = e^{\frac{\alpha_2 \pi}{\beta_2}} (v - \lambda_0 v^2 + \mathcal{O}(|v|^3)) - b(e^{\frac{\alpha_2 \pi}{\beta_2}} + 1).$$

Again we observe that the function

$$G(v, b) = P(v, b) - v = (e^{\frac{\alpha_2 \pi}{\beta_2}} - 1)v + \mathcal{O}(|v|^2) - b(e^{\frac{\alpha_2 \pi}{\beta_2}} + 1),$$

satisfy $G(0, 0) = 0$, and $\frac{\partial G}{\partial b}(0, 0) = -(e^{\frac{\alpha_2 \pi}{\beta_2}} + 1) \neq 0$, then from the Implicit Function Theorem there is a function

$$b = g(v) = c_1 v + \mathcal{O}(|v|^2) \text{ where } c_1 = \frac{e^{\frac{\alpha_2 \pi}{\beta_2}} - 1}{e^{\frac{\alpha_2 \pi}{\beta_2}} + 1} = \begin{cases} < 0 & \text{if } \alpha_2 < 0, \\ > 0 & \text{if } \alpha_2 > 0, \end{cases}$$

such that $G(v, g(v)) = 0$ for each $v < 0$ sufficiently small. In other words, for each b sufficiently small with $b\Lambda_0 < 0$, there exists $v < 0$ such that $P(v, g(v)) = v$. That is the unfolding (3) has a crossing limit cycle. Finally to determine the stability of the limit cycle we observe that

$$\frac{\partial}{\partial v} P(v, b) = e^{\frac{\alpha_2 \pi}{\beta_2}} (1 - 2\lambda_0 v + \mathcal{O}(|v|^2)) = \begin{cases} < 1 & \text{if } \Lambda_0 < 0, \\ > 1 & \text{if } \Lambda_0 > 0. \end{cases}$$

In this case the stability of the limit cycle only depends on the stability of the boundary focus.

5.3.2. Assume $r_1 = 0$ and $r_2 < 0$. Then $S_1 = 0 \Leftrightarrow h_1(v) = -e^{\frac{\alpha_1 \pi}{\beta_1}} v$, and $S_2 = 0 \Leftrightarrow h_2(u) = -2b - u - \delta_0(b+u)^2 - \delta_0^2(b+u)^3 + \dots$. Then, the Poincaré map is given by

$$P(v, b) = h_2(h_1(v)) = e^{\frac{\alpha_1 \pi}{\beta_1}} (1 + \mathcal{O}(|b|))v + e^{\frac{2\alpha_1 \pi}{\beta_1}} (-\delta_0 + \mathcal{O}(|b|))v^2 + \mathcal{O}(|v|^3) + g_0(b),$$

where $g_0(b) = -2b + \mathcal{O}(|b|^2)$. Again observe that the function

$$G(v, b) = P(v, b) - v = -v + e^{\frac{\alpha_1 \pi}{\beta_1}} (1 + \mathcal{O}(|b|))v + \mathcal{O}(|v|^2) + g_0(b),$$

satisfy $G(0, 0) = 0$, and $\frac{\partial G}{\partial b}(0, 0) = -2$, then from the Implicit Function Theorem there is a function

$$b = g(v) = c_1 v + \mathcal{O}(|v|^2), \text{ where } c_1 = \frac{1}{2}(e^{\frac{\alpha_1 \pi}{\beta_1}} - 1) = \begin{cases} < 0 & \text{if } \alpha_1 < 0, \\ > 0 & \text{if } \alpha_1 > 0, \end{cases}$$

such that $G(v, g(v)) = 0$ for each $v < 0$ sufficiently small. In other words, for each b sufficiently small with $b\Lambda_0 < 0$, there exists $v < 0$ such that $P(v, g(v)) = v$. That is, the unfolding (3) has a crossing limit cycle. Finally, to determine the stability of the limit cycle observe that

$$\begin{aligned} \frac{\partial}{\partial v} P(v, b) &= e^{\frac{\alpha_1 \pi}{\beta_1}} (1 + \mathcal{O}(|b|)) + \mathcal{O}(|v|) \\ &= e^{\frac{\alpha_1 \pi}{\beta_1}} (1 + \mathcal{O}(|v|)) + \mathcal{O}(|v|) = \begin{cases} < 1 & \text{if } \Lambda_0 < 0, \\ > 1 & \text{if } \Lambda_0 > 0. \end{cases} \end{aligned}$$

As in the previous case, the stability of the boundary focus determines the stability of the limit cycle. This completes the proof of Theorem 2.

6. FINAL REMARKS

We have established under which conditions a family of DPWL systems with a discontinuity line, which satisfy the generic condition of having a fold point in each zone, undergoes the pseudo-Hopf bifurcation. Although this phenomenon has been studied in several articles, it has always been in the context of searching for multiple crossing limit cycles, and as far as we know, a similar result had not been established previously.

ACKNOWLEDGEMENTS

The second author is partially supported by MINECO grants MTM2016-77278-P and MTM2013-40998-P, an AGAUR grant number 2014SGR-568, and the grant FP7-PEOPLE-2012-IRSES 318999.

REFERENCES

- [1] Akhmet M.U. *Perturbations and Hopf bifurcation of the planar discontinuous dynamical system*, Nonlinear Anal. **60** (2005), 163–178.
- [2] Artés J.C., Llibre J., Medrado J.C. & Teixeira M.A. *Piecewise linear differential systems with two real saddles*, Math. Comput. Simulation **95** (2013), 1–22.
- [3] Buzzi C., Pessoa M. & Torregrosa J. *Piecewise linear perturbations of a linear center*, Discrete Continuous Dyn. Syst. **33** (2013), 3915–3936.

- [4] Euzébio R.D. & Llibre J. *On the number of limit cycles in discontinuous piecewise linear differential systems with two pieces separated by a straight line*, J. Math. Anal. Appl. **424** (2015), 475–486
- [5] Filippov, A.F. *Differential Equations with Discontinuous Right-hand Sides*, Kluwer Academic Publishers, Dordrecht. 1988.
- [6] Freire E., Ponce E. & Torres F. *Hopf-like bifurcations in planar piecewise linear systems*, Publ. Mat. **41** (1997), 135–148.
- [7] Freire E., Ponce E., Rodrigo F. & Torres F. *Bifurcation sets of continuous piecewise linear systems with two zones*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **8** (1998), 2073–2097.
- [8] Freire E., Ponce E. & Torres F. *Canonical Discontinuous Planar Piecewise Linear Systems*, SIAM J. Appl. Dyn. Syst. **11** (2012), 181–211.
- [9] Freire E., Ponce E. & Torres F. *Planar Filippov systems with maximal crossing set and piecewise linear focus dynamics*, Progress and Challenges in Dynamical Systems, Springer, 2013, pp 221–232.
- [10] Freire E., Ponce E. & Torres F. *A general mechanism to generate three limit cycles in planar Filippov systems with two zones*, Nonlinear Dynam. **78** (2014), 251–263.
- [11] Freire E., Ponce E. & Torres F. *On the critical crossing cycle bifurcation in planar Filippov systems*, J. Differential Equations **259** (2015), 7086–7107.
- [12] Guardia M., Seara T.M. & Teixeira M.A. *Generic bifurcations of low codimension of planar Filippov Systems*, J. Differential Equations **250** (2011), 1967–2023.
- [13] Guckenheimer J. & Holmes P. *Nonlinear oscillations, Dynamical systems, and bifurcations of vector fields*, Applied Mathematical Sciences, 1993.
- [14] Han M. & Zhang W. *On Hopf bifurcation in non-smooth planar systems*, J. Differential Equations **248** (2010), 2399–2416.
- [15] Huan S.M. & Yang X.S. *The number of limit cycles in general planar piecewise linear systems*, Discrete Contin. Dyn. Syst. **32** (2012), 2147–2164.
- [16] Huan S.M. & Yang X.S. *Existence of limit cycles in general planar piecewise linear systems of saddle-saddle dynamics*, Nonlinear Anal. **92** (2013), 82–95.
- [17] Huan, S.M. & Yang X.S. *On the number of limit cycles in general planar piecewise linear systems of node-node types*, J. Math. Anal. Appl. **411** (2014), 340–353.
- [18] Kuznetsov Yu.A., Rinaldi S. & Gragnani A. *One parameter bifurcations in planar Filippov systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 2157–2188.
- [19] Kuznetsov Yu.A. *Elements of Applied Bifurcation Theory*, Springer-Verlag, 2004.
- [20] Liping Li. *Three crossing limit cycles in planar piecewise linear systems with saddle-focus type*, Electron. J. Qual. Theory Differ. Equ. **70** (2014), 1–14.
- [21] Llibre J., Ponce E. & Torres F. *On the existence and uniqueness of limit cycles in Liénard differential equations allowing discontinuities*, Nonlinearity **21** (2008), 2121–2142.
- [22] Llibre J. & Ponce E. *Three nested limit cycles in discontinuous piecewise linear differential systems*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **19** (2012), 325–335.
- [23] Llibre J., Teixeira M.A. & Torregrosa J. *Lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems with a straight line of separation*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **23** (2013), 1350066-1-10.
- [24] Llibre J., Novaes D.D., & Teixeira M.A. *Limit cycles bifurcating from the periodic orbits of a discontinuous piecewise linear differentiable center with two zones*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **17** (2015), 1550144-1-11.
- [25] Medrado J.C & Torregrosa J. *Uniqueness of limit cycles for sewing planar piecewise linear systems*, J. Math. Anal. Appl. **431** (2015), 529–544.
- [26] Simpson D.J.W. & Meiss J.D. *Andronov-Hopf bifurcations in planar, Piecewise-smooth, Continuous flows*, Phys. Lett. A **371** (2007), 213–220.
- [27] Simpson D.J.W. & Meiss J.D. *Unfolding a codimension two, Discontinuous, Andronov-Hopf bifurcation*, Chaos **18** (2008), 033125.

¹ DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE SONORA, HERMOSILLO, MEXICO

E-mail address: verduzco@mat.uson.mx

² DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, BELLATERRA,
08193 BARCELONA, CATALONIA, SPAIN.

E-mail address: `jllibre@mat.uab.cat`